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Spherical Π-Type Operators in Clifford Analysis and Applications

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Abstract The Π-operator (Ahlfors–Beurling transform) plays an important role in solving the Beltrami equation. In this paper we define two Π-operators on the n-sphere. The first spherical Π-operator is shown to be an L^2 isometry up to isomorphism. To improve this, with the help of the spectrum of the spherical Dirac operator, the second spherical Π operator is constructed as an isometric L^2 operator over the sphere. Some analogous properties for both Π-operators are also developed. We also study the applications of both spherical Π-operators to the solution of the spherical Beltrami equations.

Keywords Singular integral operator · Π-Operator · Spectrum · Beltrami equation

1 Introduction

The Π-operator is one of the tools used to study smoothness of functions over Sobolev spaces and to solve some first order partial differential equations such as the Beltrami

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This paper is dedicated to Franciscus Sommen on the occasion of his 60th birthday.

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equation which describes quasi-conformal mappings. In one dimensional complex analysis, the Beltrami equation is the partial differential equation:

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z}$$

where $\mu = \mu(z)$ is a given complex function, and $z = x + iy \in \mathbb{C}$, $\partial_z = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$, $\partial_{\overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. It can be transformed to a fixed-point equation

$$h = q(z)(I + \Pi_\Omega h)$$

where

$$\Pi_\Omega h(z) = -\frac{1}{\pi i}\int_\Omega \frac{h(\xi)}{(\xi - z)^2} d\xi_1 d\xi_2$$

is the complex Π-operator. This singular integral operator acts as an isometry from $L^2(\mathbb{C})$ to $L^2(\mathbb{C})$ with the L_p-norm being a long standing conjecture by Iwaniec.

With the help of Clifford algebras, the classical Beltrami equation and Π-operator with some well known results can be generalized to higher dimensions. Abundant results in Euclidean space have been found(see [4,8,9]). In order to generate results in Euclidean space to the unit sphere, we define two Π-operators related to the conformally invariant spherical Dirac operator. The idea to consider the n-sphere is not only motivated by being the classic example of a manifold and being invariant under the conformal group, but also by the fact that in the case of $n = 3$ due to the recently proved Poincaré conjecture there is a wide class of manifolds which are homeomorphic to the 3-sphere. This makes our results much more general and valid for any simply connected closed 3-manifold. In particular, results on local and global homeomorphic solutions of the sperical Beltrami equation carry over to such manifolds.

This paper is organized as follows: In Sect. 2, we briefly introduce Clifford algebras, Clifford analysis, the Euclidean Dirac operator, and some well known integral formulas. In Sect. 3, we review the construction and some properties for the Π-operator in Euclidean space. In Sect. 4, we construct the Π-operator in a generalized spherical space and solve the Beltrami equation with a singular integral operator $\Pi_{s,0}$. In the last section, we will investigate the spectra of several spherical Dirac type operators and the spherical Laplacian, and construct the isometric spherical Π-operator $\Pi_{s,1}$.

2 Preliminaries

Let $e_1, \cdots, e_n$ be an orthonormal basis of $\mathbb{R}^{n+1}$. The Clifford algebra Cl_n is the algebra over $\mathbb{R}^n$ generated by the relation

$$x^2 = -||x||^2 e_0$$

where e_0 is the identity of $\mathcal{C}l_n$. These algebras were introduced by Clifford in 1878 in [6]. Each element of the algebra $\mathcal{C}l_n$ can be represented in the form

$$x = \sum_{A \subset \{1,\cdots,n\}} x_A e_A$$

where x_A are real numbers. The norm of a Clifford number x is defined as

$$\|x\|^2 = \sum_{A \subset \{1,\cdots,n\}} x_A^2.$$

If the set A contains k elements, then we call e_A a *k-vector*. Likewise, we call each linear combination of k-vectors a k-vector. The vector space of all k-vectors is denoted by $\Lambda^k \mathbb{R}^n$. Obviously, $\mathcal{C}l_n$ is the direct sum of all $\Lambda^k \mathbb{R}^n$ for $k \leq n$. The following anti-involutions are well known:

- Reversion:

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$

  where $|A|$ is the cardinality of A. In particular, $\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$. Also $\widetilde{ab} = \tilde{b}\tilde{a}$ for $a,\ b \in \mathcal{C}l_n$.
- Clifford conjugation:

$$a^\dagger = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A,$$

  satisfying $e_{j_1} \cdots e_{j_r}{}^\dagger = (-1)^r e_{j_r} \cdots e_{j_1}$ and $(ab)^\dagger = b^\dagger a^\dagger$ for $a,\ b \in \mathcal{C}l_n$.
- Clifford involution:

$$\bar{a} = \tilde{a}^\dagger = \widetilde{a^\dagger}.$$

In the following we identify the Euclidean space $\mathbb{R}^{n+1}$ with the direct sum $\Lambda^0 \mathbb{R}^n \oplus \Lambda^1 \mathbb{R}^n$. For all that follows let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. Then functions f defined in Ω with values in $\mathcal{C}l_n$ are considered. These functions may be written as

$$f(x) = \sum_{A \subseteq \{e_1, e_2, \ldots e_n\}} e_A f_A(x), \quad (x \in \Omega).$$

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_A(x),\ (A \subseteq \{e_1, e_2, \ldots e_n\})$. The spaces $C^k(\Omega, \mathcal{C}l_n)$, $L_p(\Omega, \mathcal{C}l_n)$ are defined as right Banach modules with the corresponding traditional norms. The space $L_2(\Omega, \mathcal{C}l_n)$ is a right Hilbert module equipped with a $\mathcal{C}l_n$-valued sesquilinear form

$$(u, v) = \int_\Omega \overline{u(\eta)} v(\eta)\, d\Omega_\eta.$$

Furthermore, $W_p^k(\Omega, \mathcal{C}l_n)$, $k \in \mathbb{N} \cup \{0\}$, $1 \le p < \infty$ denotes the Sobolev spaces as the right module of all functionals whose derivatives belong to $L_p(\Omega, \mathcal{C}l_n)$, with norm

$$\|f\|_{W_p^k(\Omega, \mathcal{C}l_n)} := \left(\sum_A \sum_{\|\alpha\| \le k} \|D_w^\alpha f_A\|^p_{L_p(\Omega, \mathcal{C}l_n)} \right)^{1/p}.$$

The closure of the space of test functions $C_0^\infty(\Omega, \mathcal{C}l_n)$ in the W_p^k-norm will be denoted by $\overset{\circ}{W}{}_p^k (\Omega, \mathcal{C}l_n)$.

