Received: 20 May 2017
DOI: 10.1002/mma.4614

RESEARCH ARTICLE

WILEY

Detection of C^2-singularities using uniform spline approximations

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Communicated by: K. Gürlebeck

MOS Classification: 41A15; 41A27; 41A36; 41A40

For the detection of C^2-singularities, we present lower estimates for the error in Schoenberg variation-diminishing spline approximation with equidistant knots in terms of the classical second-order modulus of smoothness. To this end, we investigate the behaviour of the iterates of the Schoenberg operator. In addition, we show an upper bound of the second-order derivative of these iterative approximations. Finally, we provide an example of how to detect singularities based on the decay rate of the approximation error.

KEYWORDS

inverse theorem, Schoenberg operator, singularity detection, spline approximation

1 | INTRODUCTION

The detection of singularities is one of the central topics in many areas of applications such as signal and image processing, computer-aided geometric design and tomography; see, eg, previous studies.[1-7] Even more general, the problem of an efficient estimation of the local regularity of a function is necessary for the choice of suitable numerical algorithms. Since most algorithms work stable with C^2-smooth data, it is paramount to be able to detect C^2-singularities. Our motivation comes from the development of a method to detect such singularities for a robust estimation of the digital curvature of piecewise smooth curves in images; see Nagler.[8] To this end, we use lower estimates in terms of moduli of smoothness. As usual, only discrete data are available; we provide an effective method that combines these estimates with smooth approximations of the data. More concretely, we look at this problem using variation-diminishing uniform spline approximations. While there are more general approaches using decay rates of coefficients in some basis expansion, here, we are looking for a fast and stable method that allows us to get also concrete constants of the estimates for numerical evaluation. Besides, the variation-diminishing property guarantees that no additional oscillations are introduced in the approximation.

1.1 | Schoenberg variation-diminishing spline operator

To introduce the spline operator, we consider integers $n, k > 0$, equidistant knots $\{x_j = \frac{j}{n}\}_{j=0}^{n}$ that induce a partition of $[0, 1]$ and the extended knot sequence $\Delta_n = \{x_j\}_{j=-k}^{n+k}$, where

$$x_{-k} = \cdots = x_0 = 0 < x_1 < \cdots < x_n = \cdots = x_{n+k} = 1.$$

Definition 1. The variation-diminishing Schoenberg operator of degree k with respect to the knots $\{x_j\}_{j=-k}^{n+k}$ is defined for $f \in C([0, 1])$ by

$$S_{n,k} f(x) := \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x), \quad x \in [0, 1],$$

with the Greville nodes, see the supplement in Schoenberg,[9]

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}, -k \le j \le n-1,$$

and the normalized B-splines

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

Hereby, for $f \in C([0,1])$ and points $y_0, \ldots, y_k \in [0,1]$, the divided difference $[y_0, \ldots, y_k]f$ is defined to be the coefficient of x^k in the unique polynomial of degree k or less that interpolates f at the points $y_0, \ldots, y_k$. The placeholder notation $\cdot$ is used to indicate that the divided difference is applied to the function $(t-x)_+^k$, where $x \in [0,1]$ is fixed. Note that to guarantee that the the Schoenberg operator can be evaluated on the whole interval $[0,1]$, the B-splines are chosen here to be right-continuous at the knots $x_1, \ldots, x_{n-1}$, while at the point $x_n = 1$, they are chosen to be left-continuous; see de Boor[10] for more details. Also note that k denotes the degree of the spline and not the order as often used in the literature. Therefore, the splines considered here are for all $k > 0$ continuous functions.

This spline operator samples a continuous function at the so called Greville nodes, named after T.N.E Greville who introduced these nodes, and yields a variation-diminishing smooth approximation of this function by a linear combination of the splines basis functions, where the smoothness is depending on the degree of the spline. The Schoenberg operator reproduces constants since the normalized B-splines form a partition of unity

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1. \tag{1}$$

Moreover, the Schoenberg operator can reproduce linear functions, ie,

$$\sum_{j=-k}^{n-1} \xi_{j,k} N_{j,k}(x) = x, \tag{2}$$

due to the Greville nodes. For more properties of this operator, see, eg, Schoenberg,[9] Marsden and Schoenberg,[11] and Marsden.[12] We note that the reference for the Greville nodes and the Schoenberg operator is dated by 1967, while the conference where the result has first been published has been held in 1965. A comprehensive overview of direct inequalities for this operator can be found in Beutel et al.[13]

1.2 | Lower estimates

One aim of this article is to establish lower estimates in terms of classical moduli of smoothness of the following form: There exists constants $C_1, C_2 > 0$ independent on n such that

$$C_1 \cdot \omega_2(f, \delta_n) \le \left\| S_{n,k} f - f \right\|$$

holds for all $f \in C([0,1])$ and $\delta_n \to 0$ for $n \to \infty$, where the second-order modulus of smoothness $\omega_2 : C([0,1]) \times \left(0, \frac{1}{2}\right] \to [0, \infty)$ is defined by

$$\omega_2(f,t) := \sup_{0<h<t} \sup_{x \in [0,1-2h]} |f(x) - 2f(x+h) + f(x+2h)|;$$

see Butzer and Berens[14] and Johnen and Scherer.[15] Although there exist already several methods to derive such estimates for positive linear operators, see, eg, Ditzian and Ivanov,[16] Knoop and Zhou,[17,18] and Totik,[19] these methods still require many restrictions and are not applicable for Schoenberg operator.

Error estimates of uniform spline approximations on a finite interval have already been discussed amongst others by de Boor[20] using the first-order modulus of continuity and have been later improved by Marsden.[21] These estimates provide naturally a lower bound for the approximation with Schoenberg variation-diminishing splines on uniformly distributed knots. As the second-order modulus of smoothness annihilates constant and linear functions, it better reflects the behaviour of the spline approximation error. Thus, lower estimates in terms of the second-order modulus of smoothness may lead to sharper bounds of the approximation error and are more suitable to detect C^2-singularities. Furthermore, there exists upper bounds for the approximation error with the second-order modulus of smoothness, see eg, Esser.[22]

Recently, such estimates have been shown in Nagler et al[23] and Zapryanova and Tachev,[24] while the results in both articles do not provide computable constants. In this article, we show lower estimates of the uniform spline approximation error in terms of the second-order modulus of smoothness. Furthermore, we provide computable constants.

