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Complex Analysis and Operator Theory

Numerical Null-Solutions to Iterated Dirac Operator on Bounded Domains

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Abstract The main purpose of this paper is to study numerical null-solutions to the iterated Dirac operator on bounded domains by using methods of discrete Clifford analysis. First, we study the properties of discrete Euler operators, introduce its inverse operators, and construct a discrete version of the Almansi-type decomposition theorem for the iterated discrete Dirac operator. Then, we give representations of numerical null-solutions to the iterated Dirac operator on a bounded domain in terms of its Taylor series. Finally, in order to illustrate our numerical approach, we present a simple numerical example in form of a discrete approximation of the Stokes' equation, and show its convergence to the corresponding continuous problem when the lattice constant goes to zero.

Keywords Discrete Dirac operator · Almansi-type decomposition · Taylor series · Numerical solutions

Mathematics Subject Classification Primary 15A66 · 35K05 · 35K08 · 39A12

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1 Introduction

The Dirac equations, Maxwell equations, iterated Laplace equations, and iterated Schrödinger equations in the higher-dimensions (cf. [1–4]) describe many physical phenomena. It is known that these equations and their related boundary values problems (cf. [2–4]) can be studied by a factorization approach in virtue of a pair of ladder operators constructed in Clifford analysis, which is an elegant generalization of complex function theory to higher dimensions. Besides this, it is important to find their numerical solutions, particularly when one has physical applications and computational mechanics in mind. To this end, there are many contributions on how to construct the discrete Dirac operator (cf. [2,5,6]), but these are based essentially on potential theoretical arguments (cf. [2,7,8]). In [5], an alternative approach was proposed based on one difference operator, i.e., either forward difference operator or backward difference operator, but here the drawback is that it is impossible to factorize the star-Laplacian (see Sect. 2). In [5,7,8], although the corresponding function theory could be build up using polynomials, there are neither basic polynomials nor Taylor series.

In order to overcome the shortcomings and to develop the discrete function theory in the higher dimensions, to the authors' knowledge, discrete Clifford analysis has arisen as a novel branch of Clifford analysis quite recently, as can be seen in, e.g., [9–16]. It focuses on null-solutions to the discrete Dirac operator, the so-called discrete monogenic functions, which is a generalization of discrete analytic functions on the complex plane (cf. [17,18]). Because the discrete Dirac operator (cf. [9,11,12,14]) by combining both forward and backward difference operators and the splitting of the basis element into forward and backward basis elements, factorizes the star-Laplacian it represents a refinement of discrete harmonic analysis. Moreover, by using the same algebra as Hermitean Clifford analysis, its corresponding theory of discrete monogenic functions has been developed in several papers, like [12–16]. In [12], the authors introduced "skew" Weyl relations, and the Lie algebra consisting of the discrete Dirac operator, its discrete vector variable operator, and the discrete Euler operator. In [13, 15,16], the authors further constructed discrete monogenic polynomials, and studied Taylor series expansions of discrete monogenic functions. In [14,16], the authors gave fundamental solutions to the discrete Dirac operator, which play the same role than fundamental solutions of the continuous Dirac operator, presented the discrete Cauchy formula for discrete monogenic functions on half space, discussed the discrete Hilbert transform, and proved the convergence of the results for the discrete Dirac operator to the continuous case. As applications, this provides a way of studying numerical null-solutions to the Dirac equations. Thus, the natural question arises as what numerical null-solutions to the iterated discrete Dirac equations would look like. This will be not only a purely theoretical question, but also linked to discrete physical applications like Ising models or problems in the computational mechanics [16]. Especially, since to obtain their numerical null-solutions we need to reduce the continuous problems again to discrete ones.

Motivated by the above arguments, our aim is to study the structure of numerical null-solutions to the iterated Dirac operator on bounded domains. Our main contribution is to consider a discrete version of the Almansi-type decomposition theorem with respect to null-solutions to the iterated discrete Dirac operator, and derive explicit numerical null-solutions to the iterated Dirac operator in terms of the Taylor series. When the functions considered are restricted to discrete monogenic homogeneous polynomials, the results are similar to those contained in [5,6,11,13]. Moreover, compared to the results in [5,6,11,13], one of the novelties of ours is to pay attention to explicit numerical null-solutions to the iterated Dirac operator, and show the convergence theorem of our numerical null-solutions to the continuous ones. The method we use is to first introduce the inverse of the discrete Euler operator which is more general than the one studied in [5,6,11], and discuss its intertwining properties. This allows us to get the Almansi-type decomposition theorem for null-solutions to the iterated discrete Dirac operator. Applying it, we give representations of numerical null-solutions to the iterated Dirac operator on a bounded domain in virtue of Taylor series. As a special case, numerical null-solutions to the iterated Laplace operator on a bounded domain of the higher dimensional Euclidean space are obtained. Next we illustrate our numerical approach by means of a simple numerical example based on a laminar flow modeled by the Stokes' equation, and provide approximation and convergence results when the lattice constant goes to zero. Finally, we discuss and evaluate our results. For the sake of completeness we recall some basic facts about the discrete Dirac operator and their underlying algebraic structure in Sect. 2.

2 Preliminaries

Let us begin with some basic facts from discrete function theory which will be needed in the sequel. For more details refer the reader to the literature, e.g. [9,10,14].

Let $\mathbb{R}^l$ be the l-dimensional Euclidean space and $\{e_j, j = 1, 2, \ldots, l\}$ be an orthonormal basis. The grid $h\mathbb{Z}^l$ of $\mathbb{R}^l$ is denoted by $h\mathbb{Z}^l = \left\{\sum_{j=1}^{l} hm_j e_j \middle| \sum_{j=1}^{l} m_j e_j \in \mathbb{Z}^l\right\}$, where $h > 0$ denotes the lattice constant (mesh size). The standard forward and backward differences $\Delta_j^{\pm}$ are defined by

$$\Delta_j^+ f(mh) = h^{-1}(f(mh + e_j h) - f(mh)),$$
$$\Delta_j^- f(mh) = h^{-1}(f(mh) - f(mh - e_j h)),$$

with $hm = h(m_1e_1 + \ldots + m_l e_l) \in h\mathbb{Z}^l$.

