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Complex Variables, Vol. 39, pp. 115–135
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On $\mathbf{Q}_p$-Spaces of Quaternion-Valued Functions

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Communicated by Klaus Habetha

(Received 10 October 1997; In final form 9 January 1998)

We consider a definition of $\mathbf{Q}_p$-spaces for quaternion-valued functions of three real variables and study some of its basic properties.

Keywords: Hypercomplex analysis; function spaces; $\mathbf{Q}_p$-spaces

MOS Subject Classification: 30G35, 46E15

1 INTRODUCTION

A new class of complex-valued functions, the scale of so-called $\mathbf{Q}_p$-spaces, has been recently introduced and studied intensively by several authors (see e.g. [1–3]).

Let $\Delta = \{z\colon |z| < 1\}$ be the complex unit disk. Then the well-known Bloch space

$$\mathbf{B} = \{f\colon f \text{ analytic in } \Delta \text{ and } B(f) = \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty\}$$

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and the Dirichlet space

$$\mathbf{D} = \left\{ f\colon\ f \text{ analytic in } \Delta \text{ and } \int_\Delta |f'(z)|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}$$

are introduced. Applying the Möbius transform $\varphi_a(z) = (a-z)/(1-\bar{a}z)$, which maps the unit disk Δ onto itself, and the fundamental solution of the two-dimensional real Laplacian a function $g(z,a) = \ln|(1-\bar{a}z)/(a-z)|$ is defined. Obviously, this function has a logarithmic singularity at $a \in \Delta$. Then the spaces

$$\mathbf{Q}_p = \left\{ f\colon\ f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \int_\Delta |f'(z)|^2 g^p(z,a) \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}$$

are defined. The idea of these $\mathbf{Q}_p$-spaces is to find a (continuous) scale of spaces with $\mathbf{D}$ and $\mathbf{B}$, respectively, "at the both end points" of the scale.

Indeed, a lot of essential results are already known as for instance

$$\begin{aligned} &\mathbf{D} \subset \mathbf{Q}_p \subset \mathbf{Q}_q \subset BMOA \quad 0 < p < q < 1 \qquad [3] \\ &\mathbf{Q}_1 = BMOA \qquad [3] \\ &\mathbf{Q}_p = \mathbf{B} \quad \forall p > 1 \qquad [1] \end{aligned}$$

This means that the spaces $\mathbf{Q}_p$ form a scale as desired and for special values of the scale parameter p these spaces are connected with other known and important spaces of analytic functions. Another special property of these spaces is the conformal invariance under Möbius transformations.

There are several attempts to generalize these ideas and the corresponding approach to higher dimensions [4,9,10,12]. Independently on method these approaches treat the case of the unit ball in $\mathbb{C}^n$ and not the case of the unit ball in $\mathbb{R}^n$. Basic ideas are to replace the derivative f' by the complex gradient of f and the measure $\mathrm{d}x\,\mathrm{d}y$ by a weighted measure $\mathrm{d}\lambda(z) = \mathrm{d}\nu/(1-|z|^2)^{n+1}$, where $\mathrm{d}\nu$ stands for the usual Lebesgue measure. Using an invariant Green's function some results similar to the complex one-dimensional case were proved. The most important results are that

$$\mathbf{Q}_p = \mathbf{B} \quad \text{for } 1 < p < n/(n-1) \quad \text{and} \quad \mathbf{Q}_1 = BMOA(\partial B),$$

where ∂B is the surface of the unit ball in $\mathbb{C}^n$. But, for $p \notin ((n-1)/n, n/(n-1))$ all $\mathbf{Q}_p$-spaces are trivial, i.e., only constant functions belong to $\mathbf{Q}_p$.

This is one of the reasons to look for other possibilities to generalize the complex (one-dimensional) ideas. Furthermore, using the $\mathbb{C}^n$-approach it is impossible in principle to consider $\mathbf{Q}_p$-spaces in odd real dimensions of the Euclidean space.

In this paper we study hypercomplex generalizations of $\mathbf{Q}_p$-spaces. Instead of holomorphic functions in the unit disk we study hyperholomorphic functions $f: \mathbb{R}^n \mapsto C\ell_{0,n-1}$ (i.e., solutions of generalized Cauchy–Riemann systems), which are a higher-dimensional generalization of holomorphic functions also in the case of odd real dimensions of the Euclidean space. Such important function classes like the solutions of the *div–rot* system are included in the theory of hyperholomorphic functions. These functions can be considered in all real space dimensions.

With the generalized Cauchy–Riemann operator D, its adjoint $\bar{D}$, the hypercomplex Möbius transformation $\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$, and a modified fundamental solution g of the real Laplacian we consider generalized $\mathbf{Q}_p$-spaces defined by

$$\mathbf{Q}_p = \left\{ f \in \ker D\colon\ \sup_{a\in B_1(0)} \int_B |\bar{D}f(x)|^2 (g(\varphi_a(x)))^p \, \mathrm{d}x < \infty \right\},$$

where $B_1(0)$ stands for the unit ball in $\mathbb{R}^n$. This definition seems to be natural because

- it has a deep structural analogy with the complex (one-dimensional) definition;
- all the used items generalize definitions (analyticity, derivative, Möbius transformations and Green's functions) from the complex one-dimensional case;
- the generalized $\mathbf{Q}_p$-spaces have analogous properties as the complex spaces.