To prove lower estimates, we first characterize the behaviour of the iterates of Schoenberg spline operator and show an upper bound for the second-order derivative of these iterative spline approximations. The essential idea depends on uniformly distributed knots and the resulting translation invariant basis functions in the interior of the interval. Finally, we provide an illustrative example of this approach to detect singularities of a piecewise smooth function using only a limited number of samples.

1.3 | Notation

Throughout this paper, we will consider the Banach space $C([0,1])$, ie, the space of real-valued continuous functions on the interval $[0,1]$, endowed with the supremum norm $\|\cdot\|_{[0,1]}$,

$$\|f\|_{[0,1]} = \sup\{|f(x)| \ : \ x \in [0,1]\}, \qquad f \in C([0,1]).$$

The space of bounded linear operators on $C([0,1])$ will be denoted by $\mathcal{B}(C([0,1]))$ equipped with the usual operator norm $\|\cdot\|_{op}$.

By $\mathcal{S}(n,k)$, we denote the spline space of degree k with respect to the knot sequence $\Delta_n = \{x_j\}_{j=-k}^{n+k}$,

$$\mathcal{S}(n,k) := \left\{ \sum_{j=-k}^{n-1} c_j N_{j,k} \ : \ c_j \in \mathbb{R},\ j \in \{-k, \dots, n-1\} \right\} \subset C^{k-1}([0,1]).$$

The spline space is an $n+k$-dimensional subspace of $C([0,1])$, as the $n+k$ basis functions $N_{j,k}$ are linearly independent. Since $\mathcal{S}(n,k)$ is finite-dimensional, $\mathcal{S}(n,k)$ is a Banach space with the inherited norm $\|\cdot\|_{[0,1]}$. The maximal distance between 2 knots will be denoted by the mesh gauge $\|\Delta_n\| := 1/n$. For more information on spline spaces and spline approximations, we refer to, eg, de Boor.[10]

2 | THE ITERATES OF THE SCHOENBERG OPERATOR

In the following, we discuss some basic properties of the iterates of the Schoenberg operator. For $m \in \mathbb{N}$, we define

$$(S_{n,k}^m f) := (S_{n,k}^{m-1}(S_{n,k} f)).$$

Lemma 1. *We can write the mth iterate of the Schoenberg operator as*

$$\begin{aligned} S_{n,k}^m f &= S_{n,k}^{m-1}\left(\sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}\right) \\ &= \sum_{j_1,\dots j_m=-k}^{n-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k}. \end{aligned}$$

Proof. Induction over m. □

2.1 | The first and second derivative of the iterates

In this section, we consider the derivatives and give explicit representations. For that, we define a discrete backward difference operator ∇_l by

$$\nabla_l f(\xi_{j,k}) := \frac{f(\xi_{j,k}) - f(\xi_{j-1,k})}{\xi_{j,l} - \xi_{j-1,l}}.$$

This operator is only defined at the discrete evaluations of f at the nodes and can be seen as the usual backward difference operator defined at these nodes but additionally weighted by the node difference of degree l. With this, we can state:

Lemma 2. *Let $f \in C([0,1])$. The first and the second derivative of the approximation of f with the Schoenberg operator can be represented by*

$$DS_{n,k} f = \sum_{j=1-k}^{n-1} \nabla_k f(\xi_{j,k}) N_{j,k-1}$$

and

$$D^2 S_{n,k} f = \sum_{j=2-k}^{n-1} \nabla_{k-1} \nabla_k f(\xi_{j,k}) N_{j,k-2},$$

respectively.

Proof. This lemma follows directly from the representation of the appropriate derivatives given by Marsden.[12] Using lemma 1 on page 32, we obtain for the first derivative

$$DS_{n,k} f = \sum_{j=1-k}^{n-1} \frac{f(\xi_{j,k}) - f(\xi_{j-1,k})}{\xi_{j,k} - \xi_{j-1,k}} N_{j,k-1},$$

while the second-order derivative can be written by Lemma 2 on page 35 as

$$D^2 S_{n,k} f = \sum_{j=2-k}^{n-1} \frac{\frac{f(\xi_{j,k}) - f(\xi_{j-1,k})}{\xi_{j,k} - \xi_{j-1,k}} - \frac{f(\xi_{j-1,k}) - f(\xi_{j-2,k})}{\xi_{j-1,k} - \xi_{j-2,k}}}{\xi_{j,k-1} - \xi_{j-1,k-1}} N_{j,k-2}.$$

Applying the definition of the discrete backward difference operator ∇_k and ∇_{k-1} gives the required representation. □

Now, we give an analogous representation for the iterates of the Schoenberg operator.

Corollary 1. *Let $f \in C([0,1])$. The first- and second-order derivative of $S^m_{n,k} f$ is of the form*

$$DS^m_{n,k} f = \sum_{j_m=1-k}^{n-1} \sum_{j_1, \dots j_{m-1}=-k}^{n-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_k N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-1}$$

and

$$D^2 S^m_{n,k} f = \sum_{j_m=2-k}^{n-1} \sum_{j_1, \dots j_{m-1}=-k}^{n-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_{k-1} \nabla_k N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-2}.$$

Proof. Applying Lemmas 1 and 2 to $S^{m-1}_{n,k} f$ yields the result. □

2.2 | An upper bound for the second derivative of the iterates

Now, we proceed to show that the above given representation of the first- and second-order derivative of $S^m_{n,k} f$ is equivalent to other representations and the number of these equivalent representations depends on m. This will allow us to use the arithmetic mean of these representations for our purposes.

Hereby, our idea is to work with the shift invariant basis functions $N_{j,k}, j \in \{0, \dots, n-k-1\}$, ie, to stay away from the boundary of the interval $[0,1]$. Then, the Schoenberg operator acts like a convolution operator and techniques for this kind of operators can be applied. We will show that the backward difference operator as placed in Corollary 1 commutes with the discrete evaluation of the spline basis functions at the nodes and, as consequence, can be also applied to all of the values $N_{j_1,k}(\xi_{j_2,k}), \dots, N_{j_{m-1},k}(\xi_{j_m,k})$. Note that this technique is only possible in the case of uniformly distributed knots.

