# CONTINUOUS WAVELET TRANSFORM AND WAVELET FRAMES ON THE SPHERE USING CLIFFORD ANALYSIS 

P. Cerejeiras, M. Ferreira, U. Kähler and F. Sommen<br>Department of Mathematics<br>University of Aveiro, P-3810-193 Aveiro, Portugal


#### Abstract

In this paper we construct a continuous wavelet transform (CWT) on the sphere $S^{n-1}$ based on the conformal group of the sphere, the Lorentz group $\operatorname{Spin}(1, n)$. For this purpose, we present a short survey on the existing techniques of continuous wavelet transform and of conformal transformations on the unit sphere. We decompose the conformal group into the maximal compact subgroup of rotations $\operatorname{Spin}(n)$ and the set of Möbius transformations of the form $\varphi_{a}(x)=(x-a)(1+a x)^{-1}$, where $a \in B^{n}$ and $B^{n}$ denotes the unit ball in $\mathbb{R}^{n}$. Based on a study of the influence of the parameter $a$ arising in the definition of dilations/contractions on the sphere we define a class of local conformal dilation operators and consequently a family of continuous wavelet transforms for the Hilbert space of square integrable functions on the sphere $L_{2}\left(S^{n-1}\right)$ and the Hardy space $H^{2}$. In the end we construct Banach frames for our wavelets and prove Jackson-type theorems for the best $n$-point approximation.


1. Introduction. In recent years the research in Fourier Analysis and Approximation Theory has been extended from the classical setting, i.e., from the investigation in $\mathbb{R}^{n}$ and $\mathbb{T}$, respectively, to the investigation of manifolds. One of the most important examples is the case of the unit sphere. Here, spherical harmonics and its derivates, in particular inner spherical monogenics, present a clear advantage. But while in a purely analytic setting it is not so easy to imagine situations where spherical harmonics - as polynomials - are not the best choice for approximation, this is not the same in cases where the approximation should lead to a numerical algorithm. In fact there are several problems arising for spherical monogenics, among others the fast increasing of the number of spherical monogenics of the same degree, the stability of the numerical algorithms for the orthonormalization process and the badly conditioned matrices for the best approximation.

On the other hand all these problems do not exist (or are easily solvable) in classical wavelet theory (c.f. [27, [17]), therefore, it seems only natural to construct wavelets on the sphere. A number of attempts have been made to extend wavelets to the sphere, mainly via stereographic projection or using reproducing kernel approximations based on spherical harmonics (c.f. 20] and references therein). The first satisfactory approach is that of Holschneider ([22]), introducing an abstract parameter that plays the role of dilations but has to fulfill a number of assumptions and is therefore difficult to compute. But, it is possible to introduce local dilations in a quite natural way on the sphere if one uses the conformal group, that

[^0]is, the Lorentz group $S O_{0}(1, n)$. In [3], the authors use, for the 2 -sphere, the Iwasawa decomposition of $S O_{0}(1,3)$ (or $K A N$-decomposition, where $K$ is the maximal compact subgroup, $A=S O_{0}(1,1) \cong \mathbb{R} \cong \mathbb{R}_{*}^{+}$is the subgroup of Lorentz boosts in the $z$-direction and $N \cong \mathbb{C}$ is a two dimensional abelian subgroup). They use the parameter space $X \cong S O_{0}(1,3) / N \cong S O(3) \cdot \mathbb{R}_{*}^{+}$, i.e., the product of $S O(3)$ for motions and $\mathbb{R}_{*}^{+}$for dilations on $S^{2}$. Here we want to remark that using this decomposition some information is lost. A generalization of this approach for the $n$-sphere is presented in [4]. For a more complete treatment of the spherical wavelet transform in this framework and the correspondence between spherical and Euclidean wavelets we refer to [32]. More recently, in [28], the authors extend the case of isotropic dilations on the 2 -sphere to the case of anisotropic dilations defined on the 2 -sphere in two orthogonal directions, obtaining a generalization of the CWT defined by Antoine and Vandergheynst [3]. They develop also fast algorithms for performing the directional continuous wavelet analysis.

Nevertheless, this approach has two main drawbacks. First, due to the Iwasawa decomposition it is limited to dilations centered at the north pole and second it does not have a nice geometric description. But this nice geometric description exists in the language of Clifford analysis.

Therefore, we want to define a CWT on the unit sphere which makes use of the conformal group of the sphere without restricting to the Iwasawa decomposition. In this way we will generalize the dilation operator defined in [4]. Our conformal dilation operator is clearly different from the dilation operator defined in the anisotropic case, which is no longer conformal (see [28]). Our approach will use some well known facts in Clifford Analysis and from wavelet theory. Therefore, we will omit some of the proofs which can be found in the literature.

In this area the study of the invariance group of null solutions of the Euclidean Dirac operator is of major importance (see [5], 18]). In the case of the sphere this group coincides with the group of Möbius transformations leaving the unit ball invariant. One possible description of this group is in terms of a projective identification of the points in the Euclidean space $\mathbb{R}^{n}$ with the rays in the null cone in $\mathbb{R}^{1, n+1}$ (see [23, [12, [10]), [11]). Hence, the Möbius group is identified with the group $\operatorname{Spin}(1, n+1)$. This identification has been the main theme of several works on Clifford Analysis (see [12], [19]). Also related with this approach is the study of Clifford Analysis on hyperbolic spaces, due to the fact that the subgroup $\operatorname{Spin}(1, n)$ of Möbius transformations leaving the unit sphere invariant is the isometry group of these non-Euclidean geometries. For an overview of the function theory in the hyperbolic unit ball we refer the work of D. Eelbode (19). Furthermore, for the connection between wavelet theory and Clifford Analysis we also would like to refer to 6], 7], 13, 14, and (24.

In the end we aim to establish frames for our continuous wavelet transform as an alternative to the use of spherical harmonics as approximating functions on the unit sphere $S^{n-1}$. We will construct such frames based on the abstract approach by Dahlke, Steidl, and Teschke [15] via convenient representations of the conformal group and we establish a Jackson-type theorem for nonlinear approximations, which also includes the case of best approximation in the $L_{2}$-space.
2. Preliminaries. We denote by $\mathbb{R}^{p, q}$ the $n$-dimensional vectorial space over $\mathbb{R}$ $(n=p+q)$ endowed with an orthonormal basis $e_{i}, i=1, \ldots, n$, and with signature $(p, q)$ induced by the non-degenerate bilinear form $B(x, y)$ such that $B\left(e_{i}, e_{i}\right)=-1$
for $1 \leq i \leq p$ and $B\left(e_{i}, e_{i}\right)=1$ for $p<i \leq n$. We define $\mathbb{R}_{p, q}$ as the universal real algebra generated by $\mathbb{R}^{p, q}$ which preserves the bilinear form $B(x, y)$. Hence we have $e_{i}^{2}=-B\left(e_{i}, e_{i}\right), i=1, \ldots, n$ and $e_{i} e_{j}+e_{j} e_{i}=0, i \neq j$. For a vector $x$ we have that $x^{2}=-B(x, x)$ is real valued. A vector is said to be invertible if and only if it is non-isotropic. In $\mathbb{R}^{0, n}$ we have that each non-zero vector $y$ is invertible. From now on we consider $e_{1}, \ldots, e_{n}$ as the canonical basis in $\mathbb{R}^{n}$ for a more simple geometrical interpretation. In this framework the Euclidean Dirac operator $\partial_{x}=\sum_{i=1}^{n} e_{i} \partial_{x_{i}}$ arises as a natural Clifford-valued first-order operator. Functions, which are annihilated by the Dirac operator $\partial_{x}$, i.e. $\partial_{x} f=0$, are called left-monogenic functions.

We define the Clifford conjugation $a \mapsto \bar{a}$ by $\overline{a b}=\bar{b} \bar{a}, \overline{e_{i}}=-e_{i}$, and $\overline{1}=1$. As a consequence, the inverse of a vector $y$ is given by $y^{-1}=\bar{y} /|y|^{2}$. We remark that due to the non-commutative character of Clifford algebras, the inverse at left is in general different from the inverse at right. Usually we denote by $\frac{x}{y}$ the product $x y^{-1}$, there is, by means of the right-hand side inverse. The particular linear combination of basic elements $e_{i_{1}} \ldots e_{i_{k}},\left(1 \leq i_{1}<\ldots<i_{k} \leq n\right)$, with equal length $k$ is designated a $k$-vector and we shall denote by $[x]_{k}$ the $k$-vector part of $x \in \mathbb{R}_{p, q}$. The linear subspace over $\mathbb{R}$ spanned by the elements of equal length $k$ is to be called the space of $k$-vectors $\mathbb{R}_{p, q}^{k}$.

We introduce the $\operatorname{Spin}$ group $\operatorname{Spin}(p, q)$ of all even finite products of invertible vectors $s$ such that $s \bar{s}= \pm 1$. For each $s \in \operatorname{Spin}(p, q)$ we have that the mapping $\chi(s): x \mapsto \chi(s) x=s x \bar{s}^{-1}$ is a special orthogonal transformation, thus setting $\operatorname{Spin}(p, q)$ as a double covering of $S O(p, q)$.
3. Conformal group of the unit sphere. Let us now take a look at the special case of the conformal group over the sphere. In [25] the group of conformal mappings of the open unit sphere $S^{n-1}$ is represented by Vahlen matrices and is denoted by $M\left(B^{n}\right)$.

We can parameterize this group in the form $M\left(B^{n}\right) \sim S O(n) \times B^{n}$ where $S O(n)$ is the maximal compact subgroup of $M\left(B^{n}\right)$ and $B^{n}$ is identified with the left cosets $M\left(B^{n}\right) \backslash S O(n)$, which gives rise to the set of Möbius transformations

$$
\begin{equation*}
\varphi_{a}(x)=(x-a)(1+a x)^{-1}, \quad a \in \mathbb{R}^{n}:|a|<1 \tag{1}
\end{equation*}
$$

This set of Möbius transformations map the unit ball onto itself and also the unit sphere onto itself.

The composition of two Möbius transformations of type (1) is (up to a rotation) again a Möbius transformation of type (11). In fact we have $\varphi_{a} \circ \varphi_{b}(x)=$ $q \varphi_{(1-a b)^{-1}(a+b)}(x) \bar{q}$, where $q=\frac{1-a b}{|1-a b|}$. We denote by $a \times b=(1-a b)^{-1}(a+b)$ the symbol of the new Möbius transformation. The symbol satisfies the relation $(1-a b)^{-1}(a+b)=(a+b)(1-b a)^{-1}$. We notice that the neutral element under this operation is $\varphi_{0} \equiv I d$ while the inverse is given by $\varphi_{a}^{-1}(x)=\varphi_{-a}(x)$. It is wellknown that $G=\left(M\left(B^{n}\right), \circ\right.$ ) is a (non-abelian) locally compact group [1], [29]. We can make an isomorphism between the subset of Möbius transformations of type (11) mapping the unit ball onto itself and the set of points $G^{*}=\left(B^{n}, \times\right)$ by means of an identification of each $\varphi_{a} \leftrightarrow a \in B^{n}$ and $\varphi_{a} \circ \varphi_{b} \leftrightarrow a \times b=(1+a \bar{b})^{-1}(a+b)$. Moreover, for each $a \in B^{n}$ the points $a /|a|$ and $-a /|a|$ are the fixed points of $\varphi_{a}$.