Each basis element e_j, $j = 1, \ldots, l$, will be split into two basis elements e_j^+ and e_j^-, i.e., $e_j = e_j^+ + e_j^-$, corresponding to the forward and backward directions. These new basis elements satisfy

$$\begin{cases} e_j^- e_k^- + e_k^- e_j^- = 0, \\ e_j^+ e_k^+ + e_k^+ e_j^+ = 0, \\ e_j^+ e_k^- + e_k^- e_j^+ = \delta_{jk}, \end{cases} \tag{1}$$

where δ_{jk} is the Kronecker delta. These basis elements generate a free algebra which is isomorphic to the complexified Clifford algebra $\mathbb{C}_l$ [1]. Any Clifford number a in

$\mathbb{C}_l$ may thus be written as $a = \sum_A a_A e_{\mathcal{A}}, a_A \in \mathbb{R}$, where e_A is a basis element of $\mathbb{C}_l$ with $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, l\}, j_1 < \cdots < j_k$, while for $A = \emptyset$, one puts $e_\emptyset = 1$, the identity element.

This discrete Dirac operator D is given by

$$D = \sum_{j=1}^{l} e_j^+ \Delta_j^+ + e_j^- \Delta_j^-,$$

which converges to the continuous Dirac operator ∂_x as $h \to 0$ with $x = \sum_{j=1}^{l} e_j x_j (x_j \in \mathbb{R}), \partial_x = \sum_{j=1}^{l} \partial_{x_j}, \partial_{x_j} = \frac{\partial}{\partial x_j}$. This operator factorizes the star-Laplacian $\Delta_h^* = \sum_{j=1}^{l} \Delta_j^+ \Delta_j^-$, i.e.,

$$D^2 = \Delta_h^*,$$

which converges to the Laplacian $\Delta = \sum_{j=1}^{l} \partial_{x_j^2}^2$ as $h \to 0$.

Let $G(\neq \emptyset) \subset h\mathbb{Z}^l$ be a bounded domain containing the origin such that for all $x \in G$ the corresponding rectangle $\{y : |y_j| \leq |x_j|\} \subseteq G$. Furthermore, let $\Omega = \cup_{x \in G}\{y : |y_j| \leq |x_j| + h\}$. Next, we consider functions defined in Ω and taking values in the algebra $\mathbb{C}_l$. They are of the form $f = \sum_{\mathcal{A}} f_{\mathcal{A}} e_{\mathcal{A}}$, where $f_{\mathcal{A}}$ are real-valued. A property such as l_p-summability $(1 \leq p < \infty)$ and so forth, are defined for a $\mathbb{C}_l$-valued function by being ascribed to each component of f, i.e., all the components $f_{\mathcal{A}}$ possess the cited property. The corresponding spaces are denoted, respectively, by $l_p(\Omega, \mathbb{C}_l)$ $(1 \leq p < +\infty)$, $\mathcal{C}(\Omega, \mathbb{C}_l)$, $\mathcal{C}^1(\Omega, \mathbb{C}_l)$ and so on.

Definition 2.1 A function $f : \Omega \to \mathbb{C}_l$ is called discrete poly-monogenic if and only if $D^k f = 0, x \in \Omega$, where $D^k f \triangleq D^{k-1}(Df), k \in \mathbb{N}$. When $k = 1$, it reduces to the discrete monogenic case [13,15,16]. In the sequel we consider that $k \in \mathbb{N}, k \geq 2$ unless otherwise stated. In what follows, $\ker D^j$ denotes the kernel of D^j, $j = 1, 2, \ldots, k-1$, i.e., all functions being annihilated by D^j and defined in Ω.

We introduce the vector variable operator X (cf. [12]) corresponding to the discrete Dirac operator D as

$$X = \sum_{j=1}^{l} (e_j^+ X_j^- + e_j^- X_j^+), \tag{2}$$

where, for a fixed lattice constant $h > 0$, $X_j^\pm$ are scalar raising operators, which act on the j-th coordinate and correspond to operators $\Delta_j^\pm (j = 1, 2, \ldots, l)$ by satisfying the Sommen–Weyl relation

$$\Delta_j^+ X_j^+ - X_j^- \Delta_j^- = \mathcal{I},$$
$$\Delta_j^- X_j^- - X_j^+ \Delta_j^+ = \mathcal{I},$$

with $\mathcal{I}$ being the identity operator.

Definition 2.2 A discrete polynomial P_r is called homogeneous of degree r ($r \in \mathbb{N} \cup \{0\}$) if and only if it is an eigenfunction with eigenvalue r of the discrete Euler operator, i.e., $EP_r = rP_r$, where the discrete Euler operator is defined by

$$E = \sum_{j=1}^{l} \left(e_j^+ e_j^- X_j^- \Delta_j^- + e_j^- e_j^+ X_j^+ \Delta_j^+ \right). \tag{3}$$

3 Several lemmas

In this section we will present several lemmas before starting with our main results.

Lemma 3.1 *Let X, D, l, and E be as defined in Sect. 2, and denote $E+\nu$ by E_ν, $\nu > 0$. Then we have*

$$DX + XD = 2E_{\frac{l}{2}}, \tag{4}$$

$$EX - XE = X, \tag{5}$$

$$DE - ED = D. \tag{6}$$

Proof By direct calculation or checking in [12]. □

Lemma 3.2 *Let D and E_ν, $\nu > 0$, be as defined in Lemma 3.1. Then one has*

$$DE_\nu = E_{\nu+1}D. \tag{7}$$

Proof By $E_\nu = E + \nu$, $\nu > 0$, following Lemma 3.1, we get

$$DE_\nu = D\left(E + \nu\mathcal{I}\right) = ED + D + \nu\mathcal{I}D = \left(E + (1+\nu)\mathcal{I}\right)D.$$

□

Definition 3.3 Let $\mathcal{P}_r$ be the set of all discrete homogeneous polynomials of degree r, $r \in \mathbb{N} \cup \{0\}$. For arbitrary $P_r \in \mathcal{P}_r$, we define the operator $I_\nu : \mathcal{P}_r \to \mathcal{P}_r$, $\nu > 0$, via

$$I_\nu P_r \mapsto \frac{1}{r+\nu} P_r.$$

Lemma 3.4 (cf. [15]) *Let Ω be as defined in Sect. 2 and f be defined in Ω. If $f \in \ker D$ then we have*

$$f(hm) = \sum_{r=0}^{+\infty} P_r f(hm), \quad P_r f \in \mathcal{P}_r, \quad hm \in \Omega, \tag{8}$$

where the discrete homogeneous polynomial of degree r is given by

$$P_r f(hm) = \sum_{|\alpha|=r} \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!} V_\alpha(mh)\partial_l^{\alpha_l}\ldots\partial_1^{\alpha_1} f(0), \tag{9}$$

