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Fischer Decomposition and Cauchy Kernel for Dunkl–Dirac operators

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This paper is dedicated to the memory of our friend and colleague Jarolim Bureš

Abstract. In this paper we present a Fischer decomposition for Dirac operator and an explicit construction of a Cauchy kernel for Dunkl-monogenic functions.

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1. Introduction

Classical hypercomplex function theory is strongly linked to rotation groups. One of the main properties of the Dirac operator is its invariance under rotations. In fact, there is a correspondence between monogenic functions and irreducible representations of Spin groups. But in many applications it would be advantageous to have a function theory which is based on reflection groups instead of rotation groups, for instance in the analysis of quantum many-body systems of Calogero–Moser–Sutherland type in mathematical physics [2] or in the study of the crystallographic Radon transform. However, there exists one major obstacle. While partial derivatives are invariant under rotations this is not the case for reflections. The way out seems to be to consider differential-difference operators [4], also called Dunkl operators. These operators are invariant under reflections and, additionally, are pairwise commuting.

This allowed the authors in [1] to introduce a Dirac operator based on such differential-difference operators which is invariant under reflection groups and also factorizes the Dunkl Laplacian. Starting from the latter operator they obtained a

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Stokes theorem, a Borel–Pompeiu formula and a Cauchy integral formula for the former operator. The authors also gave a method to construct a Cauchy kernel for a Dirac operator of a given finite reflection group, but it requires the explicit knowledge of the Dunkl transform which is known in a few particular cases only.

In this paper we will present another method of calculating the Cauchy kernel without prior knowledge of the Dunkl transform. To this end we prove a Fischer decomposition for Dunkl-monogenic functions. Although a Fischer decomposition already exists for Dunkl harmonic functions (see [4]) there is a fundamental difference between the hamonic and the monogenic case. While the basic building block for the decomposition into Dunkl harmonic polynomials, $|x|^2$, is invariant under reflection, i.e. $\Delta_h|x|^2 = \Delta|x|^2$, this is no longer true in the monogenic case. This requires a different treatment of the projectors onto the space of Dunkl monogenic polynomials of degree k as it will be shown in Section 4. Furthermore, we will show a bi-orthogonality between Dunkl-monogenic polynomials and usual monogenic polynomials. The study of the Fischer decomposition in Section 4 together with the existence of a reproducing kernel for it will allows us to construct a Cauchy kernel for the unit disk without the explicit knowledge of the Dunkl transform. Moreover, the universality of this kernel ensures its validity for any simply connected star-like domain.

2. Preliminaries

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ satisfying the anti-commutation relationships $e_ie_j + e_je_i = -2\delta_{i,j}$. We define the universal real valued Clifford algebra $\mathcal{C}\ell_{0,n}$ as the 2^n-dimensional associative algebra with basis given by $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset N = \{1, \ldots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Hence, each element $x \in \mathcal{C}\ell_{0,n}$ will be represented by $x = \sum_A x_Ae_A$, $x_A \in \mathbb{R}$.

In what follows, $\mathrm{sc}[x] = x_0$ will denote the scalar part of $x \in \mathcal{C}\ell_{0,n}$, while an element $x = (x_1, \cdots, x_n)$ of $\mathbb{R}^n$ will be identified with $x = \sum_{i=1}^n x_ie_i$. Also, we need the anti-involution $\bar{\cdot}$ defined by $\overline{e}_0 = e_0$, $\overline{e}_i = -e_i$ and $\overline{e_ie_k} = \overline{e}_k\overline{e}_i$. An important property of the algebra $\mathcal{C}\ell_{0,n}$ is that each non-zero vector $x \in \mathbb{R}^n$ has a multiplicative inverse given by $\frac{\overline{x}}{|x|^2} = \frac{-x}{|x|^2}$. Up to a minus sign this inverse corresponds to the Kelvin inverse of a vector in Euclidean space.