The Euclidean Dirac operators D_x and D_0 arise as generalizations of the Cauchy–Riemann operator of one complex variable. As homogenous linear differential operators,

$$D_x := \sum_{i=1}^n e_i \partial_{x_i},$$

$$D_0 := e_0 \partial_{x_0} + \sum_{i=1}^n e_i \partial_{x_i} = e_0 \partial_{x_0} + D_x.$$

Note $D_x^2 = -\Delta_x$, where Δ_x is the Laplacian in $\mathbb{R}^{n+1}$, and $\Delta_{n+1} = D_0 \overline{D_0}$, where $\overline{D_0}$ is the Clifford conjugate of D_0.

Definition 1 A $\mathcal{C}l_n$-valued function $f(x)$ defined on a domain Ω in $\mathbb{R}^{n+1}$ is called left monogenic if

$$D_x f(x) = \sum_{i=1}^n e_i \partial_{x_i} f(x) = 0.$$

Similarly, f is called a right monogenic function if it satisfies

$$f(x) D_x = \sum_{i=1}^n \partial_{x_i} f(x) e_i = 0$$

Let $f \in C^1(\Omega, \mathcal{C}l_n)$, $G(x-y) = \dfrac{\overline{x-y}}{\|x-y\|^{n+1}}$ being the fundamental solution of D_0. Hence, the Cauchy transform is defined as

$$T_\Omega f(x) = \frac{1}{\omega_{n+1}} \int_\Omega G(x-y) f(y) dy,$$

where T is the generalization of the Cauchy transform in the complex plane to Euclidean space, and it is the right inverse of D_0, that is $D_0 T = I$. Also, the non-singular boundary integral operator is given by

$$F_{\partial\Omega} f(x) = \frac{1}{\omega_{n+1}} \int_{\partial\Omega} G(x-y) n(y) f(y) d\sigma(y).$$

We have the Borel–Pompeiu Theorem as follows.

Theorem 1 ([8]) *For $f \in C^1(\Omega, \mathcal{C}l_n) \cap C(\bar{\Omega})$, we have*

$$f(x) = \frac{1}{\omega_{n+1}} \int_{\partial\Omega} G(x-y) n(y) f(y) d\sigma(y) + \frac{1}{\omega_{n+1}} \int_{\Omega} G(x-y) D_0 f(y) dy,$$

In particular, if $f \in \overset{\circ}{W}{}_2^1 (\Omega, \mathcal{C}l_n)$, then

$$f(x) = \frac{1}{\omega_{n+1}} \int_{\Omega} G(x-y) D_0 f(y) dy.$$

3 Π-Operator in Euclidean Space

It is well known that in complex analysis, the Π-operator can be realized as the composition of $\partial_{\bar{z}}$ and the Cauchy transform. As the generalization to higher dimension in Clifford algebra, we have the Π-operator in $\mathbb{R}^{n+1}$ defined as follows.

Definition 2 The Π-operator in Euclidean space $\mathbb{R}^{n+1}$ is defined as

$$\Pi = \overline{D_0} T.$$

The following are some well known properties for the Π-operator.

Theorem 2 ([8]) *Suppose $f \in \overset{\circ}{W}{}_p^k (\Omega)(1 < p < \infty, k \geq 1)$, then*

1. $D_0 \Pi f = \overline{D_0} f$,
2. $\Pi D_0 f = \overline{D_0} f - \overline{D_0} F_{\partial\Omega} f$,
3. $F_{\partial\Omega} \Pi f = (\Pi - T\overline{D_0}) f$,
4. $D_0 \Pi f - \Pi D_0 f = \overline{D_0} F_{\partial\Omega} f$.

The following decomposition of $L^2(\Omega, \mathcal{C}l_n)$ helps us to observe that the Π-operator actually maps $L^2(\Omega, \mathcal{C}l_n)$ to $L^2(\Omega, \mathcal{C}l_n)$.

Theorem 3 ([8]) *($L^2(\Omega, \mathcal{C}l_n)$ **Decomposition**)*

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker \overline{D_0} \bigoplus D_0(\overset{\circ}{W}{}_2^1 (\Omega, \mathcal{C}l_n)),$$

and

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_0 \bigoplus \overline{D_0}\left(\overset{\circ}{W_2^1}(\Omega, \mathcal{C}l_n)\right).$$

Notice that, since

$$\Pi(L^2(\Omega, \mathcal{C}l_n) \bigcap Ker \overline{D_0}) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_0,$$
$$\Pi(D_0\left(\overset{\circ}{W_2^1}(\Omega, \mathcal{C}l_n))\right) = \overline{D_0}\left(\overset{\circ}{W_2^1}(\Omega, \mathcal{C}l_n)\right),$$

hence, this Π-operator is from $L^2(\Omega, \mathcal{C}l_n)$ to $L^2(\Omega, \mathcal{C}l_n)$.

