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# On some structural sets and a quaternionic ( $\varphi, \psi$ )-hyperholomorphic function theory 

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Quaternionic analysis is regarded as a broadly accepted branch of classical analysis referring to many different types of extensions of the Cauchy-Riemann equations to the quaternion skew field $\mathbb{H}$. It relies heavily on results on functions defined on domains in $\mathbb{R}^{4}$ (or $\mathbb{R}^{3}$ ) with values in $\mathbb{H}$. This theory is centred around the concept of $\psi$-hyperholomorphic functions related to a so-called structural set $\psi$ of $\mathbb{H}^{4}$ (or $\mathbb{H}^{3}$ ) respectively. The main goal of this paper is to develop the nucleus of the $(\varphi, \psi)$-hyperholomorphic function theory, i.e., simultaneous null solutions of two Cauchy-Riemann operators associated to a pair $\varphi, \psi$ of structural sets of $\mathbb{H}^{4}$. Following a matrix approach, a generalized Borel-Pompeiu formula and the corresponding Plemelj-Sokhotzki formulae are established.

## 1 Introduction

Quaternionic analysis is a an extension of classic complex analysis to the four-dimensional skew-field of quaternions. In the classic approach the complex Cauchy-Riemann operator is replaced by the generalized CauchyRiemann operator. Based on this operator since the 1930'th this theory was widely developed and applied (see, e.g., [4], [5]). Later on a modification of the generalized Cauchy-Riemann operator by using a general orthonormal basis in $\mathbb{R}^{4}$ (called structural sets) instead of the standard basis was proposed independently by Naser, Nôno, Shapiro and Vasilevsky, see [13]-[15], [21], [22]. This theory called $\psi$-hyperholomorphic quaternion valued functions by itself is not much of a novelty since it can be reduced by an orthogonal transformation to the standard case.

In 1998 Gürlebeck, et al. [8] showed that the class of $\psi$-hyperholomorphic functions is more than what we get by orthogonal transformation, when the picture changed completely in the study of a $\Pi$-operator which involve a pair of different orthonormal basis. First of all there is no more a single orthogonal transformation which reduces it to the standard case. Second we have that for two different structural sets they factorize a second order operator which is not anymore a scalar operator. As it turns out several important questions are linked with and uniformized when two different structural sets take part. For instance, the question of so-called monogenic constants which is raised in the study of the conjugate Cauchy-Riemann operator as derivative for monogenic functions is a special case of the question of studying the intersection between the kernels of two generalized Cauchy-Riemann operators, denoted by ${ }^{\psi} D$ and ${ }^{\varphi} D$ or the question of two-monogenic functions as a special case of the study of the kernel of the second order operator ${ }^{\psi} D^{\psi} D$. The former is crucial for the study of quasiconformal monogenic mappings.

Furthermore, this problem is also closely connected to practical problems where the application of different frames are required by either the material or the vector fields themselves. Other links include the Cimmino system

[^0]and the study of the ${ }^{\psi, \varphi} \Pi$-operator as a generalization of the Ahlfors-Beurling transform from Complex Analysis with its importance in the study of quaternionic Beltrami equations and quaternionic quasiconformal mappings [3], [8]. In the later case we have to point out that a quaternionic Beltrami equation in the simplest case has the form ${ }^{\psi} D u=q^{\varphi} D u$ where $\varphi$ does not have to correspond to $\bar{\psi}$, see, e.g., [19]. Since the intersection of the kernels of ${ }^{\psi} D$ and ${ }^{\varphi} D$ is non-trivial it is necessary to study functions in this intersection, which are sometimes called monogenic constants, particularly in the case of $\varphi=\bar{\psi}$.

Here, we want closely investigate the question of the kernel $\operatorname{ker}^{\psi} \mathrm{D} \cap \operatorname{ker}^{\varphi} \mathrm{D}$. To this end we embed our operators into a higher-dimensional space by means of a matrix embedding and construct the basic tools of a function theory in this higher-dimensional space. Unfortunately, a direct embedding for all possible transformations which map $\varphi$ into $\psi$ is not possible since they can be of a quite different nature. Essentially, we have either a rotation or a reflection each of which requires a different type of embedding. In the first case we will use the embedding into the ring of quaternionic $2 \times 2$ circular matrices while in the latter case we will use the algebra generated by the identity matrix and the complex structure $J=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ as well as the algebra generated by the parabolic element $\tilde{J}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$.

In each case we study under which conditions the corresponding Cauchy-Riemann operator factorizes the matrix Laplacian and construct the corresponding fundamental solution. Afterwards we establish the necessary integral formulae for a function theory.

## 2 Elements of quaternionic analysis

In this section we present brievely the basic definitions and results of quaternionic analysis which are necessary for our purpose. We mention in particular, without claim of completeness the works [4]-[8], [10]-[12], [14]-[16], [18]-[21] and the literature therein.

Let $\mathbb{H}$ be the set of real quaternions generated by a real unit (denoted by 1 ) and the standard quaternionic units $\{i, j, k\}$. This means that any element $x$ from $\mathbb{H}$ is of the form $x=x_{0}+x_{1} i+x_{2} j+x_{3} k$, where $x_{m} \in \mathbb{R}, m \in$ $\mathbb{N}_{3} \cup\{0\} ; \mathbb{N}_{3}:=\{1,2,3\}$. Sometimes we will write $x_{0}$ as $\operatorname{Re}\{x\}$ and $x-x_{0}$ as $V e c\{x\}$.

In this paper we denote the generators by $1=: e_{0}, i=: e_{1}, j=: e_{2}$, and $k=: e_{3}$ subject to the multiplication rules

$$
\begin{aligned}
e_{m}^{2} & =-1, \quad m \in \mathbb{N}_{3} \\
e_{1} e_{2} & =-e_{2} e_{1}=e_{3} ; e_{2} e_{3}=-e_{3} e_{2}=e_{1} ; e_{3} e_{1}=-e_{3} e_{1}=e_{2}
\end{aligned}
$$

more suited for a future extension.
Embeddings of reals and complex numbers into quaternions are realized by:

$$
\begin{aligned}
& x_{0} \in \mathbb{R} \longrightarrow x_{0} 1+0 e_{1}+0 e_{2}+0 e_{3} \in \mathbb{H}, \\
& x_{0}+x_{1} e_{1} \in \mathbb{C} \longrightarrow x_{0} 1+x_{1} e_{1}+0 e_{2}+0 e_{3} \in \mathbb{H} .
\end{aligned}
$$

If $x=\sum_{m=0}^{3} x_{m} e_{m}$ is a quaternion then

$$
\vec{x}:=\left(\begin{array}{l}
x_{0} \\
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=\left(x_{0}, x_{1}, x_{2}, x_{3}\right)^{T} \in \mathbb{R}^{4}
$$

With natural operations of addition and multiplication $\mathbb{H}$ is a non-commutative, associative skew-field. There is the quaternionic conjugation, which plays an important role and is defined as follows:

$$
\bar{e}_{m}:=-e_{m}, \quad m \in \mathbb{N}_{3}
$$

This involution extends onto the whole $\mathbb{H}$ as an $\mathbb{R}$-linear mapping: If $x \in \mathbb{H}$ then

$$
\bar{x}:=x_{0}-x_{1} e_{1}-x_{2} e_{2}-x_{3} e_{3}
$$

For $\{x, y\} \in \mathbb{H}$ we note the formula for the multiplication:

$$
(x \cdot y)_{0}=x_{0} y_{0}-x_{1} y_{1}-x_{2} y_{2}-x_{3} y_{3},
$$

$$
\begin{aligned}
& (x \cdot y)_{1}=x_{0} y_{1}+x_{1} y_{0}+x_{2} y_{3}-x_{3} y_{2}, \\
& (x \cdot y)_{2}=x_{0} y_{2}+x_{2} y_{0}+x_{3} y_{1}-x_{1} y_{3} \\
& (x \cdot y)_{3}=x_{0} y_{3}+x_{3} y_{0}+x_{1} y_{2}-x_{2} y_{1} .
\end{aligned}
$$

In a more compact form we have
$x \cdot y=\operatorname{Re}\{x\} \operatorname{Re}\{y\}-\langle V e c\{x\}, V e c\{y\}\rangle_{\mathbb{R}^{3}}+\operatorname{Re}\{x\} V e c\{y\}+\operatorname{Re}\{y\} V e c\{x\}+V e c\{x\} \times V e c\{y\}$
where $\langle\cdot, \cdot\rangle_{\mathbb{R}^{3}}$ and $\times$ denote the standard scalar and cross product in $\mathbb{R}^{3}$, respectively.
The quaternion $x \cdot y$, coincides with the result of multiplying $\vec{y}$ by the left regular matrix representation of $x$ given by:

$$
\left(\begin{array}{cccc}
x_{0} & -x_{1} & -x_{2} & -x_{3} \\
x_{1} & x_{0} & -x_{3} & x_{2} \\
x_{2} & x_{3} & x_{0} & -x_{1} \\
x_{3} & -x_{2} & x_{1} & x_{0}
\end{array}\right) \cdot\left(\begin{array}{l}
y_{0} \\
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right)=: B_{l}(x) \cdot \vec{y} .
$$

Quite analogously, considering $y \cdot x$ we get

$$
\left(\begin{array}{cccc}
x_{0} & -x_{1} & -x_{2} & -x_{3} \\
x_{1} & x_{0} & x_{3} & -x_{2} \\
x_{2} & -x_{3} & x_{0} & x_{1} \\
x_{3} & x_{2} & -x_{1} & x_{0}
\end{array}\right) \cdot\left(\begin{array}{l}
y_{0} \\
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right)=: B_{r}(x) \cdot \vec{y} .
$$

It is easy to see (and a well-known fact) that the action of both left and right regular matrix representations of $x$ correspond to a rotation in four-dimensional Euclidean space. In fact, the first, generated by $B_{l}$, corresponds to a left-isoclinic rotation while the second, generated by $B_{r}$, represents a right-isoclinic rotation. We would like to remark that a general rotation in four dimensions can always be split into a left-isoclinic rotation $B_{l}$ and a right-isoclinic rotation $B_{r}$, i.e. Van Elfrinkhof's or Cayley's formula.

We should also mention the very important property of the quaternionic conjugation $\bar{x} \cdot \bar{y}=\bar{y} \cdot \bar{x}$ and

$$
x \cdot \bar{x}=\bar{x} \cdot x=|x|^{2} \in \mathbb{R}
$$

This norm of a quaternion coincides with the usual Euclidean norm in $\mathbb{R}^{4}$. It follows, that any $x \in \mathbb{H} \backslash\{0\}$ has a multiplicative inverse

$$
x^{-1}:=\frac{\bar{x}}{|x|^{2}}
$$

Let us also remark that for $x, y \in \mathbb{H}$ we still have $|x y|=|x||y|$. This is a rather unique property of the quaternions which is not shared by higher-dimensional Clifford algebras.

We consider functions $f$ defined in a domain $\Omega \subset \mathbb{R}^{4}$ (whose boundary, denoted by $\Gamma$, is assumed to be sufficiently smooth) and taking values in $\mathbb{H}$. Such a function may be written as $f=f_{0}+f_{1} e_{1}+f_{2} e_{2}+f_{3} e_{3}$ and each time we ascribe a property such as continuity, differentiability, integrability, and so on, to $f$ it is meant that all $\mathbb{R}$-components $f_{m}$ have to fulfill this property. This means that notations as $f \in C^{k}(\Omega, \mathbb{H}), k \in \mathbb{N} \cup\{0\}$, or $f \in L_{p}(\Omega, \mathbb{H}), p \geq 1$, may be understood component-wise or directly. For instance, $f \in C^{k}(\Omega, \mathbb{H})$ means that either $f_{k} \in C^{k}(\Omega, \mathbb{R})$ or

$$
\|f\|_{C^{k}}:=\sum_{l=0}^{k} \sum_{|\nu|=l} \sup _{x \in \Omega}\left|\partial^{\nu} f(x)\right|<\infty
$$

with $v=\left(v_{0}, \ldots, v_{3}\right),|\nu|=v_{0}+\ldots+v_{3}$ denoting a multi-index.
With the above norm $C^{k}(\Omega, \mathbb{H})$ becomes either a right-quaternionic Banach module or a left-quaternionic Banach module. Analogously, $L_{p}(\Omega, \mathbb{H}), p \geq 1$, with the norm

$$
\|f\|_{L_{p}}:=\left(\int_{\Omega}|f(x)|^{p} d x\right)^{1 / p}
$$

can also be considered as a right-quaternionic Banach module or a left-quaternionic Banach module. Since in this paper our operators act in general from the left we will work with right-quaternionic Banach modules (unless explicitly stated otherwise). We remark that all these spaces can also be considered as real-linear Banach spaces.

In the same way we can introduce $\mathcal{S}$ as the corresponding Schwartz space of rapidly decaying functions. Its dual space $\mathcal{S}^{\prime}$ is given by the continuous linear functionals is the space of tempered distributions. Again, this can be either defined component-wise or via the sesquilinear form

$$
(f, g):=\int_{\mathbb{R}^{4}} \bar{f}(x) g(x) d x
$$

but the space $S^{\prime}$ is again considered as a right-linear quaternionic module. Let us remark that strictly speaking if we consider $\mathcal{S}$ as a right-quaternionic Banach module its algebraic dual $\mathcal{S}^{\prime}$ is the space of all left-quaternionic linear functionals over $\mathcal{S}$, but it can be identified with elements of a right-linear quaternionic module in the above mentioned way.

For more details we refer to [6], [20] and, in particular, the classic book [2].
Standard notations, which we will use are:

- For a quaternion $c$ we will write the constant function as

$$
c_{\Omega}: x \in \Omega \longrightarrow c \in \mathbb{H},
$$

- $B(x, \epsilon)$ is the ball of the radius $\epsilon$ centered in $x$,
- $\Omega_{x, \epsilon}:=\Omega \backslash B(x, \epsilon)$.

Let $\psi:=\left\{\psi^{0}, \psi^{1}, \psi^{2}, \psi^{3}\right\} \in \mathbb{H}^{4}$ be a system of quaternions such that the conditions

$$
\begin{equation*}
\psi^{m} \cdot \bar{\psi}^{n}+\psi^{n} \cdot \bar{\psi}^{m}=2\left\langle\overrightarrow{\psi^{m}}, \vec{\psi}^{n}\right\rangle_{\mathbb{R}^{4}}=2 \delta_{n, m}, \quad n, m \in \mathbb{N}_{3} \cup\{0\}, \tag{2.1}
\end{equation*}
$$

be fulfilled, where $\delta_{n, m}$ is the Kronecker's symbol and $\langle\cdot, \cdot\rangle_{\mathbb{R}^{4}}$ denotes the scalar product. Observe that $\langle\vec{x}, \vec{y}\rangle_{\mathbb{R}^{4}}=$ $\operatorname{Re}\{x \cdot \bar{y}\}$ for every $x, y \in \mathbb{H}$.