and the discrete monogenic polynomials

$$V_\alpha = CK[\xi_2^{\alpha_2}\ldots\xi_l^{\alpha_l}[1]]$$

are given by the discrete CK-extension of the discrete homogeneous monomials $\xi_2^{\alpha_2}\ldots\xi_l^{\alpha_l}[1]$. *Hereby, the discrete operators and discrete homogeneous monomials are given by*

$$\partial_j = e_j^+\Delta_j^+ + e_j^-\Delta_j^-, \xi_j = e_j^+X_j^+ + e_j^-X_j^-, j = 1, 2, \ldots, l,$$

$$\xi_j^r[1](x_j) = \begin{cases} x_j\left(e_j^+ + e_j^-\right), & r = 1, \\ \left(x_j^2 + thx_j(e_j^+e_j^- - e_j^-e_j^+)\right)\prod_{i=1}^{t-1}\left(x_j^2 - h^2i^2\right), & r = 2t, \\ x_j\prod_{i=1}^{t}\left(x_j^2 - h^2i^2\right)\left(e_j^+ + e_j^-\right), & r = 2t+1. \end{cases} \tag{10}$$

Furthermore, if the series of the right side of equality (8) is normally convergent in Ω, *then it exactly represents the Taylor series expansion in* Ω *of the discrete monogenic function* f.

Remark 3.5 It is easy to check that the discrete homogeneous polynomials (10) do not satisfy the binomial identity:

$$\xi_j^r[1](x_j + y_j) = \sum_{s=0}^{r} C_r^s \xi_j^r[1](x_j)\xi_j^r[1](y_j), \quad r \in \mathbb{N}, \quad r \geq 2,$$

where C_r^s denoting the binomial coefficients and neither do the discrete Cauchy–Kovalevskaya extensions of them [13,15].

Corollary 3.6 *Let* I_ν *be as defined in Lemma 3.3. Then* I_ν *can be extended linearly onto the space of discrete monogenic functions* $\ker D$ *defined in* Ω.

Proof Observing (8), in Ω, the series of the right side of equality (8) is always normally convergent. Associating Lemma 3.4 with Definition 3.3, Corollary 3.6 follows. □

Remark 3.7 After further observation of (8), under the given conditions the series converges to the Taylor series in Theorem 11.3 of [1] when $h \to 0$.

There is an importance consequence of our observations: since the discrete Euler operator is defined by means of both forward and backward difference operators in the setting of discrete Clifford analysis, whose algebraic structure is different from that of Clifford analysis, as can be seen in, e.g., [1,19,20], it is impossible to directly follow the way of the integral operators to determine its inverse. However, noticing the fact that the Taylor series expansion of the discrete monogenic functions, which is defined in a bounded domain, is always convergent, our idea is to define the inverse operator of the discrete Euler operator by means of using the discrete Taylor series expansion (cf. [13, 15]). This is the cornerstone of this approach.

Lemma 3.8 *Let E_ν and I_ν be as defined in Lemma 3.1 and Definition 3.3. Then we have*

$$E_\nu I_\nu = I_\nu E_\nu = \mathcal{I}, \tag{11}$$

where $\mathcal{I}$ is the identity operator acting on $\ker D$.

Proof For an arbitrary function $f \in \ker D$ defined in Ω, there exists a Taylor series expansion, which is normally convergent, seen in [15],

$$f(x) = \sum_{r=0}^{+\infty} P_r(x), \quad x \in \Omega,$$

where $P_r \in \mathcal{P}_r$, with $\mathcal{P}_r$ being same as in Definition 3.3.

Hence, we get

$$E_\nu I_\nu f = \sum_{r=0}^{+\infty} E_\nu \left(I_\nu P_r\right) = f, \quad f \in \ker D.$$

Applying Lemma 3.2, since $f \in \ker D$ we get $E_\nu f \in \ker D$. Therefore, starting with Corollary 3.6 and associating it with Lemma 3.4 we obtain that in the bounded domain Ω the series $\sum\limits_{r=0}^{+\infty} r P_r$ is normally convergent to $E_\nu f$. Hence, we obtain

$$I_\nu E_\nu f = \sum_{r=0}^{+\infty} I_\nu (E_\nu P_r) = f, \quad f \in \ker D.$$

$\square$

Remark 3.9 In Lemma 3.8, we restrict the considered operators to the kernel $\ker D$. In essence, following Definition 3.3, the result of Lemma 3.8 still holds on $\mathcal{P}_r$.

Lemma 3.10 *Let I_ν and D be as defined in Definition 3.3 and Sect. 2. Then one gets*

$$D I_\nu P_r = I_{\nu+1} D P_r \text{ on } \mathcal{P}_r. \tag{12}$$

Moreover, if $f \in \ker D$ then $I_\nu f \in \ker D$.

Proof For an arbitrary discrete homogeneous polynomial P_r, i.e., $P_r \in \mathcal{P}_r$, by applying Lemma 3.8 we obtain

$$DI_\nu P_r = I_{\nu+1}E_{\nu+1}DI_\nu P_r = I_{\nu+1}DE_\nu I_\nu P_r = I_{\nu+1}DP_r.$$

Furthermore, considering $P_r \in \mathcal{P}_r \cap \ker D$ the result follows. □

Remark 3.11 Lemmas 3.8 and 3.10 characterize the properties of the discrete Euler operator and its inverse, which is similar to the continuous case as it can be seen in, e.g., [19–21].

Lemma 3.12 *Let $I_\nu, \mathcal{I}$ and D be as defined in Lemma 3.10, and define an operator $Q_k, k \in \mathbb{N}$ by*

$$Q_k = \left(\frac{1}{2}\right)^k R_k^{-1}R_{k-1}^{-1}\ldots R_1^{-1} \text{ with } R_k^{-1} = \begin{cases} \frac{1}{s}\mathcal{I}, & \text{if } k = 2s, s \in \mathbb{N}, \\ I_{s+\frac{l}{2}}, & \text{if } k = 2s+1, s \in \mathbb{N}. \end{cases} \tag{13}$$

Then, for arbitrary $k \in \mathbb{N}$, one has

$$D^k(X^k Q_k f) = f, \quad f \in \ker D. \tag{14}$$

Proof By Definition 3.3 Q_k given by (13) is well defined on $\ker D$. Moreover, by combining it with Lemma 3.8 the inverse of Q_k exists on $\ker D$ and will be denoted by Q_k^{-1}, i.e.,

$$Q_k^{-1} = 2^k R_1 \ldots R_{k-1}R_k \quad \text{with} \quad R_k = \begin{cases} s\mathcal{I}, & \text{if } k = 2s, \quad s \in \mathbb{N}, \\ E_{s+\frac{l}{2}}, & \text{if } k = 2s+1, \quad s \in \mathbb{N}. \end{cases} \tag{15}$$