Of special importance for this paper are the following two types of subgroups of $M\left(B^{n}\right)$.
I) Subgroups of dimension $n-1$ : Let $\omega \in S^{n-1}$. We consider the hyperplane defined by $<\omega, x>=0$ and we define the ball $B^{n-1}$ as the intersection of the unit ball with this hyperplane. Then we have:

Proposition 1. The set of Möbius tranformations $\varphi_{a}$ with $a \in B^{n-1}$ forms (up to rotations) a subgroup of $M\left(B^{n}\right)$.
II) Subgroups of dimension one: Let $L$ be the segment resulting from the intersection of the unit ball with the straight line passing through the origin and spanned by $\omega$. Then we have:

Proposition 2. The set of Möbius transformations $\varphi_{a}$ with $a \in L$ forms an abelian subgroup of $M\left(B^{n}\right)$ of dimension one.
4. Hyperbolic model. For the construction of a theory of wavelets the study of dilations is of foremost importance. In the case of the sphere these dilations are not given by simple Euclidean dilations through inverse stereographic projection, but by hyperbolic rotations. We consider the Clifford Algebra $\mathbb{R}_{1, n}$, together with the special identification $\epsilon:=e_{n+1}$, the vector that spans the time-axis.

A pure boost is viewed as a transformation $\mathcal{B}(\omega)$ which belongs to the Lie algebra generated by the bi-vectors of the form $\epsilon \omega$, with $\omega \in S^{n-1}$. It has the general form

$$
\begin{equation*}
s=\cosh \frac{\alpha}{2}+\epsilon \omega \sinh \frac{\alpha}{2}, \alpha \in \mathbb{R}, \omega \in S^{n-1} \tag{2}
\end{equation*}
$$

and it acts on space-time vectors according to the transformations rules $X \rightarrow X^{\prime}=$ $s X \bar{s}$, and on functions via the (Spin-invariant) $L$ or $H$-representations

$$
\begin{aligned}
& F(X) \quad \rightarrow \quad L(s) F(X)=s F(\bar{s} X s) \\
& F(X) \quad \rightarrow \quad H(s) F(X)=s F(\bar{s} X s) \bar{s}
\end{aligned}
$$

Proposition 3. Let $\xi=\sum_{i=1}^{n} \xi_{i} e_{i}$ be a point on the sphere and $s$ of the form (2). Then the boost's action $\xi^{\prime}=s \xi \bar{s}$ yields the point on the sphere

$$
\begin{equation*}
\xi^{\prime}=\sum_{i=1}^{n} \frac{\xi_{i}+((\cosh \alpha-1)<\xi, \omega>-\sinh \alpha) \omega_{i}}{\cosh \alpha-\sinh \alpha<\xi, \omega>} e_{i} \tag{3}
\end{equation*}
$$

As the fixed points of this transformation are $\omega$ and $-\omega$, we can relate transformations (11) and (3) in the following way:

Proposition 4 (see [12]). We assume, in (1), $a=t \omega$, with $-1<t<1$ and $\omega \in S^{n-1}$. Then transformations (1) and (3) are related by

$$
\begin{gathered}
\cosh \alpha=\frac{1+t^{2}}{1-t^{2}} \quad \text { and } \quad \sinh \alpha=\frac{2 t}{1-t^{2}} \\
\alpha=\ln \left(\frac{1+t}{1-t}\right) \quad \text { and } \quad t=\frac{e^{\alpha}-1}{e^{\alpha}+1}=\tanh \left(\frac{\alpha}{2}\right) .
\end{gathered}
$$

Thus we obtain an isomorphism between the subgroup of Lorentz boosts in a fixed direction $\omega \in S^{n-1}$ and the subgroup of Möbius transformations of dimension one mentioned in Proposition 2 Moreover, a pure boost $\mathcal{B}(\omega)$ can always be described via the composition $R\left(e_{n}, \omega\right) \circ \mathcal{B}\left(e_{n}\right) \circ R\left(\omega, e_{n}\right)$, where $R(\omega, \xi)$ stands for the rotation mapping $\omega \in S^{n-1}$ into $\xi \in S^{n-1}$. Therefore, it is sufficient to consider pure boosts in the $e_{n}$-direction, that is to say, $\mathcal{B}\left(e_{n}\right)$. We will identify the subgroup $\operatorname{Spin}(1,1)$
with the subgroup of Lorentz boosts in the $e_{n}$-direction. Its action on a given point (in spherical coordinates) $\omega=\left\{\underline{\theta}_{j}, \underline{\phi}\right\}_{j=1}^{n-2}$ of $S^{n-1}$ is fully determined by

$$
\begin{equation*}
\omega \mapsto \omega_{\alpha}=\left\{\left(\underline{\theta}_{j}\right)_{\alpha}, \underline{\phi}_{\alpha}\right\}_{j=1}^{n-2} \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\underline{\theta}_{j}\right)_{\alpha}=\underline{\theta}_{j}, j=1, \ldots, n-2, \quad \text { and } \quad \tan \frac{(\phi)_{\alpha}}{2}=\mathrm{e}^{\alpha} \tan \frac{\phi}{2} \tag{5}
\end{equation*}
$$

This action corresponds to a pure dilation on the sphere and it is exactly the usual Euclidean dilation lifted on $S^{n-1}$ by inverse stereographic projection (see [4]). We will show in the next section that a local dilation around the North Pole depends on two parameters (not one as in [4) if we use the whole conformal group of the sphere.

It is well known that the group $S O(1, n)$ admits two different decompositions, the so-called Iwasawa decomposition (or $K A N$-decomposition) and the Cartan decomposition (or $K A K$-decomposition) (see [26] and [30]). We now show how to obtain the $K A K$-decomposition starting from the $\operatorname{Spin}(1, n)$ group. We consider the following elements of $\operatorname{Spin}(n)$

$$
\begin{align*}
s_{i} & =\cos \frac{\theta_{i}}{2}+e_{1} e_{i+1} \sin \frac{\theta_{i}}{2} \\
s_{n-1} & =\cos \frac{\phi}{2}+e_{n} e_{1} \sin \frac{\phi}{2} \tag{6}
\end{align*}
$$

with $0 \leq \theta_{1}<2 \pi, 0 \leq \theta_{i}<\pi, i=2, \ldots, n-2$, and $0 \leq \phi \leq \pi$. We identify the element $s=s_{1} \ldots s_{n-1}$ with the element $\xi\left(\theta_{1}, \ldots, \theta_{n-2}, \phi\right)$ in $S^{n-1}=$ $\operatorname{Spin}(n) / \operatorname{Spin}(n-1)$. Then we obtain the following polar decomposition.

Lemma 4.1. We have $\varphi_{a}(x)=\varphi_{\text {sre }_{n} \bar{s}}(x)=s \varphi_{r e_{n}}(\bar{s} x s) \bar{s}$, where $r=|a| \in[0,1[$.
Thus a Möbius transformation can be described in terms of a point $a$ belonging to the intersection of the unit ball with the positive $x_{n}$-axis and a convenient rotation induced by $s$. We have also the decomposition $\varphi_{a}(x)=\varphi_{-s r \mathrm{e}_{n} \bar{s}}=s \varphi_{-r \mathrm{e}_{n}}(\bar{s} x s) \bar{s}$ where the point $a$ belongs to the intersection of the unit ball with the negative $x_{n}$-axis. If we apply to the right-hand side of this identity the rotation present in the usual $\operatorname{Spin}(1, n)$ decomposition (see [12]) we derive the $K A K$-decomposition for an arbitrary element of the $\operatorname{Spin}(1, n)$ group.

The centralizer $C$ of $A=\operatorname{Spin}(1,1)$ in $K=\operatorname{Spin}(n)$, i.e., the set $C$ of all $s \in$ $\operatorname{Spin}(n)$ such that $\bar{s} \varphi_{r e_{n}}(x) s=\varphi_{r e_{n}}(\bar{s} x s)$, corresponds to the particular subgroup of rotations around the $x_{n}$-axis. Thus, in the polar decomposition of $\varphi(a)$ only the rotation $s_{n-1}=\cos \left(\frac{\phi}{2}\right)+e_{n} e_{1} \sin \left(\frac{\phi}{2}\right)$ affects $\varphi_{r e_{n}}(x)$. In the next section we study how the parameters $r$ and $\phi$ do influence local dilations around the North Pole of the unit sphere.
5. Influence of the parameter $a$ on spherical caps. In this section we describe the influence of the parameter $a \in B^{n}$ on the new spherical cap $\varphi_{a}\left(\mathcal{U}_{h}\right)$, obtained by the application of a Möbius transformation $\varphi_{a}$ to a given cap $\mathcal{U}_{h}$. Without loss of generality we consider a spherical cap $\mathcal{U}_{h}$ centered at the North Pole, with support
in the hyperplane $x_{n}=h$ given by

$$
\left\{\begin{aligned}
x_{1} & =\cos \left(\theta_{1}^{\prime}\right) \cos \left(\theta_{2}^{\prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime}\right) \sin \left(\phi^{\prime}\right) \\
x_{2} & =\sin \left(\theta_{1}^{\prime}\right) \cos \left(\theta_{2}^{\prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime}\right) \sin \left(\phi^{\prime}\right) \\
x_{3} & =\sin \left(\theta_{2}^{\prime}\right) \cos \left(\theta_{3}^{\prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime}\right) \sin \left(\phi^{\prime}\right) \\
\vdots & \\
x_{n-1} & =\sin \left(\theta_{n-2}^{\prime}\right) \sin \left(\phi^{\prime}\right) \\
x_{n} & =\cos \left(\phi^{\prime}\right)
\end{aligned}\right.
$$

with $\theta_{1}^{\prime} \in\left[0,2 \pi\left[, \theta_{i}^{\prime} \in[-\pi / 2, \pi / 2], i \in\{2, \ldots, n-2\}\right.\right.$ and $\phi^{\prime} \in\left[0, \phi_{0}\right]$, for a fixed $\left.\phi_{0} \in\right] 0, \pi\left[\right.$ such that $h=\cos \left(\phi_{0}\right)$.

Consider now the $(n-2)$-dimensional sphere $S$ in the hyperplane $x_{n}=h$

$$
\left\{\begin{array}{rl}
x_{1}^{2}+x_{2}^{2}+\ldots+x_{n-1}^{2} & =1-h^{2} \\
x_{n} & =h
\end{array} .\right.
$$

We will consider this sphere as the support of the spherical cap $\mathcal{U}_{h}$. Obviously $\varphi_{a}(S)$ is a new sphere (say, $S_{*}$ ) and it stands for the support of the new spherical cap.