By abuse of notation, $\vec{\psi}:=\left\{\vec{\psi}^{0}, \vec{\psi}^{1}, \vec{\psi}^{2}, \vec{\psi}^{3}\right\}$ can be thought of as an orthonormal (in the usual Euclidean sense) basis in $\mathbb{R}^{4}$. In this way we obtain what will be referred to as structural set. The following is an elementary property of this concept.

Proposition 2.1 Let $\psi=\left\{\psi^{0}, \psi^{1}, \psi^{2}, \psi^{3}\right\}$ be a structural set. Then,

$$
\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}}= \pm 1
$$

The proof is a direct computation and will be omitted.
It is evident that $\psi$ and $\bar{\psi}:=\left\{\bar{\psi}^{0}, \bar{\psi}^{1}, \bar{\psi}^{2}, \bar{\psi}^{3}\right\}$ are structural simultaneously only.
If $\varphi, \psi$ are two structural sets, then the coordinates of $\overrightarrow{\psi^{n}}$ with respect to the basis $\vec{\varphi}$ are given by

$$
\begin{equation*}
\overrightarrow{\psi^{n}}=\sum_{r=0}^{3} \alpha_{r n} \overrightarrow{\varphi^{r}}=\alpha_{0 n} \overrightarrow{\varphi^{0}}+\alpha_{1 n} \overrightarrow{\varphi^{1}}+\alpha_{2 n} \overrightarrow{\varphi^{2}}+\alpha_{3 n} \overrightarrow{\varphi^{3}}, \quad n \in \mathbb{N}_{3} \cup\{0\} \tag{2.2}
\end{equation*}
$$

The matrix

$$
M_{\varphi, \psi}:=\left(\begin{array}{llll}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03}  \tag{2.3}\\
\alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33}
\end{array}\right)
$$

which feature in formulae (2.2) is called the transition matrix from the basis $\vec{\varphi}$ to the basis $\vec{\psi}$. Note that

$$
M_{\varphi, \psi}^{-1}=M_{\varphi, \psi}^{T}
$$

Since the entries of $M_{\varphi, \psi}$ are real we have

$$
\begin{equation*}
\left\langle\overrightarrow{\psi^{n}}, \overrightarrow{\varphi^{m}}\right\rangle_{\mathbb{R}^{4}}=\left\langle\sum_{r=0}^{3} \alpha_{r n} \overrightarrow{\varphi^{r}}, \overrightarrow{\varphi^{m}}\right\rangle_{\mathbb{R}^{4}}=\sum_{r=0}^{3} \alpha_{r n}\left\langle\overrightarrow{\varphi^{r}}, \overrightarrow{\varphi^{m}}\right\rangle_{\mathbb{R}^{4}}=\alpha_{m n}, \quad m, n \in \mathbb{N}_{3} \cup\{0\} . \tag{2.4}
\end{equation*}
$$

By a well-known geometric reason both left and right multiplication of a structural set $\psi$ by a unitary quaternion $h$, defines structural sets also. Let us now introduce the usual definition of equivalent structural sets.

Definition 2.2 Two structural sets $\varphi, \psi$ are said to be left equivalent (resp. right) if there exists $h \in \mathbb{H},|h|=1$ such that

$$
\psi=h \varphi \quad(\text { resp. } \quad \psi=\varphi h)
$$

This name is shortened if misunderstanding is excluded.
Remark 2.3 In geometric terms the above definition means that there exists a rotation which maps the orthonormal basis $\varphi$ into the orthonormal basis $\psi$.

Proposition 2.4 Two different structural sets $\varphi, \psi$ are left and right equivalent simultaneously if and only if $\psi=-\varphi$.

Proof. Under our assumptions, there exist $h, h^{\prime} \in \mathbb{H}$ with $|h|=\left|h^{\prime}\right|=1$ such that

$$
h \varphi^{m}=\varphi^{m} h^{\prime}, \quad m \in \mathbb{N}_{3} \cup\{0\}
$$

This is equivalent to

$$
B_{l}(h) \overrightarrow{\varphi^{m}}=B_{r}\left(h^{\prime}\right) \overrightarrow{\varphi^{m}} \Longrightarrow\left[B_{l}(h)-B_{r}\left(h^{\prime}\right)\right] \overrightarrow{\varphi^{m}}=0, \quad m \in \mathbb{N}_{3} \cup\{0\}
$$

But since $\vec{\varphi}=\left\{\overrightarrow{\varphi^{0}}, \overrightarrow{\varphi^{1}}, \overrightarrow{\varphi^{2}}, \overrightarrow{\varphi^{3}}\right\}$ is an orthonormal basis of $\mathbb{R}^{4}$ we have

$$
\left[B_{l}(h)-B_{r}\left(h^{\prime}\right)\right] \vec{x}=0 \quad \forall \vec{x} \in \mathbb{R}^{4}
$$

Then, for this to happen, we must have $B_{l}(h)-B_{r}\left(h^{\prime}\right)=0$. Writing $h=h_{0}+h_{1} e_{1}+h_{2} e_{2}+h_{3} e_{3}$ and $h^{\prime}=$ $h_{0}^{\prime}+h_{1}^{\prime} e_{1}+h_{2}^{\prime} e_{2}+h_{3}^{\prime} e_{3}$ we have

$$
\left(\begin{array}{cccc}
h_{0} & -h_{1} & -h_{2} & -h_{3} \\
h_{1} & h_{0} & -h_{3} & h_{2} \\
h_{2} & h_{3} & h_{0} & -h_{1} \\
h_{3} & -h_{2} & h_{1} & h_{0}
\end{array}\right)=\left(\begin{array}{cccc}
h_{0}^{\prime} & -h_{1}^{\prime} & -h_{2}^{\prime} & -h_{3}^{\prime} \\
h_{1}^{\prime} & h_{0}^{\prime} & h_{3}^{\prime} & -h_{2}^{\prime} \\
h_{2}^{\prime} & -h_{3}^{\prime} & h_{0}^{\prime} & h_{1}^{\prime} \\
h_{3}^{\prime} & h_{2}^{\prime} & -h_{1}^{\prime} & h_{0}^{\prime}
\end{array}\right)
$$

We thus get $h_{0}=h_{0}^{\prime}, h_{1}=h_{1}^{\prime}=-h_{1}^{\prime}, h_{2}=h_{2}^{\prime}=-h_{2}^{\prime}$ y $h_{3}=h_{3}^{\prime}=-h_{3}^{\prime}$. Then, $h_{1}=h_{1}^{\prime}=0, h_{2}=h_{2}^{\prime}=0$ and $h_{3}=h_{3}^{\prime}=0$. This gives $h=h^{\prime} \in \mathbb{R} \Rightarrow h=h^{\prime}= \pm 1$.

Since $\psi$ and $\varphi$ are different we have $h=h^{\prime}=-1 \Rightarrow \psi=-\varphi$.
Taking into the account the non-commutativity of $\mathbb{H}$, every structural set $\psi$ generates Cauchy-Riemann operators (left or right), which are defined in $C^{1}(\Omega, \mathbb{H})$ by the following equalities:

$$
\begin{equation*}
\psi D[f]:=\sum_{n=0}^{3} \psi^{n} \cdot \partial_{x_{n}}[f] ; \quad D^{\psi}[f]:=\sum_{n=0}^{3} \partial_{x_{n}}[f] \cdot \psi^{n}, \tag{2.5}
\end{equation*}
$$

where $\partial_{x_{n}}:=\frac{\partial}{\partial x_{n}}$. Let us remark that since $C^{1}(\Omega, \mathbb{H})$ is a right-quaternionic Banach module we have that ${ }^{\psi} D$ is a quaternionic linear operator from $C^{1}(\Omega, \mathbb{H})$ into $C(\Omega, \mathbb{H})$ while $D^{\psi}$ is only a real-linear operator.

Let $\Delta_{\mathbb{H}}[f]=\sum_{n=0}^{3} \Delta_{\mathbb{R}^{4}}\left[f_{n}\right] e_{n}$, where $\Delta_{\mathbb{R}^{4}}=\sum_{n=0}^{3} \partial_{x_{n}^{2}}^{2}$. Then in $C^{2}(\Omega, \mathbb{H})$ the following equalities

$$
\begin{equation*}
{ }^{\psi} D \cdot{ }^{\bar{\psi}} D={ }^{\bar{\psi}} D \cdot{ }^{\psi} D=D^{\psi} \cdot D^{\bar{\psi}}=D^{\bar{\psi}} \cdot D^{\psi}=\Delta_{\mathbb{H}} \tag{2.6}
\end{equation*}
$$

hold.
For any structural set $\psi$, the following formulas establish a direct connection between the left and right Cauchy-Riemann operators

$$
\begin{equation*}
{ }^{\psi} D[f]=\overline{D^{\bar{\psi}}[\bar{f}]} \quad \text { and } \quad D^{\psi}[f]=\overline{\bar{\psi}} D[\bar{f}] . \tag{2.7}
\end{equation*}
$$

It means that given a formula for the left (or right) Cauchy-Riemann operator, no need to repeat its proof for the right (or left) case. It is enough to substitute the relation into the given formula.

Let us remark that the standard Cauchy-Riemann operator

$$
D[f]=\sum_{n=0}^{3} e_{n} \cdot \partial_{x_{n}}[f]
$$

is invariant under rotations (see, e.g. [3]), i.e.

$$
D[f](h x)=\bar{h} D[f](x), \quad h \in \mathbb{H},
$$

or with other words if $f \in \operatorname{ker} D$ then $f(h \cdot) \in \operatorname{ker} D$. We remark that this property it is not preserved in terms of arbitrary structural sets, for instance, in case of

$$
\bar{D}[f]=\sum_{n=0}^{3} \bar{e}_{n} \cdot \partial_{x_{n}}[f]
$$

we have $\bar{D}[f](h x)=\bar{D}[h f](x)$ and, therefore, no preservation of $\bar{D}[f]=0$ under rotations.
For fixed $\psi$ and $\Omega$ we introduce the set of the so-called $\psi$-hyperholomorphic functions (left or right respectively), which are given by

$$
\begin{aligned}
& \psi \mathfrak{M}(\Omega ; \mathbb{H}):=\operatorname{ker}^{\psi} D=\left\{f \in C^{1}(\Omega ; \mathbb{H}):{ }^{\psi} D[f]=0_{\Omega}\right\} \\
& \mathfrak{M}^{\psi}(\Omega ; \mathbb{H}):=\operatorname{ker} D^{\psi}=\left\{f \in C^{1}(\Omega ; \mathbb{H}): D^{\psi}[f]=0_{\Omega}\right\} .
\end{aligned}
$$

Let us also introduce the notation ${ }^{\psi} \mathfrak{M}^{\psi}(\Omega, \mathbb{H})$ for the class of all two-sided $\psi$-hyperholomorphic functions. Thus

$$
{ }^{\psi} \mathfrak{M}^{\psi}(\Omega, \mathbb{H}):={ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H}) \cap \mathfrak{M}^{\psi}(\Omega, \mathbb{H})
$$

Let us remake that we consider ${ }^{\psi} \mathfrak{M}, \mathfrak{M}^{\psi}$, and ${ }^{\psi} \mathfrak{M}^{\psi}$ as right-quaternionic Banach modules, although they are also real linear spaces (from both sides). This has the disvantagem that $D^{\psi}$ acts on them only as a real linear operator, not a quaternionic linear operator, but in our context that will be good enough since the principal operator under consideration is ${ }^{\psi} D$.

Unless otherwise stated we only use left $\psi$-hyperholomorphic functions, henceforth called $\psi$ hyperholomorphic functions. Our results for left $\psi$-hyperholomorphic functions can be adapted to the right case with the corresponding modifications, i.e. considering left-quaternionic Banach modules.

Due to (2.6), a $\psi$-hyperholomorphic functions in $\Omega$ is harmonic in $\Omega$, in all its components.
The following notation $x_{\psi}:=\sum_{n=0}^{3} x_{n} \cdot \psi^{n}$ will also prove useful.
Let us introduce the quaternionic Cauchy kernel

$$
\begin{equation*}
K_{\psi}(x):={ }^{\bar{\psi}} D\left[\theta_{4}\right](x)=\frac{1}{2 \pi^{2}|x|^{4}} \sum_{n=0}^{3} \bar{\psi}^{n} \cdot x_{n}=\frac{1}{2 \pi^{2}|x|^{4}} \sum_{n=0}^{3} x_{n} \cdot \bar{\psi}^{n}=: D^{\bar{\psi}}\left[\theta_{4}\right](x), \tag{2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\theta_{4}(x):=-\frac{1}{4 \pi^{2}} \frac{1}{|x|^{2}} \tag{2.9}
\end{equation*}
$$

is the fundamental solution of the Laplace operator, i.e. $\theta_{4} \in \mathcal{S}^{\prime}$ with $\Delta_{\mathbb{H}} \theta_{4}=\delta$ in distributional sense, $\delta$ being the Dirac distribution. This means that $K_{\psi}$ represents a fundamental solution to both operators ${ }^{\psi} D$ and $D^{\psi}$. Since the Laplace operator as well as ${ }^{\psi} D$ is elliptic we can identify $\theta_{4}$ and $K_{\psi}$ with an infinitely many differentiable function outside the origin. Therefore, it is easily shown that $K_{\psi}$ is a two-sided $\psi$-hyperholomorphic in $\mathbb{R}^{4} \backslash\{0\}$. One of the most crucial facts of quaternionic analysis is the existence of a quaternionic Stokes formula which can be written in the form: For $f, g \in C^{1}(\Omega \cup \Gamma, \mathbb{H})$

$$
\begin{equation*}
\int_{\Gamma} g(\xi) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}=\int_{\Omega}\left(D^{\psi}[g](\xi) \cdot f(\xi)+g(\xi) \cdot{ }^{\psi} D[f](\xi)\right) d \xi \tag{2.10}
\end{equation*}
$$

where $n_{\psi}(\xi)$ denotes the outward unit normal on $\Gamma$ at the point $\xi \in \Gamma$, writing it as a quaternion and $d S_{\xi}$ stands for the surface area element.

This formula leads immediately to three important consequences, which are widely known and can be found in many sources, see for instance [6], [10], [11].

