A Wavelet Galerkin Scheme for Non-linear Elliptic Partial Differential Equations

S. Dahlke [*] M. Lindemann, G. Teschke, M. Zhariy [†] M. J. Soares [‡]
P. Cerejeiras, U. Kähler [§]

April 19, 2004

Abstract

We are concerned with the construction and numerical implementation of a wavelet based Galerkin scheme for solving non-linear elliptic partial differential equations. We proceed as follows: we choose a nested sequence of finite dimensional approximation spaces building a bi-orthogonal multi-resolution analysis (MRA). Applying the Galerkin discretization results in a finite dimensional non-linear system. In order to treat the non-linear term somehow reasonable we construct a knot oriented quadrature rule based on interpolating wavelets. Finally, we apply Newton's method to approximate the solution in the given ansatz space. Choosing iteratively finer resolution levels we obtain approximations of the solution with high efficiency.

Starting from a general theory we can show in particular the convergence of the constructed Wavelet-Galerkin-Newton scheme. Moreover, to keep the resulting system stable, we apply a wavelet preconditioner. In order to show the applicability of our method a series of numerical examples for different bi-orthogonal systems is presented.

Key Words: Non-linear elliptic PDE, Galerkin method, multi-resolution analysis, interpolating wavelets, interpolation projector, quadrature rule

1 Introduction

In recent years, wavelet analysis has become a very powerful tool in applied mathematics. While the first applications of wavelets were concerned with problems in image/signal analysis/compression, quite recently also the applications of wavelets to the numerical treatment of partial differential equations have become more and more the center of attraction. Indeed, it has turned out that the strong analytical properties of wavelets can be used to derive powerful numerical schemes including very efficient adaptive algorithms. We refer, e.g., to [Dah97, Dah01] for an overview concerning the current state of the art. The success of wavelet-based numerical scheme relies on the following fundamental properties of wavelets:

[*]Universität Marburg, Fachbereich Mathematik, Germany
[†]Universität Bremen, Fachbereich 3, Germany
[‡]Universidade do Minho, Departamento de Matemática, Portugal
[§]Universidade de Aveiro, Departamento de Matemática, Portugal

- Weighted sequence norms of wavelet expansion coefficients are equivalent in a certain range to Sobolev norms;

- for a wide class of operators their representation in the wavelet basis is nearly diagonal;

- the vanishing moments of wavelets remove the smooth part of a function.

So far, the most far-reaching results were obtained for linear, boundedly invertible operator equations; see, e.g. [CDD01, CDD00, DDHS97, DDU01]. (This list is clearly not complete). Once these problems are well–understood and almost completely solved, the next challenging task is the numerical treatment of *non-linear* problems. This paper can be viewed as one small step in this direction. Quite recently, a first strategy to attack this problem was suggested in [CDD]. After transforming the equation to a well–posed ℓ_2–problem, a locally convergent iterative scheme is applied to the (infinite dimensional) problem. The involved operators are adaptively evaluated within suitable updated error tolerances. However, in this paper, we proceed, in some sense, the other way around. By using a classical Galerkin approach, we project our problem onto an increasing sequence of approximation spaces which are spanned by wavelets. Then the computation of the actual Galerkin approximation requires the solution of non-linear equations in a (finite-dimensional) space. Although the first approach seems to be more powerful, at least in the long run, our method has the advantage that it is easier to analyze and, most of all, much easier to implement.
The basic approach can be described as follows. For simplicity, we consider semi-linear equations. This setting has the advantage that all the usual assumptions which guarantee the existence and uniqueness of solutions and the convergence of Galerkin schemes can easily be verified; see Section 2.1 and Section 4.3. Since the non-linear part can be interpreted as a perturbation of a well–defined linear equation all the wavelet preconditioning results as derived, e.g. in [DK92] carry over without any serious difficulty; see Section 5.2. After projecting our problem onto the (finite dimensional) wavelet spaces, we are faced with two basic problems, namely how to solve the resulting non-linear equation and how to evaluate the non-linear functionals of wavelet expansions induced by the non-linear perturbation. The first problem is treated by a version of Newton's method which is adapted to our problem (cf. Section 4.2). The second problem is attacked by a wavelet variant of the classical "knot oriented quadrature rules"; see, e.g. [KA00]. In the wavelet setting, an analogue can be easily derived by using *interpolating* scaling functions. Fortunately, quite recently a whole variety of interpolating refinable functions including the case of general scaling matrices and bi-orthogonal systems has been constructed; see, e.g. [Der99, DD87, DGM99, DM97, DMT03, JRS99, RS97]. It is therefore natural to try to employ these new functions now for numerical purposes. In Section 3.3, we explain the construction of the associated quadrature rules and estimate their performance. The use of interpolating scaling functions and wavelets has another important advantage. In the wavelet setting, it is by now possible to construct bi-orthogonal interpolating pairs with very high regularity; see Section 6.2. Consequently, using these functions in a Galerkin approach produces a numerical scheme with a high order of convergence, at least if the solution is smooth enough. This observations is indeed confirmed by our numerical experiments; see Section 5.3.
This paper is organized as follows. In Section 2 we describe the class of problems to which our method will be applied and collect some preliminaries. Section 3 contains a

brief review of the basic results on multi-resolution analysis and interpolating wavelets. In Section 4 we describe and analyze the proposed Wavelet-Galerkin scheme. Section 5 is devoted to the detailed analysis of a specific model problem and to the presentation of some numerical results. Finally, in the Appendix, we collect some results on interpolation projectors and on regularity of the wavelet bases.
We finish this section with some remarks concerning the philosophy of this paper.

Remark 1.1 *i) The aim of this paper is to present a first approach concerning the numerical treatment of non-linear equations by means of wavelet methods. Especially, we want to show that many of the different building blocks derived so far fit together quite nicely. One specific intention is to analyze to what extent the setting of interpolating scaling functions and wavelets can be exploited for numerical purposes.*

ii) All numerical experiments were performed for a periodic setting. This might sound artificial at first sight. However, one of our aims is to apply our scheme to problems in image processing where the use of periodic boundary conditions is preferable.

iii) In this paper, our basic approach is tested for some simple univariate examples at first. Indeed, all the building blocks have a natural generalization to the multivariate case, and therefore the results of this paper are formulated for the multivariate setting and substantiated by 2D examples. Since the investigation of numerical wavelet methods involving general scaling matrices became of special interest, e.g. for computational efficiency, we give whenever possible results for general scaling matrices. This could be done in most cases, but, there is still a lack of approximation results for that matter of concern.

iv) Another challenging task which will be studied in the near future is the treatment of problems where our perturbation arguments are no longer justified. Having these problems in mind, one might wonder if the incorporation of very smooth interpolating scaling functions is of any use at all for the following reasons. Firstly, the estimation of the smoothness of the solutions to highly non-linear problems is a delicate problem, and secondly, the solutions might have serious singularities so that the convergence order of the Galerkin scheme drops down, even for smooth ansatz functions. However, in these cases, an adaptive scheme seems to be appropriate, and quite recently, it has been shown that the convergence order of adaptive wavelet schemes depends, among other things, on the smoothness of the wavelet basis.

2 Basic Setting

In this section, we describe the class of problems to which our method will be applied and we briefly review the basic setting of Galerkin's method. We also present some preliminary results, taken from [CR97], which will be essential for establishing the convergence of the proposed wavelet Galerkin method.

2.1 Galerkin's Method

We shall be concerned with the numerical treatment of nonlinear partial differential equations of the type

$$F(u) = Lu + G(u) = 0, \tag{2.1}$$

on some bounded domain $\Omega \subset \mathbb{R}^d$, where L is an linear elliptic differential operator of second order in divergence form and G is an operator of the form

$$G(u) = g(u) - f, \tag{2.2}$$

with $g(u) \in W^{n+1}(L_\infty(\Omega))$ and $f \in L_2(\Omega)$. We restrict ourselves to the periodic setting, i.e. we take Ω to be the unit d-dimensional cube $\Omega := (0,1)^d$ and prescribe periodic boundary conditions for u. For the moment, we assume that the given problem has a unique solution, which we aim to approximate numerically. The conditions that guarantee the existence of such a solution will be discussed later.
In order to solve the given equation (2.1), we first consider its associated weak formulation

$$a(u,v) + \int_\Omega G(u)v\,dx = 0\,, \qquad \text{for all } v \in H^1_p\,, \tag{2.3}$$

where $a(\cdot,\cdot)$ is the bilinear form induced by L and the prescribed boundary conditions. Here, $H^1_p := H^1_p(\Omega)$ denotes the first order Sobolev space with periodic boundary conditions on Ω. Throughout this paper we assume that the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous

$$|a(v,w)| \leq c_1 \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \text{for all } v,w \in H^1_p, \tag{2.4}$$

and also H^1_p-coercive, that is, there exists a positive constant c_2 such that

$$a(v,v) \geq c_2 \|v\|^2_{H_1} \qquad \text{for all } \ v \in H^1_p. \tag{2.5}$$

To treat the above problem numerically, we will use a Galerkin approach. That is, we consider a nested sequence of finite–dimensional approximation spaces $\{\mathcal{V}_j\}_{j\geq 0}$, $\mathcal{V}_j \subset H^1_p$, and project (2.3) onto these spaces. Hence, we are looking for an approximation $u_j \in \mathcal{V}_j$ to the solution u of problem (2.1) by solving

$$a(u_j,v_j) + \int_\Omega G(u_j)v_j\,dx = 0\,, \qquad \text{for all } v_j \in \mathcal{V}_j. \tag{2.6}$$

After fixing a basis $\{\eta_{j,k} : k \in \Lambda_j\}$ of $\mathcal{V}_j$ (Λ_j denotes a finite index set whose cardinality depends on j), the solution u_j of (2.6) will have a representation $u_j = \sum_{k\in\Lambda_j} c_{j,k}\eta_{j,k}$ and we end up with the problem of solving

$$\sum_{k\in\Lambda_j} c_{j,k} a(\eta_{j,k},\eta_{j,l}) + \int_\Omega G(\sum_{k\in\Lambda_j} c_{j,k}\eta_{j,k})\eta_{j,l}\,dx = 0 \quad , \qquad l \in \Lambda_j. \tag{2.7}$$

This is a nonlinear system which can be shortly written as

$$\mathbf{A}_j \cdot \mathbf{c}_j + \mathbf{G}_j(\mathbf{c}_j) = 0, \tag{2.8}$$

with $\mathbf{A}_j$ the matrix whose elements are $(\mathbf{A}_j)_{kl} = a(\eta_{j,k},\eta_{j,l})$ and where the l-th component of the vector $\mathbf{G}_j(\mathbf{c}_j)$ is given by

$$(\mathbf{G}_j(\mathbf{c}_j))_l = \int_\Omega G(\sum_{k\in\Lambda_j} c_{j,k}\eta_{j,k})\eta_{j,l}\,dx.$$

2.2 Convergence Theorem

First of all, we have to ensure that the sequence of Galerkin approximations indeed converges to the exact solution of our problem. In the linear case, this is an easy consequence of Cea's Lemma. In the nonlinear case, things are slightly more complicated. For the sake of completeness, let us briefly recall the basic results as e.g. stated in [CR97].
In the sequel, let X be a Hilbert space, with norm $\|\cdot\|_X$. Also, X' denotes the dual space of X and $\langle\cdot,\cdot\rangle_{X'\times X}$ the duality pairing between X and X'. Finally, if $\mathcal{L}(X,X')$ is the set of all continuous linear operators from X into its dual space X', we denote by $\|T\|_{X;X'} := \sup_{x\in X,\|x\|_X=1}\|T(x)\|_{X'}$ the norm of $T\in\mathcal{L}(X,X')$. We then have the following result.