To prove these analogous properties is the aim of this paper. We remark that for the case of functions $f: \mathbb{R}^4 \mapsto \mathbb{H}$ it is already known from [8] that $\bar{D}$ may be interpreted as derivative. In this paper we restrict us to the case $n=3$, the lowest non-commutative case, as a model case of general Clifford analysis. Moreover, we will identify the Clifford-Algebra $C\ell_{0,2}$ with the skew-field of quaternions. Thus we consider functions $f:\mathbb{R}^3 \mapsto \mathbb{H}$.

2 PRELIMINARIES

In what follows we will work in $\mathbb{H}$, the skew-field of quaternions. This means we can write each element $z \in \mathbb{H}$ in the form

$$z = z_0 + z_1 i + z_2 j + z_3 k, \quad z_n \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $kj = -jk = i$, $ki = -ik = j$. The conjugate element $\bar{z}$ is given by $\bar{z} = z_0 - z_1 i - z_2 j - z_3 k$ and we have the property $z\bar{z} = \bar{z}z = \|z\|^2 = z_0^2 + z_1^2 + z_2^2 + z_3^2$. Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

Also, in what follows we will work in $B_1(0) \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. $B_1(0)$ is a bounded, simply connected domain with a C^∞-boundary $S_1(0)$. Moreover, we will consider functions f defined on $B_1(0)$ with values in $\mathbb{H}$. We now define the generalized Cauchy–Riemann operator by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2}$$

and it's conjugate operator by

$$\bar{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

For these operators we have that

$$D\bar{D} = \bar{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. The Cauchy–Riemann operator has a right inverse of the form

$$Tf(x) = -\frac{1}{4\pi}\int_{B_1(0)} \frac{\overline{(x-y)}}{|x-y|^3} f(y)\,\mathrm{d}B_y, \quad x \in B_1(0).$$

This operator acts continuously from $W_p^k(B_1(0))$ into $W_p^{k+1}(B_1(0))$, $1 < p < \infty, k \in \mathbb{N} \cup \{0\}$ (see [5]). Moreover, we need the following Cauchy-type integral operator:

$$F_S f(x) = \frac{1}{4\pi}\int_{S_1(0)} \frac{\overline{(x-y)}}{|x-y|^3}\alpha(y) f(y)\,\mathrm{d}S_y \quad x \in B_1(0),$$

where $\alpha(y)$ is the outward pointing normal vector to $S_1(0)$ at the point y. This operator is a continuous mapping from $W_p^{k+1/2}(S_1(0))$ into $W_p^{k+1}(B_1(0))$, $1 < p < \infty, k \in \mathbb{N} \cup \{0\}$ [5]. The above introduced operators are connected by the well-known Borel–Pompeiu formula:

$$F_S f + TDf = f.$$

Functions belonging to $\ker D$ are called hyperholomorphic or regular functions. From the Borel–Pompeiu formula we have the Cauchy formula

$$F_S f = f \quad \forall f \in \ker D \cap W_2^{1/2}(S_1(0)).$$

For more information about these topics and general quaternionic analysis we refer to [5–7,13].

3 DEFINITIONS OF SOME FUNCTIONAL SPACES

For $|a| < 1$ we will denote by

$$\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x,a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with our Möbius transform $\varphi_a(x)$. Especially, we denote for all $p \geq 0$

$$g^p(x,a) = \frac{1}{4^p\pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p.$$

Let $f: B_1(0) \mapsto \mathbb{H}$ be a hyperholomorphic function. Then we can introduce the seminorms

- $B(f) = \sup_{x\in B_1(0)}(1-|x|^2)^{3/2}|\bar{D}f(x)|$,
- $Q_p(f) = \sup_{a\in B_1(0)} \int_{B_1(0)} |\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x$,

which lead to the following definitions:

DEFINITION 3.1 The spatial (or three-dimensional) Bloch space $\mathbf{B}$ is the right $\mathbb{H}$-module of all hyperholomorphic functions $f: B_1(0) \mapsto \mathbb{H}$ with $B(f) < \infty$.

DEFINITION 3.2 The right $\mathbb{H}$-module of all quaternion-valued functions f defined on the unit ball, which are hyperholomorphic and satisfy $Q_p(f) < \infty$, is called $\mathbf{Q}_p$-space.

Remark 3.1 Because of the special structure of $g(x, a)$ the seminorms $Q_p(f)$ make sense for $p < 3$ only. Consequently, we will consider in this paper $\mathbf{Q}_p$-spaces for $p < 3$ only.

Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighbourhood of the origin U_ϵ, with an arbitrary but fixed $\epsilon > 0$, then we can add the L_1-norm of f over U_ϵ to our seminorms and $\mathbf{B}$ as well as $\mathbf{Q}_p$ will become Banach spaces. Because this additional term is independent of p we will consider in the following only the spaces with the corresponding seminorm, but we have to keep in mind, that all our results are also true in the case of the norm.