Theorem 2.5 (Borel-Pompeiu (Cauchy-Green) formula) Let $f \in C^{1}(\Omega \cup \Gamma, \mathbb{H})$. Then

$$
\int_{\Gamma} K_{\psi}(\xi-x) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}-\int_{\Omega} K_{\psi}(\xi-x) \cdot{ }^{\psi} D[f](\xi) d \xi= \begin{cases}f(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma)\end{cases}
$$

Theorem 2.6 (Cauchy integral formula) Let $f \in C^{0}(\Omega \cup \Gamma, \mathbb{H}) \cap \psi \mathfrak{M}(\Omega, \mathbb{H})$. Then

$$
\int_{\Gamma} K_{\psi}(\xi-x) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}= \begin{cases}f(x) & \text { if } x \in \Omega  \tag{2.11}\\ 0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma)\end{cases}
$$

Theorem 2.7 (Cauchy integral theorem) Let $f \in{ }^{\psi} \mathfrak{M}(\Omega \cup \Gamma, \mathbb{H})$ and $g \in \mathfrak{M}^{\psi}(\Omega \cup \Gamma, \mathbb{H})$. Then

$$
\begin{equation*}
\int_{\Gamma} g(\xi) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}=0 \tag{2.12}
\end{equation*}
$$

From the above we can see that the quaternionic Cauchy kernel generates several important integrals. Especially, the following two are important for us:

$$
\begin{aligned}
& \psi \\
& Tf](x)
\end{aligned}=-\int_{\Omega} K_{\psi}(\xi-x) \cdot f(\xi) d \xi, \quad x \in \mathbb{R}^{4}, \quad \begin{aligned}
& \varphi, \psi \\
& \\
&{ }^{\varphi}[f](x):=\int_{\Gamma} K_{\varphi}(\xi-x) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}, \quad x \notin \Gamma
\end{aligned}
$$

where $\varphi, \psi$ are two structural sets.
While the first is a generalization of the usual Teodorescu transform the second represents an operator related to $\psi$ and $\varphi$. It is clear that when $\varphi=\psi$, the "exotic" operator ${ }^{\varphi, \psi} K[f]$ reduces to the usual Cauchy transform (in the $\psi$ setting)

$$
{ }^{\psi} K[f](x):=\int_{\Gamma} K_{\psi}(\xi-x) \cdot n_{\psi}(\xi) \cdot f(\xi) d S_{\xi}
$$

The integral ${ }^{\varphi, \psi} K[f]$ has the following important properties, similar to those of ${ }^{\psi} K[f]$ :

- ${ }^{\varphi, \psi} K[f] \in{ }^{\varphi} \mathfrak{M}\left(\mathbb{R}^{4} \backslash \Gamma, \mathbb{H}\right)$,
- ${ }^{\varphi, \psi} K[f] \in C^{\infty}\left(\mathbb{R}^{4} \backslash \Gamma, \mathbb{H}\right)$.

We can now derive an essential integral formula, to be called generalized Borel-Pompeiu formula, which expresses a very profound relation between ${ }^{\varphi} D,{ }^{\psi} T$ and ${ }^{\varphi, \psi} K$.

Theorem 2.8 (Generalized Borel-Pompeiu formula) Let $f \in C^{1}(\Omega \cup \Gamma, \mathbb{H})$. Then:

$$
\begin{equation*}
\int_{\Gamma} K_{\bar{\varphi}}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}-\int_{\Omega} K_{\bar{\varphi}}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi={ }^{\varphi, \psi} \Pi[f](x), \quad x \notin \Gamma \tag{2.13}
\end{equation*}
$$

where

$$
{ }^{\varphi, \psi} \Pi:={ }^{\varphi} D^{\psi} T
$$

Remark 2.9 The operator ${ }^{\varphi, \psi} \Pi$, called quaternionic $\Pi$-operator, goes back as far as [8].
Using the above notations we can write the generalized Borel-Pompeiu formula in a shorter way as:

$$
{ }^{\bar{\varphi}, \bar{\psi}} K[f](x)+{ }^{\bar{\varphi}} T^{\bar{\psi}} D[f](x)={ }^{\varphi, \psi} \Pi[f](x), \quad x \notin \Gamma .
$$

Proof. Let $x \in \Omega$. Using the Stokes formula (2.10) in $\Omega_{x, \epsilon}$, for the structural set $\bar{\psi}$, and replacing $g(\xi)$ by $\theta_{4}(\xi-x)$ we have the equality

$$
\int_{\Gamma} \theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}-\int_{\partial B(x, \epsilon)} \theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}
$$

$$
\begin{align*}
& =\int_{\Omega_{x, \epsilon}} D_{\xi}^{\bar{\psi}}\left[\theta_{4}(\xi-x)\right] \cdot f(\xi) d \xi+\int_{\Omega_{x, \epsilon}} \theta_{4}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi \\
& =\int_{\Omega_{x, \epsilon}} K_{\psi}(\xi-x) \cdot f(\xi) d \xi+\int_{\Omega_{x, \epsilon}} \theta_{4}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi \tag{2.14}
\end{align*}
$$

Further, for every $\xi \in \partial B(x, \epsilon)$ we have

$$
\left|\theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi)\right|=\frac{|f(\xi)|}{4 \pi^{2} \epsilon^{2}}
$$

By hypothesis $f$ is a continuous function in the compact set $\Omega \cup \Gamma$ and then, there exists $M>0$ such that $|f(\xi)|<M, \forall \xi \in \Omega \cup \Gamma$. Consequently,

$$
\left|\int_{\partial B(x, \epsilon)} \theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}\right|<\frac{M}{4 \pi^{2} \epsilon^{2}} \int_{\partial B(x, \epsilon)} d S_{\xi} .
$$

It is easily seen that the right-hand side of the above inequality tends to zero as $\epsilon \rightarrow 0$. Thus

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon)} \theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}=0  \tag{2.15}\\
& -{ }^{\psi} T[f](x)=\int_{\Omega} K_{\psi}(\xi-x) \cdot f(\xi) d \xi:=\lim _{\epsilon \rightarrow 0} \int_{\Omega_{x, \epsilon}} K_{\psi}(\xi-x) \cdot f(\xi) d \xi \tag{2.16}
\end{align*}
$$

and

$$
\begin{equation*}
\int_{\Omega} \theta_{4}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi:=\lim _{\epsilon \rightarrow 0} \int_{\Omega_{x, \epsilon}} \theta_{4}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi \tag{2.17}
\end{equation*}
$$

Letting $\epsilon \rightarrow 0$ in (2.14) and using (2.15), (2.16) and (2.17) we see that

$$
{ }^{\psi} T[f](x)=\int_{\Omega} \theta_{4}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi-\int_{\Gamma} \theta_{4}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}
$$

Applying the operator ${ }^{\varphi} D$ in both side of the above equality we obtain:

$$
\begin{aligned}
{ }^{\varphi} D^{\psi} T[f](x) & =\int_{\Omega}{ }^{\varphi} D_{x}\left[\theta_{4}(\xi-x)\right] \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi-\int_{\Gamma}{ }^{\varphi} D_{x}\left[\theta_{4}(\xi-x)\right] \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi} \\
& =\int_{\Gamma} K_{\bar{\varphi}}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}-\int_{\Omega} K_{\bar{\varphi}}(\xi-x) \cdot{ }^{\bar{\psi}} D[f](\xi) d \xi
\end{aligned}
$$

and (2.13) is proved.
The proof for the case $x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma)$, is similar to the previous one. First, we use again formula (2.10) but now in $\Omega$ and then the task is only to apply the operator ${ }^{\varphi} D$ in both side of the resulting equality.

Corollary 2.10 Let $f \in{ }^{\bar{\psi}} \mathfrak{M}(\Omega, \mathbb{H}) \cap C^{0}(\Omega \cup \Gamma, \mathbb{H})$. Then ${ }^{\varphi, \psi} \Pi[f] \in^{\bar{\varphi}} \mathfrak{M}(\Omega, \mathbb{H})$ and

$$
\begin{equation*}
\int_{\Gamma} K_{\bar{\varphi}}(\xi-x) \cdot n_{\bar{\psi}}(\xi) \cdot f(\xi) d S_{\xi}={ }^{\varphi, \psi} \Pi[f](x), \quad x \notin \Gamma . \tag{2.18}
\end{equation*}
$$

## 3 Holomorphic maps and classes of hyperholomorphy

In order to deal with the classical theory of holomorphic $\mathbb{C}$-valued functions of two complex variables, we will identify the space $\mathbb{C}^{2}$ with the set $\mathbb{H}$ by means of the mapping that associates the pair $\left(z_{1}, z_{2}\right):=\left(x_{0}+\right.$ $x_{1} e_{1}, x_{2}+x_{3} e_{1}$ ) with the quaternion $x=x_{0}+x_{1} e_{1}+\left(x_{2}+x_{3} e_{1}\right) e_{2}$. Then a quaternionic function $f: \Omega \rightarrow \mathbb{H}$ can be expressed as $f=u+v e_{2}$, where $u, v$ are $\mathbb{C}$-valued functions defined on $\Omega$. We will refer to the functions $u, v$ as the complex components of $f$.

Let us consider the following spaces of holomorphic maps:

$$
\operatorname{Hol}(\Omega, \mathbb{C}):=\left\{w \in C^{1}(\Omega, \mathbb{C}): \partial_{\overline{\bar{z}}_{1}} w=\partial_{\bar{z}_{2}} w=0 \text { on } \Omega\right\}
$$

where $\partial_{\bar{z}_{1}}, \partial_{\bar{z}_{2}}$ are the usual complex differential operators. More generally, we can consider

$$
\operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right):=\left\{w=\left(w_{1}, w_{2}\right): w_{1}, w_{2} \in \operatorname{Hol}(\Omega, \mathbb{C})\right\}
$$

We are interested in finding conditions on $\psi$ for the inclusion

$$
\begin{equation*}
\operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right) \subset{ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H}) \tag{3.1}
\end{equation*}
$$

to be true. It is shown in [12], [16], [19] that (3.1) is true if and only if the structural set $\psi$ is left equivalent to

$$
\psi_{\theta}=\left\{1, e_{1}, e_{1} e^{e_{1} \theta} \cdot e_{2}, e^{e_{1} \theta} \cdot e_{2}\right\}
$$

for $0 \leq \theta<2 \pi$.
Remark 3.1 If $f: \Omega \rightarrow \mathbb{H}$ has $u: \Omega \rightarrow \mathbb{C}$ and $v: \Omega \rightarrow \mathbb{C}$ as complex components then the $\psi$ hyperholomorphicity of $f$ can be interpreted as the following system of equations

$$
\psi_{\theta} D[f]=0 \Leftrightarrow\left\{\begin{array}{l}
\partial_{\bar{z}_{1}} u-e_{1} e^{e_{1} \theta} \cdot \partial_{z_{2}} \bar{v}=0,  \tag{3.2}\\
e_{1} e^{-e_{1} \theta} \cdot \partial_{\bar{z}_{2}} u-\partial_{z_{1}} \bar{v}=0 .
\end{array}\right.
$$

The system above may be interpreted as a generalization of that study by the distinguished Italian mathematician Gianfranco Cimmino (12 Marzo 1908-30 Mayo 1989), which can be recovered from (3.2) if we take $\theta=\frac{\pi}{2}$. In [1] an in-depth study of Cimmino system using quaternionic analysis has been presented.

As the basic for the development of our theory, we need to point out that the class of all equivalent structural sets defines the same hyperholomorphicity, then we should in fact further restrict our analysis to consider only structural sets not to be equivalent from the same side of hyperholomorphicity we are assumed.

First of all, a proof of the fact that equivalence of two structural sets $\psi$ and $\varphi$ yields

$$
\begin{equation*}
{ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H})={ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H}) \tag{3.3}
\end{equation*}
$$

is obtained in [18]. Moreover, the following result offers the necessity of the condition and then provides a criterion on two structural sets under which (3.3) holds.

Proposition 3.2 Let $\varphi, \psi$ be two arbitrary structural sets in $\mathbb{H}^{4}$. If (3.3) holds, then $\varphi, \psi$ are left equivalent.
Proof. Assume that ${ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H})={ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H}), \psi=\left\{\psi^{0}, \psi^{1}, \psi^{2}, \psi^{3}\right\}$ and $\varphi=\left\{\varphi^{0}, \varphi^{1}, \varphi^{2}, \varphi^{3}\right\}$. Let $h^{0}, h^{1}, h^{2}, h^{3} \in \mathbb{H}$ such that $\varphi^{m}=h^{m} \cdot \psi^{m}, m \in \mathbb{N}_{3} \cup\{0\}$, hence $\left|h^{m}\right|=1$. Direct calculation yields that

$$
f_{m}(x)=\overline{\psi^{m-1}} x_{m-1}-\overline{\psi^{m}} x_{m}, \quad m \in \mathbb{N}_{3}
$$

are left $\psi$-hyperholomorphic functions. We thus get $f_{m} \in{ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H}), m \in \mathbb{N}_{3}$. Then

$$
\begin{aligned}
0={ }^{\varphi} D\left[f_{m}\right] & =\sum_{n=0}^{3} \varphi^{n} \partial_{x_{n}}\left[f_{m}\right] \\
& =h^{m-1} \cdot \psi^{m-1} \cdot \overline{\psi^{m-1}}-h^{m} \cdot \psi^{m} \cdot \overline{\psi^{m}}=h^{m-1}-h^{m}, \quad m \in \mathbb{N}_{3}
\end{aligned}
$$

This clearly forces $h_{0}=h_{1}=h_{2}=h_{3}=: h$, which complete the proof.
We summarize the above reasoning in the following theorem.
Theorem $3.3{ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H})={ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H})$ if and only if $\varphi, \psi$ are left equivalent.
Remark. Note that we have used, in the proof of Proposition 3.2, only the fact that ${ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H}) \subset{ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H})$. We have thus proved

Corollary 3.4 Let $\varphi, \psi$ be two structural sets. Then ${ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H}) \subset{ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H})$ if and only if

$$
{ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H})={ }^{\varphi} \mathfrak{M}(\Omega, \mathbb{H})
$$

The following result asserts that the quaternionic constants are the only $\psi$-hyperholomorphic functions for all structural set $\psi$.

## Theorem 3.5

$$
\bigcap_{\psi \in \mathcal{E}} \psi^{\psi} \mathfrak{M}(\Omega, \mathbb{H})=\left\{f=c_{\Omega}: c \in \mathbb{H}\right\},
$$

where $\mathcal{E}:=\left\{\psi \in \mathbb{H}^{4}: \psi\right.$ is a structural set $\}$.
Proof. Consider the standard structural set

$$
\psi_{s t}:=\left\{1, e_{1}, e_{2}, e_{3}\right\}
$$

as well as the associated ones $\left\{1,-e_{1}, e_{2}, e_{3}\right\},\left\{1, e_{1},-e_{2}, e_{3}\right\}$ and $\left\{1, e_{1}, e_{2},-e_{3}\right\}$. This choice involves no loss of generality.