Theorem 2.1 *Let $F : X \to X'$ be a C^1-mapping from X into its dual and u be a certain point in X such that*

$$F(u) = 0. \tag{2.9}$$

Also, let $\{X_j\}$ be a family of finite dimensional subspaces of X. Furthermore, assume that the following conditions hold:

A1. *The Fréchet derivative of F at u, $DF(u)$, is an isomorphism from X onto X' and DF is Lipschitzian in a neighborhood of u;*

A2. *The bilinear form $b : X\times X \to \mathbb{R}$ defined by*

$$b(x,y) := \langle DF(u)x, y\rangle_{X'\times X} \tag{2.10}$$

satisfies

$$\inf_{\substack{x\in X_j\\ ||x||_X=1}} \sup_{\substack{y\in X_j\\ ||y||_X=1}} b(x,y) \geq \beta_j, \qquad \text{for all} \;\; j,$$

for a certain number $\beta_j > 0$;

A3.

$$\lim_{j\to\infty} \inf_{x_j\in X_j} \beta_j^{-2}\|u - x_j\|_X = 0.$$

Then, there exist two constants $j_0 > 0$ and $\delta_0 > 0$ such that, for all $j \geq j_0$, there exists a unique solution $u_j \in V_j$ to the problem

$$\langle F(u_j), y_j\rangle_{X'\times X} = 0\,, \quad \text{for all } y_j \in X_j,$$

in the closed ball $\overline{B}_{\delta_j}(u)$ with $\delta_j = \delta_0\beta_j$. Moreover, the following estimates hold:

$$\|u - u_j\|_X \leq C\beta_j^{-1} \inf_{x_j\in X_j} \|u - x_j\|_X, \tag{2.11}$$

with the constant C independent of j, and

$$\|u - u_j\|_X \leq 2\,\|DF(u_j)^{-1}\|_{X'\times X}\|F(u_j)\|_{X'}. \tag{2.12}$$

3 Multi-resolution Analysis

Our goal is to derive an efficient Galerkin scheme for the approximate solution of (2.3). However, in contrast to conventional finite element discretizations, we will work with trial spaces that not only exhibit the usual approximation and good localization properties, but in addition lead to *expansions* of any element in the underlying Hilbert spaces in terms of *multi-scale* or *wavelet* bases with certain stability properties. Therefore, in this section we briefly recall the basic properties and construction principles of wavelets.

3.1 Stable Multi-scale Bases

Let us first introduce some notation. Let M be an integer $d \times d$ matrix which is *expanding*, that is, all its eigenvalues have modulus larger than one. A finite family of functions $\{\psi^i : i \in \mathcal{I}\}$ is said to be a set of *mother wavelets* (with scaling matrix M) if the $\mathbb{Z}^d$ translates of M dilates of these functions form a Riesz basis of $L_2(\mathbb{R}^d)$. Recall that a Riesz basis is a set of functions $\{f_\lambda : \lambda \in \Lambda,\ \Lambda \text{ a countable index set}\}$ in $L_2(\mathbb{R}^d)$ which is ℓ_2-stable, that is,

$$\|c\|_{\ell_2(\Lambda)} \sim \| \sum_{\lambda \in \Lambda} c_\lambda f_\lambda \|_{L_2(\mathbb{R}^d)}, \tag{3.1}$$

for any sequence $c = \{c_\lambda\} \in \ell_2(\Lambda)$, and its linear span is dense in $L_2(\mathbb{R}^d)$.
Here and throughout the paper, $a \sim b$ means $a \lesssim b$ and $b \lesssim a$, with the latter relation expressing that b can be bounded by some constant times a uniformly in any parameters on which a and b may depend.
More specifically, the members of the wavelet basis are the functions given by

$$\psi^i_{j,k}(x) := m^{j/2} \psi^i(M^j x - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ i \in \mathcal{I},$$

where $m = |\det M|$. It can also be shown that the minimum number of basic wavelets associated with the scaling matrix M is $m - 1$.
A wavelet basis is usually constructed by means of a *multi-resolution analysis*. This is a nested sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L_2(\mathbb{R}^d)$ whose union is dense in $L_2(\mathbb{R}^d)$, while their intersection is zero. All the spaces are related by M-dilation i.e.

$$f \in V_j \quad \text{if and only if} \quad f(M\cdot) \in V_{j+1}. \tag{3.2}$$

Finally, there is a function ϕ, the so-called *generator* of the multi-resolution analysis, whose translates $\{\phi(\cdot - k) :\ k \in \mathbb{Z}^d\}$ form a Riesz basis for V_0.
Under the above assumptions, it can be shown that the generating function ϕ is $(\mathbf{h}, M)$-*refinable*, i.e. it satisfies a *refinement equation*

$$\phi(x) = \sum_{k \in \mathbb{Z}^d} h_k \phi(Mx - k) \tag{3.3}$$

for a certain *refinement mask* $\mathbf{h} = \{h_k\}_{k\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$. Any function $\phi \in L_2(\mathbb{R}^d)$ satisfying a two-scale-relation (3.3) is called a *scaling function*.
The case where the scaling matrix is $M = 2I$ is often referred to as a *dyadic* multi-resolution analysis and is, by far, the most well studied case, with many available results.

Remark 3.1 *In all practical applications, we will work with spaces V_j satisfying certain approximation and suitable localization properties. These properties are, of course, dependent on the generating function ϕ. Throughout the remainder of this paper, we shall always assume that the scaling function $\phi \in L_2(\mathbb{R}^d)$ satisfies the following properties:*

(P1) $\int \phi(x)dx = 1$;

(P2) $\phi \in C_c(\mathbb{R}^d)$;

(P3) $\phi \in W^{n+1}(L_\infty(\mathbb{R}^d))$, $n \in \mathbb{N}$;

Let W_j be a subspace complementing V_j in V_{j+1}, i.e., W_j is such that $V_{j+1} = V_j \oplus W_j$, where $\oplus$ denotes a direct sum. From (3.2) and the further properties of the multi-resolution analysis it can be checked that $\bigoplus_{j\in\mathbb{Z}} W_j = L_2(\mathbb{R}^d)$, where W_j is defined by

$$W_j = \{f : \ f = g(M^j \cdot), \ g \in W_0\}.$$

Hence, if we can find functions $\{\psi^i : i \in \mathcal{I}\}$ whose $\mathbb{Z}^d$ translates form a stable basis of W_0, then the collection $\{\psi^i_{j,k} : i \in \mathcal{I}, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}$ will be a good candidate for a wavelet basis. Of course, there is a continuum of possible choices of such complement spaces W_j. Orthogonal decompositions would lead to orthonormal wavelet bases. However, orthogonality often interferes with locality and the actual computation of orthonormal bases might be too expensive. Moreover, in certain applications orthogonal decompositions are actually not best possible [DPS94]. These limitations motivated the search for bi-orthogonal wavelets. In this case, we look for two wavelet bases $\{\psi^i_{j,k} : i \in \mathcal{I}, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}$ and $\{\widetilde{\psi}^i_{j,k} : i \in \mathcal{I}, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ which satisfy

$$\left\langle \psi^i_{j,k}, \widetilde{\psi}^{i'}_{j',k'} \right\rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}, \quad \text{for all } j, j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^d \text{ and } i, i' \in \mathcal{I}. \tag{3.4}$$

These bi-orthogonal wavelet bases can be associated with two multi-resolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ which are "connected" in the following manner:

$$\left\langle \phi(\cdot - k), \widetilde{\phi}(\cdot - k') \right\rangle = \delta_{k,k'}, \quad \text{for all } k, k' \in \mathbb{Z}^d, \tag{3.5}$$

where ϕ and $\tilde{\phi}$ are the generating functions of the sequences $\{V_j\}$ and $\{\widetilde{V}_j\}$, respectively. Two such multi-resolution analysis are said to be *dual* or *bi-orthogonal*. In this case, we can define the detail spaces W_j and $\widetilde{W}_j$ by

$$V_{j+1} = V_j \oplus W_j, \ W_j \perp \tilde{V}_j \qquad \text{and} \qquad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j, \ \tilde{W}_j \perp V_j. \tag{3.6}$$

For the univariate case and dyadic scaling there exist quite canonical approaches to construct functions $\{\psi^i : i \in \mathcal{I}\}$ and $\{\widetilde{\psi}^i : i \in \mathcal{I}\}$ such that the sets of functions $\{\psi^i_{j,k}\}$ and $\{\widetilde{\psi}^i_{j,k}\}$ are two Riesz bases of $L_2(\mathbb{R}^d)$ satisfying the bi-orthogonality relations (3.4), see e.g. [CDF92]. However, for the multivariate case things are much more complicated. Nevertheless, for dyadic scalings in the multivariate setting a wavelet basis can always be constructed by means of tensor products, see [CHR00, JRS99, RS92, Dah97, DDM96, DM90, JM91, RS97] for further details. The relationship of the scaling functions ϕ and $\widetilde{\phi}$ to wavelets generated via the multi-resolution analysis paradigm is given by

$$\psi^i(x) = \sum_{k\in\mathbb{Z}^d} g^i_k \phi(Mx - k) \ , \qquad \widetilde{\psi}^i(x) = \sum_{k\in\mathbb{Z}^d} \tilde{g}^i_k \widetilde{\phi}(Mx - k) \ , \quad i \in \mathcal{I}, \tag{3.7}$$

where $\{g^i_k\}_{k\in\mathbb{Z}^d}$ and $\{\tilde{g}^i_k\}_{k\in\mathbb{Z}^d}$ are certain sequences in $\ell_2(\mathbb{Z}^d)$. These sequences, together with the masks $\{h_k\}$ and $\{\tilde{h}_k\}$ of the scaling functions are the basis of fast decomposition

and reconstruction algorithms for computing with wavelets. In many cases it is also possible to choose finite sequences $\{h_k\}$ and $\{\tilde{h}_k\}$, hence obtaining wavelet functions ψ^i and $\widetilde{\psi}^i$ of compact support.
Let P_j denote the bi-orthogonal projector of $L_2(\mathbb{R}^d)$ onto V_j given by

$$P_j f = \sum_{k \in \mathbb{Z}^d} \left\langle f, \tilde{\phi}_{jk} \right\rangle \phi_{j,k} \tag{3.8}$$

Also, let $Q_j := P_{j+1} - P_j$. Note that Q_j is a projector onto W_j which can be written as

$$Q_j f = \sum_{i \in \mathcal{I}} \sum_{k \in \mathbb{Z}^d} \left\langle f, \tilde{\psi}^i_{j,k} \right\rangle \psi^i_{j,k}. \tag{3.9}$$

Each function $f \in L_2(\mathbb{R}^d)$ will thus have a *multi-scale* representation, as

$$f = P_{j_0} f + \sum_{j \geq j_0} Q_j f = \sum_k \left\langle f, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} + \sum_i \sum_{j \geq j_0} \sum_k \left\langle f, \tilde{\psi}^i_{j,k} \right\rangle \psi^i_{j,k}, \tag{3.10}$$

where j_0 denotes a certain coarsest level. Sometimes, it will also be convenient to adopt a simplified notation for the nodal and wavelet bases. We will set

$$\Gamma_j := \{\gamma = (i, j, k) : i \in \mathcal{I}, k \in \mathbb{Z}^d\}, \quad j \in \mathbb{N}_0, \tag{3.11}$$

and

$$\psi_\gamma := \psi^i_{j,k}, \quad \text{for} \quad \gamma = (i, j, k) \in \Gamma_j.$$