A point $y$ of the sphere $S$ is given by

$$
\left\{\begin{align*}
y_{1} & =\sqrt{1-h^{2}} \cos \left(\theta_{1}^{\prime \prime}\right) \cos \left(\theta_{2}^{\prime \prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime \prime}\right)  \tag{7}\\
y_{2} & =\sqrt{1-h^{2}} \sin \left(\theta_{1}^{\prime \prime}\right) \cos \left(\theta_{2}^{\prime \prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime \prime}\right) \\
y_{3} & =\sqrt{1-h^{2}} \sin \left(\theta_{2}^{\prime \prime}\right) \cos \left(\theta_{3}^{\prime \prime}\right) \cdots \cos \left(\theta_{n-2}^{\prime \prime}\right) \\
\vdots & \\
y_{n-1} & =\sqrt{1-h^{2}} \sin \left(\theta_{n-2}^{\prime \prime}\right) \\
y_{n} & =h
\end{align*}\right.
$$

where $\theta_{1}^{\prime \prime} \in\left[0,2 \pi\left[\right.\right.$ and $\theta_{i}^{\prime \prime} \in[-\pi / 2, \pi / 2], i \in\{2, \ldots, n-2\}$. The support sphere $S_{*}$ of the new spherical cap has the following parametrization:

$$
g_{i}\left(\theta_{1}^{\prime \prime}, \theta_{2}^{\prime \prime}, \cdots, \theta_{n-2}^{\prime \prime}\right)=\frac{\left(1-|a|^{2}\right) y_{i}+2(<a, y>-1) a_{i}}{1+|a|^{2}-2<a, y>}, \quad i=1,2, \ldots, n
$$

and it has its center in the point:

$$
\left[\begin{array}{c}
\frac{2 a_{1}\left(a_{n}-h\right)\left(h\left(|a|^{2}+1\right)-2 a_{n}\right)}{k}  \tag{8}\\
\frac{2 a_{2}\left(a_{n}-h\right)\left(h\left(|a|^{2}+1\right)-2 a_{n}\right)}{k} \\
\vdots \\
\frac{2 a_{n-1}\left(a_{n}-h\right)\left(h\left(|a|^{2}+1\right)-2 a_{n}\right)}{k} \\
\frac{\left(2 a_{n}-h\left(|a|^{2}+1\right)\right)\left(|a|^{2}-1-2 a_{n}\left(a_{n}-h\right)\right)}{k}
\end{array}\right]
$$

where $k=4\left(a_{n}-h\right)^{2}\left(a_{1}^{2}+a_{2}^{2}+\cdots+a_{n-1}^{2}\right)+\left(1-|a|^{2}+2 a_{n}\left(a_{n}-h\right)\right)^{2}$. It is easy to see that $k>0$. If $a_{n}-h=0$ then $k=\left(1-|a|^{2}\right)^{2}>0$ because $|a|<1$; while if $a_{1}=a_{2}=\cdots=a_{n-1}=0$ we would have $k=\left(a_{n}^{2}-2 h a_{n}+1\right)^{2}>0$ and due to $|a|<1$, we have $a_{n} \neq 1$. Hence, $k$ can never assume the zero value.

The sphere $S_{*}$ is in the hyperplane with equation:

$$
\begin{array}{r}
2 a_{1}\left(a_{n}-h\right) x_{1}+2 a_{2}\left(a_{n}-h\right) x_{2}+\cdots+2 a_{n-1}\left(a_{n}-h\right) x_{n-1}+ \\
\left(1-|a|^{2}+2 a_{n}\left(a_{n}-h\right)\right) x_{n}=\left(1+|a|^{2}\right) h-2 a_{n} \tag{9}
\end{array}
$$

and has radius $\tau$ given by:

$$
\begin{equation*}
\tau=\frac{\left(1-h^{2}\right)^{1 / 2}\left(1-|a|^{2}\right)}{k^{1 / 2}} \tag{10}
\end{equation*}
$$

Moreover, the projection of the center of the new spherical cap (in the unit sphere) is given by:

$$
\begin{equation*}
\left(\frac{2 a_{1}\left(a_{n}-h\right)}{k^{1 / 2}}, \cdots, \frac{2 a_{n-1}\left(a_{n}-h\right)}{k^{1 / 2}}, \frac{1-|a|^{2}+2 a_{n}\left(a_{n}-h\right)}{k^{1 / 2}}\right) . \tag{11}
\end{equation*}
$$

We can easily see that this point belongs to the unit sphere. The distance between the points (8) and (11) is

$$
\begin{equation*}
\text { dist }=\frac{\left(2 a_{n}-h\left(1+|a|^{2}\right)\right)}{\sqrt{k}}+1 . \tag{12}
\end{equation*}
$$

Indeed, we can rewrite $k$ as $k=4\left(a_{n}-h\right)\left(a_{n}-h|a|^{2}\right)+\left(1-|a|^{2}\right)^{2}$. Then a simple calculation shows that

$$
\begin{aligned}
\text { dist }^{2}= & \sum_{i=1}^{n-1}\left(\frac{2 a_{i}\left(a_{n}-h\right)\left(h\left(1+|a|^{2}\right)-2 a_{n}\right)}{k}-\frac{2 a_{i}\left(a_{n}-h\right)}{k^{1 / 2}}\right)^{2}+ \\
& +\left(\frac{\left(2 a_{n}-h\left(1+|a|^{2}\right)\right)\left(|a|^{2}-1-2 a_{n}\left(a_{n}-h\right)\right)}{k}-\frac{1-|a|^{2}+2 a_{n}\left(a_{n}-h\right)}{k^{1 / 2}}\right)^{2} \\
= & \left(4\left(a_{1}^{2}+\ldots+a_{n-1}^{2}\right)\left(a_{n}-h\right)^{2}+\left(1-|a|^{2}+2 a_{n}\left(a_{n}-h\right)\right)^{2}\right) \\
& \cdot \frac{\left(\left(2 a_{n}-h\left(1+|a|^{2}\right)\right) \sqrt{k}+k\right)^{2}}{k^{3}} \\
= & \frac{\left(\left(2 a_{n}-h\left(1+|a|^{2}\right)\right) \sqrt{k}+k\right)^{2}}{k^{2}}, \quad \text { by definition of } \mathrm{k} \\
= & \left(\frac{2 a_{n}-h\left(1+|a|^{2}\right)}{\sqrt{k}}+1\right)^{2} .
\end{aligned}
$$

For each $h \in[-1,1]$ we can prove that $-1 \leq \frac{2 a_{n}-h\left(1+|a|^{2}\right)}{\sqrt{k}} \leq 1$. Finally we obtain that dist $=\frac{2 a_{n}-h\left(1+|a|^{2}\right)}{\sqrt{k}}+1$. Thus, $0 \leq d i s t \leq 2$, as it was expected.

We consider now the point $a \in B^{n}$ described in spherical coordinates

$$
\left\{\begin{align*}
a_{1} & =r \cos \theta_{1} \cos \theta_{2} \cdots \cos \theta_{n-2} \sin \phi  \tag{13}\\
a_{2} & =r \sin \theta_{1} \cos \theta_{2} \cdots \cos \theta_{n-2} \sin \phi \\
a_{3} & =r \sin \theta_{2} \cos \theta_{3} \cdots \cos \theta_{n-2} \sin \phi \\
\vdots & \\
a_{n-1} & =r \sin \theta_{n-2} \sin \phi \\
a_{n} & =r \cos \phi
\end{align*}\right.
$$

with $r \in\left[0,1\left[, \theta_{1} \in\left[0,2 \pi\left[\right.\right.\right.\right.$ and $\theta_{2}, \ldots \theta_{n-2} \in[-\pi / 2, \pi / 2], \phi \in[0, \pi]$. We can rewrite the expressions (10) and (12) in the following way:

$$
\begin{equation*}
\tau=\frac{\left(1-h^{2}\right)^{1 / 2}\left(1-r^{2}\right)}{\sqrt{k_{1}}}, \quad \quad \text { dist }=\frac{2 r \cos \phi-h\left(1+r^{2}\right)}{\sqrt{k_{1}}}+1 \tag{14}
\end{equation*}
$$

where $k_{1}=4 r^{2}(r \cos \phi-h)^{2} \sin ^{2} \phi+\left(1-r^{2}+2 r \cos \phi(r \cos \phi-h)\right)^{2}$. We can observe that these expressions are independent of the parameters $\theta_{1}, \theta_{2}, \cdots \theta_{n-2}$. We will return to this fact later.

Definition 5.1. The image of the North Pole under the action of $\varphi_{a}$ will be called attractor point and it will be denoted by A. It is given by

$$
\begin{equation*}
A=\left(\frac{2 a_{1}\left(a_{n}-1\right)}{1+|a|^{2}-2 a_{n}}, \cdots, \frac{2 a_{n-1}\left(a_{n}-1\right)}{1+|a|^{2}-2 a_{n}}, \frac{1-|a|^{2}+2 a_{n}\left(a_{n}-1\right)}{1+|a|^{2}-2 a_{n}}\right) . \tag{15}
\end{equation*}
$$

Given an initial spherical cap $\mathcal{U}_{h}$, its image $\varphi_{a}\left(\mathcal{U}_{h}\right)$ is a new spherical cap, say $\mathcal{U}_{h, a}$, centered, in general, in a point of the sphere different of the North Pole. Moreover it represents a dilation or a contraction of the initial cap $\mathcal{U}_{h}$. Applying a convenient rotation to each $\mathcal{U}_{h, a}$ we can center all spherical caps in an arbitrary desired point of the sphere. In this way we obtain a family of neighborhoods $\left\{\mathcal{U}_{h, r, \phi}^{\theta_{1}, \cdots, \theta_{n-2}}: r \in\left[0,1\left[, \theta_{1} \in\left[0,2 \pi\left[, \theta_{2}, \ldots \theta_{n-2} \in[-\pi / 2, \pi / 2], \phi \in[0, \pi]\right\}\right.\right.\right.\right.$ that will generate our local analysis on a given point of the sphere. For instance, in case of $n=3$ (the sphere $S^{2}$ ), if we consider $s_{a}=\cos \beta / 2+w \sin \beta / 2 \in \operatorname{Spin}(3)$ with

$$
w=\left(-\frac{a_{2}}{\left(a_{1}^{2}+a_{2}^{2}\right)^{1 / 2}}, \frac{a_{1}}{\left(a_{1}^{2}+a_{2}^{2}\right)^{1 / 2}}, 0\right) \quad \text { and } \quad \cos \beta=\frac{1-|a|^{2}+2 a_{3}\left(a_{3}-h\right)}{k^{1 / 2}}
$$

where $w$ is the axis of the rotation and $\beta$ is the angle of the rotation, then the set $\left\{s_{a} \mathcal{U}_{h, a} \overline{s_{a}}: a \in B^{n}\right\}$ stands for a family of neighborhoods centered at the North Pole. We remark that the axis of the rotation is only defined when the parameter $a$ does not belong to the $x_{3}$-axis. If $a$ belongs to the $x_{3}$-axis then the North Pole is a fixed point and the cap remains centered at the North Pole. This is what happens in the case presented in (3) and [4].