Let $f \in \bigcap_{\psi \in \mathcal{E}}{ }^{\psi} \mathfrak{M}(\Omega, \mathbb{H})$, then we have

$$
\left\{\begin{array}{l}
\partial_{x_{0}} f+e_{1} \partial_{x_{1}} f+e_{2} \partial_{x_{2}} f+e_{3} \partial_{x_{3}} f=0,  \tag{3.4}\\
\partial_{x_{0}} f-e_{1} \partial_{x_{1}} f+e_{2} \partial_{x_{2}} f+e_{3} \partial_{x_{3}} f=0, \\
\partial_{x_{0}} f+e_{1} \partial_{x_{1}} f-e_{2} \partial_{x_{2}} f+e_{3} \partial_{x_{3}} f=0, \\
\partial_{x_{0}} f+e_{1} \partial_{x_{1}} f+e_{2} \partial_{x_{2}} f-e_{3} \partial_{x_{3}} f=0 .
\end{array}\right.
$$

Note that to solve (3.4) is equivalent to solving the four real systems with a non-zero determinant and consequently

$$
\partial_{x_{0}} f=e_{1} \partial_{x_{1}} f=e_{2} \partial_{x_{2}} f=e_{3} \partial_{x_{3}} f=0_{\Omega} \Longrightarrow \partial_{x_{0}} f=\partial_{x_{1}} f=\partial_{x_{2}} f=\partial_{x_{3}} f=0_{\Omega}
$$

Since $f \in C^{1}(\Omega, \mathbb{H})$ it follows the differentiability on $\Omega$ of all the real components of $f$ and hence $f=c_{\Omega}$, where $c \in \mathbb{H}$ is a constant.

Remark 3.6 The most successful notion extending holomorphy to the quaternionic setting is the one of regularity in the sense of Cauchy-Fueter, which is established directly by the $\psi_{s t}$-hyperholomophy. The best modern reference here, in their general form, is [20].

Observe that

$$
\psi_{s t} \notin \mathcal{E}_{\theta}:=\left\{\psi_{\theta}: \theta \in[0,2 \pi)\right\}
$$

Our next theorem, which is given in [12] although without all the details, is an analogue of the latter if we restrict the class $\mathcal{E}$ to the special case $\mathcal{E}_{\theta}$. It ensures that the only $\psi_{\theta}$-hyperholomorphic functions for all $\theta \in[0,2 \pi)$ are those whose complex components are holomorphic.

## Theorem 3.7

$$
\bigcap_{\psi_{\theta} \in \mathcal{E}_{\theta}} \psi_{\theta} \mathfrak{M}(\Omega, \mathbb{H})=\operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right)
$$

Proof. On account of the fact that

$$
\operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right) \subset{ }^{\psi_{\theta}} \mathfrak{M}(\Omega, \mathbb{H}), \quad \text { for all } \quad \theta \in[0,2 \pi)
$$

we shall have established the theorem if we prove the following proposition.
Proposition 3.8 Let $\theta, \vartheta \in[0,2 \pi), \theta \neq \vartheta$. Then

$$
{ }^{\psi_{\theta}} \mathfrak{M}(\Omega, \mathbb{H}) \cap{ }^{\psi_{\vartheta}} \mathfrak{M}(\Omega, \mathbb{H})=\operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right)
$$

Proof. Assume that $f=u+v e_{2} \in{ }^{\psi_{\theta}} \mathfrak{M}(\Omega, \mathbb{H}) \cap{ }^{\psi_{\vartheta}} \mathfrak{M}(\Omega, \mathbb{H})$. From (3.2) it follows that

$$
\left\{\begin{array} { l } 
{ \partial _ { \overline { 1 } _ { 1 } } u - e _ { 1 } e ^ { e _ { 1 } \theta } \cdot \partial _ { z _ { 2 } } \overline { v } = 0 , } \\
{ e _ { 1 } e ^ { - e _ { 1 } \theta } \cdot \partial _ { \overline { \overline { z } } _ { 2 } } u - \partial _ { z _ { 1 } } \overline { v } = 0 , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
\partial_{\bar{z}_{1}} u-e_{1} e^{e_{1} \vartheta} \cdot \partial_{z_{2}} \bar{v}=0 \\
e_{1} e^{-e_{1} \vartheta} \cdot \partial_{\bar{z}_{2}} u-\partial_{z_{1}} \bar{v}=0
\end{array}\right.\right.
$$

Therefore

$$
e_{1} e^{e_{1} \theta} \cdot \partial_{z_{2}} \bar{v}=e_{1} e^{e_{1} \vartheta} \cdot \partial_{z_{2}} \bar{v}
$$

and

$$
e_{1} e^{-e_{1} \theta} \cdot \partial_{\bar{z}_{2}} u=e_{1} e^{-e_{1} \vartheta} \cdot \partial_{\bar{z}_{2}} u
$$

Since $e^{e_{1} \theta} \neq e^{e_{1} \vartheta}$ we conclude that $\partial_{z_{2}} \bar{v}=\partial_{\bar{z}_{2}} u=0$, hence that $\partial_{\overline{\overline{1}}_{1}} u=\partial_{z_{1}} \bar{v}=0$. Then,

$$
u, v \in \operatorname{Hol}(\Omega, \mathbb{C}) \Rightarrow f \in \operatorname{Hol}\left(\Omega, \mathbb{C}^{2}\right)
$$

and the proof is complete.
Now we state and proof a remainder result.
Proposition 3.9 Let $\varphi, \psi$ be two non-equivalent structural sets, then there exists a non-constant function $f: \Omega \rightarrow \mathbb{H}$ such that $f \in \psi \mathfrak{M}(\Omega, \mathbb{H}) \cap^{\varphi} \mathfrak{M}(\Omega, \mathbb{H})$. Thus

$$
\psi \mathfrak{M}(\Omega, \mathbb{H}) \cap^{\varphi} \mathfrak{M}(\Omega, \mathbb{H}) \neq\left\{f=c_{\Omega}: c \in \mathbb{H}\right\}
$$

Proof. Writing $\varphi^{m}=h^{m} \cdot \psi^{m}, m \in \mathbb{N}_{3} \cup\{0\}$, give $h^{m} \neq h^{n}$ for at least one pair $m, n \in \mathbb{N}_{3} \cup\{0\}$. Without restriction of generality we can assume $h^{0} \neq h^{1}$, then directly we may see that the function

$$
\begin{aligned}
f(x)= & \overline{\psi^{0}}\left\{\left[\left(h^{1}-h^{0}\right)^{-1}\left(h^{2}-h^{0}\right)-1\right] \psi^{2}+\left[\left(h^{1}-h^{0}\right)^{-1}\left(h^{3}-h^{0}\right)-1\right] \psi^{3}\right\} x_{0} \\
& -\overline{\psi^{1}}\left(h^{1}-h^{0}\right)^{-1}\left[\left(h^{2}-h^{0}\right) \psi^{2}+\left(h^{3}-h^{0}\right) \psi^{3}\right] x_{1}+x_{2}+x_{3}
\end{aligned}
$$

is both $\varphi$ - and $\psi$-hyperholomorphic. Combining this example with Teorema 3.3 the assertion follows.

## 4 Notion of $(\varphi, \psi)$-hyperholomorphic function. Matrix approach rotational case

The above preliminaries now allow for defining the notion of $(\varphi, \psi)$-hyperholomorphy.
Definition 4.1 A function $f: \Omega \rightarrow \mathbb{H}$, is said to be $(\varphi, \psi)$-hyperholomorphic in $\Omega$ if $f \in C^{1}(\Omega ; \mathbb{H})$ and moreover satisfies the equations

$$
{ }^{\varphi} D[f]=0 \quad \text { and } \quad \psi D[f]=0
$$

In order to develop the crucial facts of this proper theory, a matrix approach will be followed, leading to the concept of left or right $(\varphi, \psi)$-hyperholomorphic functions, introduced as circulant $2 \times 2$ matrix functions, which are left or right null solutions of a $2 \times 2$ circulant matrix Cauchy-Riemann operator, having those ${ }^{\varphi} D$ and ${ }^{\psi} D$ as its entries.

Let $\mathbf{C} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$ be the ring of circulant $(2 \times 2)$-matrices over $\mathbb{H}$.

$$
\mathbf{C M}_{\mathbb{H}}^{2 \times 2}=\left\{\left(\begin{array}{ll}
x & y \\
y & x
\end{array}\right): x, y \in \mathbb{H}\right\} .
$$

It is easily checked that for $\mathbf{A}, \mathbf{B} \in \mathbf{C M}_{\mathbb{H}}^{2 \times 2}, \mathbf{A}+\mathbf{C}$ and $\mathbf{A B}$ both belong to $\mathbf{C} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$. Moreover, defining for

$$
\mathbf{A}=\left(\begin{array}{ll}
x & y \\
y & x
\end{array}\right) \in \mathbf{C M}_{\mathbb{H}}^{2 \times 2}
$$

its conjugate $\overline{\mathbf{A}}$ in the classical way by

$$
\overline{\mathbf{A}}=\left(\begin{array}{ll}
\bar{x} & \bar{y} \\
\bar{y} & \bar{x}
\end{array}\right),
$$

then clearly $\overline{\overline{\mathbf{A}}}=\mathbf{A} ; \overline{\mathbf{A} \lambda}=\bar{\lambda} \overline{\mathbf{A}}$ for $\lambda \in \mathbb{H} ; \overline{\mathbf{A}+\mathbf{B}}=\overline{\mathbf{A}}+\overline{\mathbf{B}}$ and $\overline{\mathbf{A B}}=\overline{\mathbf{B A}}$. In other words, $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$ is a right linear quaternionic space which is also a real algebra with involution.

Notice that $\mathbb{H}$ may be embedded into $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$ by identifying $x \in \mathbb{H}$ with

$$
\left(\begin{array}{cc}
x & 0 \\
0 & x
\end{array}\right) \in \mathbf{C M}_{\mathbb{H}}^{2 \times 2}
$$

On $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$ let us define the non-negative function

$$
\|\mathbf{A}\|=\max \{|x|,|y|\}, \mathbf{A}=\left(\begin{array}{ll}
x & y \\
y & x
\end{array}\right) .
$$

It can be easily seen that this function fulfills the axioms of a norm. Together with this norm $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$ becomes a right-linear quaternionic normed space, in fact a right-linear quaternionic Banach module.

Spaces of functions defined over a domain $\Omega \subset \mathbb{H}$ with values in $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$ will be introduced in the usual way via its entries. In particular, $C^{k}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$ is a right-linear quaternionic space equipped with the norm

$$
\|\mathbf{F}\|_{C^{k}}=\sum_{l=0}^{k} \sum_{|\nu|=l} \sup _{x \in \Omega}\left\|\partial^{\nu} \mathbf{F}(x)\right\|
$$

Let ${ }^{\varphi, \psi} \mathbb{D}$ and $\mathbb{D}^{\varphi, \psi}$ be the circulant (left and right) $(2 \times 2)$-matrix Cauchy-Riemann operators given by

$$
{ }^{\varphi, \psi} \mathbb{D}=\left(\begin{array}{ll}
\varphi^{\varphi} D & \psi^{2}  \tag{4.1}\\
\psi D & \varphi^{\varphi} D
\end{array}\right) \quad \text { and } \quad \mathbb{D}^{\varphi, \psi}=\left(\begin{array}{cc}
D^{\varphi} & D^{\psi} \\
D^{\psi} & D^{\varphi}
\end{array}\right) .
$$

Extending Definition 4.1 onto the $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$-valued continuously differentiable functions we arrive to what will be called here $(\varphi, \psi)$-hyperholomorphy.

Definition 4.2 A $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}$-valued continuously differentiable function

$$
\mathbf{F}=\left(\begin{array}{ll}
f_{1} & f_{2} \\
f_{2} & f_{1}
\end{array}\right)
$$

in $\Omega \subset \mathbb{R}^{4}$ is said to be left (resp. right) $(\varphi, \psi)$-hyperholomorphic in $\Omega$ if

$$
{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}]=\mathbf{0} \quad\left(\text { resp. } \mathbb{D}^{\varphi, \psi}[\mathbf{F}]=\mathbf{0}\right) \quad \text { in } \quad \Omega
$$

Denote by

$$
\begin{aligned}
& \Phi, \Psi \mathfrak{M}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right):=\operatorname{ker}^{\varphi, \psi} \mathbb{D}(\Omega)=\left\{\mathbf{F} \in C^{1}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right):{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}]=0 \text { in } \Omega\right\}, \\
& \mathfrak{M}^{\Phi, \Psi}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right):=\operatorname{ker} \mathbb{D}^{\varphi, \psi}(\Omega)=\left\{\mathbf{F} \in C^{1}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right): \mathbb{D}^{\varphi, \psi}[\mathbf{F}]=0 \text { in } \Omega\right\},
\end{aligned}
$$

the set of left (resp. right) $(\varphi, \psi)$-hyperholomorphic functions in $\Omega$. As before, these spaces will be considered as right-linear quaternionic normed spaces.

This again makes ${ }^{\varphi, \psi} \mathbb{D}$ a right-linear quaternionic operator while $\mathbb{D}^{\varphi, \psi}$ is only a real-linear operator.
Invoking (4.1) we have

$$
\begin{equation*}
\varphi, \psi \mathbb{D}[\mathbf{F}]=\overline{\mathbb{D}^{\bar{\varphi}, \bar{\psi}}[\overline{\mathbf{F}}]} \quad \text { and } \quad \mathbb{D}^{\varphi, \psi}[\mathbf{F}]=\overline{\bar{\varphi}, \bar{\psi}} \mathbf{D}[\overline{\mathbf{F}}] . \tag{4.2}
\end{equation*}
$$

$\operatorname{Denote}^{\Phi, \Psi} \mathfrak{M}^{\Phi, \Psi}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right):={ }^{\Phi, \Psi} \mathfrak{M}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right) \cap \mathfrak{M}^{\Phi, \Psi}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$.

## Remark 4.3

- The $(\varphi, \psi)$-hyperholomorphy of the matrix function $\mathbf{F}$ does not imply that its entry functions $f_{1}$ and $f_{2}$ are both $\varphi$ - and $\psi$-hyperholomorphic. However, choosing $f_{1}=f$ and $f_{2}=0$, the $(\varphi, \psi)$-hyperholomorphy of the corresponding diagonal matrix, denoted as $\mathbf{F}_{\mathbf{0}}=\left(\begin{array}{cc}f & 0 \\ 0 & f\end{array}\right)$, is equivalent to the $f$ being both $\varphi$ - and $\psi$-hyperholomorphic.
- Direct calculation yields

$$
\mathbf{F} \in \operatorname{ker}^{\varphi, \psi} \mathbb{D} \Longleftrightarrow \mathbf{F} \in \operatorname{ker}^{\psi, \varphi} \mathbb{D}
$$

for $\mathbf{F} \in C^{1}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$.

- Notions of continuity (in particular Hölder continuity), differentiability of $\mathbf{F}$ are introduced by means of the corresponding notions for its entries. In particular, the notations $C^{0, \beta}\left(\Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$ and $C^{p}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right), p \in \mathbb{N} \cup\{0\}$ will be used.