We shall also write

$$\Lambda_j := \{\lambda = (j, k) : k \in \mathbb{Z}^d\}, \quad j \in \mathbb{N}_0, \tag{3.12}$$

and

$$\phi_\lambda := \phi_{j,k}, \quad \text{for} \quad \lambda = (j, k) \in \Lambda_j.$$

(Obviously, we will adopt similar notations for the duals). Then, the multi-scale representation for f can be written as

$$f = \sum_{\lambda \in \Lambda_0} \left\langle f, \tilde{\phi}_\lambda \right\rangle \phi_\lambda + \sum_{j \geq 0} \sum_{\gamma \in \Gamma_j} \left\langle f, \tilde{\psi}_\gamma \right\rangle \psi_\gamma. \tag{3.13}$$

If we further adopt the conventions $\Gamma_{-1} := \Lambda_0$ and $\psi_\gamma := \phi_\gamma$, if $\gamma \in \Gamma_{-1}$, the above representation reads as

$$f = \sum_{j \geq -1} \sum_{\gamma \in \Gamma_j} \left\langle f, \tilde{\psi}_\gamma \right\rangle \psi_\gamma. \tag{3.14}$$

With the above notations, $\Phi_j := \{\phi_\lambda : \lambda \in \Lambda_j\}$ and $\Psi_j := \{\psi_\gamma : \gamma \in \Gamma_l, -1 \leq l < j\}$ are the nodal and multi-scale basis of V_j, respectively.
For the numerical application we have in mind, it is essential to estimate the approximation power of the bi-orthogonal multi-resolution analysis. Usually, the order of approximation that can be achieved depends on the (Sobolev) smoothness of the approximated function and of the generator. The following result is known for the case where $(V_j, \widetilde{V}_j)$ form a pair of bi-orthogonal multi-resolution analyses obtained by *dyadic* scalings, with scaling functions ϕ and $\tilde{\phi}$ satisfying properties **P1**–**P3** (with parameter $\tilde{n}$ for the dual function); see, e.g. [Coh00, Chapter 3].

Theorem 3.1 *Let $0 < s < t < \nu$ and assume that $\phi \in H^\nu(\mathbb{R}^d)$. Then, we have*

$$\|f - P_j f\|_{H^s(\mathbb{R}^d)} \lesssim 2^{-j(t-s)} |f|_{H^t(\mathbb{R}^d)}, \tag{3.15}$$

where $|\cdot|_{H^t(\mathbb{R}^d)}$ denotes the usual semi-norm for the Sobolev space $H^t(\mathbb{R}^d)$.

The generalization of this result to other types of multi-scale decompositions (e.g, for the case where M is a general expanding matrix) is currently under consideration, see [Lin03]. The results above were given for spaces defined on the whole Euclidean space. As already mentioned, in this paper we are interested in the case where the domain Ω is the d-cube $(0,1)^d$ and we are looking for 1-periodic solutions of the differential equation (2.1), i.e. for functions u such that $u(x) = u(x+k)$ and $\frac{\partial u}{\partial x} = \frac{\partial u(\cdot + k)}{\partial x}$ for all $k \in \mathbb{Z}^d$. To deal with this periodic setting, we can simply proceed as indicated, e.g., in [DPS94]. Let $\mathcal{T}^d$ be the d-dimensional torus $\mathcal{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and identify all 1-periodic functions with functions defined on $\mathcal{T}^d$. For any function g (compactly supported or rapidly decaying), let $[g]$ denote the function obtained from g by periodization, i.e.

$$[g](x) := \sum_{l \in \mathbb{Z}^d} g(x + l).$$

For a given pair of dual multi-resolution analyses of $L_2(\mathbb{R}^d)$, with a dual pair of scaling functions ϕ and $\tilde{\phi}$, consider the periodized functions

$$\begin{aligned} [\phi_{j,k}] &= m^{j/2} \sum_{l \in \mathbb{Z}^d} \phi(M^j(\cdot + l) - k), \\ [\tilde{\phi}_{j,k}] &= m^{j/2} \sum_{l \in \mathbb{Z}^d} \tilde{\phi}(M^j(\cdot + l) - k) \end{aligned}$$

and define the spaces

$$\begin{aligned} \mathcal{V}_j &= [V_j] := \mathrm{Span}(\{[\phi_{j,k}] : \ k \in \mathbb{Z}^{d,j}\}), \\ \tilde{\mathcal{V}}_j &= [\tilde{V}_j] := \mathrm{Span}(\{[\tilde{\phi}_{j,k}] : \ k \in \mathbb{Z}^{d,j}\}), \text{ respectively,} \end{aligned}$$

where

$$\mathbb{Z}^{d,j} := \mathbb{Z}^d/(M^j \mathbb{Z}^d).$$

Then, it can easily be shown that these finite dimensional spaces (with dimension m^j) are nested and dense in $L_2(\mathcal{T}^d)$. We will say that they constitute a *periodic multi-resolution analysis* generated by ϕ and $\tilde{\phi}$. It can be shown that the orthogonality relations (3.5) and (3.4) carry over to the periodic setting in the usual way, i.e.

$$\left\langle [\psi^i_{j,k}], [\widetilde{\psi}^{i'}_{j',k'}] \right\rangle = \delta_{i,i'} \delta_{j,j'} \delta_{k,k'}, \quad \text{for all } j, j' \in \mathbb{N}_0, k \in \mathbb{Z}^{d,j}, k' \in \mathbb{Z}^{d,j'} \text{ and } i, i' \in \mathcal{I}. \tag{3.16}$$

Finally, it is important to observe that the direct estimate (3.15) for the dyadic case remains valid in this periodic setting.

3.2 Interpolating Scaling Functions and Duals

For our purpose, it will be convenient to work with *interpolating* refinable functions, i.e., one requires that ϕ is continuous and satisfies

$$\phi(k) = \delta_{0,k}, \qquad k \in \mathbb{Z}^d. \tag{3.17}$$

As already stated, we only consider compactly supported scaling functions. Furthermore, we would like ϕ to have a certain smoothness property. In recent studies, several examples of refinable functions satisfying the above conditions have been constructed; see, e.g., [DGM99, DM97, Der99, DDD91, DD87, Dub86, RS97]. In particular, in all our numerical computations, we select the interpolating scaling functions from the family of the so-called Deslauriers-Dubuc *fundamental functions*. These functions, which are obtained as auto-correlation of the well-known compactly supported orthogonal Daubechies scaling functions, have very attractive properties; see, e.g., [DD87, Dub86]. In fact, if we denote by $\phi := \phi^{2N}$ the Deslauriers-Dubuc function of order $2N$ (obtained as autocorrelation of the Daubechies function associated with the parameter N), then ϕ has compact support and is interpolating; moreover the smoothness of ϕ^{2N} increases with N and ϕ^{2N} has polynomial exactness $2N-1$.

Algorithms for constructing a dual scaling function $\tilde{\phi}$ for a given interpolating scaling function ϕ was developed in [JRS99]. A brief description of this algorithm can be seen in the appendix.

3.3 Quadrature Rules via Interpolating Wavelets

Later on, we want to use wavelets as basis functions for a Galerkin scheme. Then, as discussed subsequently in more detail, the problem of evaluating the nonlinear term applied to a wavelet expansion arises, i.e. we have to compute

$$\int_\Omega G\left(\sum_{j\geq -1}\sum_{\gamma\in\Gamma_j} d_\gamma[\psi_\gamma]\right)[\psi_{\gamma'}]\ dx. \tag{3.18}$$

Later on, we will see that due to the functional equation (3.7), it is sufficient for our setting to treat expansions in terms of the nodal basis

$$\int_\Omega G\left(\sum_{\lambda\in\Lambda_j} c_\lambda[\phi_\lambda]\right)[\phi_{\lambda'}]\ dx. \tag{3.19}$$

In this paper, we suggest to approximate the integral in (3.19) by a version of the so-called knot oriented quadrature rules used in the finite element setting. The general approach can be described as follows: given a function v, the integral $\int_\Omega v(x)dx$ is approximated by $\int_\Omega (\pi v)(x)dx$ where π is a general interpolation operator. A widely adopted strategy is to use, e.g. an interpolation operator based on polynomials or Lagrange finite elements. Here we focus on an *interpolation projector* Π_j induced by an interpolating scaling function ϕ. For some 1-periodic v this projector is defined by

$$\Pi_j v := \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}\,k)[\phi_{jk}]. \tag{3.20}$$

Replacing $\int_\Omega v$ by $\int_\Omega \Pi_j v$ we obtain

$$
\begin{aligned}
\int_\Omega (\Pi_j\, v)(x)dx &= \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k)\int_\Omega [\phi_{jk}](x)dx \\
&= \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k)\int_\Omega \sum_{l\in\mathbb{Z}^d}\phi(M^j(x-l)-k)dx \qquad (3.21)\\
&= \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k)\sum_{l\in\mathbb{Z}^d}\int_\Omega \phi(M^j(x-l)-k)dx \qquad (3.22)\\
&= \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k)\sum_{l\in\mathbb{Z}^d}\int_{\Omega+l} \phi(M^j x-k)dx \qquad (3.23)\\
&= \sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k)\int_{\mathbb{R}^d} \phi(M^j x-k)dx \qquad (3.24)\\
&= m^{-j}\sum_{k\in\mathbb{Z}^{d,j}} v(M^{-j}k). \qquad (3.25)
\end{aligned}
$$

We first state the following results concerning the interpolation operator Π_j. For the proof, the reader is referred to the appendix.

Theorem 3.2 *Let Π_j be the operator* (3.20) *associated with a $(\mathbf{h}, M)$–refinable ℓ_2–stable interpolating scaling function $\phi \in W^n(L_1(\mathbb{R}^d))$ for $0 < n \in \mathbb{N}$, $\phi \in C_c(\mathbb{R}^d)$, where M is an isotropic scaling matrix, that is, M is similar to a diagonal matrix* $\operatorname{diag}\{\lambda_1, \ldots, \lambda_d\}$ *with $|\lambda_1| = \ldots = |\lambda_d|$. If $v \in W^{n+1}(L_\infty(\mathcal{T}^d))$, then*

$$\|v - \Pi_j v\|_{L_\infty(\mathcal{T}^d)} \lesssim r(M)^{-j(n+1)}, \qquad (3.26)$$

where $r(M)$ denotes the spectral radius of the scaling matrix M.

Remark 3.2 *The assumption $\phi \in W^n(L_1(\mathbb{R}^d))$ is not an additional requirement on ϕ. It follows directly from property* **P3**.

Applying Theorem 3.2, we immediately obtain the following error estimate for the quadrature rule (3.21).

Corollary 3.1 *Under the conditions of Theorem* 3.2*, the following error estimate holds*

$$\left|\int_\Omega v(x)dx - \int_\Omega (\Pi_j v)(x)dx\right| \lesssim r(M)^{-j(n+1)} \ .$$

4 Wavelet-Galerkin Scheme

We are now ready to construct a Galerkin scheme based on wavelets where the quadrature rule explained above allows an efficient treatment of the nonlinear part.