These caps will constitute the basis for our local analysis on the sphere. A dilation (in our sense) around an arbitrary point $\omega \in S^{n-1}$ can be obtained by combining the dilation around the North Pole just described above with an appropriate rotation.

We illustrate the facts above with some examples in $\mathbb{R}^{3}$ (see figure 1):


Figure 1. Spherical caps for $h=\cos (\pi / 6)=\sqrt{3} / 2$ and different values of $a=(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi): 1-a=(0,0,0)$ $\left(\mathcal{U}_{\sqrt{3} / 2}\right), \quad \mathbf{2}-r=1 / 2, \theta=5 \pi / 3, \phi=\pi / 6, \quad \mathbf{3}-r=3 / 10, \theta=$ $5 \pi / 3, \phi=7 \pi / 9$.

The cap 2 is a dilation of $\mathcal{U}_{\sqrt{3} / 2}$, whereas the cap 3 is a contraction of $\mathcal{U}_{\sqrt{3} / 2}$. It is possible to define for each fixed $h \in]-1,1$ [ two different regions on the unit ball that will be called dilation region and contraction region respectively. The two regions are separated by a surface of revolution obtained by considering the revolution around the $x_{n}$-axis of the arc defined by

$$
\begin{equation*}
\vec{\gamma}(r)=\left(r\left(1-(h r)^{2}\right)^{1 / 2}, 0, \ldots, 0, r^{2} h\right), \quad r \in[0,1[ \tag{16}
\end{equation*}
$$

In fact if we substitute $\cos \phi=h r$ and $\sin \phi=\left(1-(h r)^{2}\right)^{1 / 2}$ (from 16) in the expression (14) of the distance we obtain:

$$
\begin{aligned}
\text { dist } & =\frac{2 r \cos \phi-h\left(1+r^{2}\right)}{\sqrt{4 r^{2}(r \cos \phi-h)^{2} \sin ^{2} \phi+\left(1-r^{2}+2 r \cos \phi(r \cos \phi-h)\right)^{2}}}+1 \\
& =\frac{h\left(r^{2}-1\right)}{1-r^{2}}+1 \\
& =1-h, \quad \forall r \in[0,1[,
\end{aligned}
$$

which shows that the dist remains the same. We remark that the spherical cap $\mathcal{U}_{h}$ has support on the hyperplane $x_{n}=h$, and therefore its distance to the North Pole is $1-h$.

Then we obtain the following parametrization for the surface of revolution:

$$
\mathcal{S}_{h}:\left\{\begin{align*}
s_{1} & =r \cos \theta_{1} \cos \theta_{2} \cdots \cos \theta_{n-2}\left(1-(h r)^{2}\right)^{1 / 2}  \tag{17}\\
s_{2} & =r \sin \theta_{1} \cos \theta_{2} \cdots \cos \theta_{n-2}\left(1-(h r)^{2}\right)^{1 / 2} \\
s_{3} & =r \sin \theta_{2} \cos \theta_{3} \cdots \cos \theta_{n-2}\left(1-(h r)^{2}\right)^{1 / 2} \\
\vdots & \\
s_{n-1} & =r \sin \theta_{n-2}\left(1-(h r)^{2}\right)^{1 / 2} \\
s_{n} & =r^{2} h
\end{align*}\right.
$$

where $r \in\left[0,1\left[, \theta_{1} \in\left[0,2 \pi\left[, \theta_{2}, \ldots, \theta_{n-2} \in[-\pi / 2, \pi / 2]\right.\right.\right.\right.$.
For example, in $\mathbb{R}^{3}$, with respectively $h=1 / 2$ and $h=-1 / 2$, we can observe a projection of $\mathcal{S}_{h}$ in the $x z$-plane.



Figure 2. Projection of $S_{h}$ in the $x z$-plane: $\mathcal{S}_{1 / 2}$ (left) and $\mathcal{S}_{-1 / 2}$ (right).
The dilation region is the region in the unit ball above the surface $\mathcal{S}_{h}$ and the contraction region is the region in the unit ball bellow the surface $\mathcal{S}_{h}$.

The fact that the $\mathcal{S}_{h}$ is a surface of revolution is related with the result obtained in (14). In fact, the distance considered there is independent on the parameters $\theta_{1}, \ldots, \theta_{n-2}$. All the spherical caps obtained by the application of $\varphi_{a}$, where $a \in \mathcal{S}_{h}$, have then the same area. However they differ in the localization of the attractor point on the spherical cap.
Proposition 5. Consider $a \in \mathcal{S}_{h}$ with $h$ fixed. For each $r$, the corresponding attractor point lies in the intersection of the sphere of equation $A_{1}^{2}+A_{2}^{2}+\ldots+A_{n-1}^{2}=$ $\frac{4 r^{2}\left(1-(h r)^{2}\right)\left(r^{2} h-1\right)^{2}}{\left(1+r^{2}-2 r^{2} h\right)^{2}}$ with the hyperplane of equation $A_{n}=\frac{1-r^{2}+2 r^{2} h\left(r^{2} h-1\right)}{1+r^{2}-2 r^{2} h}$.

We remark that, with $h$ and $r$ fixed, the point $a$ belongs to the intersection of the sphere described by $s_{1}^{2}+s_{2}^{2}+\ldots+s_{n-1}^{2}=r^{2}\left(1-(h r)^{2}\right)$ with the hyperplane $s_{n}=r^{2} h$.

Hence, we can conclude that only the parameters $r$ and $\phi$ induce a local dilation on the sphere. We are interested in the study of the distance (14) as a function of these parameters, and this independent of the dimension considered. As an example, for $h=1 / 2$ we obtain the following picture:


Figure 3. Variation of the distance considered in (14).

From now on, we assume the parameter $a$ as $d=(r \sin \phi, 0, \ldots, 0, r \cos \phi)=$ $s_{n-1} r e_{n} \overline{s_{n-1}}$ (cf. Lemma 4.1). As we approach the boundary of the unit ball we obtain a discontinuous jump, corresponding to dist $=2$ if $\phi \leq \arccos (h)$ or dist $=0$ if $\phi>\arccos (h)$. We denote this limit angle as $\phi_{\lim }=\arccos (h)$, the critical angle. It is related with the separation between the dilation and contraction regions near the boundary of the unit ball (see fig. 2)

The particular case of [3] and [4] is obtained assuming the values $\phi=0$ (intersection of $B^{n}$ with the positive $x_{n}$-axis - dilation region) and $\phi=\pi$ (intersection of $B^{n}$ with the negative $x_{n}$-axis - contraction region). In the first case we have dist $=\frac{(1-h)(r+1)^{2}}{1+r^{2}-2 r h}$ and in the second case we have dist $=\frac{(1-h)(r-1)^{2}}{1+r^{2}+2 r h}$. These two half axes can be used to generate/construct a sequence of approximation spaces. However, it is possible to choose other domains for the parameter's variation and thus we would obtain different approximation spaces.
6. Characterization of the local dilation. In the case of the subgroup $\operatorname{Spin}(1,1)$ it is possible to describe its action on each point of the sphere in terms of (5). We can describe the action of the set of Möbius transformations $\varphi_{d}$ on $\mathcal{U}_{h}$ by the relation:

$$
\begin{equation*}
\varphi_{d}\left(\mathcal{U}_{h}\right)=\varphi_{s_{n-1} r e_{n} \overline{s_{n-1}}}\left(\mathcal{U}_{h}\right)=s_{n-1} \varphi_{r e_{n}}\left(\overline{s_{n-1}} \mathcal{U}_{h} s_{n-1}\right) \overline{s_{n-1}} \tag{18}
\end{equation*}
$$

We shall restrict our attention to the most interesting case, the one of the sphere $S^{2}$. We would like to remark that the following results can easily be generalized to higher dimensions.

The mapping of a point $x=\left(x_{1}, x_{2}, x_{3}\right) \equiv(\underline{\theta}, \underline{\phi}) \in \mathcal{U}_{h}$ onto the point $x_{r, \phi, \underline{\theta}}=$ $\left(\underline{\theta}_{\phi, \underline{\phi}}, \underline{\phi}_{r, \phi, \underline{\theta}}\right)$ via the action of $\varphi_{r e_{3}}\left(\bar{s}_{2} x s_{2}\right)$, where $\bar{s}_{2}=s_{2}(\phi)$ denotes the rotation (c.f. (6) with $s_{n-1}=s_{2}$ ), is given by

$$
\begin{align*}
\tan \frac{\underline{\phi}_{r, \phi, \underline{\theta}}}{2} & =\frac{1+r}{1-r} \sqrt{\frac{1-\left(\cos \phi x_{3}-\sin \phi x_{1}\right)}{1+\cos \phi x_{3}-\sin \phi x_{1}}} \\
& =\frac{1+r}{1-r} \sqrt{\frac{1-(\cos \phi \cos \underline{\phi}-\sin \phi \cos \underline{\theta} \sin \underline{\phi})}{1+\cos \phi \cos \underline{\phi}-\sin \phi \cos \underline{\theta} \sin \underline{\phi}}} \tag{19}
\end{align*}
$$

and

$$
\begin{equation*}
\tan \underline{\theta}_{\phi, \underline{\phi}}=\frac{x_{2}}{\cos \phi x_{1}+\sin \phi x_{3}}=\frac{\sin \underline{\theta} \sin \underline{\phi}}{\cos \phi \cos \underline{\sin } \underline{\phi}+\sin \phi \cos \underline{\phi}} . \tag{20}
\end{equation*}
$$

By comparing these relations with the relations obtained in the anisotropic case [28], we observe that the parameters involved are different and, therefore, they produce different neighborhoods. Our case is conformal whereas the anisotropic case [28] corresponds to a deformation of spherical caps to "elliptic" caps.

We have the advantage of being able to choose a preferable contraction inside the cap conveniently choosing the position of the attractor point. The parameters $\theta_{1}, \ldots, \theta_{n-2}$ contribute to the localization of the attractor point inside the spherical cap.

We will require the following formula:

- the arc-length between the attractor point and the center of the cap:

$$
\begin{equation*}
d_{2}:=\arccos \left(\frac{4 r^{2} \cos \phi^{2}-2 r\left(1+r^{2}\right)(h+1) \cos \phi+1+r^{4}+2(2 h-1) r^{2}}{\left.\sqrt{k_{1}}\left(1+r^{2}-2 r \cos \phi\right)\right)}\right) \tag{21}
\end{equation*}
$$

The importance of this formula is that it provides information about the geometry of the caps (compact support of our future wavelets) under the action of a Möbius transformation and in particular about the dilation/contraction effects inside that same cap.
7. Continuous wavelet transform on the unit sphere. We will consider two different Hilbert spaces. The first is the Hilbert module $L_{2}\left(S^{n-1}\right)$ of square integrable functions on the sphere, and the second is the monogenic Hardy space $H^{2} \subset L_{2}\left(S^{n-1}\right)$, that is, the space of all functions which can be considered as boundary values of monogenic functions on the unit ball.