To this end, let $x \in I_{\Delta_n,k} := [x_{2k+2}, x_{n-2k-2}]$. Then, we have

$$x \notin \bigcup_{j=-k}^{k+1} \operatorname{supp} N_{j,k} \text{ and } x \notin \bigcup_{j=n-2k-2}^{n-1} \operatorname{supp} N_{j,k}, \tag{3}$$

because supp $N_{j,k} \subset [x_j, x_{j+k+1}]$. Besides, we can simplify the notation of the iterates of the Schoenberg operator for $x \in I_{\Delta_n,k}$ to

$$S^m_{n,k} f(x) = \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k}(x),$$

ie, we do not need to evaluate the sums outside of the interval $[x_{2k+2}, x_{n-2k-2}]$. This is useful for the calculation with derivatives, as we can avoid to increase the lower index by the order of the differentiation. Now, we show that the basis functions $\{N_{j,k}\}_{j=0}^{n-k-1}$ are translates of each other with respect to the Greville nodes. This idea goes back to Schoenberg,[25] where a basis of translates of each other has been considered on uniform knots.

Lemma 3. *The $N_{j,k}$ with $j \in \{0, \dots, n-k-1\}$ is shift invariant at the nodes, in particular,*

$$N_{j+1,k}(\xi_{i,k}) = N_{j,k}(\xi_{i-1,k}),$$

and supp span$_{j\in\{0,\dots,n-k-1\}}$ $N_{j,k} = [0,1]$.

Proof. As supp $N_{j,k} \subset [x_j, x_{j+k+1}]$ all corresponding knots x_i, $i \in \{j, \dots, j+k+1\}$ are distinct from each other. Explicitly, we have $x_i = \frac{i}{n}$. Now, let $h = 1/n$. Then, we get

$$\begin{aligned}
N_{j+1,k}(\xi_i) &= (x_{j+k+2} - x_{j+1})[x_{j+1}, \dots, x_{j+k+2}](\cdot - \xi_i)^k_+ \\
&= (x_{j+k+1} - x_j)\frac{1}{h^k \cdot k!} \sum_{l=j+1}^{j+k+2} \binom{k+1}{l-j-1} (-1)^{j+k+2-l}(x_l - \xi_i)^k_+ \\
&= (x_{j+k+1} - x_j)\frac{1}{h^k \cdot k!} \sum_{l=j}^{j+k+1} \binom{k+1}{l-j} (-1)^{j+k+1-l}(x_{l+1} - \xi_i)^k_+ \\
&= (x_{j+k+1} - x_j)\frac{1}{h^k \cdot k!} \sum_{l=j}^{j+k+1} \binom{k+1}{l-j} (-1)^{j+k+1-l}(x_l - \xi_{i-1})^k_+ \\
&= N_{j,k}(\xi_{i-1}).
\end{aligned}$$

The last line holds, because

$$x_{l+1} - \xi_i = \frac{k \cdot (l+1) - \sum_{j=1}^k (i+j)}{nk} = \frac{k \cdot l - \sum_{j=1}^k (i+j-1)}{nk} = x_l - \xi_{i-1}.$$

Finally, note that supp $N_{0,k} = [x_0, x_{k+1}]$ and supp $N_{n-k-1,k} = [x_{n-k-1}, x_n]$. □

With Lemma 3, we can represent the first derivative of the mth iterate in $m-1$ equivalent ways and the second derivative in $\frac{m(m-1)}{2}$ ways.

Theorem 1. *For $m \in \mathbb{N}$ and $x \in I_{\Delta_n,k}$, we get*

$$\begin{aligned}
DS^m_{n,k} f(x) &= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_k N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-1}(x) \\
&= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots \nabla_k N_{j_{m-2},k}(\xi_{j_{m-1},k}) N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-1}(x) \\
&\qquad \vdots \\
&= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) \nabla_k N_{j_1,k}(\xi_{j_2,k}) N_{j_2,k}(\xi_{j_3,k}) \cdots N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-1}(x),
\end{aligned}$$

ie, the backward difference operator can be applied to $N_{j,k}$ for every index j. Thus, we have $m-1$ possibilities to represent the first derivative of the iterated Schoenberg operator applied to the function f.

Regarding the term $D^2 S^m_{n,k} f$, we can proceed in the same way, ie,

$$\begin{aligned}
D^2 S^m_{n,k} f(x) &= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_{k-1} \nabla_k N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-2}(x) \\
&= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots \nabla_{k-1} N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_k N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-2}(x) \\
&\qquad \vdots \\
&= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots \nabla_{k-1} N_{j_l,k}(\xi_{j_{l+1},k}) \cdots \nabla_k N_{j_s}(\xi_{j_m}) \cdots N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-2}(x) \\
&\qquad \vdots \\
&= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) \nabla_{k-1} \nabla_k N_{j_1,k}(\xi_{j_2,k}) N_{j_2,k}(\xi_{j_3,k}) \cdots N_{j_{m-1}}(\xi_{j_m}) N_{j_m,k-2}(x).
\end{aligned} \tag{4}$$

Similar to $DS^m_{n,k} f$, we have $\frac{m(m-1)}{2}$ possibilities to represent the second derivative of the mth iterate of the Schoenberg operator applied to the function f.

Proof. We will only show that for a given m, it is equivalent to apply the discretely defined backward difference operator ∇_k to $N_{j_{m-1},k}$ or to $N_{j_{m-2},k}$. Concretely, we will show that the equality

$$DS^m_{n,k} f(x) = \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_k N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-1}(x) \tag{5}$$

$$= \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots \nabla_k N_{j_{m-2},k}(\xi_{j_{m-1},k}) N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-1}(x) \tag{6}$$

holds. It is then easy to see that also the other cases hold true.