Applying Lemmas 3.10 and 3.2 we get

$$Q_k \ker D = \ker D, \text{ i.e., } DQ_k f = 0 \text{ for arbitrary } f \in \ker D, \tag{16}$$
$$Q_k^{-1} \ker D = \ker D, \text{ i.e., } DQ_k^{-1} f = 0 \text{ for arbitrary } f \in \ker D. \tag{17}$$

When $k \in \mathbb{N}$ for arbitrary $f \in \ker D$ we get

$$D(X^k Q_k f) = \begin{cases} 2sX^{k-1}Q_k f, & \text{if } k = 2s, \quad s \in \mathbb{N} \\ 2X^{k-1}E_{s+\frac{l}{2}}Q_k f, & \text{if } k = 2s+1, \quad s \in \mathbb{N} \end{cases} = X^{k-1}(R_k Q_k f), \tag{18}$$

by direct calculation and using (15) we obtain

$$D^k(X^k Q_k f) = 2^k R_1 \ldots R_{k-1}R_k Q_k f = f. \tag{19}$$

The proof of the lemma is complete. □

Remark 3.13 The technique used in Lemma 3.12 is of the same flavor as the one which appeared in [6,20–22]. Here, the difference lies in the fact that the discrete Dirac operator is different from the one studied in [5,6,11]. This means that the Euler operator defined in Definition 2.2 is different from the Euler operator discussed in [5,6,11].

4 Main Results

In this section we will present the main results of this paper.

Theorem 4.1 *If f is defined in Ω belonging to $h\mathbb{Z}^l$ and is a null-solution to D^k, i.e., $D^k f = 0\,(k \in \mathbb{N}, k \geq 2)$, then there exits uniquely defined discrete monogenic functions $f_j\,(j = 0, 1, 2, \ldots, k-1)$, satisfying the decomposition*

$$f(hm) = f_0(hm) + Xf_1(hm) + X^2 f_2(hm) + \ldots + X^{k-1} f_{k-1}(hm). \tag{20}$$

The discrete monogenic functions $f_j\,(j = 0, 1, 2, \ldots, k-1)$ can be expressed by

$$\begin{cases} f_{k-1} = Q_{k-1} D^{k-1} f, \\ f_{k-2} = Q_{k-2} D^{k-2} (\mathcal{I} - X^{k-1} Q_{k-1} D^{k-1}) f, \\ f_{k-3} = Q_{k-3} D^{k-3} (\mathcal{I} - X^{k-2} Q_{k-2} D^{k-2})(\mathcal{I} - X^{k-1} Q_{k-1} D^{k-1}) f \\ \vdots \qquad \vdots \\ f_1 = Q_1 D (\mathcal{I} - X^2 Q_2 D^2) \ldots (\mathcal{I} - X^{k-2} Q_{k-2} D^{k-2})(\mathcal{I} - X^{k-1} Q_{k-1} D^{k-1}) f, \\ f_0 = (\mathcal{I} - X Q_1 D)(\mathcal{I} - X^2 Q_2 D^2) \ldots (\mathcal{I} - X^{k-2} Q_{k-2} D^{k-2})(\mathcal{I} - X^{k-1} Q_{k-1} D^{k-1}) f. \end{cases} \tag{21}$$

Hereby, X is defined in (2), $X^k f \triangleq X^{k-1}(Xf)$, $Q_j, j = 1, 2, \ldots, k-1\,(k \in \mathbb{N}, k \geq 2)$, $\mathcal{I}$ are given by (13) of Sect. 3. With other words we have

$$\ker D^k = \ker D \oplus X \ker D \oplus X^2 \ker D \oplus \cdots \oplus X^{k-1} \ker D. \tag{22}$$

Conversely, if $f_j \in \ker D\,(j = 0, 1, 2, \ldots, k-1)$, $k \in \mathbb{N}, k \geq 2$, then $\sum_{j=0}^{k-1} X^j f_j \in \ker D^k$.

Remark 4.2 When $k = 2s, s \in \mathbb{N}$, $\ker D^{2s}$ is in fact $\ker(\Delta_h^*)^s = \{f : \Omega \to \mathbb{C}_l \big| (\Delta_h^*)^s f = D^{2s} f = 0\}$. Hereby, we have the convergence of the discrete operator $(\Delta_h^*)^s \to \Delta^s$, the iterated Laplace operator, as $h \to 0$. If $f \in \ker(\Delta_h^*)^s$ Theorem 4.1 provides a unique decomposition of discrete poly-harmonic functions defined in Ω, i.e., the null-solutions to $(\Delta_h^*)^s$, into discrete monogenic functions. When the dimension of the space is $l = 2$ then the discrete Dirac operator D reduces to $D_2 = \sum_{j=1}^2 e_j^+ \Delta_j^+ + e_j^- \Delta_j^-$ and converges to the Dirac operator $\sum_{j=1}^2 e_j \partial_{x_j}$, as $h \to 0$, where $\mathbb{R}^2 = \left\{\sum_{j=1}^2 e_j x_j : x_j \in \mathbb{R}, j = 1, 2\right\} \cong \mathbb{C}$. This implies that Theorem 4.1 provides a unique decomposition of the discrete poly-analytic functions defined in Ω, i.e., the null-solutions to $D_2^k, k \in \mathbb{N}, k \geq 2$, whose limits are poly-analytic functions (cf. [19]), into discrete analytic functions (cf. [17,18]), i.e., the null-solutions to the discrete Dirac operator $D_2 = \sum_{j=1}^2 e_j^+ \Delta_j^+ + e_j^- \Delta_j^-$.

Remark 4.3 Although similar results can be observed in [5,6,11], there is a difference: in our case Theorem 4.1 holds for any discrete monogenic function defined on a bounded domain while in [6] its existence depends on the domain of convergence of the series $f = \sum_{s=0}^{+\infty} f_s$ with respect to a Hilbert space, where f_s is a Clifford-valued homogeneous polynomial of degree s (see B.3. Proof of Lemma 3.3 in [6]). The dependence on the domain of convergence of the given series happens in [5,11], too.