One key property of the Π-operator is that it is an L^2 isometry, in other words,

Theorem 4 ([4]) *For functions in $L^2(\Omega, \mathcal{C}l_n)$, we have*

$$\Pi^*\Pi = I.$$

To complete this section, we give the classic example of the Π-operator solving the Beltrami equation. Let $\Omega \subseteq \mathbb{R}^{n+1}$, $q : \Omega \to \mathcal{C}l_n$ a bounded measurable function and $\omega : \Omega \to \mathcal{C}l_n$ be a sufficiently smooth function. The generalized Beltrami equation

$$D_0\omega = q\overline{D_0}\omega$$

could be transformed into an integral equation

$$h = q(\overline{D_0}\phi + \Pi h)$$

where $\omega = \phi + Th$, which could have a unique solution if $\|q\| \leq q_0 < \dfrac{1}{\|\Pi\|}$, see [8], with q_0 being a constant. This tells us that the existence of a unique solution to the Beltrami equation depends on the norm estimate for the Π-operator.

4 Construction and Properties of Spherical Π-Type Operator with Generalized Spherical Dirac Operator

Recall that in one dimensional complex analysis, the Π-operator is defined as

$$\Pi f(z) := \partial_{\bar{z}} T f(z) = \partial_{\bar{z}} \int_\Omega \frac{f(z)}{\eta - z} dz,$$

where $z = x + iy \in \mathbb{C}$ and $\partial_{\bar{z}} = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$. This suggests us to generalize the Π-operator, we need to consider a variable z with "real" and "imaginary" parts, so we can take conjugate of ∂_z to define the Π-operator.

4.1 Spherical Π-Type Operator with Generalized Spherical Dirac Operator

Let $\mathbb{S}^n$ be the n-unit sphere. The spherical Dirac operator D_s on $\mathbb{S}^n$ is defined as follows.

$$\overline{x}D_0 = \sum_{j=1}^{n} e_0e_j\left(x_0\partial_{x_j} - x_j\partial_{x_0}\right) - \sum_{i=1,j>i}^{n} e_ie_j\left(x_i\partial_{x_j} - x_j\partial_{x_i}\right) + \sum_{j=0}^{n}\left(x_j\partial_{x_j}\right).$$

Denote $\Gamma_0 = \sum_{j=1}^{n} e_0e_j\left((x_0\partial_{x_j} - x_j\partial_{x_0})\right) - \sum_{i=1,j>i}^{n} e_ie_j\left((x_i\partial_{x_j} - x_j\partial_{x_i})\right)$. Hence,

$$D_s = \overline{x}^{-1}\overline{x}D_s = \frac{x}{\|x\|^2}(E_r + \Gamma_0) = \xi\left(D_r + \frac{\Gamma_0}{r}\right),$$

where $rD_r = E_r$, $r = \|x\|$ and $\xi \in \mathbb{S}^n$.

In particular, we have the conformally invariant spherical Dirac operator as follows,

$$D_s = w\left(\Gamma_0 - \frac{n}{2}\right).$$

Similarly, we have $\overline{D_s} = \overline{\xi}\left(D_r + \frac{\overline{\Gamma_0}}{r}\right)$, and since $\overline{D_s}$ is also conformally invariant, we have $\overline{D_s} = \overline{w}(\overline{\Gamma_0} - \frac{n}{2})$, where

$$\overline{\Gamma_0} = -\sum_{j=1}^{n} e_0e_j\left(w_0\partial_{w_j} - w_j\partial_{w_0}\right) - \sum_{i=1,j>i}^{n} e_ie_j\left(w_i\partial_{w_j} - w_j\partial_{w_i}\right).$$

Here $\overline{D_s}$ is the Clifford involution of D_s.

Lemma 1

$$\Gamma_0\overline{w} = n\overline{w} - \overline{w}\overline{\Gamma_0};$$
$$\overline{\Gamma_0}w = nw - w\Gamma_0.$$

Proof The proof is similar to Theorem 3 in [10]. □

Theorem 5

$$D_s\overline{w} = -w\overline{D_s}, \ \overline{D_s}w = -\overline{w}D_s.$$

Proof Applying the last Lemma, a straight forward calculation completes the proof. □

Theorem 6 *Since D_s and $\overline{D_s}$ are both conformally invariant, we have their fundamental solutions as follows:*

$$D_s G_s(w-v) = D_s \frac{\overline{w-v}}{\|w-v\|^n} = \delta(v),$$
$$\overline{D_s G_s(w-v)} = \overline{D_s} \frac{w-v}{\|w-v\|^n} = \delta(v),$$

where $w, v \in \mathbb{S}^n$.

Proof The proof is similar to Proposition 4 in [10]. □

Let Ω be a bounded smooth domain in $\mathbb{S}^n$ and $f \in C^1(\Omega, \mathcal{C}l_n)$, we have the Cauchy transforms for both D_s and $\overline{D_s}$,

$$T_\Omega f(w) = \frac{1}{\omega_n}\int_\Omega G_s(w-v)f(v)dv = \int_\Omega \frac{\overline{w-v}}{\|w-v\|^n} f(v)dv,$$
$$\overline{T}_\Omega f(w) = \frac{1}{\omega_n}\int_\Omega \overline{G_s(w-v)}f(v)dv = \int_\Omega \frac{w-v}{\|w-v\|^n} f(v)dv.$$

Also, the non-singular boundary integral operators are given by

$$F_{\partial\Omega} f(w) = \frac{1}{\omega_n}\int_{\partial\Omega} G_s(w-v)n(v)f(v)d\sigma(v),$$
$$\overline{F}_{\partial\Omega} f(w) = \frac{1}{\omega_n}\int_{\partial\Omega} \overline{G_s(w-v)}n(v)f(v)d\sigma(v)$$

Then we have Borel–Pompeiu Theorem as follows.

