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$2 + 2 = 4$

$$(x,y) = 0 \quad \Leftrightarrow \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

$$E = mc^2 \quad | \quad E = h\nu$$

Future light cone $\simeq \mathbb{C}^2$ | *Empty space-time* $\simeq$ *self-adjoint subset of* $\mathbb{C}^2 \otimes \mathbb{C}^2$

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Clifford Analysis on Projective Hyperbolic Space

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and
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Abstract. In this paper we present a new model for Clifford analysis over the hyperbolic unit ball, which is identified with the manifold of rays within the future null cone. By making use of the available Clifford algebra structure there, we arrive at the definition of Dirac operators on sections of homogeneous line bundles.

1. Introduction

The starting basis for Clifford analysis is a Clifford algebra generated by an orthonormal basis $\mathbf{e}_1, \cdots, \mathbf{e}_m$ satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$. In this framework, Dirac operators in Euclidean spaces arises as natural Clifford-valued first order operators. The book [BDS] dealing with the function theory of this operator can be viewed as a first major monograph in this research field. A main problem in the

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development of this theory is the study of the invariance group of null solutions of the Dirac operator. This invariance group is the group of Möbius transformations which can be described by Vahlen matrices (see [V], [A],[M]). Another description of this group makes use of the identification of points in Euclidean space $\mathbb{R}^m$ with rays in the null cone in $\mathbb{R}^{m+1,1}$ (see e.g. [LKH]). In this picture the Möbius group corresponds to the group $\mathbf{Spin}(m+1,1)$. This identification has been the main theme of several works on Clifford analysis (see [Cn], [Hs], [M]). Strongly related to these themes is the development of Clifford analysis on the hyperbolic unit ball or the hyperbolic half plane and this is due to the fact that the subgroup $\mathbf{Spin}(m,1)$ of Möbius transformations leaving the unit sphere invariant is the isometry group of these non-Euclidean geometries.

Many attempts have been made to define a version of Clifford analysis on these spaces, in the first place by Heinz Leutwiler and his students (see [L], [EL], [Hm]). This research field is usually called modified Clifford analysis and contains a Clifford algebra description of the Riesz system which was later generalized to a Hodge system on the hyperbolic unit ball (see [CC], [Cn]).

The above attempts do not really generalize the Dirac operator for $\mathbf{Spin}\frac{1}{2}$ fields, in fact they correspond to $\mathbf{Spin}1$ fields. In this paper we are going to establish a true generalization of the Dirac operator for $\mathbf{Spin}\frac{1}{2}$ fields. Hereby, we make use of a homogeneous description of the hyperbolic unit ball whereby points are identified with rays in the future null cone and the isometry group corresponds to the Lorentz group $\mathbf{Spin}(m,1)$. Monogenic functions on the hyperbolic space are defined as homogeneous solutions of the space-time Dirac operator.

The ideas presented in this paper follow closely the ones already established in [SvL] and [vL].

2. The conformal embedding of $\mathbb{R}^m$

Let $x = (x_1, \ldots, x_m)$ be an element of $\mathbb{R}^m$.

We introduce an embedding of $\mathbb{R}^m$ in $\mathbb{R}^{m+2}$ defined by

$$\begin{aligned} x \mapsto X &= (x_1, \ldots, x_m, \frac{1-r^2}{2}, \frac{1+r^2}{2}) \\ &= (X_1, \ldots, X_m, X_{m+1}, X_{m+2}), \end{aligned}$$

where $r = |x|$.

Therefore, the above transformation takes the form

$$\begin{aligned} X_j &= x_j \quad j = 1, \ldots, m \\ X_{m+1} &= \frac{1-r^2}{2} \\ X_{m+2} &= \frac{1+r^2}{2} \end{aligned} \tag{1}$$

and, with respect to the standard Euclidean basis, it can be viewed as conic surface (henceforward, null cone NC) in $\mathbb{R}^{m+2}$. The elements $X \in$ NC satisfy the quadratic equation $X_1^2 + \ldots + X_{m+1}^2 = X_{m+2}^2$.