As we have uniformly placed knots and we only look at the basis functions that are translation invariant with respect to the nodes, the backward difference operator ∇_k applied to $N_{j,k}(\xi_{i,k})$ is independent on k and simplifies to

$$\nabla_k N_{j,k}(\xi_{i,k}) = \frac{N_{j,k}(\xi_{i,k}) - N_{j,k}(\xi_{i-1,k})}{\xi_{i,k} - \xi_{i-1,k}} = \|\Delta_n\|^{-1} \left(N_{j,k}(\xi_{i,k}) - N_{j,k}(\xi_{i-1,k})) \right),$$

and by the same argument, we get

$$\nabla_{k-1} \nabla_k N_{j,k}(\xi_{i,k}) = \nabla_k \nabla_{k-1} N_{j,k}(\xi_{i,k}) = \|\Delta_n\|^{-2} \left(N_{j,k}(\xi_{i,k}) - 2N_{j,k}(\xi_{i-1,k}) + N_{j,k}(\xi_{i-2,k}) \right).$$

Applying Lemma 3 to the concrete situation given here in (5), we get the difference of the B-spline $N_{j_{m-1},k}$ evaluated at $\xi_{j_m,k}$:

$$\nabla_k N_{j_{m-1},k}(\xi_{j_m,k}) = \|\Delta_n\|^{-1} \left(N_{j_{m-1},k}(\xi_{j_m,k}) - N_{j_{m-1}+1,k}(\xi_{j_m,k}) \right).$$

Therefore, the first line in (5) can be rewritten as

$$\begin{aligned}
&\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) \nabla_k N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-1}(x) \\
&= \|\Delta_n\|^{-1} \Bigg(\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k-1}(x) \\
&\quad - \sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1},k}) N_{j_{m-1}+1,k}(\xi_{j_m,k}) N_{j_m,k-1}(x) \Bigg).
\end{aligned}$$

Using an index shift regarding the index j_{m-1} and the relation $N_{j_{m-2},k}(\xi_{j_{m-1},k})N_{j_{m-1}+1,k}(\xi_{j_m,k}) = N_{j_{m-2},k}(\xi_{j_{m-1}-1,k})N_{j_{m-1},k}(\xi_{j_m,k})$, we further obtain

$$\begin{aligned}
&\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots N_{j_{m-2},k}(\xi_{j_{m-1},k})\nabla_k N_{j_{m-1},k}(\xi_{j_m,k})N_{j_m,k-1}(x)\\
&\quad= \|\Delta_n\|^{-1}\Bigg(\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots N_{j_{m-2},k}(\xi_{j_{m-1},k})N_{j_{m-1},k}(\xi_{j_m,k})N_{j_m,k-1}(x)\\
&\qquad-\sum_{j_{m-1}=1}^{n-k}\sum_{j_1,\dots j_{m-2}j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots N_{j_{m-2},k}(\xi_{j_{m-1}-1,k})N_{j_{m-1},k}(\xi_{j_m,k})N_{j_m,k-1}(x)\Bigg)\\
&\quad=\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots \nabla_k N_{j_{m-2},k}(\xi_{j_{m-1},k})N_{j_{m-1},k}(\xi_{j_m,k})N_{j_m,k-1}(x)\\
&\qquad+\|\Delta_n\|^{-1}\Bigg(\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots N_{j_{m-2},k}(\xi_{0,k})N_{0,k}(\xi_{j_m,k})N_{j_m,k-1}(x)\Bigg)\\
&\qquad-\|\Delta_n\|^{-1}\Bigg(\sum_{j_1,\dots j_m=0}^{n-k-1} f(\xi_{j_1,k})N_{j_1,k}(\xi_{j_2,k})\cdots N_{j_{m-2},k}(\xi_{n-k,k})N_{n-k,k}(\xi_{j_m,k})N_{j_m,k-1}(x)\Bigg).
\end{aligned}$$

The last 2 sums vanish as $x \in I_{\Delta_n,k} = [x_{2k+2}, x_{n-2k-2}]$ and $\xi_{0,k} \notin \operatorname{supp}(N_{j,k})$ for $j \in \{k+2, \dots, n-2k-2\}$ and $\xi_{n-k,k} \notin \operatorname{supp}(N_{j,k})$ for $j \in \{k+2, \dots, n-2k-2\}$; see (3). Hence, the last line reduces to (6), and the proof is complete. □

In the following, we are interested in an upper bound of the second-order derivative of the iterates that guarantees the convergence of the series.

For the convenience of the reader, we will abbreviate the right-hand side in (4) by

$$D^2\, S^m_{n,k} f(x) = \sum_{j_1,\dots j_m=-k}^{n-1} f(\xi_{j_1,k})\cdot P(j_1,\dots,j_m;x)\cdot I_{l_1,l_2}(j_1,\dots,j_{m-1};x), \tag{7}$$

where

$$P(j_1,\dots,j_m;x) := \left[\prod_{l=1}^{m-1} N_{j_l,k}(\xi_{j_{l+1},k})\right] N_{j_m,k-2}(x),$$

and for $l_1, l_2 \in \{1,\dots,m-1\}$, $l_1 \le l_2$,

$$I_{l_1,l_2}(j_1,\dots,j_{m-1};x) := \begin{cases} \dfrac{\nabla_{k-1}N_{l_1,k}(x)\cdot\nabla_k N_{l_2,k}(x)}{N_{l_1,k}(x)\cdot N_{l_2,k}(x)}, & \text{for } l_1 \ne l_2,\\[2ex] \dfrac{\nabla_{k-1}\nabla_k N_{l_1,k}(x)}{N_{l_1,k}(x)}, & \text{for } l_1 = l_2. \end{cases}$$

The term I_{l_1,l_2} sets the position of the 2 backward difference operators such that ∇_k is always applied in front of ∇_{k-1} as $l_1 \le l_2$. With this notation, we are now able to give an upper bound for the second-order derivative of the iterated spline approximation:

Theorem 2. *Let $f \in C([0,1])$ and let $n > 0, k \ge 3, m > 1$ be fixed integers. The pointwise upper bound*

$$\left|D^2 S^m_{n,k} f(x)\right| \le \frac{2\varepsilon_{n,k}}{(m-1)^{3/2}\,\|\Delta_n\|}\cdot\|f\|_{[0,1]} \tag{8}$$

holds for $x \in I_{\Delta_n,k}$, where

$$\varepsilon^2_{n,k} := \sup_i \sum_{j=-k}^{n-1} \frac{(N_{j,k}(\xi_{i,k}) - 2N_{j,k}(\xi_{i-1,k}) + N_{j,k}(\xi_{i-2,k}))^2}{\overline{N}_{j,k}(\xi_{i,k})} \tag{9}$$

with

$$\overline{N}_{j,k}(\xi_{i,k}) := \begin{cases} N_{j,k}(\xi_{i,k}), & \text{if } N_{j,k}(\xi_{i,k}) \neq 0, \\ 1, & \text{if } N_{j,k}(\xi_{i,k}) = 0. \end{cases}$$

Remark 1. We introduce the modified B-splines $\overline{N}_{j,k}$ to avoid formal 0 divisions.