Theorem 7 ([10]) *(Borel–Pompeiu Theorem)*
For $f \in C^1(\Omega) \cap C(\bar{\Omega})$, we have

$$f(w) = \frac{1}{\omega_n}\int_{\partial\Omega} G_s(w-v)n(v)f(v)d\sigma(v) + \frac{1}{\omega_n}\int_\Omega G_s(w-v)D_s f(v)dv,$$

in other words, $f = F_{\partial\Omega} f + T_\Omega D_s f$. Similarly, $f = \overline{F}_{\partial\Omega} f + \overline{T}_\Omega \overline{D_s} f$

$$f(w) = \frac{1}{\omega_n}\int_{\partial\Omega} \overline{G_s(w-v)}n(v)f(v)d\sigma(v) + \frac{1}{\omega_n}\int_\Omega \overline{G_s(w-v)D_s} f(v)dv,$$

If f is a function with compact support, then $TD_s = \overline{TD_s} = I$.

Since the conformally invariant spherical Laplace operator Δ_s has the fundamental solution $H_s(w-v) = -\frac{1}{n-2}\frac{1}{\|w-v\|^{n-2}}$, see [10]. We have factorizations of Δ_s as follows.

Theorem 8 $\Delta_s = \overline{D_s}(D_s + w) = D_s(\overline{D_s} + \overline{w})$.

Proof The proof is similar to Proposition 5 in [10]. □

We also have the dual of D_s as follows.

Theorem 9 $D_s^* = -\overline{D_s}$.

Proof Let $f, g : \Omega \to \mathcal{C}l_n$ both have compact supports,

$$
\begin{aligned}
&< D_s f, g >=< w\left(\Gamma_0 - \frac{n}{2}\right) f, g >=< \left(\Gamma_0 - \frac{n}{2}\right) f, \overline{w} g > \\
&\quad =< \Gamma_0 f, \overline{w} g > -\frac{n}{2} < f, \overline{w} g >=< f, \Gamma_0 \overline{w} g > -\frac{n}{2} < f, \overline{w} g > \\
&\quad =< f, (n\overline{\omega} - \overline{\omega}\overline{\Gamma_0}) g > -\frac{n}{2} < f, \overline{w} g >=< f, -\overline{w}(\overline{\Gamma_0} - \frac{n}{2}) g > \\
&\quad =< f, -\overline{D_s} g > .
\end{aligned}
$$

□

Definition 3 Define the generalized spherical Π-type operator as

$$\Pi_{s,0} f = (\overline{D_s + w}) T f.$$

We have some properties of $\Pi_{s,0}$ as follows.

Proposition 1

$$
\begin{aligned}
D_s \Pi_{s,0} &= \overline{D_s - w}, \\
\Pi_{s,0} D_s &= \overline{D_s + w}.
\end{aligned}
$$

Proof

$$
\begin{aligned}
D_s \Pi_{s,0} &= D_s (\overline{D_s + w}) T = (\overline{D_s - w}) D_s T = \overline{D_s - w}, \\
\Pi_{s,0} D_s &= (\overline{D_s + w}) T D_s = \overline{D_s + w}.
\end{aligned}
$$

□

From the proposition above, we can have decompositions of $L^2(\Omega, \mathcal{C}l_n)$ as follows.

Theorem 10

$$
\begin{aligned}
L^2(\Omega, \mathcal{C}l_n) &= L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}) \bigoplus D_s \left(\overset{\circ}{W}{}_2^1 (\Omega, \mathcal{C}l_n)\right), \\
L^2(\Omega, \mathcal{C}l_n) &= L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s \bigoplus (\overline{D_s + w}) \left(\overset{\circ}{W}{}_2^1 (\Omega, \mathcal{C}l_n)\right).
\end{aligned}
$$

Notice that

$$\Pi_{s,0}(L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s,$$
$$\Pi_{s,0} D_s \left(\overset{\circ}{W_2^1} (\Omega, \mathcal{C}l_n) \right) = (\overline{D_s + w}) \left(\overset{\circ}{W_2^1} (\Omega, \mathcal{C}l_n) \right).$$

Hence, $\Pi_{s,0}$ operator is from $L^2(\Omega, \mathcal{C}l_n)$ to $L^2(\Omega, \mathcal{C}l_n)$. The proof is similar to Theorem 1 in [8].

Definition 4 We define the Π_s^+ operator as

$$\Pi_s^+ f = \overline{D_s} T^+ f,$$

where $T^+ f = \frac{1}{\omega_n} \int_\Omega G^+(w - v) f(v) dv,$

$$G^+(w - v) = G_s(w - v) + w H_s(w - v) - 2G_s^{(3)}(w - v),$$

and

$$G_s^{(3)}(w - v) = \frac{1}{(n - 2)(n - 4)} \frac{\overline{w - v}}{\|w - v\|^{n-4}}.$$

Notice that $G_s^{(3)}(w-v)$ is actually the reproducing kernel of $D_s^{(3)} = (D_s - w)\overline{D_s}(D_s + w)$ and the proof is similar to a proof in [10].