We will proceed now with the homogenization of our space: let us consider the equivalence relation $X \sim \lambda X, \lambda \neq 0$. Hence, the rays in NC

$$ray(X) = \{Y = \lambda X | \lambda \neq 0\}$$

appear as the equivalent classes of the above relation. We shall denote by ray(NC) the set of all such equivalent classes, i.e. $ray(\mathrm{NC}) = \mathrm{NC}/\sim$.

To come back from each $ray(X)$ in NC to its unique generator $x \in \mathbb{R}^m$ we need to identify the special point in $ray(X)$ which satisfy (1); thus, for every $Y \in ray(X)$, we have $X = \nu Y = \frac{1}{\lambda} Y$. Then we have ν such that

$$X_{m+1} + X_{m+2} = \nu(Y_{m+1} + Y_{m+2}) = 1,$$

i.e. $\nu = 1/(Y_{m+1} + Y_{m+2})$.

Obviously, we have exceptional points in the case of $Y_{m+1} + Y_{m+2} = 0$, that is, $Y_{m+1} = -Y_{m+2}$. In this situation, the NC-equation

$$\sum_{i=1}^{m+1} Y_i^2 = Y_{m+2}^2 \tag{2}$$

implies that $\sum_{i=1}^{m} Y_i^2 = 0$. Hence the correspondent variable in the homogeneous space is $ray(\underline{0}, -1, 1)$, where $\underline{0}$ denotes the zero-vector in $\mathbb{R}^m$. This particular ray can be viewed as the point at infinity ∞ in $\mathbb{R}^m$.

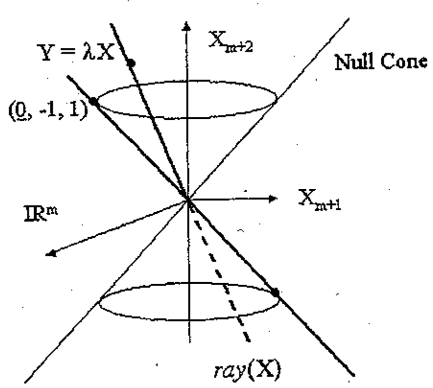

Fig. 1 - Null cone

The remaining elements of $\mathbb{R}^m$ will be obtained, from the value $\lambda \neq 0$, as $x_j = X_j, j = 1, \ldots, m$; we then get, for $r = |x|$, that

$$\begin{aligned} r^2 &= \sum_{i=1}^{m} x_i^2 \\ &= \sum_{i=1}^{m} X_i^2 \\ &= X_{m+1}^2 - X_{m+2}^2 \\ &= (X_{m+1} - X_{m+2})(X_{m+1} + X_{m+2}) \end{aligned}$$

hence $X_{m+1} = \frac{1-r^2}{2}$ and $X_{m+2} = \frac{1+r^2}{2}$. Henceforward, the special elements $X \in \mathrm{NC}$ satisfying $X_{m+1} + X_{m+2} = 1$ will be denoted as the *ray generator*.

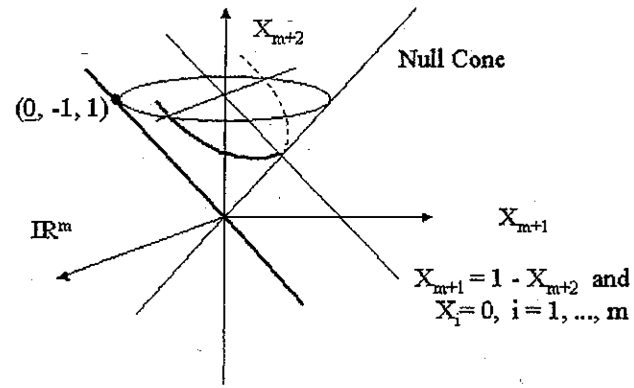

Fig. 2 - Rays generators

Remark. The mapping $x \mapsto ray(X)$ defines projective coordinates from $\mathbb{R}^m \cup \{\infty\} = S^m$ on $ray(\mathrm{NC})$. Obviously, such embedding privileged radial forms in the initial space $\mathbb{R}^m$; moreover, the above imbedding arises as natural projective coordinates for S^m on radial algebras ([SvL]).