We use the standard inner product and norm

$$
\begin{equation*}
<f, g>_{L_{2}}=\int_{S^{n-1}} \overline{f(x)} g(x) d S(x), \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|^{2}=2^{n} \int_{S^{n-1}}[\overline{f(x)} f(x)]_{0} d S(x) \tag{23}
\end{equation*}
$$

where $[\lambda]_{0}$ denotes the real part of the Clifford number $\lambda$ and $d S(x)$ is the normalized $\operatorname{Spin}(n)$ - invariant measure on $S^{n-1}$.

We consider the following unitary operators:

- the Spin invariant rotation operator

$$
\begin{equation*}
R_{1}(s) f(x)=s f(\bar{s} x s), \quad s \in \operatorname{Spin}(n) \tag{24}
\end{equation*}
$$

- and the dilation operator

$$
\begin{equation*}
D_{1}(d) f(x)=\left(\frac{1-|d|^{2}}{|1-d x|^{2}}\right)^{\frac{n-1}{2}} f\left(\varphi_{-d}(x)\right), \tag{25}
\end{equation*}
$$

with $d=(r \sin \phi, 0, \cdots, 0, r \cos \phi)$ a bi-dimensional parameter, where $r \in[0,1[$ and $\phi \in[0, \pi]$. The motivation for the definition of the dilation operator comes from the results of section 5 and Lemma 4.1 The parameters $\theta_{1}, \ldots, \theta_{n-2}$ only give us information about the localization of the attractor point, which can also be obtained by the analysis of the Spin group.

We remark that in the cases where $\phi=0$ and $\phi=\pi$, (i.e., where we have $d=t \mathrm{e}_{n}$, with $t \in]-1,1\left[\right.$, if we perform the change of variables $t=\frac{u-1}{u+1}$, with $u>0$, we obtain the following description for the operator (25),

$$
\begin{equation*}
D_{1}(u) f(\omega)=\lambda_{1}(u, \underline{\phi}) f\left(\omega_{1 / u}\right), \quad \omega=\omega\left(\underline{\theta}_{1}, \ldots, \underline{\theta}_{n-2}, \underline{\phi}\right) \tag{26}
\end{equation*}
$$

where

$$
\lambda_{1}(u, \underline{\phi})=\left(\frac{4 u^{2}}{\left(\left(u^{2}-1\right) \cos \underline{\phi}+\left(u^{2}+1\right)\right)^{2}}\right)^{\frac{n-1}{4}}
$$

and $\omega_{1 / u}$ is the notation used in (5) with $\alpha=\ln (1 / u)$ and $0 \leq \underline{\phi} \leq \pi$. This operator is the same used in [4].

Based on the two operators defined above we consider the representation

$$
\begin{equation*}
\pi_{1}(s, d) f(x)=R_{1}(s) \circ\left(D_{1}(d) f(x)\right)=s\left(\frac{1-|d|^{2}}{|1-d \bar{s} x s|^{2}}\right)^{\frac{n-1}{2}} f\left(\varphi_{-d}(\bar{s} x s)\right) \tag{27}
\end{equation*}
$$

The representation $\pi_{1}$ is unitary. It only remains to check whether it is square integrable, i.e. to find a nonzero function $\psi$ in the Hilbert space under consideration such that

$$
\int_{\operatorname{Spin}(n)} \int_{0}^{1} \int_{0}^{\pi}\left|<\pi_{1}(s, d) \psi, \psi>_{L_{2}}\right|_{0}^{2} d \mu(d) d \mu(s)<\infty
$$

where $d \mu(d)=\frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi$ is the restriction to the bidimensional parameter $d$ of the invariant measure for the group of Möbius transformations (see [8]), and $d \mu(s)$ is the invariant measure in $\operatorname{Spin}(n)$. This is also known as the admissibility condition for a wavelet (see [21]).

We follow the ideas of [4]. We denote by $N(n, k)$ the dimension of the subspace $\mathcal{H}_{k}$ of all linearly independent homogeneous harmonic polynomials of degree $k$ in $n$ variables (see [5]). We consider an orthonormal basis of spherical harmonics $\left\{H_{k}^{(i)}, i=1, \ldots, N(n, k)\right\}_{k=0}^{\infty}$ with the property $<H_{k}^{(i)}, H_{l}^{(j)}>_{L_{2}}=\delta_{k, l} \delta_{i, j}$. Thus, a function $f \in L_{2}\left(S^{n-1}\right)$ has a Fourier expansion

$$
\begin{equation*}
f=\sum_{k=0}^{\infty} \sum_{i=1}^{N(n, k)} H_{k}^{(i)} a_{k}^{(i)} \tag{28}
\end{equation*}
$$

where we denote formally $a_{k}^{(i)}=<H_{k}^{(i)}, f>_{L_{2}}$ as the Fourier coefficients of $f$.
Theorem 7.1. A non-zero function $\psi \in L_{2}\left(S^{n-1}, \mathbb{R}_{0, n}\right)$ is admissible if there exists a finite constant $c>0$ such that for all $k$ it holds

$$
\begin{equation*}
\sum_{m=1}^{N(n, k)} \int_{0}^{\pi} \int_{0}^{1}\left|a_{k}^{(m)}(d)\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi<c \tag{29}
\end{equation*}
$$

where $a_{k}^{(m)}(d)=<H_{k}^{(m)}, D_{1}(d) \psi>_{L_{2}}$ are the Fourier coefficients of $D_{1}(d) \psi$.
Proof. Let us write

$$
\pi_{1}(s, d) \psi(x)=\sum_{k=0}^{\infty} \sum_{i} H_{k}^{(i)}(x) b_{k}^{(i)}(s, d)
$$

with $b_{k}^{(i)}(s, d)=<H_{k}^{(i)}, R_{1}(s) \circ\left(D_{1}(d) \psi\right)>_{L_{2}}$ and

$$
\psi(x)=\sum_{l=0}^{\infty} \sum_{j} H_{l}^{(j)}(x) c_{l}^{(j)}
$$

with $c_{l}^{(j)}=<H_{l}^{(j)}, \psi>$. Then, by the orthogonality property of spherical harmonics we have

$$
\begin{aligned}
<\pi_{1}(s, d) \psi, \psi>_{L_{2}} & =\sum_{k, l=0}^{\infty} \sum_{i, j} \overline{b_{k}^{(j)}(s, d)} \int_{S^{n-1}} \overline{H_{k}^{(i)}(x)} H_{l}^{(j)}(x) d S(x) c_{l}^{(j)} \\
& =\sum_{k=0}^{\infty} \sum_{i} \overline{b_{k}^{(i)}(s, d)} c_{k}^{(i)}
\end{aligned}
$$

Now, for each $k$ and $i$ fixed we have

$$
\begin{aligned}
b_{k}^{(i)}(s, d)= & <H_{k}^{(i)}, R_{1}(s) \circ\left(D_{1}(d) \psi\right)>_{L_{2}} \\
= & <R_{1}(\bar{s}) \circ H_{k}^{(i)}, D_{1}(d) \psi>_{L_{2}} \\
= & <\sum_{m} H_{k}^{(m)} g_{k}^{(m)}(s), D_{1}(d) \psi>_{L_{2}} \\
= & \sum_{m} \overline{g_{k}^{(m)}(s)}<H_{k}^{(m)}, D_{1}(d) \psi>_{L_{2}} \\
= & \sum_{m} \overline{g_{k}^{(m)}(s)} a_{k}^{(m)}(d)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I & =\int_{\operatorname{Spin}(n)} \int_{0}^{\pi} \int_{0}^{1}\left|<\pi_{1}(s, d) \psi, \psi>_{L_{2}}\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi d \mu(s) \\
& =\int_{\operatorname{Spin}(n)} \int_{0}^{\pi} \int_{0}^{1}\left|\sum_{k=0}^{\infty} \sum_{i} \overline{\sum_{m} \overline{g_{k}^{(m)}(s)} a_{k}^{(m)}(d)} c_{k}^{(i)}\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi d \mu(s) \\
& \leq\left.\left.\int_{\operatorname{Spin}(n)} \int_{0}^{\pi} \int_{0}^{1} \sum_{k=0}^{\infty} \sum_{i} \sum_{m}\left|a_{k}^{(m)}(d)\right|_{0}^{2}\left|g_{k}^{(m)}(s)\right|_{0}^{2}\right|_{k} ^{(i)}\right|_{0} ^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi d \mu(s) \\
& =C \sum_{k=0}^{\infty} \sum_{i} \sum_{m}\left|c_{k}^{(i)}\right|_{0}^{2} \int_{0}^{\pi} \int_{0}^{1}\left|a_{k}^{(m)}(d)\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi
\end{aligned}
$$

Putting

$$
S_{k}=\sum_{i=1}^{N(n, k)}\left|c_{k}^{(i)}\right|_{0}^{2}, \quad \text { and } \quad T_{k}=\sum_{m=1}^{N(n, k)} \int_{0}^{\pi} \int_{0}^{1}\left|a_{k}^{(m)}(d)\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi
$$

we see that $\left(S_{k}\right)_{k \in \mathbb{N}} \in l^{1}$, since $\sum_{k=0}^{\infty} S_{k}=\|\psi\|^{2}$. Then the admissibility condition becomes

$$
\sum_{k=0}^{\infty} S_{k} T_{k}<\infty
$$

Finally, this series converges absolutely if and only if $\left(T_{k}\right)_{k \in \mathbb{N}} \in l^{\infty}$. Thus, the function $\psi$ is admissible if and only if

$$
\sum_{m=1}^{N(n, k)} \int_{0}^{\pi} \int_{0}^{1}\left|a_{k}^{(m)}(d)\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi<c, \quad \text { uniformly in } k
$$

with $c$ being a finite constant.

Now, this condition is complicated to use in practice since it requires the evaluation of nontrivial Fourier coefficients. However it is possible to derive a necessary condition which turns out be a generalization of the necessary condition obtained in 4].