For all what follows let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$, whose complement contains a non-empty open set. Then a $\mathcal{C}\ell_{0,n}$-valued function f in Ω has a representation $f = \sum_A e_Af_A$, with components $f_A : \Omega \to \mathbb{R}$. Thus, notations such as $f \in C^k(\Omega, \mathcal{C}\ell_{0,n})$, $k \in \mathbb{N} \cup \{0\}$, and $f \in L_p(\Omega, \mathcal{C}\ell_{0,n})$, $1 \leq p$, will be understood co-ordinatewisely. For instance, $f \in L_p(\Omega, \mathcal{C}\ell_{0,n})$ means that $f_A \in L_p(\Omega)$ for all A. In the following we use the short notation $L_p(\Omega)$, $C^k(\Omega)$, etc., instead of $L_p(\Omega, \mathcal{C}\ell_{0,n})$, $C^k(\Omega, \mathcal{C}\ell_{0,n})$.

We now introduce the Dirac operator $\partial = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}$. In particular we have that $\partial^2 = -\Delta$, where Δ is the n-dimensional Laplacian. A function $f : \Omega \to \mathcal{C}\ell_{0,n}$

is said to be left-monogenic (resp. right-monogenic) if it satisfies the equation $(\partial f)(x) = 0$ (resp. $(f\partial)(x) = 0$) for each $x \in \Omega$. Basic properties of the Dirac operator and left-monogenic functions can be found in [6], [7] and [3].

3. Reflection Groups and Monogenicity

The reflection $\sigma_\nu(x)$ of a given vector $x \in \mathbb{R}^n$ with respect to the hyperplane orthogonal to $\nu \neq 0$ is given, in Clifford notation, by

$$\sigma_\nu x := -\nu x \nu^{-1}.$$

A root system R is a finite set of non-zero vectors in $\mathbb{R}^n$ such that $\sigma_\nu R = R$ and $R \cap \mathbb{R}\nu = \{\pm\nu\}$ for all $\nu \in R$. The Coxeter group $\mathcal{G}$ (or finite reflection group) generated by the root system R is the subgroup of the orthogonal group $O(n)$ generated by $\{\sigma_\nu : \nu \in R\}$. Standard examples are the groups A_{n-1} and B_n (see e.g. [5], [1]).

A multiplicity function κ_ν is a $\mathcal{G}$-invariant complex-valued function defined on R, i.e., $\kappa_\nu = \kappa_{g\nu}$ for all $g \in \mathcal{G}$. A positive subsystem R_+ is any subset of R satisfying $R = R_+ \cup (-R_+)$. This implies that R_+ and $-R_+$ are separated by a hyperplane passing through the origin.

For a chosen positive subsystem R_+ we introduce the index

$$\gamma_\kappa = \sum_{\nu \in R_+} \kappa_\nu$$

and the weight function

$$h_\kappa(x) = \Pi_{\nu \in R_+} | < \nu, x > |^{\kappa_\nu}.$$

For each fixed positive subsystem R_+ and multiplicity function κ_ν we have, as invariant operators, the differential-difference operators (also called Dunkl operators):

$$D_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\nu \in R_+} \kappa_\nu \frac{f(x) - f(\sigma_\nu x)}{< x, \nu >} \nu_j.$$

These operators are pairwise commutative: $D_i D_j = D_j D_i$. This property allows us to define a Dirac operator for Coxeter groups

$$Df = \sum_{j=1}^{n} e_j D_j f = \partial f + \sum_{\nu \in R_+} \kappa_\nu \nu \frac{f(x) - f(\sigma_\nu x)}{< x, \nu >}. \tag{3.1}$$

We would like to remark that the Dirac operator factorizes the so-called Dunkl Laplacian $D^2 = \Delta_h := D_1^2 + \ldots + D_n^2$ (see [4] and [5]).

A C^1 function which is annihilated by the Dirac operator D from the left (or right) will be called a left- (right-) Dunkl-monogenic function with respect to the corresponding Coxeter group. For simplicity we will restrict ourselves to the case of left-Dunkl-monogenic functions. In the sequel such a function will simply be called a Dunkl-monogenic function. From now on let $\mathrm{Re}\kappa_\nu \geq 0$ and let Ω be a sufficiently smooth domain, invariant under the action of $\mathcal{G}$, and $\Gamma = \partial\Omega$ its boundary.