4.1 Wavelet Based Galerkin Ansatz

Again, we consider the Galerkin approach for solving our problem as described in Section 2.1 and choose as approximating spaces $\mathcal{V}_j$, the finite-dimensional spaces of a periodic multi-resolution analysis $\{[V_j]\}_{j\geq 0}$ associated with a certain interpolating scaling function

ϕ. We will use the simplified notation introduced in Section 3.1 for the nodal and multiscale bases, i.e. we let $\Phi_j := \{[\phi_\lambda] : \lambda \in \Lambda_j\}$ and $\Psi_j := \{[\psi_\gamma] : \gamma \in \Gamma_l, -1 \le l < j\}$. In this case, we obtain the following system for computing the coefficient vector $\mathbf{c}_j := (c_\lambda)_{\lambda\in\Lambda_j}$ of the approximate solution $u_j \in [V_j]$ in the nodal basis Φ_j:

$$\sum_{\lambda\in\Lambda_j} c_\lambda a([\phi_\lambda],[\phi_{\lambda'}]) + \int_\Omega g(\sum_{\lambda\in\Lambda_j} c_\lambda[\phi_\lambda])[\phi_{\lambda'}]\,dx - \int_\Omega f[\phi_{\lambda'}]\,dx = 0\ , \qquad \lambda' \in \Lambda_j,$$

where we used the expression $G(u) = g(u) - f$ for the operator G.
To deal with the nonlinear term, we can now use the quadrature rule (3.21) associated with the operator π_j. Setting

$$v = g\left(\sum_{\lambda\in\Lambda_j} c_\lambda[\phi_\lambda]\right)[\phi_{\lambda'}]$$

and using the interpolation property of ϕ yields to the following approximation

$$\int_\Omega g\left(\sum_{\lambda\in\Lambda_j} c_\lambda[\phi_\lambda]\right)[\phi_{\lambda'}]\,dx \approx m^{-j/2} g(m^{j/2}c_{\lambda'}).$$

This leads to a modified Galerkin system which can be written as

$$\mathbf{A}_j \cdot \mathbf{c}_j + \widetilde{\mathbf{G}}_j(\mathbf{c}_j) = 0, \tag{4.1}$$

where the $\lambda'-$th component of $\widetilde{\mathbf{G}}_j(\mathbf{c}_j)$ is given by

$$\left(\widetilde{\mathbf{G}}_j(\mathbf{c}_j)\right)_{\lambda'} = m^{-j/2} g(m^{j/2}c_{\lambda'}) - \int_\Omega f[\phi_{\lambda'}]\,dx. \tag{4.2}$$

¿From (4.1) we can also obtain a system for determining the coefficient vector $\mathbf{d}_j$ of the approximate solution in the *multilevel* basis Ψ_j. Let $\mathbf{L}_j$ denote the matrix that takes the coefficients relative to Ψ_j into those relative to the nodal basis Φ_j. Then, we have

$$\mathbf{L}_j^{-1}\mathbf{A}_j\mathbf{L}_j\mathbf{L}_j^{-1}\mathbf{c}_j + \mathbf{L}_j^{-1}\widetilde{\mathbf{G}}_j(\mathbf{c}_j) = 0,$$

or

$$\mathbf{B}_j\mathbf{d}_j + \mathbf{L}_j^{-1}\widetilde{\mathbf{G}}_j(\mathbf{L}_j\mathbf{d}_j) = 0, \tag{4.3}$$

where $\mathbf{B}_j = \mathbf{L}_j^{-1}\mathbf{A}_j\mathbf{L}_j$ is the stiffness matrix of the linear part with respect to the multilevel basis and $\mathbf{d}_j = \mathbf{L}_j^{-1}\mathbf{c}_j$. Details of the construction of the matrix $\mathbf{L}_j$ can be found e.g. in [DK94].

4.2 Iterative Newton Scheme

So far, we have seen that one fundamental step for solving our partial differential equation (2.1) is the solution of the nonlinear system (4.3). Here, we propose to use Newton's method for solving this system. With $J = \#\mathbb{Z}^{d,j} = m^j$, let $F_j : \mathbb{R}^J \to \mathbb{R}^J$ be the mapping defined by

$$F_j(\xi) := \mathbf{B}_j\ \xi + \mathbf{L}_j^{-1}\widetilde{\mathbf{G}}_j(\mathbf{L}_j\xi). \tag{4.4}$$

Then, we apply the following iterative scheme, starting with an appropriate initial vector $\mathbf{d}_j^{(0)}$:

$$\begin{cases} F_j'(\mathbf{d}_j^{(n)})\ \zeta^{(n+1)} = F_j(\mathbf{d}_j^{(n)}) \\ \mathbf{d}_j^{(n+1)} = \mathbf{d}_j^{(n)} - \zeta^{(n+1)}. \end{cases} \tag{4.5}$$

To obtain the starting vector of length mJ for the next finer level $j+1$ one appends $m^j(m-1)$ zeros. The Jacobian matrix of F_j is naturally given by

$$F_j'(\xi) = \mathbf{B}_j + \mathbf{L}_j^{-1}\ \mathcal{G}_j\ \mathbf{L}_j \tag{4.6}$$

with

$$\mathcal{G}_j = \begin{pmatrix} g'(m^{j/2}(\mathbf{L}_j\xi)_1) & & 0 \\ & \ddots & \\ 0 & & g'(m^{j/2}(\mathbf{L}_j\xi)_J) \end{pmatrix}. \tag{4.7}$$

In every Newton step one has to solve a linear system of equations which has to be done efficiently. Since all the matrices in (4.6) are sparse, iterative schemes such as the CG algorithm suggest themselves. Therefore it is necessary to employ suitable precondition strategies. This problem will be discussed in detail in Section 5.2.

4.3 Convergence Result

We now establish some results concerning the convergence of our wavelet-Galerkin approximation to the nonlinear partial differential equation (2.1).
Our aim is to estimate the error

$$\|u - \tilde{u}_j\|_{H^1}, \tag{4.8}$$

where $\tilde{u}_j$ denotes an approximation obtained by a possible *exact* solution of the nonlinear system (4.1). Hence, for the moment we do not take into account the error due to the application of Newton's method, i.e. we assume that the iteration scheme is applied up to machine precision. As it is well known, the quadratic convergence of this method is guaranteed provided the initial approximation is chosen sufficiently close to the solution. To obtain an estimate for the error (4.8) we use Theorem 2.1 for the case $X = H_p^1(\Omega)$ with norm $\|.\|_{H_p^1} = \|.\|_{H_1}$, $X_j = V_j$, with V_j the spaces of the chosen multi-resolution analysis and $F : X \to X'$ the function defined by

$$\langle F(u), v\rangle_{X'\times X} = a(u,v) + \int_\Omega G(u)v\ , \quad \text{for all } u, v \in H_p^1. \tag{4.9}$$

We assume that the problem $F(u) = 0$ or equivalently

$$\langle F(u), v\rangle_{X'\times X} = a(u,v) + \int_\Omega G(u)v\ = 0\ , \quad \text{for all } v \in H_p^1, \tag{4.10}$$

has a unique solution $u \in H_p^1$. Then, the function u_j is the solution of the problem

$$\langle F(u_j), v_j\rangle_{X'\times X} = 0\,, \quad \text{for all } v_j \in V_j. \tag{4.11}$$

Then, we have the following result:

Theorem 4.1 *Let u be the solution of the problem* (4.10) *and assume that the conditions of Theorem* 2.1 *and* 3.2 *hold, so that there exists a unique solution u_j of the wavelet Galerkin system* (4.11) *in a sufficiently small neighborhood of u. Also, let $\tilde{u}_j$ denote the solution obtained from solving exactly the approximated nonlinear system* (4.1)*. Furthermore, let us assume that*

1. $g(u), \phi \in W^{n+1}(L_\infty(\mathbb{R}^d))$ *with* $0 < n \in \mathbb{N}$
2. *the function g induces a monotone operator in the following way*

$$\int_\Omega (g(v) - g(w))\,(v - w)\,dx \geq 0, \quad \textit{for all}\; v, w \in H^1_p. \tag{4.12}$$

Then, we have the following error bound for the wavelet-Galerkin scheme

$$||u - \tilde{u}_j||_{H^1} \leq C_1 \beta_j^{-1} ||u - P_j u||_{H^1} + C_2 (r(M))^{-j\frac{n+1}{2}}, \tag{4.13}$$

where $r(M)$ is the spectral radius of the scaling matrix M, β_j is the constant referred to in Assumptions **A2**–**A3** *of Theorem* 2.1 *and P_j is the oblique projector into the approximating space V_j defined by* (3.8)*.*

Proof : In order to obtain the error bound (4.13), we will use the decomposition

$$\|u - \tilde{u}_j\|_{H^1} \leq \|u - u_j\|_{H^1} + \|u_j - \tilde{u}_j\|_{H^1}. \tag{4.14}$$

A direct application of Theorem 2.1 immediately gives

$$\|u - u_j\|_{H^1} \leq C_1 \beta_j^{-1} \inf_{x_j \in V_j} ||u - x_j||_{H^1} \leq C_1 \beta_j^{-1} ||u - P_j u||_{H^1}.$$

Now we establish a bound for the second component of the error (4.14). Looking at the initial problem

$$a(u_j, v) + \int_\Omega g(u_j)\, v\, dx - \int_\Omega f v dx = 0\ , \quad \text{for all } v \in V_j$$

and the perturbed problem arising by application of the quadrature rule

$$a(\tilde{u}_j, v) + \int_\Omega \Pi_j\left(g(\tilde{u}_j)\, v\right) - \int_\Omega f v dx = 0\ , \quad \text{for all } v \in V_j\ ,$$

we obtain

$$a(u_j - \tilde{u}_j, v) + \int_\Omega \Big(g(u_j)\, v - \Pi_j\left(g(\tilde{u}_j)\, v\right) \Big)\, dx = 0, \quad \text{for all } v \in V_j,$$

so that, in particular

$$a(u_j - \tilde{u}_j, u_j - \tilde{u}_j) + \int_\Omega \Big(g(u_j)(u_j - \tilde{u}_j) - \Pi_j\left(g(\tilde{u}_j)(u_j - \tilde{u}_j)\right) \Big)\, dx \quad = \quad 0.$$

Hence, we have

$$a(u_j - \tilde{u}_j, u_j - \tilde{u}_j) + \int_\Omega \left(g(u_j) - g(\tilde{u}_j)\right)(u_j - \tilde{u}_j)\, dx$$

$$= \int_\Omega \Big(\Pi_j \left(g(\tilde{u}_j)(u_j - \tilde{u}_j) \right) - g(\tilde{u}_j)(u_j - \tilde{u}_j) \Big) dx.$$

By assumption *1.* on g and ϕ, we can apply Theorem 3.2 to the function $g(\tilde{u}_j)(u_j - \tilde{u}_j)$. Therefore, by use of the coercivity of the bilinear form a, we obtain

$$c_2 \|u_j - \tilde{u}_j\|_{H^1}^2 + \int_\Omega \left(g(u_j) - g(\tilde{u}_j) \right) (u_j - \tilde{u}_j) dx \leq C'(r(M))^{-j(n+1)}$$

where c_2 is the constant in (2.5). Hence, employing the property (4.12) of g, we get the following convenient upper estimate for $\|u_j - \tilde{u}_j\|_{H^1}$:

$$c_2 \|u_j - \tilde{u}_j\|_{H^1}^2 \leq C'(r(M))^{-j(n+1)}$$

or

$$\|u_j - \tilde{u}_j\|_{H^1} \leq \sqrt{\frac{C'}{c_2}} [r(M)]^{-j\frac{n+1}{2}} = C_3 (r(M))^{-j\frac{n+1}{2}}. \tag{4.15}$$

■

In the particular case of a dyadic multi-resolution analysis satisfying the assumptions of Theorem 3.1, we obtain the following error bound:

Corollary 4.1 *Under the conditions of Theorem* 4.1*, if the approximating wavelet spaces correspond to the choice $M = 2I$ and satisfy* (3.15)*, we have the following error estimate*

$$||u - \tilde{u_j}||_{H^1} \leq C_1 \beta_j^{-1} 2^{-j(t-1)} |u|_{H^t} + C_2 2^{-j\frac{\alpha}{2}}, \tag{4.16}$$

for $u \in H^t$.