Defining the matrix Laplacian by

$$
\Delta=\left(\begin{array}{cc}
\Delta_{\mathbb{H}} & 0 \\
0 & \Delta_{\mathbb{H}}
\end{array}\right)
$$

we may call the matrix function $\mathbf{F}$ harmonic in the domain $\Omega$ if and only if it satisfies the equation $\Delta \mathbf{F}=\mathbf{0}$ in $\Omega$. Actually the $(\varphi, \psi)$-hyperholomorphic functions are harmonic for every structural sets $\varphi, \psi$ with the additional conditions

$$
\begin{equation*}
{ }^{\varphi} D^{\bar{\psi}} D+{ }^{\psi} D^{\bar{\varphi}} D=0, \quad{ }^{\bar{\varphi}} D^{\psi} D+{ }^{\bar{\psi}} D^{\varphi} D=0 \tag{4.3}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\left\langle\overrightarrow{\varphi^{n}}, \overrightarrow{\psi^{m}}\right\rangle_{\mathbb{R}^{4}}+\left\langle\overrightarrow{\varphi^{m}}, \overrightarrow{\psi^{n}}\right\rangle_{\mathbb{R}^{4}}=\operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\}=0, \quad n, m \in \mathbb{N}_{3} \cup\{0\} . \tag{4.4}
\end{equation*}
$$

It is easy to see that given a structural set $\varphi$, then a $\psi$ ones which satisfy the relations (4.4) always exists. So the class of pair of structural sets connected by the relations (4.4) is wide enough.

Theorem 4.4 On $C^{2}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$ the equalities

$$
\begin{equation*}
{ }^{\varphi, \psi} \mathbb{D} \cdot{ }^{\bar{\varphi}, \bar{\psi}} \mathbb{D}={ }^{\bar{\varphi}, \bar{\psi}} \mathbb{D} \cdot{ }^{\varphi, \psi} \mathbb{D}=\mathbb{D}^{\varphi, \psi} \cdot \mathbb{D}^{\bar{\varphi}, \bar{\psi}}=\mathbb{D}^{\bar{\varphi}, \bar{\psi}} \cdot \mathbb{D}^{\varphi, \psi}=2 \Delta \tag{4.5}
\end{equation*}
$$

hold if and only if the structural sets $\varphi, \psi$ satisfy the relations (4.4).
Proof. Note that

$$
{ }^{\varphi, \psi} \mathbb{D} \cdot{ }^{\bar{\varphi}, \bar{\psi}} \mathbb{D}=\left(\begin{array}{cc}
\varphi \\
& \psi^{\psi} D \\
\psi D & { }^{\varphi} D
\end{array}\right) \cdot\left(\begin{array}{cc}
\bar{\varphi} D & { }^{\bar{\psi}} D \\
\bar{\psi} D & \bar{\varphi} D
\end{array}\right)=\left(\begin{array}{cc}
2 \Delta & { }^{\varphi} D^{\bar{\psi}} D+{ }^{\psi} D^{\bar{\varphi}} D \\
\psi D^{\bar{\varphi}} D+{ }^{\varphi} D^{\bar{\psi}} D & 2 \Delta
\end{array}\right) .
$$

Then,

$$
\begin{equation*}
{ }^{\varphi, \psi} \mathbb{D} \cdot \bar{\varphi}, \bar{\psi} \mathbf{D}=2 \Delta \Longleftrightarrow{ }^{\varphi} D^{\bar{\psi}} D+{ }^{\psi} D^{\bar{\varphi}} D=0 \Longleftrightarrow \operatorname{Re}\left\{{ }^{\varphi} D^{\bar{\psi}} D\right\}=0 . \tag{4.6}
\end{equation*}
$$

Similarly, one can see that

$$
{ }^{\bar{\varphi}, \bar{\psi}} \mathbb{D} \cdot{ }^{\varphi, \psi} \mathbb{D}=2 \Delta \Longleftrightarrow{ }^{\bar{\varphi}} D^{\psi} D+{ }^{\bar{\psi}} D^{\varphi} D=0 \Longleftrightarrow \operatorname{Re}\left\{{ }^{\varphi} D^{\bar{\psi}} D\right\}=0
$$

Thus, the relations (4.3) completely determines (4.5).
Observe that:

$$
\begin{aligned}
\operatorname{Re}\left\{{ }^{\varphi} D^{\bar{\psi}} D\right\} & =\operatorname{Re}\left\{\sum_{n=0}^{3} \varphi^{n} \cdot \overline{\psi^{n}} \cdot \partial_{x_{n}^{2}}^{2}+\sum_{0 \leq n<m \leq 3}\left(\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right) \cdot \partial_{x_{n} x_{m}}^{2}\right\} \\
& =\sum_{n=0}^{3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n}^{2}}^{2}+\sum_{0 \leq n<m \leq 3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n} x_{m}}^{2}
\end{aligned}
$$

Because the equality in (4.6) means that ${ }^{\varphi} D^{\bar{\psi}} D$ is a pure vectorial operator, then for all $f \in C^{2}(\Omega, \mathbb{R})$ we have

$$
\begin{aligned}
0 & =\operatorname{Re}\left\{{ }^{\varphi} D^{\bar{\psi}} D[f]\right\} \\
& =\sum_{n=0}^{3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n}^{2}}^{2}[f]+\sum_{0 \leq n<m \leq 3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n} x_{m}}^{2}[f] .
\end{aligned}
$$

Considering in particular the real-valued functions

$$
f_{n, m}(x)=x_{n} x_{m} \in C^{2}(\Omega, \mathbb{H}), \quad n, m \in \mathbb{N}_{3} \cup\{0\}
$$

we have

$$
0=\operatorname{Re}\left\{{ }^{\varphi} D^{\bar{\psi}} D\left[f_{n, m}\right]\right\}= \begin{cases}\operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{n}}\right\}, & n=m \\ \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\}, & n \neq m\end{cases}
$$

Then, (4.5) gives (4.4) and the necessity follows. The sufficiency of the conditions is easy to check.
We have proved that (4.4) are necessary and sufficient conditions for

$$
\varphi, \psi \mathbb{D} \cdot \bar{\varphi}, \bar{\psi} \mathbb{D}={ }^{\bar{\varphi}, \bar{\psi}} \mathbb{D} \cdot{ }^{\varphi, \psi} \mathbb{D}=2 \Delta .
$$

In the same manner we can see that

$$
\mathbb{D}^{\varphi, \psi} \cdot \mathbb{D}^{\bar{\varphi}, \bar{\psi}}=2 \Delta \Longleftrightarrow D^{\varphi} D^{\bar{\psi}}+D^{\psi} D^{\bar{\varphi}}=0 \Longleftrightarrow \operatorname{Re}\left\{D^{\varphi} D^{\bar{\psi}}\right\}=0,
$$

and

$$
\mathbb{D}^{\bar{\varphi}, \bar{\psi}} \cdot \mathbb{D}^{\varphi, \psi}=2 \Delta \Longleftrightarrow D^{\bar{\varphi}} D^{\psi}+D^{\bar{\psi}} D^{\varphi}=0 \Longleftrightarrow \operatorname{Re}\left\{D^{\varphi} D^{\bar{\psi}}\right\}=0
$$

But

$$
\begin{aligned}
\operatorname{Re}\left\{D^{\varphi} D^{\bar{\psi}}\right\} & =\sum_{n=0}^{3} \operatorname{Re}\left\{\overline{\psi^{n}} \cdot \varphi^{n}\right\} \cdot \partial_{x_{n}^{2}}^{2}+\sum_{0 \leq n<m \leq 3} \operatorname{Re}\left\{\overline{\psi^{m}} \cdot \varphi^{n}+\overline{\psi^{n}} \cdot \varphi^{m}\right\} \cdot \partial_{x_{n} x_{m}}^{2} \\
& =\sum_{n=0}^{3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n}^{2}}^{2}+\sum_{0 \leq n<m \leq 3} \operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\} \cdot \partial_{x_{n} x_{m}}^{2}
\end{aligned}
$$

The rest of the proof runs as before.
Denote by $\Theta \in \mathcal{S}^{\prime}$ the fundamental solution of the matrix Laplace operator $\Delta$ given by

$$
\Theta:=\left(\begin{array}{cc}
\theta_{4} & 0 \\
0 & \theta_{4}
\end{array}\right)
$$

i.e., $\Delta[\Theta]=\left(\begin{array}{ll}\delta & 0 \\ 0 & \delta\end{array}\right)$ in distributional sense, where $\delta$ denotes the Dirac delta distribution.

This leads again to the Cauchy kernel $\mathcal{K}_{\varphi, \psi}$ via

$$
\begin{aligned}
\mathcal{K}_{\varphi, \psi} & :=\left(\begin{array}{ll}
K_{\varphi} & K_{\psi} \\
K_{\psi} & K_{\varphi}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\bar{\varphi} D & \bar{\psi} D \\
\bar{\psi} D & \bar{\varphi} D
\end{array}\right) \cdot\left(\begin{array}{cc}
\theta_{4} & 0 \\
0 & \theta_{4}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\bar{\varphi} D\left[\theta_{4}\right] & \bar{\psi} D\left[\theta_{4}\right] \\
\bar{\psi} D\left[\theta_{4}\right] & \bar{\varphi} D\left[\theta_{4}\right]
\end{array}\right)={ }^{\bar{\varphi}, \bar{\psi}} \mathbf{D}[\Theta]
\end{aligned}
$$

as the fundamental solution of ${ }^{\varphi, \psi} \mathbb{D}$, i.e.

$$
{ }^{\varphi, \psi} \mathbb{D} \mathcal{K}_{\varphi, \psi}={ }^{\varphi, \psi} \mathbb{D}^{\bar{\varphi}, \bar{\psi}} \mathbb{D}[\Theta]=2 \Delta \cdot\left(\begin{array}{cc}
\theta_{4} & 0 \\
0 & \theta_{4}
\end{array}\right)=\left(\begin{array}{cc}
\Delta_{\mathbb{H}}\left[\theta_{4}\right] & 0 \\
0 & \Delta_{\mathbb{H}}\left[\theta_{4}\right]
\end{array}\right)=\left(\begin{array}{ll}
\delta & 0 \\
0 & \delta
\end{array}\right)
$$

in distributional sense. Furthermore, since the involved operators are elliptic we can identify $\mathcal{K}_{\varphi, \psi}$ with a function which belongs to the kernel of both ${ }^{\varphi, \psi} D$ and $D^{\varphi, \psi}$ in $\mathbb{R}^{4} \backslash\{0\}$, i.e. outside the origin.

## 5 On the admissible structural sets

As we have shown above it seems to be natural to consider structural sets with the relation (4.4), but firstly they are assumed to be non-equivalent. Then it is necessary to look more closely at this class of admissible structural sets, which makes our new theory allowable. To describe such class is the goal of this section.

We begin by introducing the notion of compatible structural sets.
Definition 5.1 Two structural sets $\varphi, \psi$ satisfying the relation (4.4) are said to be compatible.
Proposition 5.2 [Compatibility of two equivalent structural sets] Two equivalent structural sets $\varphi, \psi$, either $\varphi=h \psi$ or $\varphi=\psi h, h \in \mathbb{H},|h|=1$, are compatible if and only if $\operatorname{Re}\{h\}=0$.

Proof. Let us assume that $\varphi, \psi$ are left equivalent. Applying (2.1), we can assert that $\varphi, \psi$ are compatible if and only if

$$
\begin{aligned}
0 & =\operatorname{Re}\left\{\varphi^{n} \cdot \overline{\psi^{m}}+\varphi^{m} \cdot \overline{\psi^{n}}\right\} \\
& =\operatorname{Re}\left\{h\left(\psi^{n} \cdot \overline{\psi^{m}}+\psi^{m} \cdot \overline{\psi^{n}}\right)\right\}=2 \delta_{n, m} \operatorname{Re}\{h\}= \begin{cases}2 \operatorname{Re}\{h\} & \text { if } n=m, \\
0 & \text { if } n \neq m,\end{cases}
\end{aligned}
$$

and the proposition follows. The same proof works for the right equivalence case.
Proposition 5.3 [Matrix characterization of compatibility] Two structural sets $\varphi, \psi$ are compatible if and only if the transition matrix $M_{\varphi, \psi}$ is either of the form

$$
M_{\varphi, \psi}^{1}:=\left(\begin{array}{cccc}
0 & -a & -b & -c  \tag{5.1}\\
a & 0 & -c & b \\
b & c & 0 & -a \\
c & -b & a & 0
\end{array}\right) \quad \text { or } \quad M_{\varphi, \psi}^{2}:=\left(\begin{array}{cccc}
0 & -a & -b & -c \\
a & 0 & c & -b \\
b & -c & 0 & a \\
c & b & -a & 0
\end{array}\right),
$$

where $a, b, c \in \mathbb{R}$ such that $a^{2}+b^{2}+c^{2}=1^{1}$.
Proof. Combining (2.4) and (4.4), the structural sets $\varphi, \psi$ are compatible if and only if

$$
\begin{equation*}
\alpha_{m n}+\alpha_{n m}=0, \quad m, n \in \mathbb{N}_{3} \cup\{0\} \tag{5.2}
\end{equation*}
$$

That is,

$$
M_{\varphi, \psi}=\left(\begin{array}{cccc}
0 & -\alpha_{10} & -\alpha_{20} & -\alpha_{30}  \tag{5.3}\\
\alpha_{10} & 0 & -\alpha_{21} & -\alpha_{31} \\
\alpha_{20} & \alpha_{21} & 0 & -\alpha_{32} \\
\alpha_{30} & \alpha_{31} & \alpha_{32} & 0
\end{array}\right)
$$

With the notations $a:=\alpha_{10}, b:=\alpha_{20}, c:=\alpha_{30}, \mathfrak{c}:=\alpha_{21}, \mathfrak{b}:=\alpha_{31}$ and $\mathfrak{a}:=\alpha_{32}$, the compatibility of $\varphi, \psi$ is described by

$$
\left(\begin{array}{cccc}
0 & -a & -b & -c \\
a & 0 & -\mathfrak{c} & -\mathfrak{b} \\
b & \mathfrak{c} & 0 & -\mathfrak{a} \\
c & \mathfrak{b} & \mathfrak{a} & 0
\end{array}\right) \cdot\left(\begin{array}{cccc}
0 & a & b & c \\
-a & 0 & \mathfrak{c} & \mathfrak{b} \\
-b & -\mathfrak{c} & 0 & \mathfrak{a} \\
-c & -\mathfrak{b} & -\mathfrak{a} & 0
\end{array}\right)=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

what means that

$$
\begin{align*}
a^{2}+b^{2}+c^{2} & =1,  \tag{5.4}\\
a^{2}+\mathfrak{c}^{2}+\mathfrak{b}^{2} & =1,  \tag{5.5}\\
b^{2}+\mathfrak{c}^{2}+\mathfrak{a}^{2} & =1,  \tag{5.6}\\
c^{2}+\mathfrak{b}^{2}+\mathfrak{a}^{2} & =1,  \tag{5.7}\\
b \mathfrak{c}+c \mathfrak{b} & =0,  \tag{5.8}\\
b c+\mathfrak{c b} & =0,  \tag{5.9}\\
-a \mathfrak{c}+c \mathfrak{a} & =0,  \tag{5.10}\\
a c-\mathfrak{c a} & =0,  \tag{5.11}\\
a \mathfrak{b}+b \mathfrak{a} & =0, \tag{5.12}
\end{align*}
$$

[^1]\[

$$
\begin{equation*}
a b+\mathfrak{b a}=0 \tag{5.13}
\end{equation*}
$$

\]

Combining (5.4) and (5.5) gives

$$
\begin{equation*}
b^{2}+c^{2}=\mathfrak{c}^{2}+\mathfrak{b}^{2} \tag{5.14}
\end{equation*}
$$

and from (5.6) and (5.7)

$$
\begin{equation*}
b^{2}+\mathfrak{c}^{2}=c^{2}+\mathfrak{b}^{2} \tag{5.15}
\end{equation*}
$$

Then substracting, both hand-side of (5.14) and (5.15) yields

$$
\begin{equation*}
c^{2}-\mathfrak{c}^{2}=\mathfrak{c}^{2}-c^{2} \Longrightarrow c^{2}=\mathfrak{c}^{2} . \tag{5.16}
\end{equation*}
$$

Substituting (5.16) into (5.14) we get

$$
\begin{equation*}
b^{2}=\mathfrak{b}^{2} \tag{5.17}
\end{equation*}
$$

Moreover, by (5.16), (5.4) and (5.6) we have

$$
\begin{equation*}
a^{2}=\mathfrak{a}^{2} \tag{5.18}
\end{equation*}
$$

It follows directly from (5.4) that at least one of the numbers $a, b$ or $c$ is non-zero. Without loss of generality we can assume $a \neq 0$, then two cases appear:

- If $a=\mathfrak{a}$, then $(5.12) \Rightarrow \mathfrak{b}=-b$ and from (5.10) we get $\mathfrak{c}=c$. It suffices to prove that

$$
\begin{equation*}
(a, b, c, \mathfrak{c}, \mathfrak{b}, \mathfrak{a})=(a, b, c, c,-b, a) \tag{5.19}
\end{equation*}
$$

is a solution of (5.4)-(5.13).