Also, the transformation group leaving the $ray(\mathrm{NC})$ invariant is the rotation group in $\mathbb{R}^{m+1,1}$, the so-called $SO(m+1,1)$. It is well known that it stands for a representation of the Möbius group acting on $\mathbb{R}^m$. For a detailed proof, see ([Cn]).

The next goal will be to find a correct generalization of functions on $\mathbb{R}^m$ to functions on the projective space rayNC, in a suitable way, i.e. the generalization will preserve a given homogeneity condition.

For that purpose, let us consider an arbitrary real-valued function f on a subset Ω of $\mathbb{R}^m$. We will denote by F the function acting on $ray(\mathrm{NC})$ which corresponds to f.

For a ray generator X we have $X_{m+1} + X_{m+2} = 1$, so we can set

$$F(X) = f(x) = c,$$

whereby $x_1 = X_1, \ldots, x_m = X_m$.

For the extension to $ray(X)$ we consider the homogeneous line bundles over $ray(\mathrm{NC}) = S^m = \mathbb{R}^m \cup \{\infty\}$. For each $X \in \mathrm{NC}$ and $c \in \mathbb{R}$, consider $\{(X,c) : c \in \mathbb{R}\}$ the line over the base point X which is also the trivial line bundle in NC $\times \mathbb{R}$.

Construct the equivalence relation

$$(X,c) \sim (\lambda X, \lambda^\alpha c), X \in \mathrm{NC}_+, \lambda > 0 \tag{3}$$

where $\mathrm{NC}_+ = \{X \in \mathrm{NC} | X_{m+2} > 0\}$. The equivalent classes of this relation define the line bundle L_α over $ray_\alpha(\mathrm{NC}_+)$. In such way, each $F(Y), Y \in \mathrm{NC}_+$ corresponds to a section of L_α or, equivalently,

$$F(\lambda X) = \lambda^\alpha F(X) = \lambda^\alpha f(x), \lambda > 0. \tag{4}$$

Hence, we extend the previous $F(X) = f(x)$, with $X_{m+1} + X_{m+2} = 1$, to a section of L_α by setting

$$F(Y) = (Y_{m+1} + Y_{m+2})^\alpha F(\frac{Y}{Y_{m+1} + Y_{m+2}}), \quad Y \in ray(X),$$

which is always well defined; we complete the point at ∞ by taking the limit to that point, if such limit exists. Remark that relation (4) can be viewed as setting F acting on $ray(\mathrm{NC})$ through homogenization of degree α of the original values of f in $\mathbb{R}^m$.

3. Cliffordization

In this Section we want to "Cliffordize" the above idea of a conformal embedding, i.e. describe it in a Clifford algebra setting.

For this we consider the Clifford algebra $\mathbb{R}_{m+1,1}$ generated as the free algebra over the vector space $\mathbb{R}^{m+1,1}$ with signature $m+1,1$ modulo the relation

$$X^2 = Q(X)\mathbf{e}_0$$

with $X \in \mathbb{R}^{m+1,1}$, $\mathbf{e}_0$ being the identity of the algebra, and $Q(X) = \sum_{j=1}^{m+1} X_j^2 - X_{m+2}^2$ (see [BDS]).