DEFINITION 3.3 The right $\mathbb{H}$-module of all quaternion-valued functions f defined on the unit ball, which are hyperholomorphic and satisfy the condition

$$\int_{B_1(0)} |\bar{D}f(x)|^2\,\mathrm{d}B_x < \infty,$$

is called spatial (or three-dimensional) Dirichlet space $\mathbf{D}$.

Remark 3.2 Since $g(x, a)$ is non-negative in $B_1(0)$ we have, obviously,

$$\mathbf{D} \subset \mathbf{Q}_p, \quad 0 \leq p < 3.$$

4 PROPERTIES OF $\mathbf{Q}_p$-SPACES

In this section we will show that the $\mathbf{Q}_p$-spaces are in fact a scale of (with our additional term added to the seminorm) Banach $\mathbb{H}$-modules, which connects the spatial Dirichlet space with the spatial Bloch space. For doing this we need several lemmas.

LEMMA 4.1 *Let f be hyperholomorphic in the unit ball. Then we have for all $r < 1$*

$$\int_{S_r(0)} |\bar{D}f(x)|\,\mathrm{d}S_x \geq 4\pi r^2|\bar{D}f(0)|,$$

where $S_r(0)$ is the surface of the ball $B_r(0)$ with center at 0 and radius r.

Proof Let $f \in \ker D(B_1(0))$ and $S_r(0) = \partial B_r(0)$. Then we know from Cauchy's integral formula that

$$f(y) = \int_{S_r(0)} K(x-y)\alpha(x)f(x)\,\mathrm{d}S_x, \quad \forall y \in B_r(0),$$

where $K(x-y) = \frac{1}{4\pi}(\overline{x-y})/(|x-y|^3)$ is the usual Cauchy kernel and $\alpha(x)$ the outward pointing normal vector at the point x. For the Cauchy kernel we have, for $x \in S_r(0)$,

$$|K(x)| = \frac{1}{4\pi r^2}.$$

Now we have:

$$|\bar{D}f(0)| = \left| \int_{S_r(0)} K(x)\alpha(x)\bar{D}f(x)\,\mathrm{d}S_x \right| \le \int_{S_r(0)} |K(x)||\alpha(x)||\bar{D}f(x)|\,\mathrm{d}S_x,$$
$$= \frac{1}{4\pi r^2}\int_{S_r(0)} |\bar{D}f(x)|\,\mathrm{d}S_x.$$

LEMMA 4.2 *Under the same conditions as in Lemma 4.1 we have that for any fixed $R<1$*

$$\int_{B_R(0)} |\bar{D}f(x)|^2\,\mathrm{d}x \ge \frac{4\pi R^3}{3}|\bar{D}f(0)|^2$$

holds.

Proof We know from Lemma 4.1 that for all $r<R$

$$|\bar{D}f(0)|^2 \le \frac{1}{16\pi^2 r^4}\left(\int_{S_r(0)} |\bar{D}f(x)|\,\mathrm{d}S_x\right)^2$$
$$\le \frac{1}{16\pi^2 r^4}\left(\int_{S_r(0)} |\bar{D}f(x)|^2\,\mathrm{d}S_x\right)\left(\int_{S_r(0)} \mathrm{d}S\right)$$
$$= \frac{1}{4\pi r^2}\left(\int_{S_r(0)} |\bar{D}f(x)|^2\,\mathrm{d}S_x\right),$$

because $\int_{S_r(0)} \mathrm{d}S = 4\pi r^2$. If we multiply both sides by r^2 and integrate then we get

$$|\bar{D}f(0)|^2 \int_0^R r^2\,\mathrm{d}r \le \frac{1}{4\pi}\left(\int_0^R \int_{S_r(0)} |\bar{D}f(x)|^2\,\mathrm{d}S_x\,\mathrm{d}r\right),$$

or in other words

$$|\bar{D}f(0)|^2 \frac{4\pi R^3}{3} \le \int_{B_R(0)} |\bar{D}f(x)|^2\,\mathrm{d}B_x,$$

which leads to our statement.

PROPOSITION 4.1 *Let f be hyperholomorphic and $0<p<3$, then we have*

$$(1-|a|^2)^3|\bar{D}f(a)|^2 \le C_1 \int_{B_1(0)} |\bar{D}f(x)|^2\left(\frac{1}{|\varphi_a(x)|}-1\right)^p \mathrm{d}B_x,$$

where the constant C_1 does not depend on a and f.

Proof Let $R<1$ and $U(a,R)=\{x\colon |\varphi_a(x)|<R\}$ be the pseudohyperbolic ball with radius R. Then

$$\int_{B_1(0)} |\bar{D}f(x)|^2\left(\frac{1}{|\varphi_a(x)|}-1\right)^p \mathrm{d}B_x \ge \int_{U(a,R)} |\bar{D}f(x)|^2\left(\frac{1}{|\varphi_a(x)|}-1\right)^p \mathrm{d}B_x$$
$$\ge \left(\frac{1}{R}-1\right)^p \int_{U(a,R)} |\bar{D}f(x)|^2\,\mathrm{d}B_x. \tag{1}$$