- If $a=-\mathfrak{a}$, then $(5.12) \Rightarrow \mathfrak{b}=b$ and from (5.10) we have $\mathfrak{c}=-c$. It remains to prove that

$$
\begin{equation*}
(a, b, c, \mathfrak{c}, \mathfrak{b}, \mathfrak{a})=(a, b, c,-c, b,-a) \tag{5.20}
\end{equation*}
$$

is a solution of (5.4)-(5.13),
and this is precisely the assertion of the proposition.
Remark 5.4 We have thus proved that for every compatible structural sets $\varphi, \psi$ the following equalities hold:

$$
M_{\psi, \varphi}=M_{\varphi, \psi}^{T}=-M_{\varphi, \psi}
$$

Consequently, if $M_{\varphi, \psi}$ is of the form $M_{\varphi, \psi}^{1}\left(\operatorname{resp} . M_{\varphi, \psi}^{2}\right)$ then $M_{\psi, \varphi}$ is also of the form $M_{\varphi, \psi}^{1}$ (resp. $M_{\varphi, \psi}^{2}$ ). It makes sense to introduce the following definition.

Definition 5.5 Call two structural sets $\varphi$, $\psi$ compatible of first type (resp. second type) if they are compatible and the transition matrix $M_{\varphi, \psi}$ is of the form $M_{\varphi, \psi}^{1}\left(\right.$ resp. $\left.M_{\varphi, \psi}^{2}\right)$.

Let us mention two important consequences of the Proposition 5.3, which may be proved following a direct computations.

Proposition 5.6 Let $\varphi, \psi$ be two structural sets. Then, the following conditions are equivalent:
(i) $\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}}=-1$.
(ii) $\varphi, \psi$ are compatible of first type only if they are also left equivalent.
(iii) $\varphi, \psi$ are compatible of secound type only if they are also right equivalent.

Proposition 5.7 Let $\varphi, \psi$ be two structural sets. Then, the following conditions are equivalent:
(i) $\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}}=1$.
(ii) $\varphi, \psi$ are compatible of first type only if they are also right equivalent.
(iii) $\varphi, \psi$ are compatible of secound type only if they are also left equivalent.

Remark 5.8 These propositions insinuate that both left and right equivalence with a structural set $\psi$ keep unalterable the product $\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}}$. Indeed, for example if $\varphi, \psi$ are left equivalent we have

$$
\varphi^{0} \cdot \overline{\varphi^{1}} \cdot \varphi^{2} \cdot \overline{\varphi^{3}}=h \psi^{0} \cdot \overline{\psi^{1}} \bar{h} \cdot h \psi^{2} \cdot \overline{\psi^{3}} \bar{h}=h\left(\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}}\right) \bar{h}=\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}} .
$$

Likewise, if $\varphi=\psi h$ gives

$$
\varphi^{0} \cdot \overline{\varphi^{1}} \cdot \varphi^{2} \cdot \overline{\varphi^{3}}=\psi^{0} h \cdot \bar{h} \overline{\psi^{1}} \cdot \psi^{2} h \cdot \bar{h} \overline{\psi^{3}}=\psi^{0} \cdot \overline{\psi^{1}} \cdot \psi^{2} \cdot \overline{\psi^{3}} .
$$

From what has been already disposed, and making use of Proposition 5.2, we set the following
Theorem 5.9 Two structural sets $\varphi, \psi$ are compatible if and only if there exists $h \in \mathbb{H}$ with $|h|=1$ and $\operatorname{Re}\{h\}=0$, such that either $\varphi=h \psi$ or $\varphi=\psi h$.

We conclude this section with the explicit description of the desired conditions for a pair of structural sets to be used mainly in our work. To this aim, we define the concept of admissible pair of structural sets.

Definition 5.10 Two structural sets $\varphi, \psi$ are called admissible if $\varphi=\psi h, h \in \mathbb{H},|h|=1$, such that $\operatorname{Re}\{h\}=0$.

## 6 Integral formulas for ( $\varphi, \psi$ )-hyperholomorphic functions

This section focusses on integral representation formulas for $(\varphi, \psi)$-hyperholomorphic functions. The only tools needed are Stokes formula and the fundamental solution to the Cauchy-Riemann operator $\varphi, \psi \mathbb{D}$, for every admissible structural sets $\varphi, \psi$.

With the notations

$$
\mathbf{N}_{\varphi, \psi}:=\left(\begin{array}{ll}
n_{\varphi} & n_{\psi} \\
n_{\psi} & n_{\varphi}
\end{array}\right) \quad \text { and } \quad d \mathbf{S}_{\xi}:=\left(\begin{array}{cc}
d S_{\xi} & 0 \\
0 & d S_{\xi}
\end{array}\right)
$$

we have
Theorem 6.1 (Stokes formula) Let $\mathbf{F}, \mathbf{G} \in C^{1}\left(\Omega \cup \Gamma, \mathbf{C M}_{H 1}^{2 \times 2}\right)$. Then

$$
\int_{\Gamma} \mathbf{G}(\xi) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}=\int_{\Omega}\left[\mathbb{D}^{\varphi, \psi}[\mathbf{G}](\xi) \cdot \mathbf{F}(\xi)+\mathbf{G}(\xi) \cdot{ }^{\varphi, \psi} \mathbf{D}[\mathbf{F}](\xi)\right] d \xi .
$$

Proof. Let

$$
\mathbf{F}=\left(\begin{array}{ll}
f_{1} & f_{2} \\
f_{2} & f_{1}
\end{array}\right), \quad \mathbf{G}=\left(\begin{array}{ll}
g_{1} & g_{2} \\
g_{2} & g_{1}
\end{array}\right),
$$

where $f_{1}, f_{2}, g_{1}, g_{2} \in C^{1}(\Omega \cup \Gamma, \mathbb{H})$. Hence,
$\mathbf{G} \cdot \mathbf{N}_{\varphi, \psi} \cdot \mathbf{F}=\left(\begin{array}{ll}\left(g_{1} n_{\varphi} f_{1}+g_{2} n_{\psi} f_{1}\right)+\left(g_{1} n_{\psi} f_{2}+g_{2} n_{\varphi} f_{2}\right) & \left(g_{1} n_{\varphi} f_{2}+g_{2} n_{\psi} f_{2}\right)+\left(g_{1} n_{\psi} f_{1}+g_{2} n_{\varphi} f_{1}\right) \\ \left(g_{2} n_{\varphi} f_{1}+g_{1} n_{\psi} f_{1}\right)+\left(g_{2} n_{\psi} f_{2}+g_{1} n_{\varphi} f_{2}\right) & \left(g_{2} n_{\varphi} f_{2}+g_{1} n_{\psi} f_{2}\right)+\left(g_{2} n_{\psi} f_{1}+g_{1} n_{\varphi} f_{1}\right)\end{array}\right)$.
Moreover,

$$
\mathbb{D}^{\varphi, \psi}[\mathbf{G}] \cdot \mathbf{F}=\left(\begin{array}{ll}
\left(D^{\varphi}\left[g_{1}\right] f_{1}+D^{\psi}\left[g_{2}\right] f_{1}\right)+\left(D^{\varphi}\left[g_{2}\right] f_{2}+D^{\psi}\left[g_{1}\right] f_{2}\right) & \left(D^{\varphi}\left[g_{1}\right] f_{2}+D^{\psi}\left[g_{2}\right] f_{2}\right)+\left(D^{\varphi}\left[g_{2}\right] f_{1}+D^{\psi}\left[g_{1}\right] f_{1}\right) \\
\left(D^{\psi}\left[g_{1}\right] f_{1}+D^{\varphi}\left[g_{2}\right] f_{1}\right)+\left(D^{\psi}\left[g_{2}\right] f_{2}+D^{\varphi}\left[g_{1}\right] f_{2}\right) & \left(D^{\psi}\left[g_{1}\right] f_{2}+D^{\varphi}\left[g_{2}\right] f_{2}\right)+\left(D^{\psi}\left[g_{2}\right] f_{1}+D^{\varphi}\left[g_{1}\right] f_{1}\right)
\end{array}\right),
$$

and
$\mathbf{G} \cdot \cdot^{\varphi, \psi} \mathbf{D}[\mathbf{F}]=\left(\begin{array}{ll}\left(g_{1}{ }^{\varphi} D\left[f_{1}\right]+g_{1}{ }^{\psi} D\left[f_{2}\right]\right)+\left(g_{2}{ }^{\psi} D\left[f_{1}\right]+g_{2}{ }^{\varphi} D\left[f_{2}\right]\right) & \left(g_{1} D\left[f_{2}\right]+g_{1}{ }^{\psi} D\left[f_{1}\right]\right)+\left(g_{2}{ }^{\psi} D\left[f_{2}\right]+g_{2}{ }^{\varphi} D\left[f_{1}\right]\right) \\ \left(g_{2}{ }^{\varphi} D\left[f_{1}\right]+g_{2}{ }^{\psi} D\left[f_{2}\right]\right)+\left(g_{1}{ }^{\psi} D\left[f_{1}\right]+g_{1}{ }^{\varphi} D\left[f_{2}\right]\right) & \left(g_{2}{ }^{\varphi} D\left[f_{2}\right]+g_{2}{ }^{\psi} D\left[f_{1}\right]\right)+\left(g_{1}{ }^{\psi} D\left[f_{2}\right]+g_{1}{ }^{\varphi} D\left[f_{1}\right]\right)\end{array}\right)$.
Thus, it suffices to show that in every entries of the matrices $\mathbf{G} \cdot \mathbf{N}_{\varphi, \psi} \cdot \mathbf{F}$ and $\mathbb{D}^{\varphi, \psi}[\mathbf{G}] \cdot \mathbf{F}+\mathbf{G} \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}]$ we can use the quaternionic Stokes formula (2.10) and the proof is complete.

The following version of the Cauchy integral theorem follows

Theorem 6.2 (Cauchy integral theorem) Let $\mathbf{F} \in{ }^{\Phi, \Psi} \mathfrak{M}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$ and $\mathbf{G} \in \mathfrak{M}^{\Phi, \Psi}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$. Then

$$
\int_{\Gamma} \mathbf{G}(\xi) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}=0
$$

Introducing the fundamental solution $\mathcal{K}_{\varphi, \psi}$ gives the Borel-Pompeiu formula.
Theorem 6.3 (Borel-Pompeiu formula) Let $\mathbf{F} \in C^{1}\left(\Omega \cup \Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$. Then

$$
\begin{aligned}
& \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\int_{\Omega} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \varphi, \psi \mathbb{D}[\mathbf{F}](\xi) d \xi \\
& \quad= \begin{cases}2 \mathbf{F}(x) & \text { if } x \in \Omega \\
0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma) .\end{cases}
\end{aligned}
$$

Proof. We first assume $x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma)$. Then

$$
\mathcal{K}_{\varphi, \psi}(\xi-x) \in \mathfrak{M}^{\Phi, \Psi}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right) \cap C^{0}\left(\Omega \cup \Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)
$$

Applying Theorem 6.1, yields

$$
\begin{aligned}
& \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
& \quad=\int_{\Omega}\left[\left(\mathbb{D}^{\varphi, \psi} \mathcal{K}_{\varphi, \psi}(\xi-x)\right) \cdot \mathbf{F}(\xi)+\mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbf{D}[\mathbf{F}](\xi)\right] d \xi \\
& \quad \Rightarrow \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\int_{\Omega} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi=0
\end{aligned}
$$

Suppose now that $x \in \Omega$. For $\epsilon$ small, let $B(x, \epsilon) \subset \Omega$. Applying Theorem 6.1 to the region $\Omega_{x, \epsilon}$, we see that

$$
\begin{align*}
\int_{\partial \Omega_{x, \epsilon}} & \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
= & \int_{\Omega_{x, \epsilon}}\left[\left(\mathbb{D}^{\varphi, \psi} \mathcal{K}_{\varphi, \psi}(\xi-x)\right) \cdot \mathbf{F}(\xi)+\mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi)\right] d \xi \\
= & \int_{\Omega_{x, \epsilon}} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi \tag{6.1}
\end{align*}
$$

Letting $\epsilon \rightarrow 0$, we get

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \int_{\Omega_{x, \epsilon}} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbf{D}[\mathbf{F}](\xi) d \xi=\int_{\Omega} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi \tag{6.2}
\end{equation*}
$$

On the other hand, we can write

$$
\begin{aligned}
\int_{\partial \Omega_{x, \epsilon}} & \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
\quad= & \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\int_{\partial B(x, \epsilon)} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}
\end{aligned}
$$

The task is now to calculate the last integral in the above expression. For all $\xi \in \partial B(x, \epsilon)$

$$
\mathcal{K}_{\varphi, \psi}(\xi-x)=\frac{1}{2 \pi^{2} \epsilon^{4}} \mathbf{G}_{\varphi, \psi}(\xi-x)
$$

where

$$
\mathbf{G}_{\varphi, \psi}(\xi-x):=\left(\begin{array}{ll}
(\xi-x)_{\bar{\varphi}} & (\xi-x)_{\bar{\psi}} \\
(\xi-x)_{\bar{\psi}} & (\xi-x)_{\bar{\varphi}}
\end{array}\right)
$$