For the basis $\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_{m+1}, \mathbf{e}_{m+2}$ of the vectors of $\mathbb{R}_{m+1,1}$ it holds

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, i \neq j$$

and

$$\begin{aligned} \mathbf{e}_j^2 &= +1, j = 1, \ldots, m+1 \\ \mathbf{e}_{m+2}^2 &= -1. \end{aligned}$$

Now, any Euclidean Clifford vector can be written as $\underline{x} = \sum_{j=1}^m \mathbf{e}_j x_j$ and

$$\underline{X} = \underline{x} + \mathbf{e}_{m+1}\left(\frac{1-r^2}{2}\right) + \mathbf{e}_{m+2}\left(\frac{1+r^2}{2}\right).$$

Hereby, the NC-equation corresponds to $\underline{X}^2 = 0$ in the Clifford sense. Moreover $\underline{X} \in \mathrm{NC}_+$ corresponds to $X_{m+2} > 0$ and we have the homogeneity condition as

$ray(X) = \{\underline{Y} = \lambda\underline{X}, \lambda > 0\}$. Again, we come back to $\mathbb{R}^m$ by the mappings

$$\begin{aligned} \underline{Y} &\mapsto \frac{1}{Y_{m+1}+Y_{m+2}}\underline{Y} = \underline{X} \\ \underline{X} &\mapsto \underline{x} = \sum_{j=1}^{m} e_j X_j. \end{aligned}$$

Now, starting from functions $f : \Omega \subset \mathbb{R}^m \mapsto C\ell_{0,m}$ we can consider functions

$$F(\underline{Y}) = (Y_{m+1}+Y_{m+2})^\alpha F(\underline{X}), \text{ with } \underline{X} = \frac{\underline{Y}}{Y_{m+1}+Y_{m+2}}$$

where $F(\underline{X}) = f(\underline{x}), \underline{x} = \sum_{j=1}^m X_j \mathbf{e}_j$. These functions are homogeneous of degree α.

The spin group $\mathbf{Spin}(m+1,1)$, which acts on homogeneous Clifford null vectors by $\underline{X} \mapsto s\underline{X}\bar{s}, s \in \mathbf{Spin}(m+1,1)$, and preserves the null cone, acts on homogeneous functions $F(\lambda\underline{X}) = \lambda^\alpha F(\underline{X})$ over NC_+ by

$$L(s) : F(\underline{X}) \mapsto sF(\bar{s}\underline{X}s).$$

From this we can obtain several transformations of functions on $\mathbb{R}^m$ under the Möbius group.

4. Some special Clifford analysis observations

In $\mathbb{R}^{m+1,1}$ we have the algebra of $\mathbf{Spin}(m+1,1)$-invariant operators generated by

$$\begin{aligned} F(\underline{X}) &\mapsto \underline{X}F(\underline{X}) \\ F(\underline{X}) &\mapsto \partial_{\underline{X}}F(\underline{X}) \end{aligned}$$

where $\partial_{\underline{X}} = \sum_{j=1}^{m+1} \mathbf{e}_j\partial_{X_j} - \mathbf{e}_{m+2}\partial_{X_{m+2}}$. Hereby, $F(\lambda\underline{X}) = \lambda^\alpha F(\underline{X})$ is defined in an open region $\Omega \subset \mathbb{R}^{m+1,1}$ with the property of $\lambda\Omega = \Omega, \lambda > 0$, i.e. a section of a line bundle on a sphere.

We have to restrict ourselves to operators transforming homogeneous elements to homogeneous elements, for instance $\underline{X}$, $\partial_{\underline{X}}$, $\partial_{\underline{X}} - \frac{1}{\underline{X}}$, $\underline{X}\partial_{\underline{X}} = E + \Gamma$ with $E = \sum_{j=1}^{m+2} X_j\partial_{X_j}$ being the Euler operator and $\Gamma_{\underline{X}} = \underline{X} \wedge \partial_{\underline{X}}$ the Gamma operator.

In case that F is a homogeneous polynomial of degree k we have the harmonic Fischer decomposition

$$F(\underline{X}) = H(\underline{X}) + \underline{X}^2\, R(\underline{X})$$

with $\partial_{\underline{X}}^2 H(\underline{X}) = 0$. Hence, $F(\underline{X})|_{\mathrm{NC}} = H(\underline{x})|_{\mathrm{NC}}$. Next, we have for H the decomposition $H(\underline{X}) = M_1(\underline{X}) + \underline{X}M_2(\underline{X})$ where each M_j satisfies $\partial_{\underline{X}}M_j(\underline{X}) = 0$.