Proposition 6. Let $\psi \in L_{2}\left(S^{n-1}\right)$ be a function with support on a given spherical cap $\mathcal{U}_{h}$. If $\psi$ is an admissible function then it necessarily satisfies the condition

$$
\begin{equation*}
\left.\left.\int_{\mathcal{U}_{h}} \frac{\psi(y)}{\left(1+s_{n-1} e_{n} \overline{s_{n-1}} y\right)^{\frac{n-1}{2}}} d S_{y}=0, \quad \forall \phi \in\right] \arccos (h), \pi\right] . \tag{30}
\end{equation*}
$$

Proof. We have to compute

$$
\begin{equation*}
\int_{0}^{\pi} \int_{0}^{1}\left|<H_{k}^{(m)}, D_{1}(d) \psi>_{L_{2}}\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi \tag{31}
\end{equation*}
$$

Since $\psi$ is a function with support on the spherical cap $\mathcal{U}_{h}$, we have

$$
\psi(x) \equiv \psi\left(x_{1}, \ldots, x_{n}\right)=0 \quad \text { for } \quad x_{n}<h .
$$

Then,

$$
\begin{equation*}
<H_{k}^{(m)}, D_{1}(d) \psi>_{L_{2}}=\int_{\varphi_{d}\left(\mathcal{U}_{h}\right)} \overline{H_{k}^{(m)}(x)}\left(\frac{1-|d|^{2}}{|1-d x|^{2}}\right)^{\frac{n-1}{2}} \psi\left(\varphi_{-d}(x)\right) d S_{x} \tag{32}
\end{equation*}
$$

We split the integral (31) in four parts, where parts (I) and (II) are connected to the large scales - dilations, and parts (III) and (IV) represent the small scales contractions (cf. fig. 3), that is

$$
\int_{0}^{\pi} \int_{0}^{1} \ldots d r d \phi=\underbrace{\int_{0}^{\phi_{\lim }} \int_{0}^{1-\epsilon} \ldots+\underbrace{\int_{0}^{\phi_{\lim }} \int_{1-\epsilon}^{1} \ldots}_{(I I)}+\underbrace{\int_{\phi_{\lim }}^{\pi} \int_{0}^{1-\epsilon} \ldots}_{(I I I)}+\underbrace{\int_{\phi_{\lim }}^{\pi} \int_{1-\epsilon}^{1} \ldots}_{(I V)}}_{(I)}
$$

where $\phi_{\lim }=\arccos (h)$ and $0<\epsilon<1$.
The integrals (I) and (III) can be easily handled because $D_{1}(d)$ is a strongly continuous operator and thus, by continuity of the scalar product, the integrals are bounded continuous functions on the respective domains.

Let us study the integral (II). If $\epsilon$ is small enough such that $|d|=r \approx 1$ then $\psi\left(\varphi_{d}(x)\right) \approx \psi\left(-e_{n}\right)$. Then the large scale divergence in (II) will never be reached because of the support property of $\psi$ and this ensures the convergence of the integral (II).

For the integral (IV) we consider in (32) the change of variables $y=\varphi_{-d}(x)$. Then we have for the Fourier coefficients that

$$
\int_{\mathcal{U}_{h}} \overline{H_{k}^{(m)}\left(\varphi_{d}(y)\right)}\left(\frac{1-|d|^{2}}{\left|1-d \varphi_{d}(y)\right|^{2}}\right)^{\frac{n-1}{2}} \psi(y)\left(\frac{1-|d|^{2}}{|1+d y|^{2}}\right)^{n-1} d S_{y}
$$

Using

$$
1-\varphi_{d}(y)=\left(1-|d|^{2}\right)(1+d y)^{-1}
$$

we simplify the above expression to become

$$
\int_{\mathcal{U}_{h}} \overline{H_{k}^{(m)}\left(\varphi_{d}(y)\right)}\left(\frac{1-|d|^{2}}{|1+d y|^{2}}\right)^{\frac{n-1}{2}} \psi(y) d S_{y}
$$

Passing to the variables $r$ and $\phi$ we find

$$
\int_{\mathcal{U}_{h}} \overline{H_{k}^{(m)}\left(\varphi_{r, \phi}(y)\right)}\left(\frac{1-r^{2}}{\left|1+s_{n-1} r e_{n} \overline{S_{n-1}} y\right|^{2}}\right)^{\frac{n-1}{2}} \psi(y) d S_{y} .
$$

Finally, the integral (IV) becomes:

$$
\begin{aligned}
& \int_{\phi_{\mathrm{lim}}}^{\pi} \int_{1-\epsilon}^{1}\left|\int_{\mathcal{U}_{h}} \overline{H_{k}^{(m)}\left(\varphi_{r, \phi}(y)\right)}\left(\frac{1-r^{2}}{\left|1+s_{n-1} r e_{n} \bar{s}_{n-1} y\right|^{2}}\right)^{\frac{n-1}{2}} \psi(y) d S_{y}\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)^{n}} d r d \phi \\
= & \int_{\phi_{\text {lim }}}^{\pi} \int_{1-\epsilon}^{1}\left|\int_{\mathcal{U}_{h}} \overline{H_{k}^{(m)}\left(\varphi_{r, \phi}(y)\right)}\left(\frac{1}{\left|1+s_{n-1} r e_{n} \overline{s_{n-1}} y\right|^{2}}\right)^{\frac{n-1}{2}} \psi(y) d S_{y}\right|_{0}^{2} \frac{r}{\left(1-r^{2}\right)} d r d \phi
\end{aligned}
$$

If $\epsilon \approx 0$ then $r \approx 1$. In the limit case

$$
H_{k}^{(m)}\left(\varphi_{r, \phi}(y)\right) \approx H_{k}^{(m)}\left(e_{n}\right) .
$$

which assumes the value zero for $m \neq 1$.
Therefore, the integral (IV) over small scales converges if and only if we impose the condition

$$
\begin{equation*}
\left.\left.\int_{\mathcal{U}_{h}} \frac{\psi(y)}{\left(\left|1+s_{n-1} e_{n} \overline{s_{n-1}} y\right|\right)^{n-1}} d S_{y}=0, \quad \forall \phi \in\right] \arccos (h), \pi\right] \tag{33}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\int_{\overline{s_{n-1}} \mathcal{U}_{h} s_{n-1}} \frac{\psi\left(s_{n-1} y \overline{s_{n-1}}\right)}{\left(1-y_{n}\right)^{n-1}} d S_{y}=\int_{S^{n-1}} \frac{\psi\left(s_{n-1} y \overline{s_{n-1}}\right)}{\left(1-y_{n}\right)^{n-1}} d S_{y}=0, \tag{34}
\end{equation*}
$$

for all $\phi \in] \arccos (h), \pi]$.
Let us remark that this necessary condition depends strongly on the support $\mathcal{U}_{h}$ of the admissible function $\psi$.

Moreover, if we restrict to the case of [4] we obtain a similar necessary condition:

$$
\begin{equation*}
\int_{S^{n-1}} \frac{\psi(y)}{\left(1+y_{n}\right)^{n-1}} d S_{y}=0 . \tag{35}
\end{equation*}
$$

We remark that it is also possible to use inner and outer spherical monogenics for the series expansion but these functions are essentially a refinement of the spherical harmonics.

We now turn ourselves to the problem of existence of admissible wavelets. It is difficult to prove their existence if we consider the full bi-dimensional parameter $d$. A solution to this problem is to restrict the parameter $d$ by fixing an angle $\phi$ and considering the one dimensional subgroup generated by the element $\omega=$ $(\sin \phi, 0, \ldots, 0, \cos \phi)$; hence, we use from now on the parameter $d_{\phi}=(t \sin \phi, 0, \ldots$, $0, t \cos \phi$ ) for fixed $\phi$ and $-1<t<1$. Then we obtain a family of conformal dilation operators, which depends on the considered cap $\mathcal{U}_{h}$.
Proposition 7. Consider a cap $\mathcal{U}_{h}$. When $0 \leq h<1$, the operators $\varphi_{d_{\phi}}$, with $\phi \in$ $[0, \arccos (h)[$, constitute a family of local conformal dilation/contraction operators. When $-1<h<0$ the operators $\varphi_{d_{\phi}}$, with $\phi \in[0, \pi-\arccos (h)[$ constitute a family of local conformal dilation/contraction operators (cf. fig. 4).

As a consequence the parameter $h$ acts as a freedom-degree on the choice of the local dilation/contraction operator on the sphere.

As we can see in fig. 4 , for a fixed $0 \leq h<1$ the operators $\varphi_{d_{\phi}}$, with $\phi \in$ $[\arccos (h), \pi / 2[$, do not provide a complete scale of dilations/contractions, although


Figure 4. Variation of the distance (14): $h=1 / 2$ (left) and $h=$ $-1 / 2$ (right).
they can show themselves useful, according on the application's needs. The same is true for the operators $d_{\phi}$, with $-1<h<0$ and $\phi \in[\pi-\arccos (h), \pi / 2[$. Therefore, we reformulate our admissibility condition (29) for the representation $\pi_{1}$ based on the parameter $d_{\phi}$.

Theorem 7.2. A function $\psi \in L_{2}\left(S^{n-1}, \mathbb{R}_{0, n}\right)$ is admissible if there exists a finite constant $c>0$ such that

$$
\begin{equation*}
\sum_{m} \int_{-1}^{1}\left|a_{k}^{(m)}\left(d_{\phi}\right)\right|_{0}^{2} \frac{d t}{\left(1-t^{2}\right)^{n}}<c \tag{36}
\end{equation*}
$$

where $a_{k}^{(m)}\left(d_{\phi}\right)=<H_{k}^{(m)}, D_{1}\left(d_{\phi}\right) \psi>_{L_{2}}$ are the Fourier coefficients of $D_{1}\left(d_{\phi}\right) \psi$.
Let us now study the existence of wavelets for the representation based on the operator $D_{1}\left(d_{\phi}\right)$.
Theorem 7.3. Consider the operator $D_{1}\left(d_{\phi}\right)$ in the representation $\pi_{1}$. Depending on the cap $\mathcal{U}_{h}$ we have:

1. If $\cos (\phi) \leq h<1$ then for the convergence of (36) we have two necessary conditions:

$$
\begin{equation*}
\int_{\mathcal{U}_{h}} \frac{\psi(y)}{(|1+\omega y|)^{n-1}} d S_{y}=0 \quad \text { and } \quad \int_{\mathcal{U}_{h}} \frac{\psi(y)}{(|1-\omega y|)^{n-1}} d S_{y}=0 \tag{37}
\end{equation*}
$$

with $\omega=(\sin (\phi), \ldots, \cos (\phi))$. In this case the operator is almost a contraction operator.
2. If $-\cos (\phi)<h<\cos (\phi)$ then for the convergence of (36) we have the necessary condition:

$$
\begin{equation*}
\int_{\mathcal{U}_{h}} \frac{\psi(y)}{(|1-\omega y|)^{n-1}} d S_{y}=0 \tag{38}
\end{equation*}
$$

In this case the operator is a conformal contraction/dilation operator as defined in Proposition 7.

We remark that the case $-1<h \leq-\cos (\phi)$ appears to be not interesting because we loose the localization property. This is due to the fact that this operator is then almost a dilation operator and therefore no localization is possible.

Of course, here arises an interesting question: what happens in the case where we do not have a full scale for the dilation operator? What will be the space of functions/signals which can be reconstructed? However, this question is not easily
answered and it will be studied in the future. In this paper we will restrict ourselves to the case where a complete scale exists.