Theorem 4.4 *If f is a null-solution to D^k $(k \in \mathbb{N}, k \geq 2)$ defined in Ω of $h\mathbb{Z}^l$ then we have*

$$f(hm) = \sum_{r=0}^{+\infty} \sum_{j=0}^{k-1} P_{j,r}(hm), \quad hm \in \Omega, \tag{23}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, $\alpha_p \in \mathbb{N}\,(p = 1, 2, \ldots, l)$, and for each $r \in \mathbb{N}$, $j = 0, 1, 2, \ldots, k-1$, $k \in \mathbb{N}$, $k \geq 2$,

$$P_{j,r}(hm) = \sum_{|\alpha|=r} \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!} V_\alpha(hm)\partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} Q_j D^j \prod_{t=j+1}^{k-1} (\mathcal{I} - X^t Q_t D^t) f(0).$$

Remark 4.5 Theorem 4.4 represents null-solutions to the iterated discrete Dirac operator defined in Ω in terms of basic discrete monogenic homogeneous polynomials. This means that we get numerical null-solutions to the iterated Dirac operator defined in a bounded domain in virtue of the discrete Taylor series.

Remark 4.6 The Almansi-type decomposition theorem mentioned in [5,13] can be derived for the space of discrete poly-monogenic functions. Conversely, by observing the fact that the Taylor series expansion of the discrete monogenic functions defined in a bounded domains is finite the Almansi-type decomposition theorem for the space of discrete poly-monogenic functions can be obtained by applying Fischer decomposition [13]. However, this is not described in [13]. Moreover, in the continuous case [20,21, 23] the case is a bit different: although the Almansi-type decomposition theorem for the space of poly-monogenic functions can be also derived from the Almansi-type decomposition theorem mentioned in [23], its existence might depend on the domain of convergence of the Taylor series expansion for monogenic function.

5 Proof of Our Main Results

In this section we will give the proof of our main results.
Proof of Theorem 4.1

Proof For arbitrary $k \in \mathbb{N}$, $k \geq 2$ we will first prove

$$\ker D^k = \ker D^{k-1} + X^{k-1} \ker D. \tag{24}$$

By using Lemma 3.12 for arbitrary $f \in \ker D$ we have

$$D^k(X^{k-1}f) = D[(D^{k-1}X^{k-1}Q_{k-1}Q_{k-1}^{-1}f)] = DQ_{k-1}^{-1}f = 0. \tag{25}$$

Therefore, we get

$$X^{k-1}\ker D \subset \ker D^k.$$

Since $\ker D^{k-1} \subset \ker D^k$ we have $\ker D^{k-1} + X^{k-1}\ker D \subset \ker D^k$.

Next, we will show that $\ker D^k \subset \ker D^{k-1} + X^{k-1}\ker D$.

In fact, for arbitrary $f \in \ker D^k$, we have $D(D^{k-1}f) = 0$ and

$$f = (\mathcal{I} - X^{k-1}Q_{k-1}D^{k-1})f + X^{k-1}Q_{k-1}D^{k-1}f. \tag{26}$$

Since $D^{k-1}f \in \ker D$ by (16) we obtain $Q_{k-1}D^{k-1}f \in \ker D$. Furthermore, as we have

$$D^{k-1}(\mathcal{I} - X^{k-1}Q_{k-1}D^{k-1})f = D^{k-1}f - D^{k-1}X^{k-1}Q_{k-1}D^{k-1}f = 0 \tag{27}$$

from Lemma 3.12 we get

$$(\mathcal{I} - X^{k-1}Q_{k-1}D^{k-1})f \in \ker D^{k-1}.$$

This implies that (24) holds.

Finally, we will show the uniqueness of our decomposition (24).

Let us assume that there exists another decomposition for an arbitrary $f \in \ker D^k$ such that

$$f = \widetilde{f} + X^{k-1}\widetilde{f}_{k-1} \quad \text{with} \quad \widetilde{f} \in \ker D^{k-1} \quad \text{and} \quad \widetilde{f}_{k-1} \in \ker D. \tag{28}$$

Applying D^{k-1} on both sides of (28) we get

$$D^{k-1}f = D^{k-1}\widetilde{f} + D^{k-1}X^{k-1}\widetilde{f}_{k-1} = D^{k-1}X^{k-1}\widetilde{f}_{k-1} = Q_{k-1}^{-1}\widetilde{f}_{k-1}. \tag{29}$$

Hence, we have

$$\widetilde{f}_{k-1} = Q_{k-1}D^{k-1}f,$$

and

$$\widetilde{f} = (\mathcal{I} - X^{k-1}Q_{k-1}D^{k-1})f.$$

The result now follows by simple induction.

Conversely, noting that, for $j = 0, 1, \ldots, k-1$, $D^j(X^j f) = 2^j R_1 R_2 \ldots R_j f$, we obtain

$$D^k \left(\sum_{j=0}^{k-1} X^j f \right) = 0.$$

The proof of the theorem is complete. □

Proof of Theorem 4.4

Proof Since $f \in \ker D^k$ by Theorem 4.1 we have

$$f(hm) = f_0(hm) + X f_1(hm) + X^2 f_2(hm) + \cdots + X^{k-1} f_{k-1}(hm), \quad hm \in \Omega, \tag{30}$$

where

$$f_j = Q_j D^j \prod_{t=j+1}^{k-1} (\mathcal{I} - X^t Q_t D^t) f \in \ker D, \quad j = 0, 1, 2, \ldots, k-1,$$

with $D, X, \mathcal{I}, Q_j$ as the same operators as those in Lemma 3.12, and

$$Q_0 D^0 = \mathcal{I}, \quad \prod_{t=j+1}^{k-1} (\mathcal{I} - X^t Q_t D^t) = \mathcal{I}$$

for $j = k-1$.

By using Lemma 3.4, we get

$$f(hm) = \sum_{s=0}^{+\infty} \sum_{j=0}^{k-1} X^j P_{j,s}(hm), \quad hm \in \Omega, \tag{31}$$

where, for arbitrary $hm \in \Omega$,

$$P_{j,s}(hm) = \sum_{|\alpha|=s} \frac{1}{\alpha_1! \alpha_2! \ldots \alpha_l!} V_\alpha(hm) \partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} Q_j D^j \prod_{t=j+1}^{k-1} (\mathcal{I} - X^t Q_t D^t) f(0),$$

as seen in (9). It follows the result. □

6 A Simple Numerical Example

In this section we give a simple example on how to apply the discrete Almansi-type decomposition theorem in Sect. 4 and how to obtain numerical null-solutions to the iterated Dirac operator. Afterwards we derive error estimates between null-solutions to the iterated Dirac operator and our calculated numerical null-solutions.