Theorem 4.1. *On the null cone* NC *it holds*

$$F(\underline{X}) = M(F)(\underline{X}) + \underline{X}\tilde{M}(F)(\underline{X}).$$

From this Theorem we get $\underline{X}F(\underline{X}) = \underline{X}M(F)(\underline{X})$ directly. Now, applying $\partial_{\underline{X}}$ we have

$$\partial_{\underline{X}}\underline{X}F(\underline{X}) = (m+2)F(\underline{X}) + 2E_{\underline{X}}F(\underline{X}) - \underline{X}\partial_{\underline{X}}F(\underline{X}).$$

The same holds for $M(\underline{X}) = M(F)(\underline{X})$. For the Euler operator it holds $E_{\underline{X}}F(\underline{X}) = \alpha F(\underline{X})$, therefore, we obtain

$$\partial_{\underline{X}}\underline{X}F(\underline{X}) = (2\alpha + m + 2)M(\underline{X})$$

or

$$M(\underline{X}) = \frac{\partial_{\underline{X}}\underline{X}}{2\alpha+m+2}F(\underline{X}).$$

This leads to

$$\underline{X}(2\alpha + m + 2 - \partial_{\underline{X}}\underline{X})F(\underline{X}) = 0.$$

For $\tilde{M}$ we use again $\underline{X}\partial_{\underline{X}}F(\underline{X}) = (2\alpha + m + 2 - \partial_{\underline{X}}\underline{X})F(\underline{X})$:

$$\begin{aligned} \underline{X}\tilde{M}(F)(\underline{X}) &= F(\underline{X}) - \frac{\partial_{\underline{X}}\underline{X}}{2\alpha+m+2}F(\underline{X}) \\ &= \frac{2\alpha+m+2-\partial_{\underline{X}}\underline{X}}{2\alpha+m+2}F(\underline{X}) \\ &= \frac{\underline{X}\partial_{\underline{X}}F(\underline{X})}{2\alpha+m+2}. \end{aligned}$$

Therefore, we have

$$\tilde{M}(F)(\underline{X}) = \frac{\partial_{\underline{X}} F(\underline{X})}{2\alpha + m + 2}.$$

Let us make some concluding remarks to these calculations. First, $F(\underline{X})$ is defined in a neighbourhood of the null cone NC and $\partial_{\underline{X}}^2 F = 0$, otherwise it is not valid. Second, this is the theory of monogenic functions on NC. There is no further Dirac operator on NC coming.

5. Clifford analysis on hyperbolic disk

The unit sphere S^{m-1} corresponds in $ray(\mathrm{NC})$ to the equation $X_{m+1} = 0$. The subgroup of $\mathbf{Spin}(m+1,1)$ leaving the subspace $\mathbb{R}^{m,1}$ invariant is $\mathbf{Spin}(m,1)$. It leaves also invariant the vector $\mathbf{e}_{m+1}$ and the operator $\partial_{X_{m+1}}$ corresponding to that.

The unit ball $B(1)$ in $\mathbb{R}^m$ corresponds to the region $X_{m+1} > 0$ and the group $\mathbf{Spin}(m,1)$ is the subgroup of Möbius transformations leaving $B(1)$ invariant, which is also the hyperbolic motion group or the Lorentz group acting on the velocity ball. Now a function f on $B(1)$ may be extended to a section of L_α in the previous sense denoted $F(X_1, \dots, X_m, X_{m+1}, X_{m+2})$ and defined for $X_{m+1} > 0$ and $X_1^2 + \dots + X_{m+1}^2 = X_{m+2}^2$.