Another important operator is the *intertwining operator* which allows to interchange the Dunkl derivatives with the usual partial derivatives. Let Π denote the space of homogeneous polynomials. Furthermore, let Π_k denote the space of homogeneous polynomials of degree k.

Lemma 3.1 ([5]). *If κ is such that $\cap_j \ker D_j = \mathbb{C}$, then it exists an unique linear isomorphism V_κ of Π, denoted as intertwining operator, which satisfies*

1. $V_\kappa(\Pi_k) = \Pi_k$;
2. $V_\kappa|_{\Pi_0} = id$;
3. $D_j V_\kappa = V_\kappa \partial_j$, *with* $V_\kappa(1) = 1$.

As an obvious consequence, we have $\Delta_h(V_\kappa f) = V_\kappa(\Delta f)$ and $D(V_\kappa f) = V_\kappa(\partial f)$.

4. Fischer Decomposition for Dunkl-Monogenic Functions

Let us start this section with the important observation that the previously defined Dirac operator is a homogeneous operator of degree -1. Indeed, it is easy to see that if f is a function homogeneous of degree k, then $f(\sigma_\nu(\lambda x)) = f(-\nu(\lambda x)\nu^{-1}) = \lambda^k f(-\nu x \nu^{-1}) = \lambda^k f(\sigma_\nu x)$, i.e. homogeneity is preserved under reflections. Furthermore, it means that the Dirac operator not only maps polynomials of degree k into polynomials of degree $k-1$, but homogeneous polynomials of degree k into homogeneous polynomials of degree $k-1$.

Now, let $\mathcal{M}_k$ denote the space of Dunkl-monogenic homogeneous polynomials of degree k. For two polynomials $p, q \in \Pi$ the Dunkl Fischer inner product with respect to our differential-difference operators is defined by:

$$[p, q]_h := \mathrm{sc}[(\overline{p(D)}\ q)(0)] \qquad p, q \in \Pi.$$

The Dunkl–Fischer inner product has the important property [5]:

$$[x_i p, q]_h = [p, D_i q]_h. \tag{4.1}$$

This property allows us to prove the following theorem:

Theorem 4.1. *For each $k \in \mathbb{N}$ we have*

$$\Pi_k = \mathcal{M}_k + x\Pi_{k-1}.$$

Moreover, the subspaces $\mathcal{M}_k$ and $x\Pi_{k-1}$ are orthogonal with respect to the Dunkl–Fischer inner product.

Proof. Because of $\Pi_k = x\Pi_{k-1} + (x\Pi_{k-1})^\perp$ it is enough to prove that $(x\Pi_{k-1})^\perp = \mathcal{M}_{k-1}$. For this, assume that for some $P_k \in \Pi_k$ we have

$$[xP_{k-1}, P_k]_h = 0$$

for arbitrary $P_{k-1} \in \Pi_{k-1}$.

Due to (4.1) we have $[P_{k-1}, -DP_k]_h = 0$ for all P_{k-1}. As $DP_k \in \Pi_{k-1}$ we obtain $DP_k = 0$ or $P_k \in \mathcal{M}_k$. This means that $(x\Pi_{k-1})^\perp \subset \mathcal{M}_{k-1}$. Now, let $P_k \in \mathcal{M}_k$. Then we have for each $P_{k-1} \in \Pi_{k-1}$

$$\begin{aligned} [xP_{k-1}, P_k]_h &= [P_{k-1}, -DP_k]_h \\ &= [P_{k-1}, 0]_h \\ &= 0 \end{aligned}$$

and, therefore, $(x\Pi_{k-1})^\perp = \mathcal{M}_{k-1}$. □

As a result we obtain the Fischer decomposition with respect to our Dirac operator D.

Theorem 4.2. Fischer decomposition *Let P_k be a homogeneous polynomial of degree k. Then*

$$P_k = M_k + xM_{k-1} + x^2 M_{k-2} + \ldots + x^k M_0$$

where each M_j is a homogeneous Dunkl-monogenic polynomial of degree j.