Let us consider Stokes' equation:

$$\begin{cases} \Delta u + \frac{1}{\eta}\operatorname{grad} p &= 0, \\ \operatorname{div} u &= 0, \\ u|_\Gamma &= g, \end{cases} \tag{32}$$

where the parameter u represents the velocity field and p is the pressure, both are defined over a domain U with sufficiently smooth boundary Γ. We additionally assume that U is star-like with respect to the origin. Furthermore, η denotes the viscosity. This system describes the stationary flow of a homogeneous viscous incompressible fluid for small Reynold numbers. The non-appearance of the density is due to the absence of external forces. The necessary condition for solvability is $\int_\Gamma g d\sigma = 0$.

Introducing two new auxiliary functions φ and ψ with $u = D\varphi$ and $\frac{1}{\eta}p = \Delta\psi$, we obtain the system

$$\begin{cases} D^3(\varphi + \psi) &= 0, \\ \mathrm{Sc} D\varphi &= 0. \\ D\varphi|_\Gamma &= g, \end{cases} \tag{33}$$

which leads to the question of determining a poly-monogenic function under certain conditions. Hereby, we have that ψ is scalar-valued while the second equation implies that $u = D \wedge \varphi$ and, therefore, φ is a pure vector-valued function, so that $f = \psi + \varphi$ and we arrive at the problem

$$\begin{cases} D^3 f &= 0, \\ D\underline{f}|_\Gamma &= g. \end{cases} \tag{34}$$

Hereby, f is uniquely determined up to a monogenic and a harmonic function.

We discretize our continuous domain U with an equidistant square lattice with mesh-width h, i.e., $U_h = U \cap h\mathbb{Z}$. The boundary of the discrete domain consists of all points $mh \in U_h$ which have at least one neighboring point who does not belong to U_h. This means that we can use the discrete Dirac operator defined over an equidistant lattice with mesh-width h, i.e.,

$$D^3 f_h(mh) = \frac{1}{h^3}\left(\sum_{j=1}^{l} e_j^+\Delta_j^+ + e_j^-\Delta_j^-\right)^3 f_h(mh) = 0 \tag{35}$$

for $mh \in h\mathbb{Z} \cap U$. Using a simple scaling argument this domain and the discrete function can be mapped to a bounded lattice of mesh-width one, i.e., the case considered in the previous sections. From our Almansi-type decomposition we get for the representation of our solution f_h the expression

$$f_h(mh) = f_{h,0}(mh) + Xf_{h,1}(mh) + X^2 f_{h,2}(mh).$$

We can choose the discrete monogenic function $f_{h,0}$ as being zero since it has no influence to our final solution as it belongs to the kernel of the discrete boundary value problem, i.e., the set of functions from the kernel which fulfill the boundary conditions with zero right-hand side. Thus, $D^3 f_{h,0} = 0$ on U_h and $Df_{h,0} = 0$ on the boundary. To incorporate the boundary condition we have to calculate Df. Here, we get

$$\begin{aligned} Df(mh) &= DXf_{h,1}(mh) + DX^2 f_{h,2}(mh) \\ &= 2E_{\frac{l}{2}} f_{h,1}(mh) + 2E_{\frac{l}{2}-1} Xf_{h,2}(mh). \end{aligned}$$

According to Lemma 3.4 we can expand $f_{h,1}$ and $f_{h,2}$ into the following Taylor series ($i = 1, 2$)

$$f_{h,i}(hm) = \sum_{r=0}^{+\infty} \sum_{|\alpha|=r} \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_l!} V_\alpha(hm)\partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} f_{h,i}(0).$$

Since our discrete domain is bounded, the above Taylor series is in fact a finite series

$$f_{h,i}(hm) = \sum_{r=0}^{N} \sum_{|\alpha|=r} \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!} V_\alpha(hm)\partial_l^{\alpha_l} \cdots \partial_1^{\alpha_1} f_{h,i}(0), \quad i = 1, 2, \tag{36}$$

where N denotes the discrete maximum distance of the points on the lattice $h\mathbb{Z} \cap U$ from the origin.

Applying E_ν and X we get for $i = 1, 2$

$$\begin{aligned} E_\nu f_{h,i}(hm) &= \sum_{r=0}^{N} \sum_{|\alpha|=r} (\nu + r) \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!} V_\alpha(hm)\partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} f_{h,i}(0), \\ Xf_{h,i}(hm) &= \sum_{j=0}^{l} \sum_{r=0}^{N} \sum_{|\alpha|=r} \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!} V_{(\alpha_1,\ldots,\alpha_j+1,\ldots,\alpha_l)}(hm)\partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} f_{h,i}(0). \end{aligned}$$

Substituting the above expressions in our Eq. (35) we obtain the following linear system for the determination of the unknown coefficients

$$\begin{aligned} g(mh) = {} & \frac{l}{2}[1](hm)c_{(0,\ldots,0),1} \\ & + \sum_{r=1}^{N} \sum_{|\alpha|=r} V_\alpha(hm) \left[\left(\frac{l}{2} + r\right) \frac{1}{\alpha!} c_{\alpha,1} \right. \\ & \left. + \left(\frac{l}{2} + r - 1\right) \sum_{j=1}^{l} \frac{1}{\alpha_1!\ldots(\alpha_j - 1)!\ldots\alpha_l!} c_{\alpha-j,2} \right], \end{aligned} \tag{37}$$

whereby, $c_{\alpha-j,2}$ denotes $c_{(\alpha_1,\ldots,\alpha_j-1,\ldots,\alpha_l),2}$ and $c_{\alpha,i}$ stands for the unknown coefficients $\partial_l^{\alpha_l} \ldots \partial_1^{\alpha_1} f_{h,i}(0)$.

Regarding the error estimate for the proposed method, we have a problem that there are no convergence estimates for above given discrete monogenic Taylor series in the literature. Therefore, let us start with the following lemma.

Lemma 6.1 *For a real-analytic function f over a bounded domain U, the difference between the discrete and the continuous Taylor series can be estimated by*

$$\left|\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}x^{\alpha}f^{(\alpha)}(0)\right.$$
$$\left.-\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}\xi_1^{\alpha_1}\ldots\xi_l^{\alpha_l}[1]\partial_1^{\alpha_1}\ldots\partial_l^{\alpha_l}f(0)\right| \leq C\frac{lh^{N+1}}{4(N+1)},$$

where $C = \|f\|_{C^{N+1}}$ denotes the norm of f in C^{N+1}.