5 Special Model Problem

In this section, we discuss in detail the convergence of our numerical scheme for a specific model problem and present several numerical tests. We consider the case

$$Lu = -\Delta u + u \quad \text{and} \quad G(u) = \epsilon u^3 - f, \tag{5.1}$$

for a given function $f \in L_2(\Omega)$. As before, we take $X = H_p^1(\Omega)$ with the norm $\|\cdot\|_{H_p^1(\Omega)} = \|\cdot\|_{H^1}$. Here, the bilinear form $a(\cdot,\cdot)$ associated with the operator L and the periodic boundary conditions is given by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx, \tag{5.2}$$

and it is obvious that this bilinear form is continuous and H_p^1-coercive. In this particular case we obtain

$$a(v,v) = ||v||_{H^1}^2. \tag{5.3}$$

Similarily as shown in [CR97] one can prove the following result.

Remark 5.1 *The special model problem* (5.1) *has a unique solution $u \in H_p^1(\Omega)$.*

This existence result may deduced by applying the Leray-Schauder Theorem ([Zei86], Chapter 6, p.245).

5.1 Convergence for the model problem

The mapping $F : H_p^1 \to (H_p^1)'$ is now defined by

$$\langle F(u), v\rangle_{X'\times X} = a(u,v) + \int_\Omega (\epsilon u^3 - f)v dx \ , \quad \text{for all } u,v \in X = H_p^1,$$

with $a(u,v)$ given by (5.2). We determine $u \in H_p^1$ such that, for all $v \in H_p^1$

$$a(u,v) + \int_\Omega (\epsilon u^3 - f)v \, dx = 0. \tag{5.4}$$

In order to establish the convergence of the method, we want to use Theorem 4.1. First, we establish **A2**. Observe that $DF(u)$ defines the bilinear form $b(\cdot,\cdot) : H_p^1(\Omega) \times H_p^1(\Omega) \to \mathbb{R}$ as

$$b(v,w) = \langle DF(u)v, w\rangle_{H_p^{-1}(\Omega)\times H_p^1(\Omega)} = a(v,w) + 3\int_\Omega \epsilon u^2 \, v \, w dx.$$

The first step is to show that the mapping b is continuous and H_p^1-coercive. To this end, we firstly show that $u \in H_p^2(\Omega)$: For our special setting $\Omega = [0,1]^d$, the solution u and f might be expressed my means of Fourier series. This yiels for $s = 2$

$$\sum_k \|k\|^{2s}|u_k|^2 \le \{(2\pi)^2 + 1\}^2 \|f\|^2_{L_2(\Omega)} \ ,$$

i.e. $u \in H_p^2(\Omega)$ as long as $f \in L_2(\Omega)$. Consequently, for $d = 2$ we obtain by Sobolev-embedding that $u \in C(\Omega)$. Now we are able to prove that
(i) b is continuous. Indeed, by employing Hölder's inequality we obtain

$$\begin{aligned}
|b(v,w)| &= \left| a(v,w) + 3\int_\Omega \epsilon u^2 v w dx \right| \\
&\le \|v\|_{H^1}\|w\|_{H^1} + 3\left|\int_\Omega \epsilon u^2 v w dx\right| \\
&\le \|v\|_{H^1}\|w\|_{H^1} + 3\epsilon \max|u|^2 \left|\int_\Omega v w dx\right| \\
&\le \|v\|_{H^1}\|w\|_{H^1} + 3\epsilon \max|u|^2 \|v\|_{L_2}\|w\|_{L_2} \\
&\le (1 + 3\,\epsilon \, \max|u|^2)\|v\|_{H^1}\|w\|_{H^1}.
\end{aligned}$$

(ii) b is coercive:

$$b(v,v) = a(v,v) + \epsilon\, 3\int_\Omega u^2 v^2 dx \ \ge \ a(v,v) \ = \|v\|^2_{H^1}. \tag{5.5}$$

By the Lax-Milgram Lemma, we conclude that $DF(u)$ is an isomorphism from H_P^1 onto its dual. In order to show that DF is Lipschitzian in a neighborhood of u we need to bound $\|D^2F(u+h)\|$ for $\|h\|_{H_P^1} \le \varepsilon$. First, we observe that $D(D(u+h)w)v = 6uwv$. Moreover, $u,h,w,v \in H_P^1$ implies $u,h,w,v \in L_p(\Omega)$, for $p < \infty$, $d = 2$. Hence,

$$\begin{aligned}
\|D^2F(u+h)\| &\le \sup_{\|h\|_{H_P^1}\le\varepsilon} \ \sup_{\|w\|_{H_P^1}\le 1} \ \sup_{\|v\|_{H_P^1}\le 1} \|6(u+h)wv\|_{(H_P^1)'} \\
&= \sup_{\|h\|_{H_P^1}\le\varepsilon} \ \sup_{\|w\|_{H_P^1}\le 1} \ \sup_{\|v\|_{H_P^1}\le 1} \ \sup_{\|g\|_{H_P^1}=1} 6\,\langle (u+h)wv, g\rangle \\
&\le 6\|z\|_{L_8}\|w\|_{L_8}\{\|u\|_{L_4}(1+2\varepsilon) + \varepsilon\|h\|_{L_4}\} < \infty \ .
\end{aligned}$$

From the H_p^1-coerciveness (5.5) of the bilinear form b it follows that assumption **A2** of Theorem 2.1 is fulfilled with $\beta_j = 1$. It remains to verify that $g(v) = \epsilon v^3$ satisfies (4.12). This follows by the observation

$$\int_\Omega \epsilon \left(v^3 - w^3\right)(v-w)dx = \int_\Omega \epsilon \left[\frac{v^2+w^2}{2} + \frac{(v+w)^2}{2}\right](v-w)^2 dx.$$

Hence, we can conclude that for this class of problems the method converges and the error bound (4.13) holds with $\beta_j = 1$.

5.2 Wavelet Preconditioning

Typically, the sparse stiffness matrices in the nodal basis exhibit a polynomial growth rate of the spectral condition number proportional to their size. As already stated before, to tap the full potential of the Newton method it is essential to precondition the system (4.5). There is a whole theory concerning this problem, see e.g. [DK92]. Our system (4.5) can be seen as a linear system with a nonlinear perturbation $\mathbf{E}$. Regarding the spectral condition number of the linear part we first recall the well known results from [DK92]: If the elements of the wavelet basis satisfy a H^1-stability condition, i.e. there exist some constants $0 < \gamma \leq \Gamma < \infty$ and y_λ such that

$$\gamma \sum_{\lambda\in\Lambda} |y_\lambda d_\lambda|^2 \leq \| \sum_{\lambda\in\Lambda} d_\lambda \psi_\lambda \|^2_{H^1(\Omega)} \leq \Gamma \sum_{\lambda\in\Lambda} |y_\lambda d_\lambda|^2 \tag{5.6}$$

then, there exists a constant C depending only on the fact

$$a(\cdot,\cdot) \sim \|\cdot\|^2_{H^1(\Omega)}, \tag{5.7}$$

such that

$$\kappa(\mathbf{D}_j^{-1}\mathbf{B}_j\mathbf{D}_j^{-1}) \leq C\frac{\Gamma}{\gamma}\ , \tag{5.8}$$

where the matrix $\mathbf{D}_j$ is defined by $\mathbf{D}_j := 2^j \delta_{j',j''}\delta_{i',i''}$ for $j', j'' = 0, \dots, j$ and $i', i'' \in \Lambda_j$.

Remark 5.2 *Note that* (5.6) *and* (5.7) *ensure that all eigenvalues of* $\mathbf{D}_j^{-1}\mathbf{B}_j\mathbf{D}_j^{-1}$ *are in the range between* $C\gamma$ *and* $C\Gamma$.

For the investigation of the perturbed system we use the following well known perturbation result (see e.g. [GL96][page 58]). If $\mathbf{A}$ is nonsingular and

$$r := \|\mathbf{A}^{-1}\|_2 \|\mathbf{E}\|_2 < 1, \tag{5.9}$$

then $\mathbf{A} + \mathbf{E}$ is nonsingular and

$$\|(\mathbf{A}+\mathbf{E})^{-1}\|_2 \leq \frac{\|\mathbf{A}^{-1}\|_2}{1-r}.$$

Setting $\mathbf{A} := \mathcal{B}_j = \mathbf{D}_j^{-1}\mathbf{B}_j\mathbf{D}_j^{-1}$ and $\mathbf{E} := \mathbf{E}_j = \mathbf{D}_j^{-1}\mathbf{L}_j^{-1}\mathcal{G}_j\mathbf{L}_j\mathbf{D}_j^{-1}$, we obtain the following estimate for the condition number of the perturbed system (4.5)

$$\begin{aligned} \kappa(\mathcal{B}_j + \mathbf{E}_j) &\leq (\|\mathcal{B}_j\|_2 + \|\mathbf{E}_j\|_2)\frac{\|\mathcal{B}_j^{-1}\|_2}{1-r} && (5.10)\\ &\leq \frac{\kappa(\mathcal{B}_j)}{1-r} + \frac{r}{1-r}, && (5.11)\end{aligned}$$

where $r = \|\mathcal{B}_j^{-1}\|_2 \|\mathbf{E}_j\|_2 \leq \gamma^{-1}\kappa(L_j)\|\mathcal{G}_j\|_2$. We have to ensure $r < 1$, that is equivalent to

$$\|\mathcal{G}_j\|_2 < \frac{\gamma}{\kappa(L_j)} \ .$$

According to Theorem 2.1 we choose a level j. Let $\left(\xi_i^{lin}\right)_i$ be the vector of wavelet coefficient of the solution u^{lin} of the linear problem. Moreover, we denote by $\left(\xi_i^{exact}\right)_i$ the vector of wavelet coefficient of the exact solution u. We have that $u^{lin} \in H^s$ for a certain s, see [Gri92]. Hence, by norm equivalence there exists a constant $R > 0$ such that

$$\sum_i |(L_j \xi^{lin})_i|^2 \leq R \ .$$

Moreover, we get from the difference between the linear and the non-linear problem that

$$\|u^{lin} - u\|_{H^1} \leq \epsilon \|u\|_{L_6}^{1/3} \leq \epsilon \max\{1, \|u\|_{L_6}\} \ .$$

In the case $\|u\|_{L_6} \leq 1$ we obtain

$$\|u^{lin} - u\|_{H^1} \leq \epsilon \ .$$

Using the embedding $\|u\|_{L_6} \leq C\|u\|_{H^1}$, see [Ada78] p.97, and the triangle inequality $\|u\|_{H^1} \leq \|u^{lin} - u\|_{H^1} + \|u^{lin}\|_{H^1}$ we have

$$\|u^{lin} - u\|_{H^1} \leq \frac{1}{|\lambda|_{min}} \frac{\epsilon C}{1 - \epsilon C} \|f\|_{L_2} \ ,$$

where $|\lambda|_{min}$ is the smallest eigenvalue of L. Consequently, we observe the following:

(i) For an arbitrary but fixed ϵ we have $\|u^{lin} - u\|_{H^1} \leq \delta(\epsilon)$, and thus, there exists a constant $c_1 > 0$ such that

$$\sum_i |(L_j \xi^{exact})_i|^2 \leq c_1 R \ .$$

(ii) In accordance with Theorem 2.1 we obtain for a fixed j the estimate $\|u_j - u\|_{H^1} \leq \delta_1$, and thus, there exists a constant $c_2 > 0$ such that

$$\sum_i |(L_j \xi^j)_i|^2 \leq c_1 c_2 R \ .$$

(iii) Exploiting (4.15) we have $\|u_j - \tilde{u}_j\|_{H^1} \leq \delta_2$, and thus, there exists a constant $c_3 > 0$ such that

$$\sum_i |(L_j \xi)_i|^2 \leq c_1 c_2 c_3 R \ .$$

Finally, choosing ϵ such that

$$\max_{\xi \in B_{c_1 c_2 c_3 R}(0)} \sum_i |g_\epsilon'(m^{j/2}(L_j \xi)_i|^2 \leq \left(\frac{\gamma}{\kappa(L_l)}\right)^2$$

for a fixed scale j, we obtain a condition to ensure that $r < 1$.