Hence,

$$
\begin{aligned}
& \int_{\partial B(x, \epsilon)} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
& \quad=\frac{1}{2 \pi^{2} \epsilon^{4}} \int_{\partial B(x, \epsilon)} \mathbf{G}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}
\end{aligned}
$$

Since $\mathbf{G}_{\varphi, \psi}(\xi-x) \in C^{\infty}\left(\mathbb{R}^{4}, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$, repeated application of Stokes formula enables us to write

$$
\begin{aligned}
& \int_{\partial B(x, \epsilon)} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
& \quad=\frac{1}{2 \pi^{2} \epsilon^{4}}\left[\int_{B(x, \epsilon)}\left(\mathbb{D}^{\varphi, \psi} \mathbf{G}_{\varphi, \psi}(\xi-x)\right) \cdot \mathbf{F}(\xi) d \xi+\int_{B(x, \epsilon)} \mathbf{G}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi\right]
\end{aligned}
$$

Taking $\mathbf{F} \in C^{1}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right) \Rightarrow{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}] \in C^{0}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$, then there exists $M>0$ such that

$$
\left\|^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi)\right\| \leq M, \quad \forall \xi \in B(x, \epsilon)
$$

Note that

$$
\begin{aligned}
\left\|\frac{1}{2 \pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)} \mathbf{G}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi\right\| & \leq \frac{M}{2 \pi^{2} \epsilon^{4}} \sup _{\xi \in B(x, \epsilon)}\left\|\mathbf{G}_{\varphi, \psi}(\xi-x)\right\| \frac{\pi^{2}}{2} \epsilon^{4} \\
& =\frac{M}{4} \sup _{\xi \in B(x, \epsilon)}\left\|\mathbf{G}_{\varphi, \psi}(\xi-x)\right\|
\end{aligned}
$$

Since $\mathbf{G}_{\varphi, \psi}$ is a continuous function in $\mathbb{R}^{4}$ we get $\sup _{\xi \in B(x, \epsilon)}\left\|\mathbf{G}_{\varphi, \psi}(\xi-x)\right\| \rightarrow 0$ as $\epsilon \rightarrow 0$.
It is important that $M$ still satisfies the above inequality as $\epsilon \rightarrow 0$, then we conclude that,

$$
\lim _{\epsilon \rightarrow 0} \frac{1}{2 \pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)} \mathbf{G}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbf{D}[\mathbf{F}](\xi) d \xi=0
$$

We continue this fashion to claim

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{1}{2 \pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)}\left(\mathbb{D}^{\varphi, \psi} \mathbf{G}_{\varphi, \psi}(\xi-x)\right) \cdot \mathbf{F}(\xi) d \xi=2 \mathbf{F}(x) \tag{6.3}
\end{equation*}
$$

Indeed,

$$
\begin{aligned}
\mathbb{D}^{\varphi, \psi} \mathbf{G}_{\varphi, \psi}(\xi-x) & =\left(\begin{array}{cc}
D^{\varphi} & D^{\psi} \\
D^{\psi} & D^{\varphi}
\end{array}\right) \cdot\left(\begin{array}{cc}
(\xi-x)_{\bar{\varphi}} & (\xi-x)_{\bar{\psi}} \\
(\xi-x)_{\bar{\psi}} & (\xi-x)_{\bar{\varphi}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\varphi}}\right]+D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\psi}}\right] & D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\psi}}\right]+D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\varphi}}\right] \\
D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\varphi}}\right]+D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\psi}}\right] & D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\psi}}\right]+D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\varphi}}\right]
\end{array}\right) .
\end{aligned}
$$

But $(\xi-x)_{\bar{\psi}}=\sum_{n=0}^{3}\left(\xi_{n}-x_{n}\right) \overline{\psi^{n}}$, then $\partial_{\xi_{n}}\left[(\xi-x)_{\bar{\psi}}\right]=\overline{\psi^{n}}, n \in \mathbb{N}_{3} \cup\{0\}$, we have

$$
D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\psi}}\right]=\sum_{n=0}^{3} \partial_{\xi_{n}}\left[(\xi-x)_{\bar{\psi}}\right] \cdot \psi^{n}=\sum_{n=0}^{3} \overline{\psi^{n}} \cdot \psi^{n}=4
$$

Likewise, $D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\varphi}}\right]=4$. On the other hand,

$$
\begin{aligned}
D_{\xi}^{\varphi}\left[(\xi-x)_{\bar{\psi}}\right]+D_{\xi}^{\psi}\left[(\xi-x)_{\bar{\varphi}}\right] & =\sum_{n=0}^{3}\left[\partial_{\xi_{n}}\left[(\xi-x)_{\bar{\psi}}\right] \cdot \varphi^{n}+\partial_{\xi_{n}}\left[(\xi-x)_{\bar{\varphi}}\right] \cdot \psi^{n}\right] \\
& =\sum_{n=0}^{3}\left(\overline{\psi^{n}} \cdot \varphi^{n}+\overline{\varphi^{n}} \cdot \psi^{n}\right)=\sum_{n=0}^{3} \operatorname{Re}\left\{\overline{\psi^{n}} \cdot \varphi^{n}\right\}=0
\end{aligned}
$$

Therefore,

$$
\mathbb{D}^{\varphi, \psi} \mathbf{G}_{\varphi, \psi}(\xi-x)=\left(\begin{array}{ll}
8 & 0 \\
0 & 8
\end{array}\right)
$$

and then,

$$
\begin{aligned}
& \left\|\frac{1}{2 \pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)}\left(\mathbb{D}^{\varphi, \psi} \mathbf{G}_{\varphi, \psi}(\xi-x)\right) \cdot \mathbf{F}(\xi) d \xi-2 \mathbf{F}(x)\right\| \\
& \quad=\left\|\frac{4}{\pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)} \mathbf{F}(\xi) d \xi-2 \mathbf{F}(x)\right\| \\
& \quad=\left\|\frac{4}{\pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)} \mathbf{F}(\xi) d \xi-\frac{4}{\pi^{2} \epsilon^{4}} \int_{B(x, \epsilon)} \mathbf{F}(x) d \xi\right\| \\
& \quad=\frac{4}{\pi^{2} \epsilon^{4}}\left\|\int_{B(x, \epsilon)}[\mathbf{F}(\xi)-\mathbf{F}(x)] d \xi\right\| \leq 2 \cdot\|\mathbf{F}-\mathbf{F}(x)\|_{B(x, \epsilon)}
\end{aligned}
$$

where

$$
\|\mathbf{F}-\mathbf{F}(x)\|_{B(x, \epsilon)}:=\sup _{\xi \in B(x, \epsilon)}\|\mathbf{F}(\xi)-\mathbf{F}(x)\|
$$

But $\|\mathbf{F}-\mathbf{F}(x)\|_{B(x, \epsilon)} \rightarrow 0$ when $\epsilon \rightarrow 0$ since $\mathbf{F}$ is continuous at $x \in \Omega$.
Therefore, (6.3) is true and then

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0} \int_{\partial \Omega_{x, \epsilon}} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
& \quad=\int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\lim _{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon)} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
& \quad=\int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-2 \mathbf{F}(x) \tag{6.4}
\end{align*}
$$

Substituting (6.2) and (6.4) into (6.1) we conclude the proof.
The Borel-Pompeiu formula immediately implies the following analogue of the Cauchy integral formula.
Theorem 6.4 (Cauchy integral formula) Let $\mathbf{F} \in C^{0}\left(\Omega \cup \Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right) \cap{ }^{\Phi, \Psi} \mathfrak{M}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$. Then

$$
\int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}= \begin{cases}2 \mathbf{F}(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma)\end{cases}
$$

The Cauchy integral formula leads to the notion of the Cauchy type integral

$$
\begin{equation*}
\int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \tag{6.5}
\end{equation*}
$$

which, for $\mathbf{F} \in C^{0, \beta}\left(\Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$, represents a $(\varphi, \psi)$-hyperholomorphic function in $\mathbb{R}^{4} \backslash \Gamma$. Introducing now the temporary notation $\Omega^{+}=\Omega$ and $\Omega^{-}=\mathbb{R}^{4} \backslash(\Omega \cup \Gamma)$, the boundary behaviour on $\Gamma$ of (6.5), also
known as the Plemelj-Sokhotzki formulae:

$$
\begin{aligned}
& \lim _{x \rightarrow x_{0}, x \in \Omega^{ \pm}, x_{0} \in \Gamma} \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
= & \left( \pm \mathbf{F}\left(x_{0}\right)+\int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}\right)
\end{aligned}
$$

are true. As usual, the Cauchy singular integral

$$
\int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \quad x_{0} \in \Gamma
$$

exists in the sense of the principal value and defines a linear bounded operator on $C^{0, \beta}\left(\Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$.
Theorem 6.5 Let $\mathbf{F} \in C^{0, \beta}\left(\Gamma, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$. Then we have

- In order that $\mathbf{F}$ being a boundary value of a function belonging to ${ }^{\Phi, \Psi} \mathfrak{M}\left(\Omega, \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$, it is necessary and sufficient that

$$
\mathbf{F}\left(x_{0}\right)=\int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \quad x_{0} \in \Gamma
$$

- In order that $\mathbf{F}$ being a boundary value of a function belonging to ${ }^{\Phi, \Psi} \mathfrak{M}\left(\mathbb{R}^{4} \backslash(\Omega \cup \Gamma), \mathbf{C M}_{\mathbb{H}}^{2 \times 2}\right)$ which vanishes at infinity, it is necessary and sufficient that

$$
-\mathbf{F}\left(x_{0}\right)=\int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \quad x_{0} \in \Gamma
$$

The last theorem may be obtained by standard proof (see, c.f. [18], [19], [21]). We just remark that

$$
K_{\varphi} n_{\psi}+K_{\psi} n_{\varphi}=-h K_{\psi} n_{\psi}+h K_{\varphi} n_{\varphi}
$$

under our condition $\varphi=\psi h$.

## 7 Matrix approach of reflection type

In the Section 4 we introduced the notion of $(\varphi, \psi)$-hyperholomorphic function in terms of a Cauchy-Riemann operator based on $2 \times 2$ circulant matrices. Since in this case the factorization of the Laplacian in terms of the operator ${ }^{\varphi, \psi} \mathbb{D}$ is equivalent to the condition $\varphi=h \psi$ or $\varphi=\psi h$ with $\operatorname{Re}\{h\}=0$ we called it the rotational case. But there are examples like $\varphi=\psi$ or $\varphi=\bar{\psi}$ (the latter including the case where functions are at the same time belonging to the kernel of $D$ and of $\bar{D}$, i.e. functions which are sometimes called hyperholomorphic constants in the literature) for which the condition is not fulfilled, i.e. there is no quaternion $h$ with $\operatorname{Re}\{h\}=0$, such that $\varphi=h \varphi$ or $\varphi=h \bar{\varphi}$.

We now outline shortly how to treat these cases which we call the reflection case. The fundamental point is that circulant matrices are not the only subalgebra of two-dimensional quaternionic matrices. This allows us to overcome the above problem by modify our operator using other embeddings into two-dimensional matrices with quaternionic entries.

The principal problem hereby is to find embeddings which allow the corresponding second-order operator to be scalar, which in the previous section lead us to condition (4.4). Having in mind the above mentioned examples one can consider the matrix operators

$$
{ }^{\varphi, \psi} \mathbb{D}=\left(\begin{array}{cc}
{ }^{\varphi} D & { }^{\psi} D  \tag{7.1}\\
-{ }^{\psi} D & { }^{\varphi} D
\end{array}\right) \quad \text { and } \quad{ }^{\varphi, \psi} D=\left(\begin{array}{cc}
D^{\varphi} & D^{\psi} \\
0 & D^{\varphi}
\end{array}\right)
$$

In the first case we have an embedding into orthogonal matrices while in the second we have the upper-triangular matrices. Naturally, 2-matrices of the form

$$
A=\left(\begin{array}{cc}
x & y \\
-y & x
\end{array}\right)
$$

generate a ring over $\mathbb{H}$, denoted by $\mathbf{J} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$. In fact this ring is isomorphic to the algebra of complex quaternions. This can easily be observed from the fact that

$$
A=x I+y J, \quad x, y \in \mathbb{H}
$$

with

$$
I=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \quad \text { and } \quad J=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

Hereby, $J^{2}=-I$. In the second case we have the algebra $\tilde{\mathbf{J}} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$ generated by

$$
I=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \quad \text { and } \quad \tilde{J}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right)
$$

Hereby, $\tilde{J}^{2}=0$. This algebra corresponds to the algebra of upper triangular matrices with quaternionic scalars and is isomorphic to the algebra of parabolic numbers with quaternionic scalars.

In the first case using the usual conjugation (transposition of the matrix plus conjugation of the elements)

$$
\bar{A}=\overline{\left(\begin{array}{cc}
x & y \\
-y & x
\end{array}\right)}=\left(\begin{array}{cc}
\bar{x} & -\bar{y} \\
\bar{y} & \bar{x}
\end{array}\right)
$$

we have $\overline{A \lambda}=\bar{\lambda} \bar{A}, \lambda \in \mathbb{H}$ as well as $\overline{A B}=\bar{B} \bar{A}$. It is easy to check that in this case we have

$$
\begin{aligned}
{ }^{\varphi, \psi} \mathbb{D}^{\overline{\varphi, \psi}} \mathbf{D} & =\left(\begin{array}{cc}
{ }^{\varphi} D & \psi \\
-\psi \\
-\psi & \varphi \\
\hline
\end{array}\right)\left(\begin{array}{cc}
\bar{\varphi} D & -\bar{\psi} D \\
\bar{\psi} D & { }^{\bar{\varphi}} D
\end{array}\right) \\
& =\left(\begin{array}{cc}
2 \Delta_{\mathbb{H}} & -{ }^{\varphi} D^{\bar{\psi}} D+{ }^{\psi} D^{\bar{\varphi}} D \\
-{ }^{\psi} D^{\bar{\varphi}} D+{ }^{\varphi} D^{\bar{\psi}} D & 2 \Delta_{\mathbb{H}}
\end{array}\right)
\end{aligned}
$$

from which we obtain the condition

$$
\begin{equation*}
-{ }^{\psi} D^{\bar{\varphi}} D+{ }^{\varphi} D^{\bar{\psi}} D=0 \tag{7.2}
\end{equation*}
$$

for a factorization of the Laplacian. This is equivalent to

$$
{ }^{\varphi} D^{\bar{\psi}} D-\bar{\varphi} D^{\bar{\psi}} D=0
$$

or, with other words,

$$
V e c\left\{{ }^{\varphi} D^{\bar{\psi}} D\right\}=0
$$

The last formula means that ${ }^{\varphi} D^{\bar{\psi}} D$ is a scalar operator. In the second case we have

$$
{ }^{\varphi, \psi} \mathbb{D}^{\varphi,-\psi} \mathbb{D}=\left(\begin{array}{cc}
{ }^{\varphi} D & { }^{\psi} D \\
0 & { }^{\varphi} D
\end{array}\right)\left(\begin{array}{cc}
{ }^{\varphi} D & -{ }^{\psi} D \\
0 & { }^{\varphi} D
\end{array}\right)=\left(\begin{array}{cc}
{ }^{\varphi} D^{\varphi} D & -{ }^{\varphi} D^{\psi} D+{ }^{\psi} D^{\varphi} D \\
0 & { }^{\varphi} D^{\varphi} D
\end{array}\right)
$$

which leads to the condition

$$
\begin{equation*}
-{ }^{\psi} D^{\varphi} D+{ }^{\varphi} D^{\psi} D=0 \tag{7.3}
\end{equation*}
$$

Let us remark that while in the first case we have $\varphi=\psi$ as an example in the second case $\psi=\bar{\varphi}$ is a possible choice.