Proof. First, suppose that $x \in I_{\Delta_n,k}$. Then, by Theorem 1, we have $\frac{m(m-1)}{2}$ possibilities to express $D^2 S^m_{n,k} f(x)$. This allows us to write (7) as the following mean:

$$\begin{aligned} D^2\, S^m_{n,k} f(x) &= \frac{1}{(m-1)^2} \sum_{l_1 \le l_2 = 1}^{m-1} D^2\, S^m_{n,k} f(x) \\ &= \frac{1}{(m-1)^2} \sum_{j_1, \dots j_m = 0}^{n-k-1} \left(f(\xi_{j_1,k}) \cdot P(j_1, \dots, j_m; x) \cdot \sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2}(j_1, \dots, j_{m-1}; x) \right). \end{aligned}$$

Since P is positive, we can split P into $P = P^{1/2} P^{1/2}$, where $P^{1/2}$ is the positive root. Then, we apply the Cauchy-Schwarz inequality and get in abbreviated notation the following pointwise inequality for $x \in [x_{2k+2}, x_{n-2k-2}]$:

$$\begin{aligned} \left| D^2 S^m_{n,k} f \right| &\le \frac{1}{(m-1)^2} \left\{ \sum_{j_1, \dots j_m = 0}^{n-k-1} |f|^2 P \right\}^{\frac{1}{2}} \left\{ \sum_{j_1, \dots j_m = 0}^{n-k-1} P \left(\sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2} \right)^2 \right\}^{\frac{1}{2}} \\ &\le \frac{1}{(m-1)^2} \left(\|f\|_{[0,1]} \cdot 1 \right) \left(\sum_{j_1, \dots j_m = 0}^{n-k-1} P \left(\sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2} \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

Here, we used the partition of unity property of the B-splines, namely, that $\sum_{j=-k}^{n-1} N_{j,k}(x) = 1$ holds for all $x \in [0,1]$. Summation by parts, beginning with $j_1, j_2, \dots$, leads to

$$\sum_{j_1, \dots j_m = 0}^{n-k-1} P(j_1, \dots, j_m; x) = \sum_{j_m = 0}^{n-k-1} N_{j_m,k-2}(x) \sum_{j_{m-1} = 0}^{n-k-1} N_{j_{m-1},k}(\xi_{j_m,k}) \cdots \sum_{j_1 = 0}^{n-k-1} N_{j_1,k}(\xi_{j_2,k}) = 1.$$

Finally, we take the supremum norm of f and obtain the inequality used for the first term.

Next, we discuss the second product in (10). For the term $\left(\sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2} \right)^2$, we get formally

$$\left(\sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2} \right)^2 = \sum_{l=1}^{m-1} I_{l,l}^2 + \sum_{\substack{1 \le l_1 < l_2 \le m \\ 1 \le s_1 \le s_2 \le m}} I_{l_1,l_2} I_{s_1,s_2}.$$

Here, the first expression sums up all the squared values, where both backward difference operators are applied at the same position, while in the last sum, there is at least 1 index not equal to the others.

We show that the last sum vanishes. Note that for any indices $i, j \in \{0, \dots, n-k-1\}$, we have

$$\sum_{j=-k}^{n-1} \nabla_k N_{j,k}(\xi_i) = \frac{1}{\|\Delta_n\|} \left(\sum_{j=-k}^{n-1} N_{j,k}(\xi_{i,k}) - \sum_{j=-k}^{n-1} N_{j,k}(\xi_{i-1,k}) \right) = 0, \tag{11}$$

and similarly,

$$\sum_{j=-k}^{n-1} \nabla_{k-1} \nabla_k N_{j,k}(\xi_i) = \frac{1}{\|\Delta_n\|^2} \left(\sum_{j=-k}^{n-1} N_{j,k}(\xi_{i,k}) - 2 \sum_{j=-k}^{n-1} N_{j,k}(\xi_{i-1,k}) + \sum_{j=-k}^{n-1} N_{j,k}(\xi_{i-2,k}) \right) = 0, \tag{12}$$

because of the partition of unity (1). That means, if the difference operator ∇_k or $\nabla_{k-1}\nabla_k$ is applied to $N_{j,k}$ without being squared, the whole sum vanishes. Concretely, let us consider the positions $1 \le s = s_1 = s_2 < l_1 < l_2 \le m$ of the

backward difference operators. We obtain

$$\begin{aligned}
&\sum_{j_1,\dots j_m=0}^{n-k-1} P(j_1,\ \dots\ ,j_m;x) I_{l_1,l_2}(j_1,\ \dots\ ,j_m;x) I_{s_1,s_2}(j_1,\ \dots\ ,j_m;x) \\
&\quad = \sum_{j_m=0}^{n-k-1} N_{j_m,k-2}(x) \cdots \sum_{j_{l_2}=-k}^{n-1} \nabla_k N_{j_{l_2},k}(\xi_{j_{l_2}+1,k}) \cdots \sum_{j_{l_1}=-k}^{n-1} \nabla_{k-1} N_{j_{l_1},k}(\xi_{j_{l_1}+1,k}) \\
&\qquad \cdots \sum_{j_s=-k}^{n-1} \nabla_{k-1}\nabla_k N_{j_s,k}(\xi_{j_s+1,k}) \cdots \sum_{j_1=0}^{n-k-1} N_{j_1,k}(\xi_{j_2,k}) = 0.
\end{aligned}$$