We now extend it to the whole set

$$(X_1, \dots, X_{m+2}) : X_{m+2}^2 - X_1^2 - \dots - X_m^2 > 0$$

by assuming the $\mathbf{Spin}(m,1)$-invariant condition

$$\partial_{X_{m+1}} F(\underline{X}) = 0$$

and this function is determined by its restriction to $X_{m+1} = 0$ given by

$$F(X_1, \dots, X_m, X_{m+2}) = F(X_1, \dots, X_m, 0, X_{m+2}).$$

Hence, there is a one to one correspondence between $\mathbb{R}_{m,1}$ (or $\mathbb{C}_{m,1}$-valued functions on the hyperbolic space $B(1)$ and homogeneous functions $F(\lambda \underline{X}) = \lambda^\alpha F(\underline{X})$, α

fixed and

$$\begin{aligned} \underline{X} &= X_1 \mathbf{e}_1 + \dots + X_m \mathbf{e}_m + X_{m+2} \mathbf{e}_{m+2} \\ &= X_1 \mathbf{e}_1 + \dots + X_m \mathbf{e}_m + T\epsilon, \end{aligned}$$

whereby $T > R = \sqrt{\sum_{j=1}^m X_j^2}$.

Indeed, the direct embedding of $B(1)$ is given by $X_j = x_j, j = 1, \dots, m$ and $T = \frac{1+r^2}{2}, r^2 = \sum_{j=1}^m x_j^2$.

It is a paraboloid P with the equation $2T = 1 - \sum_{j=1}^m X_j$ and coordinates $x_j = X_j,\ j = 1, \dots, m$.

Hence, for any function near P, it holds

$$F|_P(x_1, \dots, x_m) = F\left(x_1, \dots, x_m, \frac{1+r^2}{2}\right)$$

so that

$$\partial_{x_j} F|_P = (\partial_{x_j} F + x_j \partial_T F)|_P.$$

Now in the solid null cone $T > R$ we consider α-homogeneous $\mathbb{R}_{m,1}$ ($\mathbb{C}_{m,1}$)-valued functions

$$F(\lambda \underline{X}, \lambda T) = \lambda^\alpha F(\underline{X}, T)$$

which are equivalent to imposing the Euler equation

$$\left(\sum_{j=1}^m X_j \partial_{X_j} - \alpha \right) F = T \partial_T F. \tag{5}$$

Apart from this we may assume F to satisfy all kinds of $\mathbf{Spin}(m,1)$-invariant equations, the algebra of which is generated by

1. Dirac operator

$$\partial_{\underline{X}+\epsilon T} = -\partial_{\underline{X}} + \epsilon \partial_T, \partial_{\underline{X}} = \sum_{j=1}^m \mathbf{e}_j \partial_{X_j}$$

2. Vector multiplication

$$F \mapsto (\underline{X} + \epsilon T)F$$

3. Pseudoscalar multiplication

$$F \mapsto \mathbf{e}_1 \dots \mathbf{e}_m \epsilon F$$

and we have to restrict ourself to homogeneous operators such as

$$\partial_{\underline{X}+\epsilon T}, (\underline{X} + \epsilon T)\partial_{\underline{X}+\epsilon T}, \partial_{\underline{X}+\epsilon T} + \frac{1}{\underline{X}+\epsilon T}, \cdots$$

For each such operator we eliminate the time derivative using the Euler equation (5) (which e.g. turns the hyperbolic Dirac operator $\partial_{\underline{X}+\epsilon T}$ into an elliptic operator) and then we use the change of coordinates formula on the paraboloid P

$$\partial_{x_j} = \partial_{X_j} + X_j \partial_T,\ j = 1, \cdots, m$$

to arrive at the corresponding system in the hyperbolic space $B(1)$.