Let us remark that the Fischer decomposition can also be obtained from the Fischer decomposition for Dunkl-harmonic functions (see [4, p.178]); all we need to do is to decompose each Dunkl-harmonic polynomial H_k as $H_k = M_k + xM_{k-1}$. The dimension of the vector space of homogeneous Dunkl-monogenic polynomials of degree k is $\binom{n+k-2}{n-2}2^n$, just as in the usual monogenic case.

Also, for polynomials p, q in Π_k the standard Fischer inner product

$$< p, q >_k := sc[\overline{p(\partial)}q(0)]$$

has a reproducing kernel given by $\frac{<x,y>^k}{k!}$. Accordingly, the reproducing kernel for the Dunkl–Fischer inner product is given by $K_k(x, y) := \frac{1}{k!} V_\kappa[< \cdot, y >^k](x)$ (see [3], [4] for details).

The Almansi decomposition for Dunkl-polyharmonic polynomials is a special case of the results presented in [9], and also generalizes to the case of Dunkl-polymonogenic polynomials (*i.e.* polynomials in the kernel of D^l): the fact that $(x^l \Pi_{k-l})^\perp = \{P_k \in \Pi_k : \ D^l P_k = 0\}$ and the uniqueness of the Fischer decomposition above lead to the conclusion that $D^l P_k = 0$ if and only if $P_k = M_k + xM_{k-1} + x^2 M_{k-2} + \ldots + x^{l-1} M_{k-l+1}$, for $M_j \in \mathcal{M}_j$. Of course, this decomposition extends to real-analytic Dunkl-polymonogenic functions.

5. Monogenic Polynomials and Cauchy Kernels

In [1] the authors constructed the Cauchy integral formula for Dunkl monogenic functions and showed a method how to construct the Cauchy kernel based on the knowledge of the Dunkl transform. Unfortunately, the Dunkl transform is explicitly known only in some special cases. In this section we will show another way how to construct the Cauchy kernel based solely on the knowledge of the intertwining operator and the Dunkl-monogenics.

The starting point are the following identities.

Proposition 5.1. *For $p \in \Pi_k$ and $q \in \mathcal{M}_k$ it holds*

$$\overline{p(D)}q(x) = c_h \int_{\mathbb{R}^n} \overline{q} p h_\kappa^2 \exp(-|x|^2/2)dx = 2^k(\gamma_\kappa + \frac{n}{2})_k c'_h \int_{S^{n-1}} \overline{q} h_\kappa^2 d\sigma p$$

with $c_h := (\int_{\mathbb{R}^n} h_\kappa^2 \exp(-|x|^2/2)dx)^{-1}$ and $c'_h = \sigma_{n-1}(\int_{S^{n-1}} h_\kappa^2 d\sigma)^{-1}$ being normalizing constants.

Proof. For p and q as indicated we have that $\overline{p(D)}q(x)$ is constant. Hence,

$$\begin{aligned}\overline{p(D)}q(x) &= c_h \int_{\mathbb{R}^n} \overline{p(D)}q(x) h_\kappa^2 \exp(-|x|^2/2)dx \\ &= c_h \int_{\mathbb{R}^n} \overline{q(x)} p(D^*)(1) h_\kappa^2 \exp(-|x|^2/2)dx\end{aligned}$$

where D^* represents the adjoint Dirac operator acting on the weighted L_2-space $L_2(h_\kappa^2 dx, \mathbb{R}^n)$ as $D^*p(x) = xp(x) - Dp(x)$, for $p \in \Pi$. Applying recursively this relation, we get $p(D^*)(1) = p(x) + r(x)$, with r a polynomial with degree less than k, that is

$$\overline{p(D)}q(x) = c_h \int_{\mathbb{R}^n} \overline{q(x)}[p(x) + r(x)] h_\kappa^2 \exp(-|x|^2/2)dx.$$