Proof Let us start with the estimate of

$$\left|\sum_{k=0}^{N}\xi_j^k[1](x_j)\partial_1^k f(0) - \sum_{k=0}^{\infty}x_j^k f_j^{(k)}(0)\right|,$$

where $f_j^{(k)}$ denotes the k-th partial derivative with respect to x_j. Here we need to point out that the first polynomial is interpolating at the lattice points $m_j h$ with the coefficients given by the divided differences, i.e., it is in fact a version of Newton interpolation in the variable x_j. Trivially, this leads to the following estimate. There exists a point $c \in U$ such that

$$\left|\sum_{k=0}^{N}\xi_j^k[1](x_j)\partial_1^k f(0) - \sum_{k=0}^{\infty}(x_j)^k f^{(k)}(0)\right| = \left|\frac{1}{(N+1)!}f^{(N+1)}(c)\prod_{k=0}^{N}(x_j - m_j h)\right|,$$

which can be estimated by

$$\frac{h^{N+1}}{4(N+1)}\sup_{c\in\Omega}|f^{(N+1)}(c)|.$$

Now, going iteratively over all space dimensions we have

$$\left|\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}x^{\alpha}f^{(\alpha)}(0)\right.$$
$$-\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}\xi_1^{\alpha_1}[1]x^{(\alpha_2,\ldots,\alpha_l)}\partial_1^{\alpha_1}f^{(\alpha_2,\ldots,\alpha_l)}(0)$$

$$
\begin{aligned}
&+\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}\xi_1^{\alpha_1}[1]x^{(\alpha_2,\ldots,\alpha_l)}\partial_1^{\alpha_1}f^{(\alpha_2,\ldots,\alpha_l)}(0)\\
&-\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}\xi_1^{\alpha_1}\xi_2^{\alpha_2}[1]x^{(\alpha_3,\ldots,\alpha_l)}\partial_1^{\alpha_1}\partial_2^{\alpha_2}f^{(\alpha_3,\ldots,\alpha_l)}(0)\\
&\ldots\\
&\left.-\sum_{r=0}^{\infty}\sum_{|\alpha|=r}\frac{1}{\alpha_1!\alpha_2!\ldots\alpha_l!}\xi_1^{\alpha_1}\ldots\xi_l^{\alpha_l}[1]\partial_1^{\alpha_1}\ldots\partial_l^{\alpha_l}f(0)\right|\\
\leq\ & C\frac{lh^{N+1}}{4(N+1)},
\end{aligned}
$$

where $C=\|f\|_{C^{N+1}}$. □

Let us now take a closer look at our approximation by discrete monogenic polynomials.

Lemma 6.2 *Let f be a real analytic function which is normal with respect the x_1-axis. For the difference between the discrete and continuous CK-extension we have the estimate*

$$|CK_h[f](x)-CK[f](x)|\leq Ch,$$

where $C>0$ denotes a constant.

Proof Let us remark that in the following proof C is always denoting a constant, but can be different from line to line. For the discrete CK-extension

$$CK_h[f]=\sum_{k=0}^{\infty}\frac{\xi_1^k[1](x_1)}{k!}\underline{D}^k f,$$

and the continuous CK-extension

$$CK[f]=\sum_{k=0}^{\infty}\frac{(e_1x_1)^k}{k!}\underline{\partial}^k f,$$

we have

$$CK_h[f]-CK[f]=\sum_{k=0}^{\infty}\frac{\xi_1^k[1]-(e_1x_1)^k}{k!}\underline{D}^k f+\sum_{k=0}^{\infty}\frac{(e_1x_1)^k}{k!}(\underline{D}^k-\underline{\partial}^k)f,$$

where $\underline{D}$ denotes the discrete Dirac operator of dimension $l-1$ and $\underline{\partial}$ denotes the continuous Dirac operator of dimension $l-1$. In both sums the terms for $k=0$ vanishes. Furthermore, the first sum is a finite sum with $k\leq N$ and the term for $k=1$ being zero, too. This means that for the first term we have

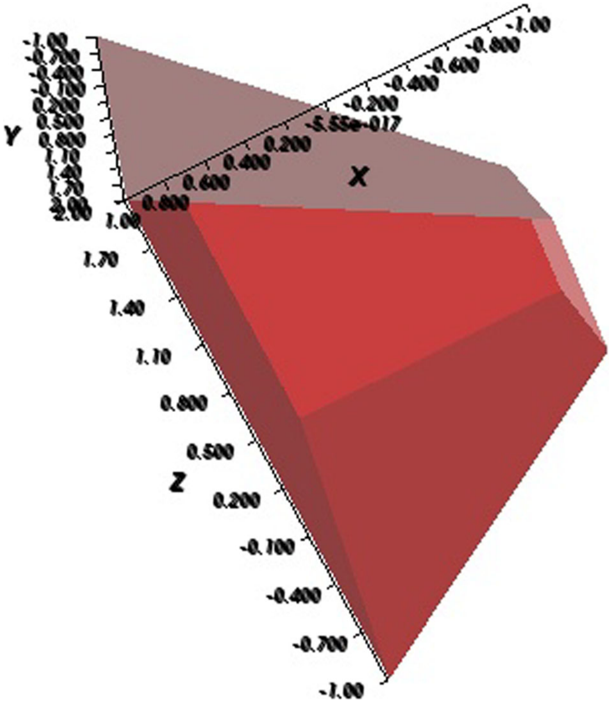

Fig. 1 Channel

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\xi_1^k[1] - (e_1 x_1)^k}{k!} \underline{D}^k f \\
&= - \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\left(x_1^2 + khx_1(e_1^+ e_1^- - e_1^- e_1^+)\right) \prod_{i=1}^{k-1} \left(x_1^2 - h^2 i^2\right) - x_1^{2k}}{2k!} \underline{D}^{2k} f \\
&\quad + \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{x_1 \prod_{i=1}^{t} \left(x_1^2 - h^2 i^2\right) e_1 - (x_1 e_1)^{2k+1}}{(2k+1)!} \underline{D}^{2k+1} f,
\end{aligned}
$$

which leads to the estimate

$$
\left| \sum_{k=0}^{\infty} \frac{\xi_1^k[1] - (e_1 x_1)^k}{k!} \underline{D}^k f \right| \leq Ch.
$$

For the second term we have the estimate $\left| \underline{D}^k f(x) - \underline{\partial}^k f(x) \right| \leq C \frac{lh^{k+1}}{4(4k+1)}$ which results in the estimate