As in the proof of Theorem 6.3, the Stokes' formula gives

$$
\begin{aligned}
& \int_{\Omega}\left[\left(f_{1} I+f_{2} J\right)\left(D^{\varphi} I+D^{\psi} J\right)\right]\left(g_{1} I+g_{2} J\right)+\left(f_{1} I+f_{2} J\right)\left[\left({ }^{\varphi} D I+{ }^{\psi} D J\right)\left(g_{1} I+g_{2} J\right)\right] d \xi \\
& \quad=\int_{\Gamma}\left(f_{1} I+f_{2} J\right)\left(n_{\varphi} I+n_{\psi} J\right)\left(g_{1} I+g_{2} J\right) d S_{\xi}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\int_{\Omega} & {\left[\left(f_{1} I+f_{2} \tilde{J}\right)\left(D^{\varphi} I+D^{\psi} \tilde{J}\right)\right]\left(g_{1} I+g_{2} \tilde{J}\right)+\left(f_{1} I+f_{2} \tilde{J}\right)\left[\left({ }^{\varphi} D I+{ }^{\psi} D \tilde{J}\right)\left(g_{1} I+g_{2} J\right)\right] d \xi } \\
& =\int_{\Gamma}\left(f_{1} I+f_{2} \tilde{J}\right)\left(n_{\varphi} I+n_{\psi} \tilde{J}\right)\left(g_{1} I+g_{2} \tilde{J}\right) d S_{\xi}
\end{aligned}
$$

and we get the corresponding Borel-Pompeiu and the Cauchy formulae for the reflection case.
By abuse of notation, we use the same letters $\mathbf{N}_{\varphi, \psi}$ and $\mathcal{K}_{\varphi, \psi}$ for the associated normal vector and Cauchy kernel matrices in the reflection case. That means $\mathbf{N}_{\varphi, \psi}(\xi)=\left(n_{\varphi}(\xi) I+n_{\psi}(\xi) J\right)$ and $\mathcal{K}_{\varphi, \psi}(\xi)=\left(K_{\varphi}(\xi) I-K_{\psi}(\xi) J\right)$ in the first case and $\mathbf{N}_{\varphi, \psi}(\xi)=\left(n_{\varphi}(\xi) I+n_{\psi}(\xi) \tilde{J}\right)$ and $\mathcal{K}_{\varphi, \psi}(\xi)=\left(K_{\varphi}(\xi) I-K_{\psi}^{*}(\xi) \tilde{J}\right)$ with

$$
K_{\psi}^{*}=\frac{x_{0} \bar{\varphi}^{0} \psi^{0}+\overline{x_{\varphi}} \psi^{0}}{|x|^{4}}-\frac{4 x_{0}^{2} \overline{x_{\varphi}} \psi^{0}}{|x|^{6}}+\frac{x_{0} \sum_{k=1}^{3} \overline{\varphi^{k}} \psi^{k}}{|x|^{4}}-\frac{4 x_{0} \overline{x_{\varphi}} \sum_{k=1}^{3} \psi^{k} x_{k}}{|x|^{6}}
$$

in the second case.
In the case of $\mathbf{J M}_{\mathbb{H}}^{2 \times 2}$ we get the following theorems.
Theorem 7.1 Let $\mathbf{F} \in C^{1}\left(\Omega, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right) \cap C^{0}\left(\Omega \cup \Gamma\right.$, $\left.\mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right)$. Then

$$
\begin{aligned}
& \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\int_{\Omega} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \varphi, \psi \mathbb{D}[\mathbf{F}](\xi) d \xi \\
& \quad= \begin{cases}2 \mathbf{F}(x) & \text { if } x \in \Omega, \\
0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma) .\end{cases}
\end{aligned}
$$

The proof is just an extension of the scalar Borel-Pompeiu and Cauchy formulae for $\psi$-hyperholomorphic functions. Since our condition (7.2) implies that ${ }^{\varphi} D^{\bar{\psi}} D$ is a scalar operator we immediately get

$$
\left({ }^{\varphi} D I+{ }^{\psi} D J\right) \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}=0, \quad x \notin \Gamma,
$$

for all functions $\mathbf{F} \in C^{0, \beta}\left(\Gamma, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$, since ${ }^{\varphi} D K_{\psi}={ }^{\psi} D K_{\varphi}$.
This represents the corresponding Cauchy type integral

$$
\begin{equation*}
\int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \tag{7.4}
\end{equation*}
$$

which is bounded if $\mathbf{F} \in C^{0, \beta}\left(\Gamma, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$. For the kernel we have $\mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi)=\left(K_{\varphi}(\xi-\right.$ $\left.x) n_{\varphi}(\xi)-K_{\psi}(\xi-x) n_{\psi}(\xi)\right) I+\left(K_{\varphi}(\xi-x) n_{\psi}(\xi)+K_{\psi}(\xi-x) n_{\varphi}(\xi)\right) J$. Using the classic Plemelj-Sokhotzki formulae and the fact that $\operatorname{Vec}\left\{x_{\varphi} x_{\bar{\psi}}\right\}=0$ we get the matrix Plemelj-Sokhotzki formulae:

$$
\begin{aligned}
& x \rightarrow x_{0}, x \in \Omega^{ \pm}, x_{0} \in \Gamma \\
& \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi} \\
&=\left( \pm \mathbf{F}\left(x_{0}\right) J+\int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}\right) .
\end{aligned}
$$

This allows us to obtain by standard proofs the following theorem.
Theorem 7.2 Let $\mathbf{F} \in C^{0, \beta}\left(\Gamma, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right), 0<\beta \leq 1$. Then we have

- In order that $\mathbf{F}$ being a boundary value of a function belonging to ${ }^{\Phi, \Psi} \mathfrak{M}\left(\Omega, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right)$, it is necessary and sufficient that

$$
\mathbf{F}\left(x_{0}\right)=\frac{1}{2}(I+J) \int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \quad x_{0} \in \Gamma .
$$

- In order that $\mathbf{F}$ being a boundary value of a function belonging to ${ }^{\Phi, \Psi} \mathfrak{M}\left(\mathbb{R}^{4} \backslash(\Omega \cup \Gamma), \mathbf{J M}_{\mathbb{H}}^{2 \times 2}\right)$ which vanishes at infinity, it is necessary and sufficient that

$$
\mathbf{F}\left(x_{0}\right)=\frac{1}{2}(I-J) \int_{\Gamma} \mathcal{K}_{\varphi, \psi}\left(\xi-x_{0}\right) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}, \quad x_{0} \in \Gamma .
$$

Using the same arguments and taking into account condition (7.3) one can easily prove the corresponding theorem for the case of $\tilde{\mathbf{J}} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$.

Theorem 7.3 Let $\mathbf{F} \in C^{1}\left(\Omega, \tilde{\mathbf{J}} \mathbf{M}_{\mathbb{H}}^{2 \times 2}\right) \cap C^{0}\left(\Omega \cup \Gamma, \tilde{\mathbf{J}} \mathbf{M}_{\mathbb{H}}^{2 \times 2}\right)$. Then

$$
\begin{aligned}
& \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}-\int_{\Omega} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot{ }^{\varphi, \psi} \mathbb{D}[\mathbf{F}](\xi) d \xi \\
& \quad= \begin{cases}\mathbf{F}(\mathbf{x}) & \text { if } x \in \Omega \\
0 & \text { if } x \in \mathbb{R}^{4} \backslash(\Omega \cup \Gamma) .\end{cases}
\end{aligned}
$$

Now, keeping in mind condition (7.3) we have

$$
\left({ }^{\varphi} D I+{ }^{\psi} D \tilde{J}\right) \int_{\Gamma} \mathcal{K}_{\varphi, \psi}(\xi-x) \cdot \mathbf{N}_{\varphi, \psi}(\xi) \cdot \mathbf{F}(\xi) d \mathbf{S}_{\xi}=0, \quad x \notin \Gamma
$$

Remark 7.4 We would like to point out that $\mathbf{C M}_{\mathbb{H}}^{2 \times 2}, \mathbf{J M}_{\mathbb{H}}^{2 \times 2}$, and $\tilde{\mathbf{J}} \mathbf{M}_{\mathbb{H}}^{2 \times 2}$ are the only interesting twodimensional algebraic embeddings into two-by-two quaternionic matrices such that the resulting matrices form an algebra over the quaternions with quaternionic dimension 2 generated by $I$ and $J$ with $I$ being the identity element. From a true linear algebra point of view the only two-dimensional real-linear subspaces (or either left-or right-linear quaternionic modules) which contain the identity and are closed under multiplication are the subspaces given by the diagonal matrices

$$
\left(\begin{array}{ll}
x & 0 \\
0 & y
\end{array}\right)
$$

as well as by the matrices of the type

$$
\left(\begin{array}{cc}
x & y \\
\lambda y & x
\end{array}\right)
$$

with fixed $\lambda$, as well as the subspace given by their transposes (to include the case of $\lambda=0$ ). From the point of view of the authors the only interesting matrices (in the sense that they are isomorphic to an embedding into a higher dimensional Clifford-type algebra) are the cases where $\lambda=0,1,-1$.

This means that we add a basis element $J$ which commutes with quaternions (seen as scalar) and its square is either $\pm I$ or 0 . The corresponding construction for the case of real numbers can be found in [9], but also in many other places. There it leads to the three choices of complex, hyperbolic, and parabolic numbers. Here, it can also seen as a "complexification" of the quaternions, although a true complexified algebra (complex quaternions or biquaternions) are only obtained with $J^{2}=-I$.

## 8 Concluding remark

We have laid the foundations of a function theory centered around the notion of $(\varphi, \psi)$-hyperholomorphic functions. Basic results such as Borel-Pompeiu and Cauchy integral representation formulae were established using the fundamental solution of an associated matrix Cauchy Riemann operator, explicitly obtained through the usage of three alternative matrix approaches. The present work offers a building block for a deep study of this function theory, which the authors will continue in forthcoming papers.

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## References

[1] R. Abreu-Blaya, J. Bory-Reyes, A. Guzmán-Adán, and B. Schneider, Boundary value problems for the Cimmino system via quaternionic analysis, Appl. Math. Comput. 219, 3872-3881 (2012).
[2] F. Brackx, R. Delanghe, and F. Sommen, Clifford Analysis (Pitman-Longman, 1982).
[3] P. Cerejeiras, K. Gürlebeck, U. Kähler, and H. Malonek, A quaternionic Beltrami-type equation and the existence of local homeomorphic solutions, Z. Anal. Anwend. 20, 17-34 (2001).
[4] R. Fueter, Die Funktionentheorie der Differentialgleichungen $\Delta u=0$ und $\Delta \Delta u=0$ mit vier reellen Variablen, Comm. Math. Helv. 7, 307-330 (1935).
[5] R. Fueter, Über die analytische Darstellung der regulären Funktionen einer Quaternionvariablen, Comm. Math. Helv. 8, 371-378 (1936).
[6] K. Gürlebeck and W. Sprößig, Quaternionic Analysis and Elliptic Boundary Value Problems (Int. Ser. Num. Math. (ISNM): Vol. 89 (Basel: Birkhäuser Verlag, 1990).
[7] K. Gürlebeck and W. Sprößig, Quaternionic and Clifford Calculus for Physicists and Engineers (John Wiley \& Sons, 1997).
[8] K. Gürlebeck, U. Kähler, and M. Shapiro, On the П-operator in hyperholomorphic function theory, Adv. Appl. Clifford Algebras 9, 23-40 (1999).
[9] V. V. Kisil and D. Biswas, Elliptic, parabolic and hyperbolic analytic function theory, ArXiv, arXiv:math/0410399.
[10] V. V. Kravchenko, Applied Quaternionic Analysis, Research and Exposition in Mathematics Vol. 28 (Heldermann, Lemgo, 2003).
[11] V. V. Kravchenko and M. V. Shapiro, Integral Representations for Spatial Models of Mathematical Physics, Pitman Res. Notes in Math. Ser., 351 (Longman, Harlow, 1996).
[12] I. M. Mitelman and M. Shapiro, Differentiation of the Martinelli-Bochner integrals and the notion of hyperderivability, Math. Nachr. 172, 211-238 (1995).
[13] M. Naser, Hyperholomorphic functions, Siberian Math. J. 12, 959-968 (1971).
[14] K. Nôno, On the quaternion linearization of Laplacian $\Delta$, Bull. Fukuoka Univ. Ed. III 35, 5-10 (1986).
[15] K. Nôno, Hyperholomorphic functions of a quaternion variable, Bull. Fukuoka Univ. Ed. III 32, 21-37 (1983).
[16] M. V. Shapiro and N. L. Vasilevsky, On the Bergman Kernel function in hyperholomorphic analysis, Acta Appl. Math. 46, 1-27 (1997).
[17] M. Shapiro, Some remarks on generalizations of the one dimensional complex analysis: Hypercomplex approach, Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations (Trieste, 1993) (World Scienti. Publ., 1995), pp. 379-401.
[18] M. V. Shapiro and N. L. Vasilevski, Quaternionic $\psi$-hyperholomorphic functions, singular integral operators and boundary value problems. I. $\psi$-hyperholomorphic function theory, Complex Var. Theory Appl. 27, 17-46 (1995).
[19] V. I. Shevchenko, About local homeomorphisms in three-dimensional space, which satisfy some elliptic system, Doklady Akademii Nauk SSSR 5, 1035-1038 (1962).
[20] A. Sudbery, Quaternionic analysis, Math. Proc. Cambridge Philos. Soc. 85, 199-225 (1979).
[21] N. L. Vasilevsky and M. V. Shapiro, Some questions of hypercomplex analysis, Complex Analysis and Applications ' 87 (Varna, 1987), (Publ. House Bulgar. Acad. Sci., Sofia, 1989), pp. 523-531.
[22] N. L. Vasilevski and M. V. Shapiro, Holomorphy, hyperholomorphy, Teoplitz operators, Russian Mathematical Surveys 44(4), 196-197 (1989).


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[^1]:    ${ }^{1}$ Note that the determinant of both matrix appeared in (5.1) is equal to $\left(a^{2}+b^{2}+c^{2}\right)^{2}=1$.