As q is a Dunkl-monogenic homogeneous polynomial of degree k we have

$$\int_{\mathbb{R}^n} \overline{q(x)} r(x) h_\kappa^2 \exp(-|x|^2/2)dx = 0$$

thus proving the first equality. The second equality is obtained by a change to spherical co-ordinates

$$\begin{aligned}&\int_{\mathbb{R}^n} \overline{q(x)} p(x) h_\kappa^2 \exp(-|x|^2/2)dx = \\ &= \frac{1}{\sqrt{(2\pi)^n}} \int_0^\infty r^{2\gamma_\kappa+2k+n-1} e^{-r^2/2} dr \int_{S^{n-1}} \overline{q} h_\kappa^2 d\sigma p \\ &= 2^{k+\gamma_\kappa} \frac{\Gamma(\gamma_\kappa + k + \frac{n}{2})}{\sigma_{n-1}\Gamma(\frac{n}{2})} \int_{S^{n-1}} \overline{q} h_\kappa^2 d\sigma p\end{aligned}$$

and the relation between the normalizing constants

$$c'_h = 2^{\gamma_\kappa} \frac{\Gamma(\gamma_\kappa + \frac{n}{2})}{\Gamma(\frac{n}{2})} c_h.$$

□

Also, we need the following additional proposition, which is an adaptation of a similar proposition for spherical harmonics (see [4]):

Proposition 5.2. *Let $p \in \Pi_k$ and let q be a monogenic homogeneous polynomial of degree k. Then*

$$c_h \int_{S^{n-1}} \overline{p} h_\kappa^2 d\sigma V_\kappa q = \frac{\left(\frac{n}{2}\right)_k}{\left(\gamma_\kappa + \frac{n}{2}\right)_k \sigma_{n-1}} \int_{S^{n-1}} \overline{p} d\sigma q.$$

The proof of the above proposition is completely similar to the proof of Proposition 5.2.8 in [4] and, therefore, it will be omitted.

Furthermore, it holds:

Corollary 5.3. *Let $\{S_\alpha\}$ be an orthonormal basis of the space of monogenic polynomials of degree k. Then $\{V_\kappa S_\alpha\}$ and $\{S_\alpha\}$ are biorthogonal systems with respect to the weighted Dunkl inner product, i.e.*

$$\int_{S^{n-1}} \overline{(V_\kappa S_\alpha)} h_\kappa^2 d\sigma S_\beta = \frac{\left(\frac{n}{2}\right)_k}{\left(\gamma_\kappa + \frac{n}{2}\right)_k \sigma_{n-1}} \delta_{\alpha,\beta}.$$

Moreover, $\{V_\kappa S_\alpha\}$ is a basis of $\mathcal{M}_k$.

This corollary allows us to construct Dunkl monogenic polynomials without the explicit knowledge of the Dunkl transform, as they can be obtained from the knowledge of the weight function h_κ^2 and the theory of special functions.

Furthermore, these propositions provide us with an alternative method to construct a Cauchy kernel. Indeed, since $K_k(x,y) = \frac{1}{k!} V_\kappa[< \cdot, y >^k](x)$ is a reproducing kernel for the Dunkl–Fischer inner product in Π_k we have for a Dunkl monogenic homogeneous polynomial q of degree k that

$$\begin{aligned} q(y) &= [K_k(\cdot,y), q(\cdot)]_h \\ &= [(\mathrm{proj}_{\mathcal{M}_k} K_k)(\cdot,y), q(\cdot)]_h \\ &= 2^k (\gamma_\kappa + \frac{n}{2})_k c_h' \int_{S^{n-1}} \overline{q} h_\kappa^2 d\sigma (\mathrm{proj}_{\mathcal{M}_k} K_k)(\cdot, y) \end{aligned}$$

where $\mathrm{proj}_{\mathcal{M}_k}$ denotes the projector of $K_k(\cdot,y) \in \Pi_k$ onto $\mathcal{M}_k$. Taking into consideration that the intertwinning operator satisfies $D(V_\kappa f) = V_\kappa(\partial f)$ we have for the reproducing kernel of Dunkl monogenic homogeneous polynomials of degree k

$$P_k(x,y) = 2^k (\gamma_\kappa + \frac{n}{2})_k (\mathrm{proj}_{\mathcal{M}_k} K_k)(x,y).$$