For the case $1 \le s = s_1 = s_2 = l_1 < l_2 \le m$, we get in the same way using (11) and (12)

$$\begin{aligned}
&\sum_{j_1,\dots j_m=0}^{n-k-1} P(j_1,\ \dots\ ,j_m;x) I_{l_1,l_2}(j_1,\ \dots\ ,j_m;x) I_{s_1,s_2}(j_1,\ \dots\ ,j_m;x) \\
&\quad = \sum_{j_m=0}^{n-k-1} N_{j_m,k-2}(x) \cdots \sum_{j_{l_2}=-k}^{n-1} \nabla_k N_{j,k}(\xi_i) \cdots \sum_{j_s=-k}^{n-1} \frac{\nabla_{k-1}\nabla_k N_{j_s,k}(\xi_{j_s+1,k}) \cdot \nabla_{k-1} N_{j_s,k}(\xi_{j_s+1,k})}{N_{j_s,k}(\xi_{j_s+1,k})} \\
&\qquad \cdots \sum_{j_1=0}^{n-k-1} N_{j_1,k}(\xi_{j_2,k}) = 0.
\end{aligned}$$

Thus, the only case where the sum does not vanish is whenever $1 \le l_1 = l_2 = s_1 = s_2 \le m$ holds, and these terms are included in the first sum. With these results, we conclude that

$$\sum_{j_1,\dots j_m=0}^{n-k-1} P\left(\sum_{l_1 \le l_2 = 1}^{m-1} I_{l_1,l_2}\right)^2 = \sum_{j_1,\dots j_m=0}^{n-k-1} P \sum_{l=1}^{m-1} I_{l,l}^2 \le \frac{(m-1)\varepsilon_{n,k}^2}{\|\Delta_n\|^4} \sum_{j_1,\dots j_{m-1}=0}^{n-k-1} P = \frac{(m-1)\varepsilon_{n,k}^2}{\|\Delta_n\|^4}.$$

Finally, we continue (10) and get for all $x \in I_{\Delta_n,k}$ the final inequality

$$\begin{aligned}
\left|D^2 S_{n,k}^m f(x)\right| &\le \frac{2}{(m-1)^2} \|f\|_{[0,1]} \left((m-1)\frac{\varepsilon_{n,k}^2}{\|\Delta_n\|^4}\right)^{\frac{1}{2}} \\
&\le \frac{2\varepsilon_{n,k}}{(m-1)^{3/2}\|\Delta_n\|^2} \cdot \|f\|_{[0,1]}.
\end{aligned}$$

In the definition (9) of $\varepsilon_{n,k}^2$, the terms $\overline{N}_{j,k}(\xi_{l,k})$ are formally needed to avoid division by 0 that can occur with the usual values of the B-splines $N_{j,k}(\xi_{l,k})$. □

We remark that $\varepsilon_{n,k}$ defined in (9) is bounded for $n \to \infty$ because of the compact support of the basis functions. Therefore, for fixed i, there are only finitely many B-splines $N_{j,k}$ that can be evaluated at the positions $\xi_{i,k}$, $\xi_{i-1,k}$ and $\xi_{i-2,k}$, and thus, the resulting series is as a finite summation bounded.

Corollary 2. *Let $k \ge 3$ be a fixed positive integer and let $m > 1$. Then, the pointwise upper bound*

$$\left|D^2 S_{n,k}^m f(x)\right| \le \frac{2\varepsilon_{n,k}}{(m-1)^{3/2}\|\Delta_n\|^2} \cdot \|f\|_{[0,1]}$$

holds for $n \to \infty$ on all compact subsets of $(0,1)$.

Proof. The restricted interval $[x_{2k+2}, x_{n-2k-2}]$ converges to $(0,1)$ for $n \to \infty$ while k is fixed. □

3 | THE LOWER BOUND OF THE APPROXIMATION ERROR OF THE SCHOENBERG OPERATOR

For $f \in C([0,1])$, let us denote by $\|f\|_{I_{\Delta_n,k}}$ the maximum norm of f restricted to the interval $I_{\Delta_n,k} := [x_{2k+2}, x_{n-2k-2}]$. In this section, we show that for $0 < t \leq \frac{1}{2}$ and $k \geq 3$, there exists a constant $M > 0$, such that

$$M \cdot \omega_2(f|_{I_{\Delta_n,k}}, t) \leq \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}},$$

and $I_{\Delta_n,k} \to [0,1]$ for $n \to \infty$. Here, the second-order modulus of smoothness $\omega_2 : C([0,1]) \times \left(0, \frac{1}{2}\right] \to [0,\infty)$ is defined by

$$\omega_2(f,t) := \sup_{0<h<t} \sup_{x\in[0,1-2h]} |f(x) - 2f(x+h) + f(x+2h)|.$$

As the modulus of smoothness is equivalent to the K-functional, the inequality

$$\omega_2(f,t) \leq 4\left\|f - S_{n,k} f\right\|_{[0,1]} + t^2\left\|D^2 S_{n,k} f\right\|_{[0,1]} \tag{13}$$

holds; for details, see Timan.[26, pp. 102] To prove our main result, we need to estimate the second term by the approximation error $\left\|f - S_{n,k} f\right\|_{[0,1]}$. In a first step, we show that the second-order differential operator D^2 is bounded on the spline space.

Lemma 4. *For $k \geq 3$, the differential operator $D^2 : \mathcal{S}(n,k) \to \mathcal{S}(n,k-2)$ is bounded with*

$$\left\|D^2\right\|_{op} \leq \frac{4d_k}{\|\Delta_n\|^2},$$

where $d_k > 0$ is a constant depending only on k.