The proof of this theorem is analogous to the proof of Proposition 6 In this way we obtain a complete classification depending on the operators $\varphi_{d_{\phi}}$. The case of 4] can be derived once again considering $\phi=0$. In this case the following proposition allows us to construct wavelets:

Proposition 8. Let $f \in L_{2}\left(S^{n-1}\right)$. Consider the operator $D_{1}\left(t e_{n}\right)$ with $\left.t \in\right]-1,1[$. Then

$$
\begin{equation*}
\int_{S^{n-1}} \frac{D_{1}\left(t e_{n}\right) f(x)}{\left(1+x_{n}\right)^{n-1}} d S_{x}=\left(\frac{1+t}{1-t}\right)^{\frac{n-1}{2}} \int_{S^{n-1}} \frac{f(x)}{\left(1+x_{n}\right)^{n-1}} d S_{x} \tag{39}
\end{equation*}
$$

It is possible to build a class of admissible functions for the operator $D_{1}\left(t \mathrm{e}_{n}\right)$. Given a square integrable function $\psi$, we define

$$
\begin{equation*}
\eta_{\psi}^{(t)}(x)=\psi(x)-\left(\frac{1-t}{1+t}\right)^{\frac{n-1}{2}} D_{1}\left(t \mathrm{e}_{n}\right) \psi(x) \tag{40}
\end{equation*}
$$

Then it is easily seen that $\eta$ satisfies the (almost) admissibility condition (35). In [3] the authors present for the sphere $S^{2}$ a difference wavelet (40) by choosing

$$
\psi(\theta, \phi)=\exp \left(-\tan ^{2}(\phi / 2)\right)
$$

which is the inverse stereographic projection of a Gaussian in the tangent plane.
Recently, in 32] it was proved a correspondence principle between spherical wavelets and Euclidean wavelets stating that the inverse stereographic projection of a wavelet on the plane leads to the definition of a wavelet in the sphere for the operator $D_{1}\left(t \mathrm{e}_{n}\right)$. Therefore, a lot of examples can be carried to the 2 -sphere such as the

- 2D Mexican hat or Marr wavelet

$$
\begin{equation*}
\psi_{H}(x)=\left(2-|x|^{2}\right) \exp \left(-1 / 2|x|^{2}\right), \quad\left(x \in \mathbb{R}^{2}\right) \tag{41}
\end{equation*}
$$

- the Gabor function

$$
\begin{equation*}
\psi_{G}(x)=\exp \left(i \vec{K}_{0} \cdot x\right) \exp \left(-1 / 2|x|^{2}\right) \tag{42}
\end{equation*}
$$

Other examples of 2D wavelets include the Morlet wavelet, the DOM filter, the optical wavelet, conical or Cauchy wavelets, multidirectional wavelets (see [2]).

It can easily be seen that these examples also provide wavelets for our case of $\varphi_{d_{\phi}}$, where $\phi$ is fixed.

For each dilation operator and an admissible function $\psi \in L^{2}\left(S^{n-1}\right)$ we define the CWT on the sphere as

$$
\begin{aligned}
V_{\psi} f\left(s, d_{\phi}\right) & :=<\pi_{1}\left(s, d_{\phi}\right) \psi, f>_{L_{2}} \\
& =\int_{S^{n-1}} s\left(\frac{1-|d|^{2}}{|1-d \bar{s} x s|^{2}}\right)^{\frac{n-1}{2}} \psi\left(\varphi_{-d}(\bar{s} x s)\right) f(x) d S(x)
\end{aligned}
$$

and thus we can generate a family of conformal spherical wavelets.
7.1. The Hardy space $H^{2}$. In the case of the Hardy space $H^{2}$, with the usual inner product on the unit sphere, we can consider operators that preserve the monogenicity of the function, namely

$$
\begin{equation*}
R_{2}(s) f(x)=s f(\bar{s} x s), \quad s \in \operatorname{Spin}(n) \tag{43}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{2}\left(d_{\phi}\right) f(x)=\left(1-\left|d_{\phi}\right|^{2}\right)^{\frac{n-1}{2}} \frac{1-x d_{\phi}}{\left|1-d_{\phi} x\right|^{n}} f\left(\varphi_{-d_{\phi}}(x)\right) . \tag{44}
\end{equation*}
$$

With these operators we define the representation

$$
\begin{align*}
\pi_{2}\left(s, d_{\phi}\right) f(x) & =R_{2}(s) \circ\left(D_{2}\left(d_{\phi}\right) f(x)\right) \\
& =\left(1-\left|d_{\phi}\right|^{2}\right)^{\frac{n-1}{2}} s \frac{1-\bar{s} x s d_{\phi}}{\left|1-d_{\phi} \bar{s} x s\right|^{n}} f\left(\varphi_{-d_{\phi}}(\bar{s} x s)\right) \tag{45}
\end{align*}
$$

For the Hardy space $H^{2}$ we consider an orthonormal basis of inner spherical monogenics $\left\{P_{k}^{(i)}, i=1, \ldots, N(n, k)\right\}_{k \in \mathbb{N}}$. Then we have:

Theorem 7.4. A function $\psi \in H^{2}$ is admissible if there exists a finite constant $c>0$ such that

$$
\begin{equation*}
\sum_{m} \int_{-1}^{1}\left|a_{k}^{(m)}\left(d_{\phi}\right)\right|_{0}^{2} \frac{d t}{\left(1-t^{2}\right)^{n}}<c \tag{46}
\end{equation*}
$$

where $a_{k}^{(m)}\left(d_{\phi}\right)=<P_{k}^{(m)}, D_{2}\left(d_{\phi}\right) \psi>_{L_{2}}$ are the Fourier coefficients of $D_{2}\left(d_{\phi}\right) \psi$.
Hence, if $\psi \in H^{2}$ is an admissible function we define the corresponding CWT on the sphere as

$$
\begin{aligned}
W_{\psi} f\left(s, d_{\phi}\right) & :=<\pi_{2}\left(s, d_{\phi}\right) \psi, f>_{L_{2}} \\
& =\int_{S^{n-1}} \overline{\left(1-\left|d_{\phi}\right|^{2}\right)^{\frac{n-1}{2}} s \frac{1-\bar{s} x s d_{\phi}}{\left|1-d_{\phi} \bar{s} x s\right|^{n}} \psi\left(\varphi_{-d_{\phi}}(\bar{s} x s)\right)} f(x) d S(x) .
\end{aligned}
$$

We stress here that although $\psi \equiv 1$ is not an admissible function if we use it in the CWT we recover the Cauchy integral formula in Clifford Analysis up to a constant factor (see [25]).

To finalize our approach to CWT on the sphere we would like to emphasize other possible choices in our parameter space. We could consider an elliptic choice for $d_{c}=(c \sin \phi, 0, \ldots, 0, \cos \phi)$ with $c \in\left[0,1[\right.$ fixed and $\phi \in] 0, \pi\left[\right.$. Then $d_{c}$ could also be seen as a conformal dilation/contraction operator in the sphere. However, this choice of parameter do not provide a subgroup of Möbius transformations, although it corresponds to a section on our parameter space and it can be also used for the definition of approximating spaces on the sphere.
8. Frames for the continuous spherical wavelet transform. In this section we would like to construct frames for our spherical wavelet transform. For the sake of simplicity let us restrict ourselves in this section to the three-dimensional case and to real-valued functions. The case of the Hardy space is completely analogous.

Let us assume that $\psi$ is a strictly admissible function, i.e. a mother wavelet. Additionally, we will consider the spaces

$$
M_{p}=\left\{F \in L_{p}\left(B^{n}\right):<F, R(g, \cdot)>=F(g)\right\}
$$

where $R(g, l)$ is the reproducing kernel, e.g. in the case of the Hardy space the Bergman kernel. For this kernel we have to infer

$$
\begin{equation*}
\int_{B^{n}}|R(g, l)| d \mu(l) \leq C_{\psi} \tag{47}
\end{equation*}
$$

for a constant $C_{\psi}<\infty$ independent of $g \in B^{n}$. In the case of the Hardy-space we have as reproducing kernel spaces the classic monogenic Bergman spaces.

Now, the next step is to derive some atomic decomposition for these spaces, i.e., we want to construct suitable Banach frames. From [15] we have the following approach.

Given some neighborhood $\mathcal{U}$ of the identity in a separable Lie group $\mathcal{G}$, a family $X=\left(h_{i}\right)_{i \in \mathcal{I}}$ in $\mathcal{G}$ is called $\mathcal{U}$-dense if $\cup_{i \in \mathcal{I}} \mathcal{U} h_{i}=\mathcal{G}$. A family $X=\left(h_{i}\right)_{i \in \mathcal{I}}$ in $\mathcal{G}$ is called relatively separated, if for any compact set $\mathcal{Q} \subset \mathcal{G}$ there exists a finite partition of the index set $\mathcal{I}$, say $\mathcal{I}=\cup_{r=1}^{r_{0}} \mathcal{I}_{r}$, such that $\mathcal{Q} h_{i} \cap \mathcal{Q} h_{j}=\emptyset$ for all $i, j \in \mathcal{I}_{r}$ with $i \neq j$.

Let $\mathcal{U}$ be an arbitrary compact neighborhood of the identity in $\mathcal{G}$. By [15], there exists a bounded uniform partition of the unity (of size $\mathcal{U}$ ), i.e., a family of continuous functions $\left(\varphi_{i}\right)_{i \in \mathcal{I}}$ on $\mathcal{G}$ such that

- $0 \leq \varphi_{i}(g) \leq 1$ for all $g \in \mathcal{G}$;
- there is an $\mathcal{U}$-dense, relatively separated family $\left(h_{i}\right)_{i \in \mathcal{I}}$ in $\mathcal{G}$ such that $\operatorname{supp} \varphi_{i} \subseteq$ $\mathcal{U} h_{i}$;
- $\sum_{i \in \mathcal{I}} \varphi_{i}(g) \equiv 1$ for all $g \in \mathcal{G}$.

Furthermore, we define the $\mathcal{U}$-oscillation with respect to the analyzing wavelet $\psi$ as

$$
\operatorname{osc}_{\mathcal{U}}(l, h):=\sup _{u \in \mathcal{U}}\left|\left\langle\psi, \pi\left(l h^{-1}\right) \psi-\pi\left(u^{-1} l h^{-1}\right) \psi\right\rangle_{H}\right| .
$$

This setting allows us to use Theorem 4.1 from [15] as well as Theorem 2 from [16.
To this end we can use the decomposition of our group into $\operatorname{Spin}(3) \times[0,1] \times$ $\operatorname{Spin}(3)$, where $[0,1]$ represents the Möbius transformations $\varphi_{\text {te }_{3}}(x)$ with $t \in[0,1]$ along the $e_{3}$-axis. Therefore, we have only to consider the cases $\operatorname{Spin}(3)$ and $t \in$ $[0,1]$.

Let us first consider the part of $\operatorname{Spin}(3)$.
It is well-known that $\operatorname{Spin}(3) \simeq S U(2)$, thereby, we can choose quaternionic coordinates for the group $S U(2)$, defined as $q=\cos (\theta / 2)+\vec{\omega} \sin (\theta / 2)$ with $\theta \in$ $[-\pi, \pi], \vec{\omega} \in S^{2}$.

Let the neighborhood $\mathcal{U}$ be given by $\mathcal{U}=(-\alpha,+\alpha) \times S^{2}$ with $\alpha \in(0, \pi)$. Writing $u=u(\theta, \omega)=u_{0}+u_{1} e_{1}+u_{2} e_{2}+u_{3} e_{3} \in \mathcal{U}$ we get from 9] as a sufficient condition for

$$
I:=\int_{S^{3}} \operatorname{osc}_{\mathcal{U}}(s, t) d \mu(s)<1 .
$$

the following estimate

$$
\begin{equation*}
I \leq K_{\psi} \frac{4 \sqrt{2 \pi}}{3} \sup _{u \in \mathcal{U}} \sqrt{1-u_{0}^{2}}+\sup _{u \in \mathcal{U}} \sqrt{2-2 u_{0}}<1 \tag{48}
\end{equation*}
$$

Passing to quaternionic coordinates we obtain sufficient estimates for the angle $\theta$ such that $u_{0}=\cos (\theta / 2)$. Let us remark that we assumed that our admissible wavelet is normalized.