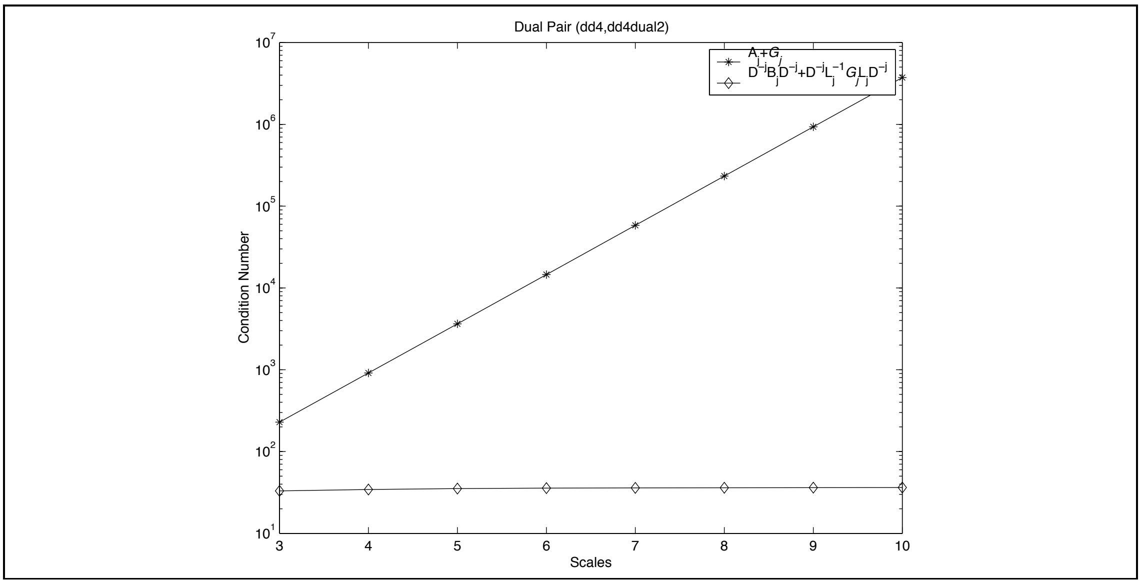

Figure 1: Condition numbers for the original system (line with slope) and wavelet preconditioned system (constant line) on a logarithmic scale.

5.3 Numerical Examples

To confirm the applicability of our approach we present some test examples that describe, e.g., a chemical inter-mixture process. The first example consists of a univariate problem whereas the second one is concerned with a problem in two dimensions. Numerical results were obtained by choosing the (periodized) Deslauriers-Dubuc interpolating scaling functions $\phi_{2N};\ N = 1,2,3,4$, with different duals $\tilde{\phi}_{2N,K};\ K = 1,2,3$, whose constructions will be briefly described in Section 6.2.

Example 5.1 To test the numerical approximation we have to know the exact solution u. Therefore, we assume that the function u is given by

$$u(x) = \sin(2\pi\omega x)\ , \tag{5.12}$$

which is a sufficiently smooth periodic function on $\Omega = [0,1]$, $\omega \in \mathbb{N}$. Then, we design the associated right–hand side f by insertion of u into (2.1)

$$f(x) = \sin^3(2\pi\omega x) + ((2\pi\omega)^2 + 1)\sin(2\pi\omega x)\ , \tag{5.13}$$

see Figure 2. The numerical approximations are displayed in Figure 3.

Example 5.2 Similar examples for 2 dimensions can be obtained by a natural generalization by means of a tensor product approach (which corresponds to the scaling matrix as $M = 2I$), see Figure 4.

In both examples ϵ is choosen to be one. It can be observed that the preconditioning applies also in this case.

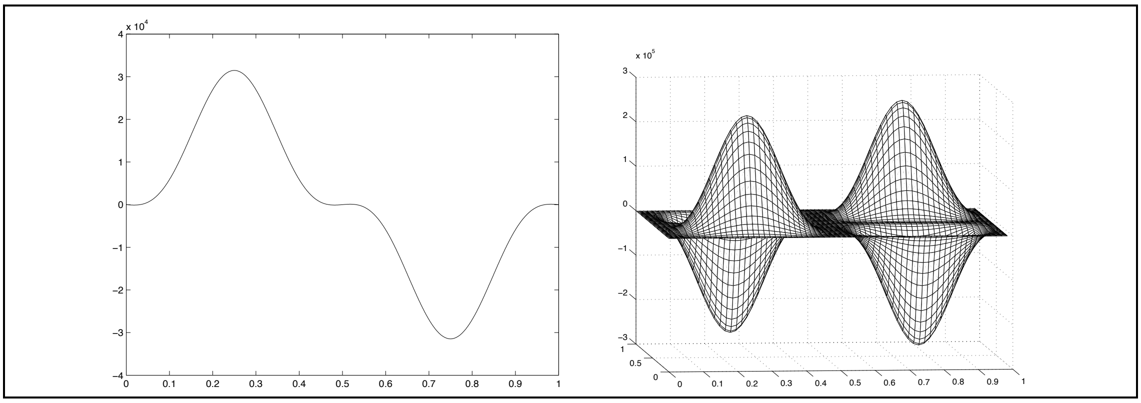

Figure 2: Right: function f for $\omega = 1$ in one dimension; Left: function f for $\omega = 1$ in two dimensions.

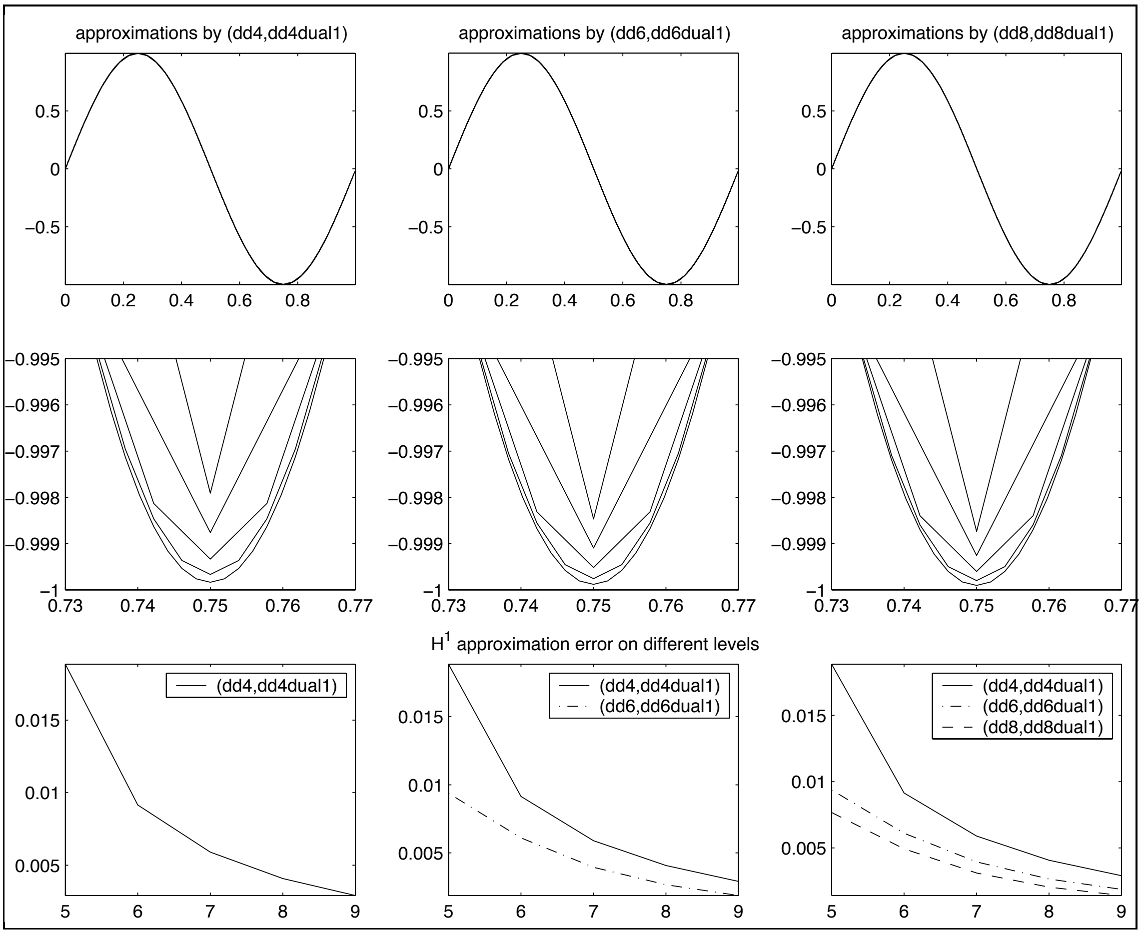

Figure 3: Approximations on successive scales $j = 5, \ldots, 9$ with functions ϕ_{2N} (denoted by `dd2*N`) of varying smoothness $N = 2, 3, 4$.

6 Appendix

6.1 Proof of Theorem 3.2

In order to establish approximation results for the interpolation projector Π_j we introduce the following mappings:

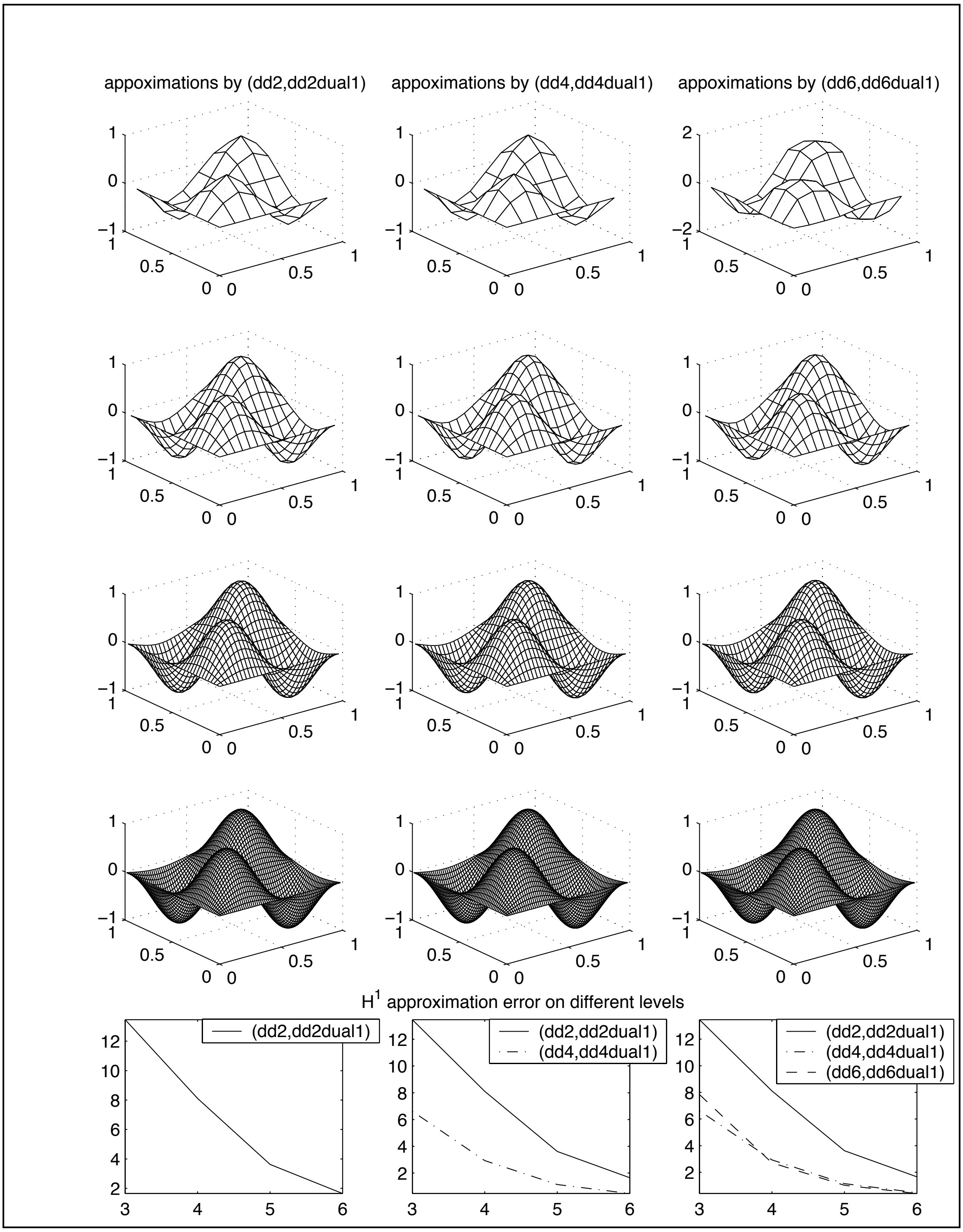

Figure 4: Approximations on successive scales $j = 3, \ldots, 6$ with functions ϕ_{2N} (denoted by dd2*N) of varying smoothness $N = 1, 2, 3$.

$$
\begin{aligned}
E &: \; L_\infty(\mathcal{T}^d) \to L_\infty(\mathbb{R}^d), \text{ with } \quad f(\cdot) \mapsto f([\cdot]) \; ; \\
R &: \; L_\infty(\mathbb{R}^d) \to L_\infty([0,1]^d), \text{ with } \quad f \mapsto 1_{[0,1]^d} f \; ; \\
P &: \; L_\infty([0,1]^d) \to L_\infty(\mathcal{T}^d), \text{ with } \quad f \mapsto \sum_l f(\cdot + l) \; .
\end{aligned}
$$

Let π_j denote the interpolation projector on $L_\infty(\mathbb{R}^d)$. We estimate the L_∞-approximation error of Π_j:

$$\begin{aligned}
\|f - \Pi_j f\|_{L_\infty(\mathcal{T}^d)} &= \|PREf - PR\pi_j Ef\|_{L_\infty(\mathcal{T}^d)} \le \|Ef - \pi_j Ef\|_{L_\infty(\mathbb{R}^d)} \\
&\le \|Ef - g\|_{L_\infty(\mathbb{R}^d)} + \|\pi_j Ef - g\|_{L_\infty(\mathbb{R}^d)} \\
&\le \|Ef - g\|_{L_\infty(\mathbb{R}^d)} + \|\pi_j Ef - \pi_j g\|_{L_\infty(\mathbb{R}^d)} \\
&\le (1 + \|\pi_j\|)\|Ef - g\|_{L_\infty(\mathbb{R}^d)} \le (1 + \|\pi_j\|) \inf_{g \in V_j} \|Ef - g\|_{L_\infty(\mathbb{R}^d)} \\
&\le (1 + \|\pi_j\|) r(M)^{-j(n+1)} |f|_{W^{n+1}(L_\infty(\mathbb{R}^d))} \ .
\end{aligned}$$

For the last estimate see [Lin03]. Consequently, the interpolation projector defined on $L_\infty(\mathcal{T}^d)$ has the following representation in $[V_j]$

$$\Pi_j = PR\pi_j E, \tag{6.1}$$

that is

$$\Pi_j f(x) = \sum_{k \in \mathbb{Z}^d / M^j \mathbb{Z}^d} f([M^j k])[\phi_{jk}](x) \ .$$

6.2 Construction of Smooth Duals

Here, we briefly recall an algorithm for constructing a dual scaling function $\tilde{\phi}$ for a given interpolating function ϕ, as developed in [JRS99].
The first step is to construct a second smoother interpolating function. Let us fix some notation. As before, let $m = |\det M|$ and let $R = \{\rho_0, \ldots, \rho_{m-1}\}$, $R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{m-1}\}$ denote complete sets of representatives of $\mathbb{Z}/M\mathbb{Z}^d$ and $\mathbb{Z}/B\mathbb{Z}^d$, $B = M^{\mathrm{T}}$, respectively. Without loss of generality, we shall always assume that $\rho_0 = \tilde{\rho}_0 = 0$. Also, let $h(z)$ be the symbol associated with the mask $\mathbf{h} = \{h_k\}$, i.e.

$$h(z) := \frac{1}{m} \sum_{k \in \mathbb{Z}^d} h_k z^k.$$

It can be shown that necessary conditions for ϕ to be interpolating is that its symbol satisfies:

C1. $h(1) = 1$;

C2. $\sum_{\tilde{\rho} \in R^T} h(\zeta_{\tilde{\rho}} e^{-iB^{-1}\omega}) = 1, \quad$ where $\zeta_{\tilde{\rho}} := e^{-2\pi i B^{-1}\tilde{\rho}}$.

The following condition, although not necessary, is easily established in many cases and is required for the construction developed in [JRS99]:

C3. $h(z) \ge 0$.

Moreover, **C3** implies that the resulting scaling function is continuous. It can also be shown that a necessary condition for $\tilde{h}$ to be the symbol of a dual scaling function $\tilde{\phi}$ for ϕ is that

$$\sum_{\tilde{\rho}} h(\zeta_{\tilde{\rho}} z)\overline{\tilde{h}(\zeta_{\tilde{\rho}})} = 1. \tag{6.2}$$

Defining

$$b_{\tilde{\rho}} = h(\zeta_{\tilde{\rho}} z), \quad \tilde{\rho} \in R^T, \tag{6.3}$$

condition **C2** may equivalently be written as

$$1 = \sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z). \tag{6.4}$$

Hence, for any integer K,

$$\left(\sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z) \right)^{Km} = \sum_{|\gamma| = mK} \left(C^{\gamma}_{mK} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) = 1. \tag{6.5}$$

By using (6.5), the following theorem was established in [JRS99].

Theorem 6.1 *Let $h(z)$ be a symbol satisfying* (6.4) *for a dilation matrix M with $m = |\det M|$. Define*

$$\begin{aligned} G_0 &:= \left\{ \gamma \in \mathbb{N}_0^m : \ |\gamma| = mK, \ \gamma_0 > K \ \text{and} \ \gamma_0 > \gamma_{\hat{\rho}}, \ \hat{\rho} \in R^T \backslash \{0\} \right\} \\ G_l &:= \left\{ \gamma \in \mathbb{N}_0^m : \ |\gamma| = qmK, \ \gamma_0 > K \ \text{and} \ \gamma_0 \geq \gamma_{\hat{\rho}}, \ \hat{\rho} \in R^T \backslash \{0\}, \ \text{with exactly } l \text{ equalities} \right\}, \\ & \qquad l = 1, \ldots, m-2, \end{aligned}$$

and define

$$H_K := \sum_{j=0}^{m-2} \frac{1}{j+1} \left(\sum_{\gamma \in G_j} C^{\gamma}_{mK} h(z)^{\gamma_0 - 1} \prod_{\hat{\rho} \in R^T \backslash \{0\}} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) + C^{(K,\ldots,K)}_{mK} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^{K}(z),$$

where C^{γ}_{qK} are the multinomial coefficients. Then the symbol $h(z)H_K(z)$ also satisfies (6.4)*.*

It can be checked that the symbol H_K can be factored as

$$H_K(z) = h(z)^K T_K(z) \tag{6.6}$$

for some suitable symbol $T_K(z)$. Consequently, the refinable function associated with $h(z)H_K(z)$ is obtained by convolving the original function $K-1$–times with itself, followed by a convolution by some distribution. Since $h(z)H_K(z)$ satisfies (6.4), it is a candidate for a symbol corresponding to an interpolating scaling function. Indeed, the following corollary was established in [JRS99].

Corollary 6.1 *Let $h(z)$ be the symbol of a continuous compactly supported interpolating refinable function and assume that $h(z)$ satisfies* **C3***. If the refinable function corresponding to $h(z)H_K(z)$ is continuous, then it is interpolating.*

This approach can now be used to construct dual functions for the given interpolating scaling function ϕ. Indeed, by recalling the necessary condition (6.4), we see that, by Theorem 6.1

$$\tilde{h}(z) := \overline{H_K(z)} = \overline{h(z)^K T_K(z)} \tag{6.7}$$

is a natural candidate for a symbol associated with a dual function. The following corollary is again taken from [JRS99].

Corollary 6.2 *If the refinable function corresponding to the mask H_K is in $L_2(\mathbb{R}^d)$, then it is stable and dual to ϕ.*

6.3 More About Regularity

In this subsection, we estimate the regularity of our constructed interpolating generator functions ϕ_{2N} and the related duals.

For simplicity we confine the considerations to the univariate case. The starting point is the Daubechies scaling function φ_N. Thus, by convolution we obtain a interpolating generator ϕ_{2N} by

$$\phi_{2N}(x) = \int \varphi_N(y)\bar{\varphi}_N(y-x)dy \ ,$$

that is

$$\phi_{2N}(k) = \delta_{0k} \ .$$

This leads to the following refinement equation

$$\hat{\phi}_{2N}(\omega) = H(\omega/2)\hat{\phi}_{2N}(\omega/2) = \prod_{j\geq 1} H(2^{-j}\omega) \ ,$$

where $H(\omega) = |H_N(\omega)|^2$ and H_N are the Daubechies symbols. Finally, the regularity of ϕ_{2N} can be estimated by the application of known results of the Daubechies scaling functions

$$|\hat{\phi}_{2N}(\omega)| = |\prod_{j\geq 1} H_N(2^{-j}\omega)|^2 \leq C_N(1+|\omega|)^{2(-1-\alpha_N-\varepsilon)} = C_N(1+|\omega|)^{-1-\beta_N-\varepsilon'} \ ,$$

where α_N are the known lower bounds for the Hölder exponents of φ_N and where $\beta_N = 2\alpha_N + 1$. Hence, we obtain the following table, see [Dau92][p. 239].