Proof. Let $s \in \mathcal{S}(n,k)$, $s(x) = \sum_{j=-k}^{n-1} c_j N_{j,k}(x)$, with $\|s\|_\infty = 1$. According to M. Marsden,[12] lemma 2 on page 35, we can calculate the second-order derivative by

$$D^2 s(x) = \sum_{j=2-k}^{n-1} \frac{\frac{c_j - c_{j-1}}{\xi_{j,k} - \xi_{j-1,k}} - \frac{c_{j-1} - c_{j-2}}{\xi_{j-1,k} - \xi_{j-2,k}}}{\xi_{j,k-1} - \xi_{j-1,k-1}} N_{j,k-2}(x).$$

Then, we obtain with the triangle inequality

$$\begin{aligned}\left\|D^2 s\right\|_{[0,1]} &= \left\|\sum_{j=2-k}^{n-1} \frac{\frac{c_j - c_{j-1}}{\xi_{j,k} - \xi_{j-1,k}} - \frac{c_{j-1} - c_{j-2}}{\xi_{j-1,k} - \xi_{j-2,k}}}{\xi_{j,k-1} - \xi_{j-1,k-1}} N_{j,k-2}(x)\right\|_{[0,1]} \\ &\leq \frac{\|c\|_{[0,1]} + 2\|c\|_{[0,1]} + \|c\|_{[0,1]}}{\|\Delta_n\|^2} \cdot \left\|\sum_{j=2-k}^{n-1} N_{j,k-2}(x)\right\|_{[0,1]},\end{aligned}$$

where

$$\|c\|_{[0,1]} = \max\left\{|c_j| \ : \ j \in \{-k, \dots, n-1\}\right\}. \tag{14}$$

According to de Boor,[27] there exists $d_k > 0$, such that

$$d_k^{-1}\|c\|_\infty \leq \left\|\sum_{j=-k}^{n-1} c_j N_{j,k}\right\|_\infty \leq \|c\|_\infty. \tag{15}$$

Rewriting the first inequality yields $\|c\|_\infty \leq d_k$, because $\|s\|_{[0,1]} = 1$. Now, we use the partition of unity (1) to derive the estimate

$$\left\|D^2 s\right\|_{[0,1]} \leq \frac{4}{\|\Delta_n\|^2} d_k.$$

Taking the supremum of all $s \in \mathcal{S}(n,k)$ with $\|s\|_{[0,1]} = 1$ yields the result. □

Now, we are able to prove our main result:

Theorem 3. *For $0 < t \leq \frac{1}{2}$ and $k \geq 3$, there exists a constant $M_{n,k} > 0$ only depending on n and k, independent of f, such that*

$$M_{n,k} \cdot \omega_2(f|_{I_{\Delta_n,k}}, t) \leq \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}}.$$

Proof. We extend $\left\|D^2 S_{n,k} f\right\|_{I_{\Delta_n,k}}$ into a telescopic series:

$$\begin{aligned}\left\|D^2 S_{n,k} f\right\|_{I_{\Delta_n,k}} &= \left\|D^2 S_{n,k} f - D^2 S_{n,k}^2 f + D^2 S_{n,k}^2 f - D^2 S_{n,k}^3 f + \dots\right\|_{I_{\Delta_n,k}} \\ &\leq \sum_{m=1}^{\infty} \left\|D^2 S_{n,k}^m (f - S_{n,k} f)\right\|_{I_{\Delta_n,k}} \\ &= \left\|D^2 S_{n,k} (f - S_{n,k} f)\right\|_{I_{\Delta_n,k}} + \sum_{m=2}^{\infty} \left\|D^2 S_{n,k}^m (f - S_{n,k} f)\right\|_{I_{\Delta_n,k}}.\end{aligned}$$

Then, we apply Theorem 2 and Lemma 4 and obtain

$$\begin{aligned}\left\|D^2 S_{n,k} f\right\|_{I_{\Delta_n,k}} &\leq \frac{4d_k \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}}}{\|\Delta_n\|^2} + \sum_{m=1}^{\infty} \frac{2\varepsilon_{n,k}}{m^{3/2} \cdot \|\Delta_n\|^2} \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}} \\ &\leq \frac{4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2})}{\|\Delta_n\|^2} \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}}.\end{aligned}$$

Finally, applying the above result to (13) yields the estimate

$$\omega_2(f|_{I_{\Delta_n,k}}, t) \leq \left(4 + t^2 \frac{(4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2}))}{\|\Delta_n\|^2}\right) \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}}.$$

□

In the limit, we obtain the lower estimate on the whole interval $[0, 1]$:

Corollary 3. *For $k \geq 3$, $n \to \infty$ and $f \in C([0, 1])$, the following uniform estimate holds:*

$$\omega_2(f|_{I_{\Delta_n,k}}, \delta_{n,k}) \leq 5 \cdot \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}},$$

where

$$\delta_{n,k} := \frac{\|\Delta_n\|}{\sqrt{(4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2}))}}$$

and $\delta_{n,k} \to 0$ as $n \to \infty$.

Corollary 4. *For $0 < t \leq \frac{1}{2}$ and $k \geq 3$, we have the equivalence*

$$\omega_2(f|_{I_{\Delta_n,k}}, t) \sim \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}}$$

in the sense that there exist constants $M_1, M_2 > 0$ independent of f and only depending on n and k such that

$$M_1 \cdot \omega_2(f|_{I_{\Delta_n,k}}, t) \leq \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}} \leq M_2 \cdot \omega_2(f|_{I_{\Delta_n,k}}, t).$$

Proof. We apply Theorem 3 to get the lower inequality, and we use the inequality

$$\left\|f - S_{n,k} f\right\|_{[0,1]} \leq \left(1 + \frac{1}{2t^2} \cdot \min\left\{\frac{1}{2k}, \frac{(k+1)\|\Delta_n\|^2}{12}\right\}\right) \cdot \omega_2(f, t),$$

from Beutel et al[13] to obtain the upper bound. □

Consequently, we have proved that the conjecture stated in[13] holds true for $n \to \infty$ if the degree k of the splines is fixed and at least three, as the restricted interval $I_{\Delta_n,k}$ converges in the limit to $[0, 1]$.