We begin with the estimation of

$$\int_{U(a,R)} |\bar{D}f(x)|^2\,\mathrm{d}B_x$$
$$= \int_{B_R} |\bar{D}f(\varphi_a(x))|^2 \frac{(1-|a|^2)^3}{|1-\bar{a}x|^6}\,\mathrm{d}B_x$$
$$= (1-|a|^2)^3 \int_{B_R} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\bar{D}f(\varphi_a(x))\right|^2 \frac{1}{|1-\bar{a}x|^2}\,\mathrm{d}B_x.$$

Obviously, we have that

$$\frac{1}{|1-\bar{a}x|^2} \ge \frac{1}{(1+R)^2}.$$

Therefore,

$$\int_{U(a,R)} |\bar{D}f(x)|^2\,\mathrm{d}B_x \ge \frac{(1-|a|^2)^3}{(1+R)^2}\int_{B_R} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\bar{D}f(\varphi_a(x))\right|^2 \mathrm{d}B_x$$
$$\ge \frac{(1-|a|^2)^3}{(1+R)^2}\frac{4\pi R^3}{3}|\bar{D}f(a)|^2.$$

If we replace this in (1), we get

$$\int_{B_1(0)} |\bar{D}f(x)|^2 \left(\frac{1}{|\varphi_a(x)|} - 1\right)^p \mathrm{d}B_x \geq \frac{4\pi}{3} \frac{(1-R)^p}{(1+R)^2} R^{3-p} (1-|a|^2)^3 |\bar{D}f(a)|^2$$

as well as

$$(1-|a|^2)^3 |\bar{D}f(a)|^2 \leq \frac{3(1+R)^4}{4\pi R^{3-p}(1-R)^p} \int_{B_1(0)} |\bar{D}f(x)|^2 \left(\frac{1}{|\varphi_a(x)|} - 1\right)^p \mathrm{d}B_x.$$

Choosing a suitable R from this we derive our estimate

$$(1-|a|^2)^3 |\bar{D}f(a)|^2 \leq C_1 \int_{B_1(0)} |\bar{D}f(x)|^2 \left(\frac{1}{|\varphi_a(x)|} - 1\right)^p \mathrm{d}B_x.$$

COROLLARY 4.1 *For* $0 \leq p < 3$ *we have* $\mathbf{Q}_p \subset \mathbf{B}$.

This corollary means, that all $\mathbf{Q}_p$-spaces are subspaces of the Bloch space.

PROPOSITION 4.2 *If f is hyperholomorphic in* $B_1(0)$ *and* $2 < p < 3$, *then for all* $|a| < 1$

$$\int_{B_1(0)} |\bar{D}f(x)|^2 g^p(x,a)\, \mathrm{d}B_x \leq J(p) B(f)^2,$$

where $J(p) = 4\pi \int_0^1 r^{2-p}/((1-r)^{3-p}(1+r)^3)\, \mathrm{d}r$ *is finite.*

Proof We know from the definition of $\mathbf{B}$ that $(1-|x|^2)^{3/2} |\bar{D}f(x)| \leq B(f)$. We estimate as follows:

$$\begin{aligned} &\int_{B_1(0)} |\bar{D}f(x)|^2 g^p(x,a)\, \mathrm{d}B_x \\ &\leq B^2(f) \int_{B_1(0)} \frac{1}{(1-|x|^2)^3} \frac{(1-|\varphi_a(x)|)^p}{|\varphi_a(x)|^p} \mathrm{d}B_x \\ &\leq B^2(f) \int_{B_1(0)} \frac{1}{(1-|\varphi_a(x)|^2)^3} \frac{(1-|x|)^p}{|x|^p} \frac{(1-|a|^2)^3}{|1-\bar{a}x|^6} \mathrm{d}B_x. \end{aligned}$$

Here, we used the fact that the Jacobian determinant is

$$\frac{(1-|a|^2)^3}{|1-\bar{a}x|^6}.$$

Now, using the equality

$$\frac{1-|\varphi_a(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{|1-\bar{a}x|^2}$$

we come to our desired result.

$$\begin{aligned} &B^2(f) \int_{B_1(0)} \frac{1}{(1-|\varphi_a(x)|^2)^3} \frac{(1-|x|)^p}{|x|^p} \frac{(1-|a|^2)^3}{|1-\bar{a}x|^6} \mathrm{d}B_x \\ &= B^2(f) \int_{B_1(0)} \frac{(1-|x|)^p}{(1-|x|^2)^3 |x|^p} \mathrm{d}B_x \\ &= B^2(f) \int_0^1 \frac{r^{2-p}}{(1-r)^{3-p}(1+r)^3} \mathrm{d}r \int_0^\pi \int_0^{2\pi} \sin\varphi_1\, \mathrm{d}\varphi_2\, \mathrm{d}\varphi_1 \\ &= B^2(f) J(p). \end{aligned}$$

THEOREM 4.1 *Let f hyperholomorphic in the unit ball. Then the following conditions are equivalent:*

1. $f \in \mathbf{B}$.
2. $Q_p(f) < \infty$ *for all* $2 < p < 3$.
3. $Q_p(f) < \infty$ *for some* $p > 2$.

Proof The implication $(1 \Rightarrow 2)$ follows from Proposition 4.2. It is obvious that $(2 \Rightarrow 3)$. From Corollary 4.1 we have that 3 implies 1.