We will work out the case of the Dirac equation

$$(-\partial_X + \epsilon\partial_T)F = 0,\quad (< X, \partial_X > + T\partial_T)F = \alpha F.$$

First multiplying with $\mathbf{e}_j$ and adding up, the transformation formulas $\partial_{x_j} = \partial_{X_j} + X_j\partial_T$, for $X_j = x_j$, $T = \frac{1+r^2}{2}$ lead to

$$\partial_{\underline{x}} F = \partial_{\underline{X}} F + \underline{X}\partial_T F$$

and multiplying now with $\underline{X} = \underline{x}$ we get

$$\underline{x}\partial_{\underline{x}} F = \underline{X}\partial_{\underline{X}} F + r^2 \partial_T F.$$

As this holds for all functions, we may take the scalar part to obtain

$$\begin{aligned} E_{\underline{x}} F &= E_{\underline{X}} F + r^2 \partial_T F \\ &= (-T\partial_t + r^2\partial_T)F + \alpha F \\ &= (\alpha - \frac{1-r^2}{2}\partial_T)F \end{aligned}$$

so that

$$\partial_T F = \frac{2}{1-r^2}(\alpha - E_{\underline{x}})F,$$

which eliminates ∂_T, while also $\partial_{\underline{X}} = \partial_{\underline{x}} - \underline{X}\partial_T F$, $\underline{X} = \underline{x}$, so that the Dirac equation expressed in the x-coordinates is given by

$$\left[-\partial_{\underline{x}} + 2\frac{\underline{X}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})\right] F = D^\alpha_{\underline{x}} F = 0, |\underline{x}| < 1.$$

We also have that

$$(-\partial_{\underline{X}} + \epsilon\partial_T)^2 = -\Delta_{\underline{X}} + \partial_T^2$$

is the wave operator while $(-\partial_{\underline{X}} + \epsilon\partial_T)F$ is homogeneous of degree $\alpha - 1$ provided $(E_{\underline{X}} + T\partial_T)F = \alpha F$.

Hence, the wave operator $(-\partial_{\underline{X}} + \epsilon\partial_T)^2 F$ for $F(\lambda\underline{X}, \lambda T) = \lambda^\alpha F(\underline{X}, T)$ has to correspond in the $\underline{x}$-coordinates to the operator

$$-\Delta^\alpha_{\underline{x}} = D^{\alpha-1}_{\underline{x}} D^\alpha_{\underline{x}}.$$

Let us compute this purely scalar operator. First

$$\begin{aligned} D^\alpha_{\underline{x}} D^\alpha_{\underline{x}} &= -\Delta_{\underline{x}} + 2\left\{\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}}), -\partial_{\underline{x}}\right\} \\ &\quad + 4\left(\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})\right)^2 \end{aligned}$$

whereby

$$\left\{\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}}), \partial_{\underline{x}}\right\} = \frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})\partial_{\underline{x}} + \partial_{\underline{x}}\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})$$

which, using the Hestenes overdot notation (see [Hs]), is given by

$$\frac{\underline{x}+\epsilon}{1-r^2}\partial_{\underline{x}} + \left\{\frac{-m}{1-r^2} + \frac{2\underline{x}(\underline{x}+\epsilon)}{(1-r^2)^2}\right\}(\alpha - E_{\underline{x}}) + \frac{\{\dot{\partial}_{\underline{x}}, \dot{\underline{x}}+\epsilon\}}{1-r^2}(\alpha - E_{\underline{x}})$$

whereby $\{\dot{\partial}_{\underline{x}}, \dot{\underline{x}} + \epsilon\} = -2E_{\underline{x}}$ so that the second term in $D^\alpha_{\underline{x}} D^\alpha_{\underline{x}}$ is given by

$$-2\frac{\underline{x}+\epsilon}{1-r^2}\partial_{\underline{x}} - \left[\frac{4\underline{x}(\underline{x}+\epsilon)}{(1-r^2)^2} - \frac{4E_{\underline{x}} + 2m}{1-r^2}\right](\alpha - E_{\underline{x}}).$$