Finally, we want to compare our result to a related result that has been shown recently in the case of non-uniform knots. Nagler et al[23] showed that for all $f \in C([0,1])$ and $n \to \infty$, the following estimate holds:

$$\omega_2\left(f, \frac{\delta_{\min}}{k} \cdot \left(\frac{1-\gamma_{\Delta_n,k}}{d_k}\right)^{1/2}\right) \leq 8 \cdot \left\|f - S_{\Delta_n,k} f\right\|_{[0,1]},$$

where $\gamma_{\Delta_n,k}$ is the second largest eigenvalue of Schoenberg variation-diminishing spline operator $S_{\Delta_n,k}$ with knot sequence $\Delta_n = \left\{x_j\right\}_{j=-k}^{n-1}$, ie,

$$\gamma_{\Delta_n,k} := \sup\left\{|\lambda| \ : \ \lambda \in \sigma(S_{\Delta_n,k}) \setminus \{1\}\right\},$$

and $\delta_{\min}$ denotes the minimal mesh length of the knots,

$$\delta_{\min} := \min\left\{(x_{j+1,k} - x_{j,k}) \ : \ j \in \{0, \dots, n-1\}\right\}.$$

Note that the term $1 - \gamma_{\Delta_n,k}$ tends to 0 for $n \to \infty$, while a concrete form of the eigenvalues of the Schoenberg operator have not been shown yet. In our result, we consider a uniform mesh, and thus, $\delta_{\min} = \|\Delta_n\|$ holds. Using the uniform mesh, we have shown the estimate

$$\omega_2\left(f|_{I_{\Delta_n,k}}, \frac{\|\Delta_n\|}{\sqrt{4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2})}}\right) \leq 5 \cdot \left\|f - S_{n,k} f\right\|_{I_{\Delta_n,k}},$$

where $\varepsilon_{n,k}$ is bounded for $n \to \infty$. Hence, for fixed degree k, we have an explicit constant independent on n, whereas the constant in the results shown in Nagler et al[23] still depends on n. Further research might improve the constants shown in both results.

4 | NUMERICAL EXAMPLE

To illustrate our method, we will consider a simple example of a function with 3 different kinds of singularites:

$$f(x) = 2\sin(6\pi \cdot x) + 15(|x-0.2|^{\frac{6}{5}} + |x-0.4|^{\frac{2}{5}} - |x-0.7|^{\frac{4}{5}}) - 15.07x, \qquad x \in [0,1].$$

This function and one of its cubic spline approximation can be seen in Figure 1. Using our approach, we are going to detect the singularities by studying the discrete approximation error between the data and the approximation in the data points. The decay rates of spline approximations with $n = 128$, $n = 256$, and $n = 512$ are shown in Figure 2.

The singularities and their types are clearly visible by the error decay, while false singularities may be detected by looking at the absolute error of only 1 spline approximation. We would like to point out that our method detects not only the strong singularities in $x = 0.4$ and $x = 0.7$ but also the weak singularity in $x = 0.2$.

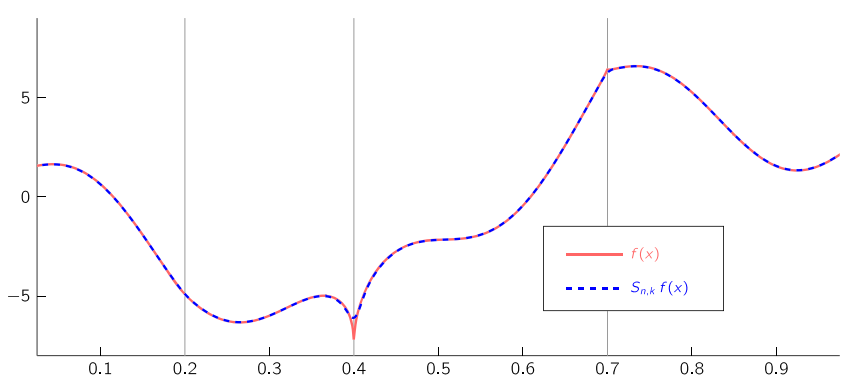

FIGURE 1 Function with 3 singularities of different orders at $x = 0.2, x = 0.4$, and $x = 0.7$ [Colour figure can be viewed at wileyonlinelibrary.com]

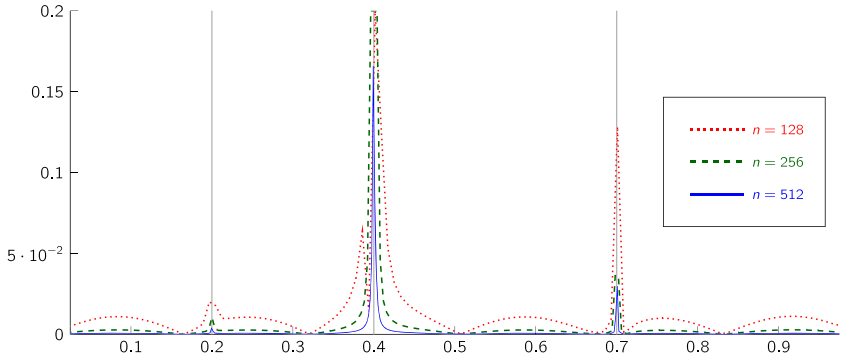

FIGURE 2 Decay rate of the cubic uniform spline approximation with 128, 256, and 512 samples. The singularities and their types are clearly visible by the error decay [Colour figure can be viewed at wileyonlinelibrary.com]

These results show that even there is only a limited number of samples available, the singularities can be computed because of the slow decay at the singularities. Of course, the number n should reflect the behaviour of the function f. The detection of the singularity is dependent on n and can only detect singularities that occur at the given scale.

5 | CONCLUSION

We presented lower estimates for the approximation error in Schoenberg variation-diminishing spline approximation with equidistant knots in terms of the classical second-order modulus of smoothness. This allows the characterization of local smoothness of a function or a curve as corresponding upper estimates exists. We have shown the principle of detecting C^2-singularities based on the decay rate of the local approximation error in a brief numerical example. Amongst other applications, this can be used for robust estimation of the digital curvature of piecewise smooth curves in images. In practical applications, one has to deal with 2 problems. Firstly, one has to compute discrete decay rates, and thus, the discrimination between a slow and fast decay rate requires a parameter. Secondly, as only a limited number of data are available, the decay rates can only be computed up to a certain resolution. Both problems need to be addressed according to the concrete problem.

ACKNOWLEDGEMENTS

The authors thank the reviewers for their valuable suggestions to improve the quality of the paper. The second author was supported by Portuguese funds through the CIDMA, Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (“FCT–Fundação para a Ciência e a Tecnologia”), within project UID/MAT/ 0416/2013.

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How to cite this article: Nagler J, Kähler U. Detection of C^2-singularities using uniform spline approximations. *Math Meth Appl Sci*. 2018;41:303–316. https://doi.org/10.1002/mma.4614