Theorem 4.1 means that all $\mathbf{Q}_p$-spaces for $p > 2$ coincide and are identical with the Bloch space.

5 ANOTHER CHARACTERIZATION OF $\mathbf{Q}_p$-SPACES

In this section we will give another possibility to characterize $\mathbf{Q}_p$-spaces, which is often easier to handle. Among others, this new characterization enables us to prove that the $\mathbf{Q}_p$-spaces are a scale of function spaces with the Dirichlet space at one extreme point and the Bloch space on the other.

LEMMA 5.1

$$\int_{B_1(0)} |\bar{D}f(x)|^2(1-|x|^2)\,\mathrm{d}B_x \simeq \int_{B_1(0)} |\bar{D}f(x)|^2 g(x,0)\,\mathrm{d}B_x.$$

We remember that $g(x,0)=1/\omega_3(1/|x|-1)$, $\omega_3=4\pi$.

Here "$\simeq$" means that there exist constants $C_1>0$, $C_2>0$ (independent on f) such that

$$C_1\int_{B_1(0)} |\bar{D}f(x)|^2 g(x,0)\,\mathrm{d}B_x \le \int_{B_1(0)} |\bar{D}f(x)|^2(1-|x|^2)\,\mathrm{d}B_x \le C_2\int_{B_1(0)} |\bar{D}f(x)|^2 g(x,0)\,\mathrm{d}B_x.$$

Proof In spherical coordinates what we need to prove is

$$\int_0^1 M_2^2(\bar{D}f,r)(1-r^2)r^2\,\mathrm{d}r \simeq \int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3}\left(\frac{1}{r}-1\right)r^2\,\mathrm{d}r,$$

where $M_2^2(\bar{D}f,r)=\int_0^\pi\int_0^{2\pi}|\bar{D}f(r,\varphi_1,\varphi_2)|^2\sin\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_1$. This means we have to show that there exist constants C_1, C_2 such that

$$C_1\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3}(r-r^2)\,\mathrm{d}r \le \int_0^1 M_2^2(\bar{D}f,r)(r^2-r^4)\,\mathrm{d}r \le C_2\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3}(r-r^2)\,\mathrm{d}r.$$

Part (a) Let us choose $C_2=2\omega_3$. Then we get

$$\int_0^1 M_2^2(\bar{D}f,r)((r^2-r^4)-2(r-r^2))\,\mathrm{d}r \le 0,$$

because

$$r^2-r^4-2(r-r^2)=(1-r)r(r(r+1)-2)\le 0\ \ \forall r\in[0,1]$$

and $M_2^2(\bar{D}f,r)\ge 0\ \forall r$. This results in

$$\int_0^1 M_2^2(\bar{D}f,r)(r^2-r^4)\,\mathrm{d}r \le 2\int_0^1 M_2^2(\bar{D}f,r)(r-r^2)\,\mathrm{d}r = 2\omega_3\int_0^1 M_2^2(\bar{D}f,r)\frac{r-r^2}{\omega_3}\,\mathrm{d}r.$$

Part (b) Now, let us choose $C_1=11\omega_3/100$, then we have to prove

$$\int_{r_0}^1 M_2^2(\bar{D}f,r)\left(r^2-r^4-\frac{11}{100}(r-r^2)\right)\mathrm{d}r - \int_0^{r_0} M_2^2(\bar{D}f,r)\left(\frac{11}{100}(r-r^2)-r^2+r^4\right)\mathrm{d}r \ge 0,$$

where $r_0=1/10$ is the solution of the equation $r^2-r^4-(11/100)(r-r^2)=0$, $0<r<1$ (This polynomial has only the zeros $-11/10, 0, 1/10, 1$). Then it is easy to see, because all integrands are positive, that

$$\begin{aligned}&\int_{6/10}^1 M_2^2(\bar{D}f,r)\left(r^2-r^4-\frac{11}{100}(r-r^2)\right)\mathrm{d}r\\ &+\int_{1/10}^{5/10} M_2^2(\bar{D}f,r)\left(r^2-r^4-\frac{11}{100}(r-r^2)\right)\mathrm{d}r\\ &+\int_{5/10}^{6/10} M_2^2(\bar{D}f,r)\left(r^2-r^4-\frac{11}{100}(r-r^2)\right)\mathrm{d}r\\ &-\int_0^{1/10} M_2^2(\bar{D}f,r)\left(\frac{11}{100}(r-r^2)-r^2+r^4\right)\mathrm{d}r \ge 0,\end{aligned}$$

due to the fact, that the integral $\int_{5/10}^{6/10} M_2^2(\bar{D}f,r)(r^2-r^4-(11/100)(r-r^2))\,\mathrm{d}r$ is greater than the integral $\int_0^{1/10} M_2^2(\bar{D}f,r)((11/100)(r-r^2)-r^2+r^4)\,\mathrm{d}r$. In particular we have $M_2^2(\bar{D}f,r_1)\ge M_2^2(\bar{D}f,r_2)$ for $r_1>r_2$ (because $\bar{D}f$ is harmonic in $B_1(0)$ and belongs to $L_2(B_r(0))\ \forall r<1$), $r_1^2-r_1^4-(11/100)(r_1-r_1^2)>8/100$ for all $r_1\in[5/10,6/10]$, and $(11/100)(r_2-r_2^2)-r_2^2+r_2^4<2/100$ for all $r_2\in[0,1/10]$. This gives our statement.