For the part of the hyperbolic rotations given by $\varphi_{t e_{3}}(x)$ along the $e_{3}$-axis, with $t \in[0,1]$ we obtain as a sufficient condition

$$
\begin{equation*}
\sup _{t \in[\epsilon, 1]}\left\|\psi(\cdot)-\psi\left(\varphi_{t e_{3}}(\cdot)\right)\right\| \ll 1 \tag{49}
\end{equation*}
$$

This leads us to the following theorem.
Theorem 8.1. Let $\mathcal{U}=\mathcal{U}_{\operatorname{Spin}(3)} \times[0, \epsilon]$ where $\mathcal{U}_{\operatorname{Spin}(3)}$ is a compact neighbourhood of the identity of $\operatorname{Spin}(3)$ which satisfies 48) and $[0, \epsilon]$ is a sufficiently small neighbourhood of the origin such that 49) is valid. Moreover, let $\psi$ be a strictly admissible function.

We suppose that $X:=\left(h_{i}\right)_{i \in \mathcal{I}}$ is a $\mathcal{U}$-dense and relatively separated family of $\mathcal{G}=M\left(B^{n}\right)$ such that for any compact neighborhood $\mathcal{Q}$ of the identity in the group $\mathcal{G}$ our wavelet function $\psi$ fulfills the following inequality

$$
\int_{B^{n}} \sup _{u \in Q}\left|\langle\pi(h) \psi, \pi(l u) \psi\rangle_{L_{2}\left(S^{2}\right)}\right| d \mu(l) \leq C
$$

with a constant $C<\infty$ independent of $h \in B^{n}$. Then every $f \in M_{p}, 1 \leq p \leq \infty$ admits the following atomic decomposition

$$
f=\sum_{i \in \mathcal{I}} \pi\left(h_{i}\right) \psi c_{i}
$$

where the sequence of coefficients $\left(c_{i}\right)_{\mathcal{I}} \in l_{p}$ depends linearly on $f$ and satisfies

$$
\left\|\left(c_{i}\right)_{\mathcal{I}}\right\|_{l_{p}} \leq \frac{1}{A}\|f\|_{M_{p}}
$$

Moreover, if $\left(c_{i}\right)_{\mathcal{I}} \in l_{p}$, then $f=\sum_{i \in \mathcal{I}} \pi\left(h_{i}\right) \psi c_{i}$ is in $M_{p}$ and

$$
\frac{1}{B}\|f\|_{M_{p}} \leq\left\|\left(c_{i}\right)_{\mathcal{I}}\right\|_{l_{p}}
$$

Furthermore, in analogy to Theorem 4.2 from [16], we have
Theorem 8.2. Under the same assumptions as in Theorem 8.1 with

$$
\begin{equation*}
\int_{B^{n}} \operatorname{osc}_{\mathcal{U}}(l, h) d \mu(l)<\frac{1}{C_{\psi}} \text { and } \int_{B^{n}} \operatorname{osc}_{\mathcal{U}}(l, h) d \mu(l)<\frac{1}{C_{\psi}} \tag{50}
\end{equation*}
$$

instead of 48), where $C_{\psi}$ is defined by 47). Then the set

$$
\begin{equation*}
\left\{\psi_{i}=\pi\left(h_{i}\right) \psi, i \in \mathcal{I}\right\} \tag{51}
\end{equation*}
$$

is a Banach frame for $M_{p}$.
This implies that
(i) there exist two constants $0<A \leq B<\infty$ such that

$$
\begin{equation*}
\frac{1}{B}\|f\|_{M_{p}} \leq\left\|\left(\left\langle f, \psi_{i}\right\rangle\right)_{i \in \mathcal{I}}\right\|_{l_{p}} \leq \frac{1}{A}\|f\|_{M_{p}} \tag{52}
\end{equation*}
$$

(ii) there exists a bounded, linear reconstruction operator $F^{*}$ from $l_{p}$ to $M_{p}$ such that
$F^{*}\left(\left(<f, \psi_{i}>\right)_{i \in \mathcal{I}}\right)=f ;$
Let $\left\{\psi_{i}=\pi\left(h_{i}\right) \psi: i \in \mathcal{I}\right\}$ denote the Banach frame constructed in Theorem8.2, i.e., we have for any $f \in M_{p}$ that

$$
\begin{equation*}
f=\sum_{i \in \mathcal{I}} \psi_{i}<f, \tilde{\psi}_{i}> \tag{53}
\end{equation*}
$$

where $\left\{\tilde{\psi}_{i}, i \in \mathcal{I}\right\}$ denotes the dual frame. In case of a tight frame, i.e. $A=B$ we have $\tilde{\psi}_{i}=\psi_{i}$.

We are now interested in the best $n$-point approximation, i.e., we want to approximate our function $f \in M_{p}$ by elements from the nonlinear manifold $\Sigma_{n}, n \in N$, which consist of all functions $S \in M_{p}$ whose expansions with respect to our frame have at most $n$ nonzero coefficients, i.e.

$$
\Sigma_{n}:=\left\{S \in M_{p}: S=\sum_{i \in \mathcal{J}} \psi_{i} a_{i}, J \subseteq \mathcal{I}, \# J \leq n\right\}
$$

Of course, we are interested in the asymptotic behavior of the error

$$
E_{n}(f)_{M_{p}}:=\inf _{S \in \Sigma_{n}}\|f-S\|_{M_{p}}
$$

To this end we can state the following Jackson-type theorem:
Theorem 8.3. Let $\left\{\psi_{i}: i \in \mathcal{I}\right\}$ be a Banach frame for $M_{p}, 1 \leq p \leq \infty$, given by Theorem 8.2. If $1 \leq p<q, \alpha:=\frac{1}{p}-\frac{1}{q}$ and $f \in L_{p}$ then

$$
\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(n^{\alpha} E_{n}(f)_{M_{q}}\right)^{p}\right)^{\frac{1}{p}} \leq C\|f\|_{M_{p}}
$$

for a constant $C<\infty$.
We would like to observe that there is no condition on the regularity on $f$, i.e. we do not assume $f$ to be in any Sobolev space $H^{s}\left(S^{2}\right), s>0$, as it is usually the case for the best approximation by spherical harmonics, c.f. 31. In fact, using Sobolev embedding theorem one can easily see that this case is enclosed in the above theorem.

Proof. Without any loss of generality we can assume that the sequence $\left(\mid<f, \tilde{\psi}_{i}>\right.$ $\mid)_{i \in \mathcal{I}}$ in (53) is given in a decreasing order, i.e.,

$$
\left|<f, \tilde{\psi}_{1}>\left|\geq\left|<f, \tilde{\psi}_{2}>\right| \geq \ldots\right.\right.
$$

Then we obtain that

$$
E_{n}(f)_{M_{q}} \leq\left\|\sum_{i=n+1}^{\infty}\left\langle f, \tilde{\psi}_{i}\right\rangle \psi_{i}\right\|_{M_{q}}
$$

due to (52) further that

$$
E_{n}(f)_{M_{q}} \leq C\left(\sum_{i=n+1}^{\infty}\left|\left\langle f, \tilde{\psi}_{i}\right\rangle\right|^{q}\right)^{\frac{1}{q}}=: C E_{n+1, q}\left(\left|\left\langle f, \tilde{\psi}_{i}\right\rangle\right|\right) \leq C E_{n, q}\left(\left|\left\langle f, \tilde{\psi}_{i}\right\rangle\right|\right)
$$

Then we can say that

$$
\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(n^{\alpha} E_{n}(f)_{M_{q}}\right)^{p}\right)^{\frac{1}{p}} \leq\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(n^{\alpha} C E_{n, q}\right)^{p}\right)^{\frac{1}{p}} \leq C\left\|\left(\left|\left\langle f, \tilde{\psi}_{i}\right\rangle\right|\right)\right\|_{l_{p}} \leq C\|f\|_{M_{p}}
$$

9. Approximation by sequences of spaces of fixed dilation. In the previous sections we established a CWT on the sphere for the cases where the admissible function $\psi$ was either in the $L_{2}$-space or in the Hardy space. Again we stress that the support of the admissible function $\psi$ - mother wavelet - must be chosen in such a way that the parameter $d_{\phi}$ defines a local conformal dilation/contraction operator.

For each $d_{\phi}=(t \sin \phi, 0, \cdots, 0, t \cos \phi),(t \in]-1,+1[)$, we choose appropriately $t=t_{0}$ fix and we perform a covering of the sphere by means of the correspondent $\operatorname{cap} \mathcal{U}_{\cos \phi, d_{\phi}}$. In such a way we obtain a discretization of the rotation parameter $s$.

Also, it is possible to choose different kinds of curves in the domain $[0,1] \times[0, \pi]$. For example, we can also consider the curve $\Gamma=(c \sin \phi, 0, \cdots, 0, \cos \phi)$ (for a fixed $c \in[0,1[$ and $\phi \in[0, \pi])$. Upon fixing a parameter $a$ is this curve we again obtain
a correspondent cap that will be used for covering the whole of the sphere. For instance, we can rewrite our representation $\pi_{1}$ in the form

$$
\begin{equation*}
\pi_{1}(s, a) f(x)=s\left(\frac{1-|a|^{2}}{|1-a s x \bar{s}|^{2}}\right)^{\frac{n-1}{2}} f\left(\bar{s} \varphi_{-a}(x) s\right) \tag{54}
\end{equation*}
$$

with $a=(c \sin \phi, 0, \ldots, 0, \cos \phi)$ (c.f. Lemma 9.1). Although the set of such parameters $a$ do not form a group, it is still possible to apply the previous results.

Lemma 9.1. Suppose we choose the parameter a in the form $a=(c \sin \phi, 0, \cdots, 0$, $\cos \phi)$ where $c \in[0,1[$ is fixed and $\phi \in] 0, \pi[$. Then we obtain a decreasing sequence of spherical caps in relation to the distance $d$ considered in (12).

Therefore, we derive an approximation space $V_{a}$ for which Theorem 8.2 still holds (restricted to the Spin-group parameters). Letting $\phi$ go to $\pi$ we obtain a sequence of spaces $V_{a}$ with $V_{a} \subset V_{a_{1}}$ with $a_{1}=\left(c \sin \phi_{1}, 0, \ldots, 0, \cos \phi_{1}\right), \phi \leq \phi_{1}$, approximating the space $L_{2}\left(S^{n-1}\right)$.

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E-mail address: pceres@mat.ua.pt
E-mail address: mferreira@mat.ua.pt
E-mail address: uwek@mat.ua.pt
E-mail address: fs@cage.UGent.be
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