Proposition 2

$$\Pi_{s,0}(L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}),$$
$$\Pi_{s,0}(\overline{D_s + w}) \left(\overset{\circ}{W_2^1} (\Omega, \mathcal{C}l_n) \right) = D_s \left(\overset{\circ}{W_2^1} (\Omega, \mathcal{C}l_n) \right).$$

Theorem 11 *Π_s is an isometry on $\overset{\circ}{W_2^1} (\Omega, \mathcal{C}l_n)$ up to isomorphism.*

Proof Let $f \in L^2(\Omega, \mathcal{C}l_n)$, then

$$\begin{aligned}
\langle \Pi_s f, \Pi_s^+ g \rangle &= \langle (\overline{D_s} + \overline{w}) T f, \overline{D_s} T^+ g \rangle \\
&= \langle T f, (-D_s + w) \overline{D_s} T^+ g \rangle \\
&= -\langle T f, (D_s - w) \overline{D_s} T^+ g \rangle \\
&= -\langle T f, \overline{D_s} (D_s + w) T^+ g \rangle \\
&= \langle D_s T f, (D_s + w) T^+ g \rangle = \langle f, g \rangle.
\end{aligned}$$

□

4.2 Application of $\Pi_{s,0}$ to the Solution of a Beltrami Equation

We have a Beltrami equation related to $\Pi_{s,0}$ as follows. Let $\Omega \subseteq \mathbb{S}^{n-1}$ be a bounded, simply connected domain with sufficiently smooth boundary, $q : \Omega \longrightarrow \mathcal{C}l_n$ a measurable function. Let $f : \Omega \longrightarrow \mathcal{C}l_n$ be a sufficiently smooth function. The spherical Beltrami equation is as follows:

$$D_s f = q(\overline{D_s + w})f.$$

It has a unique solution $f = \phi + Th$ where ϕ ia an arbitrary left-monogenic function such that $D_s\phi = 0$ and h is the solution of an integral equation

$$h = q\big((\overline{D_s + w})\phi + \Pi_{s,0}h\big).$$

By the Banach fixed point theorem, the previous integral equation has a unique solution in the case where

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_{s,0}\|},$$

with q_0 being a constant. Hence, for the rest of this section, we will estimate the L^p norm of $\Pi_{s,0}$ with $p > 1$.

Since $\overline{D_s} = \overline{w}(\overline{\Gamma} - \frac{n}{2}) = \overline{w}(w\overline{D_0} - E_r - \frac{n}{2}) = \overline{D_0} - wE_r - \frac{n}{2}\overline{w}$, then

$$\Pi_{s,0}f(w) = \overline{(D_s + w)}Tf(w) = (\overline{D}T + \overline{w}(1 - E_w)T - \frac{n}{2}T)f(w).$$

it is easy to see that

$$\frac{\partial}{\partial w_j}\int_{\mathbb{S}^n} \frac{\overline{w - v}}{\|w - v\|^n} f(v)dv = \int_{\mathbb{S}^n} \frac{\overline{e_j} - n(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w - v\|^n} f(v)dv + \omega_n\frac{\overline{e_j}}{n} f(v),$$

since

$$\frac{\partial}{\partial w_j}\frac{\overline{w - v}}{\|w - v\|^n} = \frac{\overline{e_j} - n(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w - v\|^n},$$

and using Chapter IX § 7 in [11]

$$\int_S \frac{\overline{w - v}}{\|w - v\|}\cos(r, w_j)dS = \omega_n\frac{\overline{e_j}}{n},$$

where S is a sufficiently small neighborhood of w on $\mathbb{S}^n$.

Hence, we have

$$\begin{aligned}\overline{D}Tf(w) &= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\sum\overline{e_j}^2 - n\sum(w_j - v_j)\overline{e_j}\frac{\overline{w-v}}{\|w-v\|^2}}{\|w - v\|^n} f(v)dv + \frac{\sum\overline{e_j}^2}{n}f(v)\\ &= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{(1 - n) - n\frac{\overline{w-v}^2}{\|w-v\|^2}}{\|w - v\|^n} f(v)dv + \frac{1 - n}{n}f(v)\end{aligned}$$

$$
\begin{aligned}
E_w Tf(w) &= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\sum w_j\overline{e_j} - n\sum w_j(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{\sum w_j\overline{e_j}}{n} f(v)\\
&= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{w} - n < w, w-v > \frac{\overline{w-v}}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{\overline{w}}{n} f(v).
\end{aligned}
$$

Therefore, we have an integral expression of $\Pi_{s,0}$ as follows.

Theorem 12

$$
\begin{aligned}
\Pi_{s,0} f(w) &= (\overline{D}T + \overline{w}(1 - E_w)T - \frac{n}{2}T) f(w)\\
&= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{1 - n - \overline{w}^2}{\|w-v\|^n} f(v)dv + \frac{n}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{v} - \langle w, v\rangle\overline{w}}{\|w-v\|^{n+1}} \cdot \frac{\overline{w-v}}{\|w-v\|} f(v)dv\\
&+ \left(1 - \frac{n}{2}\right)\frac{\overline{w}}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{w-v}}{\|w-v\|^n} f(v)dv + \frac{1-n}{n} f(v).
\end{aligned}
$$

Since

$$
\Pi_{s,0} = (\overline{D_s + w})T = (\overline{w}(\overline{\Gamma_0} - \frac{n}{2}) + \overline{w})T = \overline{w}\overline{\Gamma_0}T + (1 - \frac{n}{2})\overline{w}T,
$$

where $\overline{\Gamma_0} = -\sum_{j=1}^{n} e_0 e_j(x_0\partial_{x_j} - x_j\partial_{x_0}) - \sum_{i=1, j>i}^{n} e_i e_j(x_i\partial_{x_j} - x_j\partial_{x_i})$. To estimate the L^p norm of $\Pi_{s,0}$, we need the following result.

Theorem 13 *Suppose p is a positive integer and $p > 1$, then* $\|T\|_{L^p} \le \frac{\omega_{n-1}}{4}$.