LEMMA 5.2

$$\int_{B_1(0)} |\bar{D}f(x)|^2(1-|x|^2)^p \,\mathrm{d}B_x \simeq \int_{B_1(0)} |\bar{D}f(x)|^2 g^p(x,0)\,\mathrm{d}B_x$$

with $1 < p < 2.99$.

Proof We have, again, in spherical coordinates:

$$\int_0^1 M_2^2(\bar{D}f,r)(1-r^2)^p r^2\,\mathrm{d}r \simeq \int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3^p}\left(\frac{1}{r}-1\right)^p r^2\,\mathrm{d}r,$$

where $M_2^2(\bar{D}f,r)$ is as in Lemma 5.1. This means we have to show that there exist constants $C_1(p), C_2(p)$ such that

$$\begin{aligned} C_1(p)&\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3^p}(r^{-1}-1)^p r^2\,\mathrm{d}r\\ &\le \int_0^1 M_2^2(\bar{D}f,r)(1-r^2)^p r^2\,\mathrm{d}r\\ &\le C_2\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3^p}(r^{-1}-1)^p r^2\,\mathrm{d}r.\end{aligned}$$

Part (a) Let $C_2(p) = 2^p\omega_3^p$. Then

$$\int_0^1 M_2^2(\bar{D}f,r)[(1-r^2)^p r^2 - 2^p(r^{-1}-1)^p r^2]\,\mathrm{d}r \le 0$$

because $M_2^2(\bar{D}f,r) \ge 0\ \forall r \in [0,1]$ and $(1-r^2)^p r^2 - 2^p(r^{-1}-1)^p r^2 = (1-r)^p r^{2-p}((1+r)^p r^p - 2^p) \le 0\ \forall r \in [0,1]$.

From this we get

$$\begin{aligned}\int_0^1 M_2^2(\bar{D}f,r)(1-r^2)^p r^2\,\mathrm{d}r &\le \int_0^1 M_2^2(\bar{D}f,r)(r^{-1}-1)^p r^2\,\mathrm{d}r\\ &= C_2(p)\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3^p}(r^{-1}-1)^p r^2\,\mathrm{d}r.\end{aligned}$$

Part (b) Let $C_1(p) = (11/100)^p\omega_3^p$. We want to prove that

$$C_1(p)\int_0^1 M_2^2(\bar{D}f,r)\frac{1}{\omega_3^p}(r^{-1}-1)^p r^2\,\mathrm{d}r \le \int_0^1 M_2^2(\bar{D}f,r)(1-r^2)^p r^2\,\mathrm{d}r.$$

This means we have to consider the integral

$$\int_0^1 M_2^2(\bar{D}f,r)\left[(1-r^2)^p r^2 - \frac{11^p}{100^p}(r^{-1}-1)^p r^2\right]\mathrm{d}r.$$

This is equivalent to the integral

$$\int_0^1 M_2^2(\bar{D}f,r)\left[(1-r)^p r^{2-p}\left((1+r)^p r^p - \frac{11^p}{10^{2p}}\right)\right]\mathrm{d}r.$$

The important term in this integral is

$$k(r) = (1-r)^p r^{2-p}\left((1+r)^p r^p - \frac{11^p}{10^{2p}}\right).$$

It may be observed that $k(r) < 0$ for $r \in (0, 1/10)$ with a "pole" at the origin if $2-p<0$ and $k(r) > 0$ for $r \in (1/10, 1)$. Especially, for $r < 1/10$ we have

$$\left|(1-r)^p r^{2-p}\left[(1+r)^p r^p - \frac{11^p}{10^{2p}}\right]\right| < r^{2-p}\frac{11^p}{10^{2p}},$$

because of $(1/10^p + 1/10^{2p}) - 11^p/10^{2p} < 0$.

This means we have to compare the integrals $\int_0^{1/10} M_2^2(f,r)r^{2-p}(11^p/10^{2p})\,\mathrm{d}r$ and $\int_{5/10}^{6/10} M_2^2(f,r)(1-r)^p r^{2-p}[(1+r)^p r^p - 11^p/10^{2p}]\,\mathrm{d}r$. For the first integral we get that it is smaller than $(11^p M_2^2(f,1/10))/(10^{2p})(1/(3-p))(1/10)^{3-p}$. For the second integral we have the estimate

$$\begin{aligned}\int_{5/10}^{6/10} &M_2^2(f,r)(1-r)^p r^{2-p}[(1+r)^p r^p - 11^p/10^{2p}]\,\mathrm{d}r\\ &\ge \int_{5/10}^{6/10} M_2^2(f,1/10)\left(-\frac{11}{100}+\frac{111}{100}r - r^3\right)^p r^{2-p}\,\mathrm{d}r\\ &\ge \frac{M_2^2(f,1/10)}{10^{2p}}32^p\int_{5/10}^{6/10} r^{2-p}\,\mathrm{d}r\\ &\ge \frac{M_2^2(f,1/10)}{10^{2p}}32^p\frac{1}{3-p}\frac{6^{3-p}-5^{3-p}}{10^{3-p}}.\end{aligned}$$

We remark that the infimum of $-(11/100)+(111/100)r-r^3$ is 32/100 for $5/10 < r < 6/10$. Following the same lines as in the proof (Part (b)) of Lemma 5.1 we will get our estimate and our statement.