N	α_N	β_N
1		1.0000
2	0.5500	2.1000
3	1.0878	3.1756
4	1.6179	4.2358

The symbol of the dual generator functions $\phi_{2N,K}$ can be constructed by means of Theorem 6.1

$$H_{K,N}(\omega) = H^K(\omega)T_K(\omega) = |H_N(\omega)|^{2K}T_K(\omega) \ ,$$

where $T_K(\omega)$ is given for $K = 1, 2, 3$ by

$$\begin{aligned} T_1(\omega) &= 1 + 2b_1(\omega) \ , \\ T_2(\omega) &= b_0(\omega) + 4b_1(\omega) + 6b_1^2(\omega) \ , \\ T_3(\omega) &= b_0^2(\omega) + 6b_0(\omega)b_1(\omega) + 15b_1^2(\omega) + 20b_1^3(\omega) \ , \end{aligned}$$

where $b_0(\omega) = H(\omega)$ and $b_1(\omega) = H(\omega + \pi)$, compare with 6.3. By 6.4 we have

$$H(\omega) + H(\omega + \pi) = 1$$

and thus,

$$b_1(\omega) = 1 - H(\omega) \ .$$

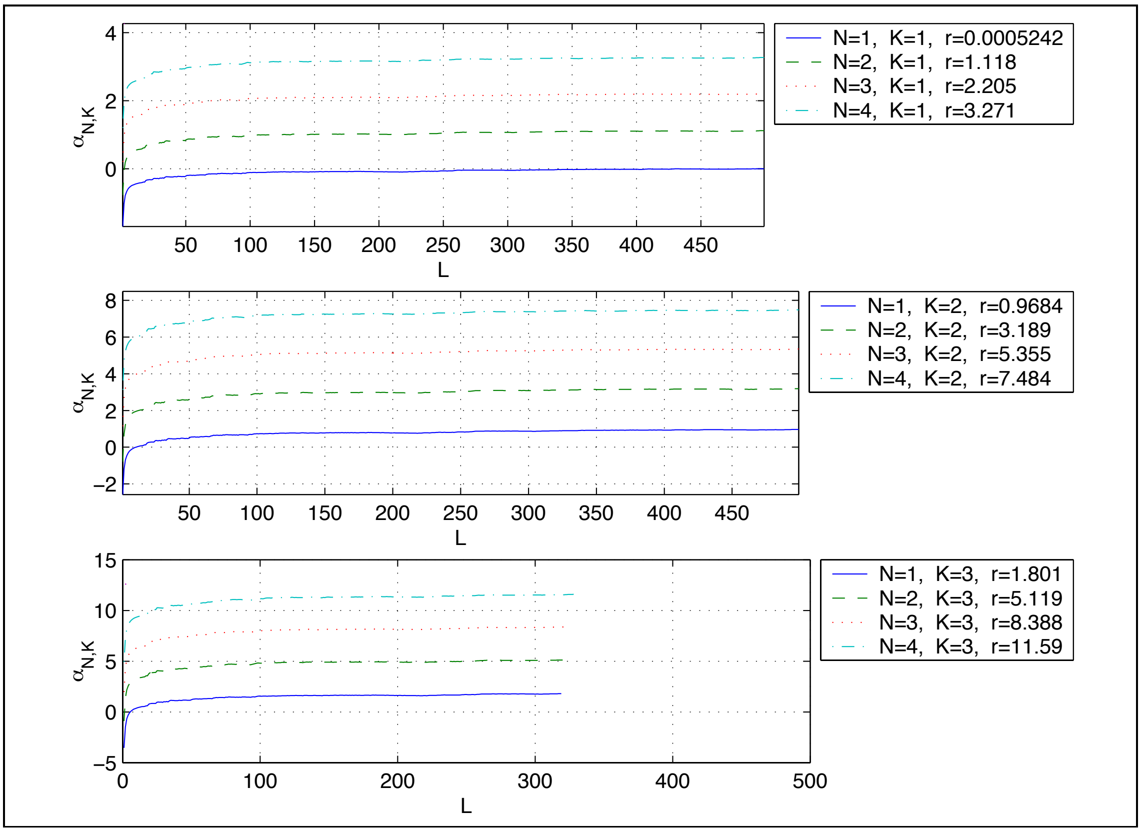

Figure 5: Estimated regularities of the duals.

To compute the regularity of the dual generators $\phi_{2N,K}$ we have to estimate

$$|\hat{\phi}_{2N,K}(\omega)| = \left(\prod_{j\geq 1} |H_N(2^{-j}\omega)|\right)^{2K} \prod_{j\geq 1} |T_K(2^{-j}\omega)| \ .$$

Thus, we have to estimate the second term only. This can be performed by following the standard technique in e.g. [Dau92] outlined. Defining

$$F_{K,L}(\omega) = \prod_{l=0}^{L-1} T_K(2^l\omega)$$

and

$$C_{K,L} = \max_{\omega\in[0,2\pi)} |F_{K,L}(\omega)|$$

we obtain for $2^{J-1} \leq |\omega| \leq 2^J$

$$\begin{aligned}\prod_{j=1}^{J} |T_K(2^{-j}\omega)| &= \prod_{j=1}^{J/L}\prod_{l=0}^{L-1} |T_K(2^{-jL+l}\omega)| = \prod_{j=1}^{J/L} |F_{K,L}(2^{-jL}\omega)| \\ &\leq (C_{K,L})^{J/L} \leq C'_{K,L}(1+|\omega|)^{\gamma_K} \ ,\end{aligned}$$

where $\gamma_K = \log C_{K,L}/\log 2^L$. Finally, we have that

$$|\hat{\phi}_{2N,K}(\omega)| \ \leq \ C(1+|\omega|)^{-1-\alpha_{N,K}-\varepsilon'} \ ,$$

where $C = C_N^{2K} C'_{K,L} e^{C'_K}$. The Hölder exponent is given by $\alpha_{N,K} = 2K(\alpha_N + 1) - \gamma_K - 1$. We obtain the following table of numerical estimates of Hölder regularities, cp Figure 5:

	$\phi_{2N} \in C^{\beta_N}$	$\phi_{2N,K} \in C^{\alpha_{N,K}}$		
	β_N	$\alpha_{N,1}$	$\alpha_{N,2}$	$\alpha_{N,3}$
N=1	1.0000	0.00005	0.968	1.801
N=2	2.1000	1.118	3.189	5.119
N=3	3.1756	2.205	5.355	8.388
N=4	4.2358	3.271	7.484	11.59

Acknowledgment

The work the authors has been supported by Project Based Personnel Exchange Programme with Portugal-Acções Integradas Luso-Alemãs (DAAD/ICCTI).

References

[Ada78] R. A. Adams. *Sobolev spaces.* Academic press, Boston, San Diego, New York, 1978.

[CR97] G. Caloz and J. Rappaz. Numerical Analysis for Nonlinear and Bifurcation Problems. In P. G. Ciarlet and J. L. Lions, editors, *Handbook of Numerical Analysis*, volume V, pages 487–637. Elsevier, 1997.

[CHR00] D.R. Chen, B. Han, and S.D. Riemenschneider. Construction of multivariate biorthogonal wavelets with arbitrary vanishing moments. *Adv. Comput. Math.*, 13(2):131–165, 2000.

[Coh00] A. Cohen. Wavelet methods in numerical analysis. In P.G.Ciarlet and J.L.Lions, editors, *Handbook of Numerical Analysis*, volume VII, pages 417–711. Elsevier, Amsterdam, 2000.

[CDD] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for nonlinear variational problems. in preparation.

[CDD00] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods II: Beyond the elliptic case, 2000.

[CDD01] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations: Convergence rates. *Mathematics of Computation*, 70(233):27–75, 2001.

[CDF92] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Commun. Pure Appl. Math.*, 45(5):485–560, 1992.

[Dah97] S. Dahlke. *Wavelets: Construction Principles and Application to the Numerical Treatment of Operator Equations*. Shaker-Verlag, Aachen, 1997.

[DDHS97] S. Dahlke, W. Dahmen, R. Hochmuth, and R. Schneider. Stable multiscale bases and local error estimation for elliptic problems. *Applied Numerical Mathematics: Transactions of IMACS*, 23(1):21–47, 1997.

[DDU01] S. Dahlke, W. Dahmen, and K. Urban. Adaptive wavelet methods for saddle point problems - optimal convergence rates. Technical Report 204, RWTH Aachen, 2001.

[DGM99] S. Dahlke, K. Gröchenig, and P. Maaß. A new approach to interpolating scaling functions. *Appl. Anal. 72(3-4)*, pages 485–500, 1999.

[DK94] S. Dahlke and A. Kunoth. A biorthogonal wavelet approach for solving boundary value problems. In *Proceedings of the 9th GAMM, Seminar Adaptive Methods: Algorithms, Theory and Applications*, volume 46 of *NN FM Series*, pages 99–115, 1994.

[DM97] S. Dahlke and P. Maaß. Interpolating refinable functions and wavelets for general scalings. *Nemer. Funct. Anal. Optim.*, 18(586):521–534, 1997.

[DMT03] S. Dahlke, P. Maaß, and G. Teschke. Interpolating scaling functions with duals. *Journal of Computational Analysis and Applications*, 5(3):361–273, 2003.

[Dah97] W. Dahmen. Wavelet and multiscale methods for operator equations. *Acta Numerica*, 6:55–228, 1997.

[Dah01] W. Dahmen. Wavelet Methods for PDEs - Some Recent Developments, 2001.

[DDM96] W. Dahmen, A. Dress, and C.A. Micchelli. On multivariate splines, matroids, and the ext-functor. *Adv. Appl. Math.*, 17(3):251–307, 1996.

[DK92] W. Dahmen and A. Kunoth. Multilevel preconditioning. *Numerische Mathematik*, 63(3):315–344, 1992.

[DM90] W. Dahmen and C.A. Micchelli. On stationary subdivision and the construction of compactly supported orthonormal wavelets. In *Multivariate approximation and interpolation*, pages 69–90. Birkhauser Verlag, 1990.

[DPS94] W. Dahmen, S. Prössdorf, and R. Schneider. Multiscale methods for pseudodifferential equations on smooth manifolds. In C.K. Chui, L. Montefuso, and L. Puccio, editors, *Wavelets: Theory, Algorithms, and Applications*, pages 385–424. Academic Press, 1994.

[Dau92] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, Philadelphia, 1992.

[Der99] J. Derado. Multivariate refineable interpolating functions. *Appl. Comp. Harm. Anal.*, 7(2):165–183, 1999.

[DDD91] G. Deslaurier, J. Dubois, and S. Dubuc. Multidimensional iterative interpolation. *Can. J. Math*, 43:297–312, 1991.

[DD87] G. Deslaurier and S. Dubuc. Interpolation dyadique. In G. Cherbit, editor, *Fractals, dimensions non entières et applications*, pages 44–55. Masson, Paris, 1987.

[Dub86] S. Dubuc. Interpolation through an iterative scheme. *J. Math. Anal. Appl.*, 114:185– 204, 1986.

[GL96] G. Golub and Charles F. Van Loan. *Matrix computations*, volume 3rd ed. Baltimore, MD: The Johns Hopkins Univ. Press. xxvii, 694 p., 1996.

[Gri92] P. Grisvard. *Singularities in boundary value problems.* Masson, Springer, Berlin, Paris, 1992.

[JRS99] H. Ji, S.D. Riemenschneider, and Z. Shen. Multivariate compactly supported fundamental refinable functions, duals, and biorthogonal wavelets. *Stud. Appl. Math.*, 102(2):173–204, 1999.

[JM91] R.Q. Jia and C.A. Micchelli. Using the refinement equations for the construction of prewavelets. II: Powers of two. In *Curves and surfaces, Pap. Int. Conf., Chamonix-Mont-Blanc/Fr.* , pages 209–246. Academic Press, 1991.

[KA00] P. Knabner and L. Angermann. *Numerik partieller Differentialgleichungen.* Springer, Berlin, 2000.

[Lin03] M. Lindemann. Phd thesis in preparation, 2003.

[MS86] G. Maz'ya and T.O. Shaposhnikova. *Multiplikatory v prostranstvakh differentsiruemykh funktsiui.* Izd-vo Leningr. un-ta, 1986.

[RS92] S.D. Riemenschneider and Z. Shen. Wavelets and pre-wavelets in low dimensions. *J. Approximation Theory*, 71(1):18–38, 1992.

[RS97] S.D. Riemenschneider and Z. Shen. Multidimensional interpolatory subdivision schemes. *SIAM Journal on Numerical Analysis*, 34(6):2357–2381, 1997.

[Zei86] E. Zeidler. *Nonlinear Functional Analysis and its Applications I.* Springer, New York, 1986.