Proof Since

$$
\begin{aligned}
\|Tf\|_{L^p}^p &= \left(\frac{1}{\omega_n}\right)^p \int_\Omega \|\int_\Omega G_s(w-v)f(v)dv^n\|^p dw^n\\
&= \left(\frac{1}{\omega_n}\right)^p \int_\Omega \|\int_\Omega G_s(w-v)^{\frac{1}{q}} G_s(w-v)^{\frac{1}{p}} f(v)dv^n\|^p dw^n\\
&\le \left(\frac{1}{\omega_n}\right)^p \int_\Omega \Big(\left(\int_\Omega \|G_s(w-v)\|dv^n\right)^{\frac{p}{q}} \cdot \int_\Omega \|G_s(w-v)\|\|f(v)\|^p dv^n\Big)dw^n\\
&\le \left(\frac{1}{\omega_n}\right)^p C_1^{\frac{p}{q}} \int_\Omega\int_\Omega \|G_s(w-v)\|\|f(v)\|^p dv^n dw^n\\
&= \left(\frac{1}{\omega_n}\right)^p C_1^{\frac{p}{q}} \int_\Omega \|f(v)\|^p \left(\int_\Omega \|G_s(w-v)\|dw^n\right) dv^n\\
&\le \left(\frac{1}{\omega_n}\right)^p C_1^{\frac{p}{q}+1} \int_\Omega \|f(v)\|^p \left(\int_\Omega \|G_s(w-v)\|dw^n\right) dv^n
\end{aligned}
$$

$$= \left(\frac{1}{\omega_n}\right)^p C_1^p \cdot \int_\Omega \|f(v)\|^p dv^n$$
$$= \left(\frac{1}{\omega_n}\right)^p C_1^p \cdot \|f\|_{L^p}^p$$

where $p,\ q > 1$ are positive integers and $\frac{1}{p} + \frac{1}{q} = 1$, where

$$C_1 \le \Big| \int_{\mathbb{S}^n} \|G_s(w-v)\| dv^n \Big| = \Big| \int_{\mathbb{S}^n} \frac{1}{\|w-v\|^{n-1}} dv^n \Big|.$$

Due to the symmetry we can choose any fixed point w, hence we choose $w = (1, 0, 0, \ldots, 0)$ and $v = (x_0, x_1, \cdots, x_n) \in \mathbb{S}^n$, i.e. $\sum_{i=0}^n \|x_i\|^2 = 1$. Let $v = \cos\theta e_0 + \sin\theta\zeta$, where ζ is a vector on $n-1$-sphere, then we have $dv^n = \sin^{n-1}\theta d\theta$,

$$\int_{\mathbb{S}^n} \frac{1}{[2(1-x_1)]^{\frac{n-1}{2}}} dv^n$$
$$= 2^{-\frac{n-1}{2}} \int_0^\pi \frac{1}{(1-\cos\theta)^{\frac{n-1}{2}}} \sin^{n-1}\theta d\theta$$
$$= 2^{-\frac{n-1}{2}} \int_0^\pi \left(2\sin^2\frac{\theta}{2}\right)^{-\frac{n-1}{2}} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{n-1} d\theta$$
$$= \int_0^\pi \cos^{n-1}\frac{\theta}{2} d\theta$$
$$= 2 \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2}+1)}$$
$$= \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Since $\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$, we have $\|T\|_{L^p} \le \frac{\omega_{n-1}}{4}$. □

Let G_0 be the operator defined by

$$G_0 g(w) = -\frac{1}{(n-1)\omega_n} \int_{\mathbb{S}^n} \frac{1}{\|w-v\|^{n-1}} g(v) dv,\ n \ge 3,$$

and $R_s = \overline{\Gamma_0} \circ G_0$ is a Riesz transformation of gradient type (see [1]). Then we have,

Proposition 3 [1], *The operator R_s is a L^p operator and the L^p norm is bounded by*

$$\frac{\pi^{1/2}}{2\sqrt{2}} \left(\frac{p}{p-1}\right)^{1/2} B_p,$$

where $B_p = C_{M,p} + C_p$, $C_{M,p}$ is the L^p norm of the maximal truncated Hilbert transformation on $\mathbb{S}^1$, and $C_p = cot\frac{\pi}{2p^*}$, $\frac{1}{p} + \frac{1}{p^*} = 1$.

Hence,

$$\begin{aligned}
&\|\overline{\Gamma_0}\frac{1}{\omega_n}\int_\Omega \frac{1}{\|w-v\|^{n-1}}\cdot\frac{\overline{w-v}}{\|w-v\|}f(v)dv\|_{L^p} \\
&\leq (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}\left(\frac{p}{p-1}\right)^{1/2}B_p\|f(v)\|_{L^p} \\
&= (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}\left(\frac{p}{p-1}\right)^{1/2}B_p\|f(v)\|_{L^p}.
\end{aligned}\tag{1}$$

Recall that $\Pi_{s,0}f = (\overline{D_s}+\overline{w})Tf = (\overline{w}(\overline{\Gamma_0}-\frac{n}{2})+\overline{w})Tf = \overline{w}\overline{\Gamma_0}Tf+(1-\frac{n}{2})\overline{w}Tf$, and by Theorem 13,

$$||\left(1-\frac{n}{2}\right)\overline{w}Tf||_{L^p} = \|\left(1-\frac{n}{2}\right)\frac{\overline{w}}{\omega_n}\int_\Omega\frac{\overline{w-v}}{\|w-v\|^n}f(v)dv\|_{L^p} \leq \left(\frac{n}{2}-1\right)\frac{\omega_{n-1}}{4}\|f\|_{L^p}. \tag{2}$$

By inequalities (1) and (2), we show that $\Pi_{s,0}$ is a bounded operator mapping from L^p space to itself, and

$$\|\Pi_{s,0}\|_{L^p} \leq (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}\left(\frac{p}{p-1}\right)^{1/2}B_p + \left(\frac{n}{2}-1\right)\frac{\omega_{n-1}}{4}.$$

Remark The spherical Π-type operator $\Pi_{s,0}$ preserves most properties of the Π operator in Euclidean space and more importantly, it is a singular integral operator which helps to solve the corresponding Beltrami equation. Unfortunately, it is also only an L^2 isometry up to isomorphism as shown in Theorem 11 . In the next section, we will use the spectrum theory of differential operators to claim that there is a spherical Π-type operator which is also an L^2 isometry.