THEOREM 5.1 *Let f be hyperholomorphic in $B_1(0)$. Then, for $1 \le p < 2.99$,*

$$f \in \mathbf{Q}_p \Leftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} |\bar{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^p \, \mathrm{d}B_x < \infty.$$

On the first view, the condition $p < 2.99$ looks strange. But we have to keep in mind that Theorem 4.1 means that all $\mathbf{Q}_p$-spaces for $p > 2$ are the same, so in fact this condition is only of technical nature.

Proof Let us consider the equivalence

$$\int_{B_1(0)} |\bar{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^p \, \mathrm{d}B_x \simeq \int_{B_1(0)} |\bar{D}f(x)|^2 g^p(x, a) \, \mathrm{d}B_x,$$

with $g(x, a) = 1/\omega_3((1/|\varphi_a(x)|) - 1)$ and $\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$ the Möbius-transform, which maps the unit ball onto itself. After a change of variables $w = \varphi_a(x)$ (the Jacobian determinant $((1 - |a|^2)/|1 - \bar{a}w|^2)^3$ has no singularities) we get

$$\int_{B_1(0)} |\bar{D}_x f(\varphi_a(w))|^2 (1 - |w|^2)^p \left(\frac{1 - |a|^2}{|1 - \bar{a}w|^2} \right)^3 \mathrm{d}B_w$$
$$\simeq \int_{B_1(0)} |\bar{D}_x f(\varphi_a(w))|^2 g^p(w, 0) \left(\frac{1 - |a|^2}{|1 - \bar{a}w|^2} \right)^3 \mathrm{d}B_w,$$

where D_x means the Cauchy–Riemann-operator with respect to x. The problem here is, that $\bar{D}_x f(x)$ is hyperholomorphic, but after the change of variables $\bar{D}_x f(\varphi_a(w))$ is not hyperholomorphic. But we know from [11] that $((1 - \bar{w}a)/|1 - \bar{a}w|^3)\bar{D}_x f(\varphi_a(w))$ is again hyperholomorphic. We also refer to [13] who studied this problem for the four-dimensional case already in 1979. Therefore, we get

$$\int_{B_1(0)} |\psi(w)|^2 (1 - |w|^2)^p \frac{1}{|1 - \bar{a}w|^2} \, \mathrm{d}B_w$$
$$\simeq \int_{B_1(0)} |\psi(w)|^2 \frac{1}{\omega_3^p} \left(\frac{1}{|w|} - 1 \right)^p \frac{1}{|1 - \bar{a}w|^2} \, \mathrm{d}B_w,$$

with $\psi(w) = ((1 - \bar{w}a)/|1 - \bar{a}w|^3)\bar{D}_x f(\varphi_a(w))$. Again, this means we have to find constants $C_1(p)$ and $C_2(p)$ with

$$C_1(p) \int_{B_1(0)} |\psi(w)|^2 \frac{1}{\omega_3^p} \left(\frac{1}{|w|} - 1 \right)^p \frac{1}{|1 - \bar{a}w|^2} \, \mathrm{d}B_w$$
$$\le \int_{B_1(0)} |\psi(w)|^2 (1 - |w|^2)^p \frac{1}{|1 - \bar{a}w|^2} \, \mathrm{d}B_w$$
$$\le C_2(p) \int_{B_1(0)} |\psi(w)|^2 \frac{1}{\omega_3^p} \left(\frac{1}{|w|} - 1 \right)^p \frac{1}{|1 - \bar{a}w|^2} \, \mathrm{d}B_w.$$

Obviously, we can set $C_2(p) = 2^p \omega_3^p$. For the first estimate we choose $C_1(p) = \omega_3^p (11^p / 100^p)$ and consider the integral

$$\int_{B_1(0)} |\psi(w)|^2 \frac{1}{|1 - \bar{a}w|^2} \left[-\left(\frac{11}{100} \frac{1}{|w|} - \frac{11}{100} \right)^p + (1 - |w|^2)^p \right] \mathrm{d}B_w.$$

To get our estimate this integral has to be greater than or equal to zero. Similarly to the proof of Lemma 5.2 we get