Straightforward calculation gives P_k in terms of Gegenbauer polynomials (see also [8]):

$$\begin{aligned} P_k(x,y) &= \frac{\Gamma(\gamma_\kappa + \frac{n}{2} + 1)}{2^k \Gamma(\gamma_\kappa + \frac{n}{2} - k + 1)} |y|^k \\ V_\kappa &\left[(2\gamma_\kappa + k + n - 2) C_k^{\gamma_\kappa + \frac{n}{2} - 1}(t) + (2\gamma_\kappa + n - 2) \frac{\cdot \wedge y}{|\cdot \wedge y|} C_{k-1}^{\gamma_\kappa + \frac{n}{2}}(t) \right](x) \end{aligned} \tag{5.1}$$

with $t = \frac{<\cdot, y>}{|y|}$ and where $|y| \leq |x| = 1$. Hereby, we would like to remark that the term $(\gamma_\kappa + \frac{n}{2})_k$ forces the upper exponent of the Gegenbauer polynomials to be $\gamma_\kappa + \frac{n}{2} - 1$ and $\gamma_\kappa + \frac{n}{2}$ instead of the usual $\frac{n}{2} - 1$ and $\frac{n}{2}$.

Now, we obtain the following theorem for the (Dunkl–)Cauchy kernel.

Theorem 5.4. *The Cauchy kernel for Dunkl monogenic functions is given by*

$$C(x,y) = V_\kappa \left[\partial \frac{1}{|\cdot - y|^{2\gamma_\kappa + n - 2}} \right](x)$$

where again V_κ is the intertwining operator corresponding to the group under consideration.

Proof. From [3] we have the decomposition of the fundamental solution for the Laplace operator in terms of Gegenbauer polynomials C_k^ν of degree k associated to ν,

$$\begin{aligned} \frac{1}{|x-y|^{n-2}} &= (1-2t|y|+|y|^2)^{-(n-2)/2} \\ &= \sum_{k=0}^{\infty} |y|^k C_k^{(n-2)/2}(t) \qquad (5.2) \\ &= \sum_{k=0}^{\infty} |y|^k \left(\sum_{i\leq k/2} \frac{((n-2)/2)_{k-i}}{i!(k-2i)!}(-1)^i(2t)^{k-2i} \right) \end{aligned}$$

where $x=\xi\in S^{n-1}$, $y=|y|\omega$ and $t=<\xi,\omega>$. Moreover, each term in this sum is a homogeneous polynomial of degree k in y. Hence, we have that the auxiliary function

$$\begin{aligned} \frac{1}{|x-y|^{2\gamma_\kappa+n-2}} &= (1-2t|y|+|y|^2)^{-(\gamma_\kappa+(n-2)/2)} \\ &= (1+|y|^2)^{-(\gamma_\kappa+(n-2)/2)} \left(1-\frac{2t|y|}{1+|y|^2}\right)^{-(\gamma_\kappa+(n-2)/2)} \\ &= (1+|y|^2)^{-(\gamma_\kappa+(n-2)/2)} \sum_{k=0}^{\infty} \frac{(\gamma_\kappa+(n-2)/2)_k 2^k}{k!(1+|y|^2)^k}|y|^k t^k. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{|x-y|^{2\gamma_\kappa+n-2}} &\leq (1+|y|^2)^{-(\gamma_\kappa+(n-2)/2)} \left(1-\frac{2t|y|}{1+|y|^2}\right)^{-(\gamma_\kappa+(n-2)/2)} \\ &\leq (1+|y|^2)^{-(\gamma_\kappa+(n-2)/2)} \end{aligned}$$

one has that $V_\kappa\left(\frac{1}{|\cdot-y|^{2\gamma_\kappa+n-2}}\right)(x)$ is defined and continuous for $|x|\leq 1$. Since $|y|\leq 1$, the result follows from $C(x,y)=\sum_{k=0}^{\infty} P_k(x,y)$, expression (5.1) and the expression of the classical kernel in terms of Gegenbauer polynomials. □

Remark 5.5. While in the above theorems the Cauchy kernel is in fact only defined for the unit ball it can be obviously extended to any simply connected star-like (with respect to the origin) domain Ω.

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