5 Eigenvectors of Spherical Dirac Type Operators

In this section, we will investigate the spectrums of several spherical Dirac type operators and the spherical Laplacian. During the investigation, we will point out there is a spherical Π-type operator which is an L^2 isometry.

Since $\Gamma_0 = \overline{x}D_0 - E_r$, it is easy to verify the fact that if p_m is a monogenic polynomial and is homogeneous with degree m, that is $D_0 f_m = 0$ and $E_r f_m = mf_m$, then $\Gamma_0 f_m = -mf_m$, so f_m is an eigenvector of Γ_0 with eigenvalue $-m$. Similarly, if $\overline{D_0}g_m = 0$, g_m is an eigenvector of $\overline{\Gamma_0}$ with eigenvalue $-m$.

Let $\mathcal{H}_k$ be the space of $\mathcal{C}l_n$-valued harmonic polynomials homogeneous of degree k and $\mathcal{M}_k$ be the $\mathcal{C}l_n$-valued monogenic polynomials homogeneous of degree k, $\overline{\mathcal{M}_k}$ is the clifford involution of $\mathcal{M}_k$. By an Almansi-Fischer decomposition [5,7], $\mathcal{H}_k = \mathcal{M}_k \bigoplus \overline{x}\overline{\mathcal{M}_{k-1}}$. Hence, for for all harmonic functions with homogeneity of degree k, there exist $p_k \in Ker D_0$, and $p_{k-1} \in Ker \overline{D_0}$ such that $h_k = p_k + \bar{x}\overline{p_{k-1}}$. Then, it is easy to get that $\Gamma_0 p_k = -kp_k$ and $\Gamma_0 \bar{x}\overline{p_{k-1}} = (n+k)\bar{x}\overline{p_{k-1}}$.

Let H_m denote the restriction to $\mathbb{S}^n$ of the space of $\mathcal{C}l_n$-valued harmonic polynomials with homogeneity of degree m. P_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $-m$ and Q_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $n+m$, $m = 0, 1, 2, \ldots$. Then we have $H_m = P_m \bigoplus Q_m$ ([3]). It is well known that $L^2(\mathbb{S}^n) = \sum_{m=0}^{\infty} H_m$ ([2]), it follows $L^2(\mathbb{S}^n) = \sum_{m=0}^{\infty} P_m \bigoplus Q_m$. If $p_m \in P_m$, since $\Gamma_0 p_m = -mp_m$, it is an eigenvector of Γ_0 with eigenvalue $-m$, and for $q_m \in Q_m$, it is an eigenvector of Γ_0 with eigenvalue $n+m$. Therefore, the spectrum of Γ_0 is $\sigma(\Gamma_0) = \{-m, m = 1, 2, \ldots\} \cup \{m+n, m = 0, 1, 2, \ldots\},$. Since $D_s = w(\Gamma_0 - \frac{n}{2})$, the spectrum of D_s is $\sigma(D_s) = \sigma(\Gamma_0) - \frac{n}{2}$, which is $\{-m - \frac{n}{2}, m = 0, 1, 2, \ldots\} \cup \{m + \frac{n}{2}, m = 0, 1, 2, \ldots\}$.

As mentioned in the previous section that $D_s T = T D_s = I$, and we know that $D_s : P_m \longrightarrow Q_m$ ([3]). Hence, we have $T : Q_m \longrightarrow P_m$ and the spectrum of T is the reciprocal of the spectrum of D_s, which is $\sigma(T) = \{\frac{1}{m+\frac{n}{2}}, m = 0, 1, 2, \ldots\} \bigcup \{\frac{1}{-m-\frac{n}{2}}, m = 0, 1, 2, \ldots\}$. Similar arguments apply for $\overline{D_s}$ and $\overline{T}$, in fact $\sigma(\overline{D_s}) = \sigma(D_s)$ and $\sigma(\overline{T}) = \sigma(T)$.

Now with similar strategy as in [3], we consider the operator $\overline{D_s}T$ which maps $L^2(\mathbb{S}^n)$ to $L^2(\mathbb{S}^n)$. If $u \in C^1(\mathbb{S}^n)$ then $u \in L^2(\mathbb{S}^n)$. It follows that

$$u = \sum_{m=0}^{\infty} \sum_{p_m \in P_m} p_m + \sum_{m=0}^{-\infty} \sum_{q_m \in Q_m} q_m,$$

where p_m and q_m are eigenvectors of Γ_0. Further the eigenvectors p_m and q_m can be chosen so that within P_m they are mutually orthogonal. The same can be done for the eigenvectors q_m. Moreover, as $u \in C^1(\mathbb{S}^n)$ then $\overline{D_s}Tu \in C^0(\mathbb{S}^n)$ and so $\overline{D_s}Tu \in L^2(\mathbb{S}^n)$. Consequently,

$$\begin{aligned}
\overline{D_s}Tu &= \sum_{m=0}^{\infty} \sum_{p_m \in P_m} \overline{D_s}Tp_m + \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} \overline{D_s}Tq_m \\
&= \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} \overline{D_s}\frac{1}{m+\frac{n}{2}}q_m + \sum_{m=0}^{\infty} \sum_{p_m \in P_m} \overline{D_s}\frac{1}{-m-\frac{n}{2}}p_m
\end{aligned}$$

and

$$\begin{aligned}
||\overline{D_s}Tu||_{L^2}^2 &= \sum_{m=0}^{\infty} \left(\frac{1}{m+\frac{n}{2}}\right)^2 \sum_{q_m \in Q_m} \|\overline{D_s}q_m\|_{L^2} + \sum_{m=0}^{\infty} \left(\frac{1}{-m-\frac{n}{2}}\right)^2 \sum_{p_m \in P_m} \|\overline{D_s}p_m||_{L^2} \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{m+\frac{n}{2}}\right)^2 (m+\frac{n}{2})^2 \sum_{p_m \in P_m} \|p_m||_{L^2} + \sum_{m=0}^{\infty} \left(\frac{1}{-m-\frac{n}{2}}\right) \left(-m-\frac{n}{2}\right)^2 \sum_{q_m \in Q_m} \|q_m||_{L^2}
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \sum_{p_m \in P_m} ||p_m||_{L^2} + \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} ||q_m||_{L^2}$$
$$= ||u||_{L^2}.$$

This shows $\overline{D_s}T$ is an $L^2(\mathbb{S}^n)$ isometry.