The third term is given by

$$4\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})$$
$$= \frac{4}{1-r^2}(\alpha - E_{\underline{x}})^2 - 4\frac{\underline{x}+\epsilon}{1-r^2}\left[\frac{\underline{x}}{1-r^2} + \frac{2r^2(\underline{x}+\epsilon)}{(1-r^2)^2}\right](\alpha - E_{\underline{x}}).$$

All together this gives

$$\Delta_{\underline{x}} - 2\frac{\underline{x}+\epsilon}{1-r^2}\partial_{\underline{x}} + \frac{4\alpha+2m}{1-r^2}(\alpha - E_{\underline{x}}) + \frac{8r^2}{(1-r^2)^2}(\alpha - E_{\underline{x}}) - \frac{8r^2}{(1-r^2)^2}(\alpha - E_{\underline{x}})$$

while also

$$D_{\underline{x}}^{\alpha-1}D_{\underline{x}}^{\alpha} - D_{\underline{x}}^{\alpha}D_{\underline{x}}^{\alpha} = -2\frac{\underline{x}+\epsilon}{1-r^2}\left(-\partial_{\underline{x}} + 2\frac{\underline{x}+\epsilon}{1-r^2}(\alpha - E_{\underline{x}})\right).$$

Hence,

$$\begin{aligned} D_{\underline{x}}^{\alpha-1}D_{\underline{x}}^{\alpha} &= -\Delta_{\underline{x}} - \frac{4}{1-r^2}(\alpha - E_{\underline{x}}) + \frac{4\alpha+2m}{1-r^2}(\alpha - E_{\underline{x}}) \\ &= -\Delta_{\underline{x}} \end{aligned}$$

in case of $\alpha = 1 - m/2$.

Secondly, note that

$$\begin{aligned} (\underline{x}+\epsilon)D_{\underline{x}}^{\alpha} &= -(\underline{x}+\epsilon)\partial_{\underline{x}} + 2(\alpha - E_{\underline{x}}) \\ &= 2E_{\underline{x}} + \partial_{\underline{x}}\underline{x} + m + \partial_{\underline{x}}\epsilon + 2\alpha - 2E_{\underline{x}} \\ &= \partial_{\underline{x}}(\underline{x}+\epsilon) + (m+2\alpha). \end{aligned}$$

Hence, for $\alpha = -\frac{m}{2}$, null solutions of $D_{\underline{x}}^{\alpha}g(\underline{x}) = 0$ are given by

$$\partial_{\underline{x}}f(\underline{x}) = 0, f(\underline{x}) = (\underline{x}+\epsilon)g(\underline{x})$$

so that we work essentially with monogenic functions. For the other values of α this is much less the case, a new function theory is to be developed.

Note that for $\alpha = -\frac{m}{2}$ we have that

$$D_{\underline{x}}^{\alpha-1}D_{\underline{x}}^{\alpha} = -\Delta_{\underline{x}} + \frac{4}{1-r^2}\left(\frac{m}{2} + E_{\underline{x}}\right)$$

which is another canonical operator on the hyperbolic disk.

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A New Characterization of Möbius Transformations by Use of $2n$ Points

Nikolas Samaris

Abstract. Using appropriate classes of conformal mappings we give a new characterization of Möbius Transformation. As a special case of this characterization we generalize to arbitrary Apollonius $2n$-gon results that were only known for $n = 1$ and $n = 2$.

1. Introduction

A $2n$-gon (not necessarily simple) on the complex plane is called Apollonius if for the consecutive vertices $z_1, z_2, \ldots, z_{2n} \in \mathbb{C}$, the following condition holds

$$A(z_1, z_2, \ldots, z_{2n}) = 1, \tag{1}$$

where

$$A(z_1, z_2, \ldots, z_{2n}) = \frac{|(z_1 - z_2)(z_3 - z_4) \cdots (z_{2n-1} - z_{2n})|}{|(z_2 - z_3)(z_4 - z_5) \cdots (z_{2n-2} - z_{2n-1})(z_{2n} - z_1)|}. \tag{2}$$

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