$$- \int_{B_{(1/10)}(0)} |\psi(w)|^2 \frac{1}{|1 - \bar{a}w|^2} \left[-\left(\frac{11}{100} \frac{1}{|w|} - \frac{11}{100} \right)^p + (1 - |w|^2)^p \right] \mathrm{d}B_w$$
$$+ \int_{B_{(5/10)}(0) \setminus B_{(1/10)}(0)} |\psi(w)|^2 \frac{1}{|1 - \bar{a}w|^2}$$
$$\times \left[-\left(\frac{11}{100} \frac{1}{|w|} - \frac{11}{100} \right)^p + (1 - |w|^2)^p \right] \mathrm{d}B_w$$
$$+ \int_{B_{(6/10)}(0) \setminus B_{(5/10)}(0)} |\psi(w)|^2 \frac{1}{|1 - \bar{a}w|^2}$$
$$\times \left[-\left(\frac{11}{100} \frac{1}{|w|} - \frac{11}{100} \right)^p + (1 - |w|^2)^p \right] \mathrm{d}B_w$$
$$+ \int_{B_1(0) \setminus B_{(6/10)}(0)} |\psi(w)|^2 \frac{1}{|1 - \bar{a}w|^2}$$
$$\times \left[-\left(\frac{11}{100} \frac{1}{|w|} - \frac{11}{100} \right)^p + (1 - |w|^2)^p \right] \mathrm{d}B_w \ge 0,$$

where $B_r(0)$ is the ball centred at zero with radius r. Obviously, the second and the fourth integral are greater than zero. Therefore, it is sufficient to compare the first and the third integral. For the further consideration it is necessary to estimate

$$\frac{1}{|1-\bar{a}w|^2}.$$

This can be done with the aid of the inequality

$$0.9 \le 1-|w| \le 1-|a||w| = |1-|\bar{a}w|| \le |1-\bar{a}w|$$

in $B_{1/10}$ and

$$|1-\bar{a}w| \le 1+|a||w| \le 1+|w| \le 1.6$$

in $B_{6/10}(0)\backslash B_{5/10}(0)$. Now, let us make a change of variables to spherical coordinates, which gives us

$$\int_0^{1/10} M_2^2(\psi,r)\frac{10^2}{9^2}(1-r)^p r^{2-p}\left(\frac{11^p}{100^p}-(1+r)^p r^p\right)\mathrm{d}r$$
$$\le \int_{5/10}^{6/10} M_2^2(\psi,r)\frac{10^2}{16^2}(1-r)^p r^{2-p}\left((1+r)^p r^p-\frac{11^p}{100^p}\right)\mathrm{d}r.$$

This can be verified in the same way as in the proof (Part (b)) of Lemma 5.2. We only remark that for $1 \le p < 2.99$

$$32^p\frac{10^2}{16^2}(6^{3-p}-5^{3-p}) \ge \frac{10^2}{9^2}.$$

Our above theorem enables us now to state the same characterization also in the case of $p<1$.

PROPOSITION 5.1 *Let f be hyperholomorphic in $B_1(0)$. Then, for* $0<p\le 1$,

$$f\in \mathbf{Q}_p \Leftrightarrow \sup_{a\in B_1(0)}\int_{B_1(0)}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)^p\,\mathrm{d}B_x<\infty.$$

Proof ("$\Rightarrow$") Let $0<p\le 1$. Due to the following relationship

$$1-|\varphi_a(x)|^2 \le \frac{2\omega_3}{\omega_3}\frac{1-|\varphi_a(x)|}{|\varphi_a(x)|} = \frac{2\omega_3}{\omega_3}\left(\frac{1}{|\varphi_a(x)|}-1\right) = 2\omega_3 g(x,a)$$

the assertion follows.

("$\Leftarrow$") We suppose that $\sup_{a\in B_1(0)}\int_{B_1(0)}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)^p\,\mathrm{d}B_x < \infty$ for $0<p\le 1$. Since

$$\int_{B_1(0)}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)\,\mathrm{d}B_x$$
$$\le \int_{B_1(0)}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)^p\,\mathrm{d}B_x<\infty$$

and Theorem 5.1, we have

$$\sup_{a\in B_1(0)}\int_{B_1(0)}|\bar{D}f(x)|^2 g(x,a)\,\mathrm{d}B_x<\infty.$$

We split the integral into two parts,

$$\int_{B_1(0)}|\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x = \int_{\{|\varphi_a(x)|\le\frac{1}{25}\}}|\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x$$
$$+\int_{\{|\varphi_a(x)|>\frac{1}{25}\}}|\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x.$$

It may be observed that

$$g(x,a)=\frac{1}{\omega_3}\left(\frac{1}{|\varphi_a(x)|}-1\right)\begin{cases}\ge 24/\omega_3>1, & |\varphi_a(x)|\le\frac{1}{25},\\ \le 25(1-|\varphi_a(x)|^2), & |\varphi_a(x)|>\frac{1}{25}.\end{cases}$$

Therefore,

$$\int_{\{|\varphi_a(x)|\le\frac{1}{25}\}}|\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x \le \int_{\{|\varphi_a(x)|\le\frac{1}{25}\}}|\bar{D}f(x)|^2 g(x,a)\,\mathrm{d}B_x$$
$$\le \int_B|\bar{D}f(x)|^2 g(x,a)\,\mathrm{d}B_x$$

and

$$\int_{\{|\varphi_a(x)|>\frac{1}{25}\}}|\bar{D}f(x)|^2 g^p(x,a)\,\mathrm{d}B_x$$
$$\le 25^p\int_{\{|\varphi_a(x)|>\frac{1}{25}\}}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)^p\,\mathrm{d}B_x$$
$$\le 25^p\int_{B_1(0)}|\bar{D}f(x)|^2(1-|\varphi_a(x)|^2)^p\,\mathrm{d}B_x.$$