By the help of the spectrum of T, we have the L^2 norm estimate of the $\Pi_{s,0}$, that is

$$\begin{aligned} \|\Pi_{s,0}u\|_{L^2} &\leq \|\overline{D_s}Tu\|_{L^2} + \|\overline{w}\|_{L^2}\|Tu\|_{L^2} \\ &= \|u\|_{L^2} + \left(\frac{1}{m+\frac{n}{2}}\right)^2 \left(\sum_{m=0}^{\infty} \sum_{p_m \in P_m} \|p_m\|_{L^2} + \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} \|q_m\|_{L^2}\right) \\ &\leq \left(1 + \frac{4}{n^2}\right) \|u\|_{L^2}. \end{aligned}$$

Hence we have $\|\Pi_{s,0}\|_{L^2} \leq 1 + \dfrac{4}{n^2}$.

By Theorem 13, $\Delta_s = \overline{D_s}(D_s + w) = (\overline{D_s} - \overline{w})D_s = D_s(\overline{D_s} + \overline{w}) = (D_s - w)\overline{D_s}$. Since $D_s = w(\Gamma_0 - \dfrac{n}{2})$, $\overline{D_s} = \overline{w}(\overline{\Gamma_0} - \dfrac{n}{2})$, a straightforward calculation shows us that

$$\begin{aligned} \Delta_s &= -\left(\Gamma_0 - \frac{n}{2}\right)^2 - \overline{w}w\left(\Gamma_0 - \frac{n}{2}\right) = -\Gamma_0^2 + (n-1)\Gamma_0 - \left(\frac{n^2}{4} - \frac{n}{2}\right) \\ &= -\left(\overline{\Gamma_0} - \frac{n}{2}\right)^2 - \overline{w}w(\overline{\Gamma_0} - \frac{n}{2}) = -\overline{\Gamma_0}^2 + (n-1)\overline{\Gamma_0} - \left(\frac{n^2}{4} - \frac{n}{2}\right). \end{aligned}$$

Since for $0 < r < 1$, any harmonic function $h_m \in B(0, r) = \{x \in \mathbb{R}^n : ||x|| < r\}$ with homogeneity degree m, we have $h_m = f_m + g_m$, where $f_m \in Ker D_0$ and $g_m \in \overline{D_0}$, they are both homogeneous with degree m (see Lemma 3 [12]). Consequently,

$$\Delta_s f_m = \left(-\Gamma_0^2 + (n-1)\Gamma_0 - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right) f_m = \left(-m^2 - m(n-1) - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right) f_m,$$

and

$$\Delta_s g_m = \left(-\overline{\Gamma_0}^2 + (n-1)\overline{\Gamma_0} - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right) g_m = \left(-m^2 - m(n-1) - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right) g_m.$$

Hence

$$\begin{aligned} \Delta_s h_m = \Delta_s(f_m + g_m) &= \left(-m^2 - m(n-1) - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right)(f_m + g_m) \\ &= \left(-m^2 - m(n-1) - \left(\frac{n^2}{4} - \frac{n}{2}\right)\right) h_m. \end{aligned}$$

Since for any function $u \in L^2(\mathbb{S}^n) : \Omega \mapsto \mathcal{C}l_n$, $u = \sum_{m=0}^{\infty} h_m$, where $h_m \in H_m$, it follows that Δ_s has spectrum $\sigma(\Delta_s) = \left\{-m^2 - m(n-1) - \left(\frac{n^2}{4} - \frac{n}{2}\right) : m = 0, 1, 2, \ldots\right\}$.

In order to preserve the property of isometry of the Π-operator on the sphere, we define the isometric spherical Π-operator as $\Pi_{s,1}$ as $\Pi_{s,1} = \overline{D_s} T$, which is isometry in L^2 space. We can solve the Beltrami equation related to $\Pi_{s,1}$ as follows.

Let $\Omega \subseteq \mathbb{S}^n$ be a bounded, simply connected domain with sufficiently smooth boundary, and $q, f : \Omega \longrightarrow \mathcal{C}l_n$, q is a measurable function, and f is sufficiently smooth. The spherical Beltrami equation is as follows:

$$D_s f = q \overline{D_s} f.$$

It has a unique solution $f = \phi + Th$ where ϕ ia an arbitrary left-monogenic function such that $D_s \phi = 0$ and h is the solution of an integral equation

$$h = q(\overline{D_s} \phi + \Pi_{s,1} h).$$

By the Banach fixed point theorem, the previous integral equation has a unique solution in the case of

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_{s,1}\|}$$

with q_0 being a constant. Hence, we can use the estimate of the L^p norm of $\Pi_{s,1}$ with $p > 1$, where

$$\|\Pi_{s,1}\|_{L^p} \leq (n-1) \frac{\pi^{1/2}}{2\sqrt{2}} \left(\frac{p}{p-1}\right)^{1/2} B_p + \frac{n}{2} \frac{\omega_{n-1}}{4}.$$

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