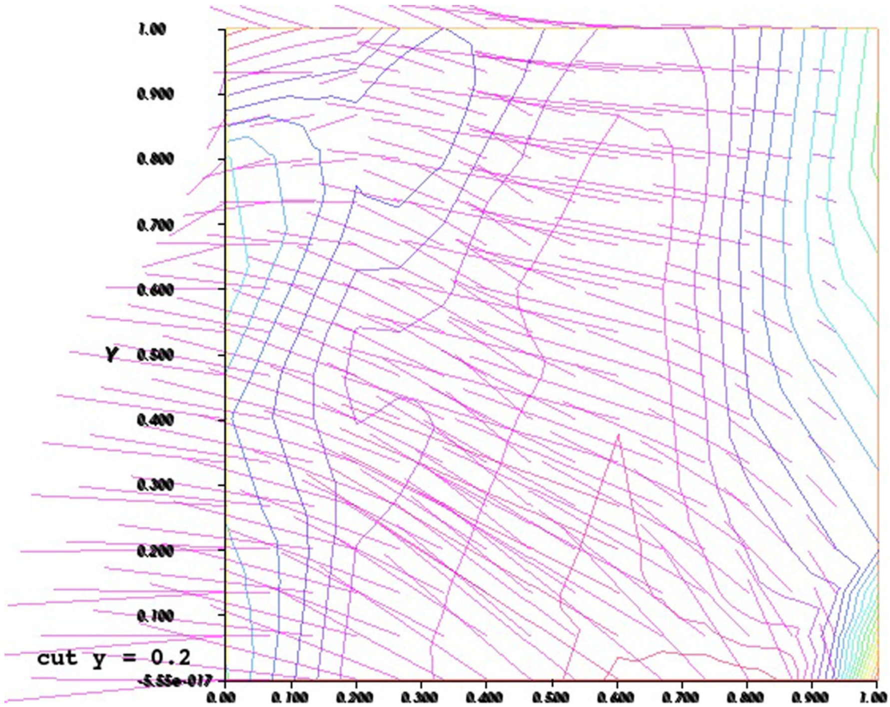

Fig. 2 2D cross-section for y = 0.2 with h = 0.2

$$\left|\sum_{k=1}^{\infty}\frac{(e_1x_1)^k}{k!}(\underline{D}^k-\underline{\partial}^k)f\right| \le C\sum_{k=0}^{\infty}\frac{|x_1|^k}{k!}\frac{lh^{k+1}}{4(4k+1)} = Clh\sum_{k=1}^{\infty}\frac{|x_1|^k}{k!}\frac{h^k}{4(4k+1)}$$
$$\le Clh\left(e^{|x_1|h}-1\right).$$

□

Additionally we can state the following lemma (cf. [14,16]).

Lemma 6.3 *Let $f \in C^1(U)$ and let us denote by X_c the continuous vector variable operator. Then we have for the difference between the discrete and the continuous vector variable operators X and X_c*

$$|Xf(x) - X_c f(x)| \le Ch,$$

where $C > 0$ denotes a constant.

Since we have $V_\alpha = CK_h[\xi_2^{\alpha_2}\ldots\xi_l^{\alpha_l}[1]]$ the above lemmas result in the following convergence result.

Theorem 6.4 *Let f be the solution of the continuous problem, i.e., $\partial^k f = 0$, and f_h the solution of the discrete problem, i.e., $D^k f = 0$, where D is defined in Sect. 2 and $\partial = \sum_{j=1}^{l} e_j \frac{\partial}{\partial_{x_j}}$. Then*

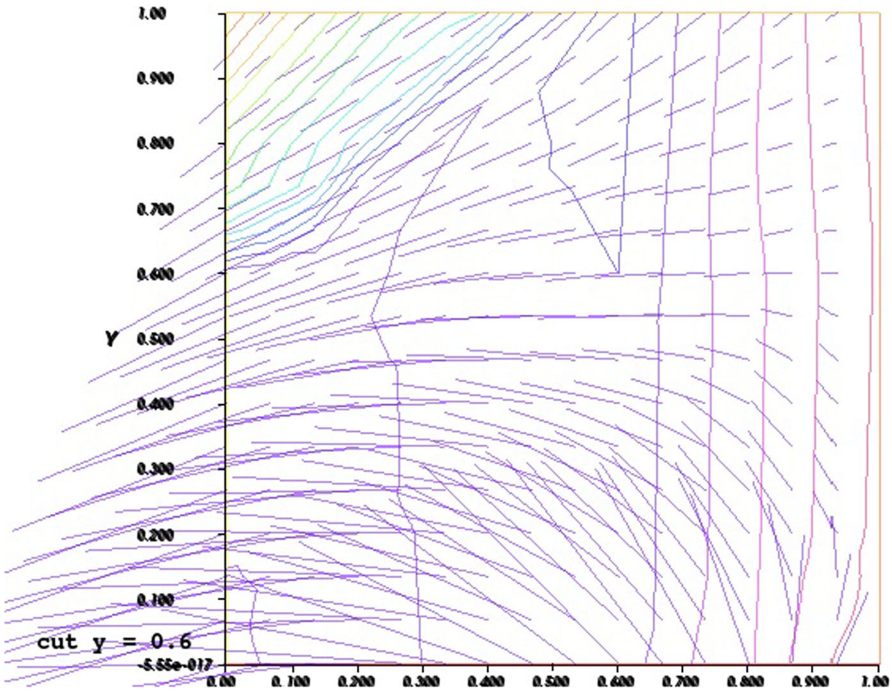

Fig. 3 2D cross-section for y = 0.6 with h = 0.2

$$|f(x) - f_h(x)| \leq Ch,$$

where $C > 0$ *denotes a constant.*

Next, before finishing this section, let us describe our numerical experiment. To present a numerical example in 3D, we consider a laminar flow in a rectangular channel with reducing cross-section diameter, where the diameter is understood in metrical sense as the maximum distance between two points of the set. The domain can be seen in Fig. 1.

The Reynolds number is being put to 1. As boundary conditions we assume the incoming velocity to be $u = (1, 0, 0)$ m/s and the outgoing velocity to be $u = (4, 0, 0)$ m/s, whereby, the physical unit m stands for meter, s for second, and m/s for meter per second, respectively. On the other boundaries we have no-slip boundary conditions. The calculations were done on a PC (processor i7, 8 GB RAM).

We discretize with $h = 0.2$ m in each dimension (10 points in each direction). The solution on the first discretization level can be seen in Figs. 2 and 3 as a 2D-cross-section for $y = 0.2$ m/s and $y = 0.6$ m/s, respectively. The velocity field is represented as vectors while the pressure is shown as continuous lines.

The results for a higher discretization where the step-length h was halved are given by Figs. 4 and 5.

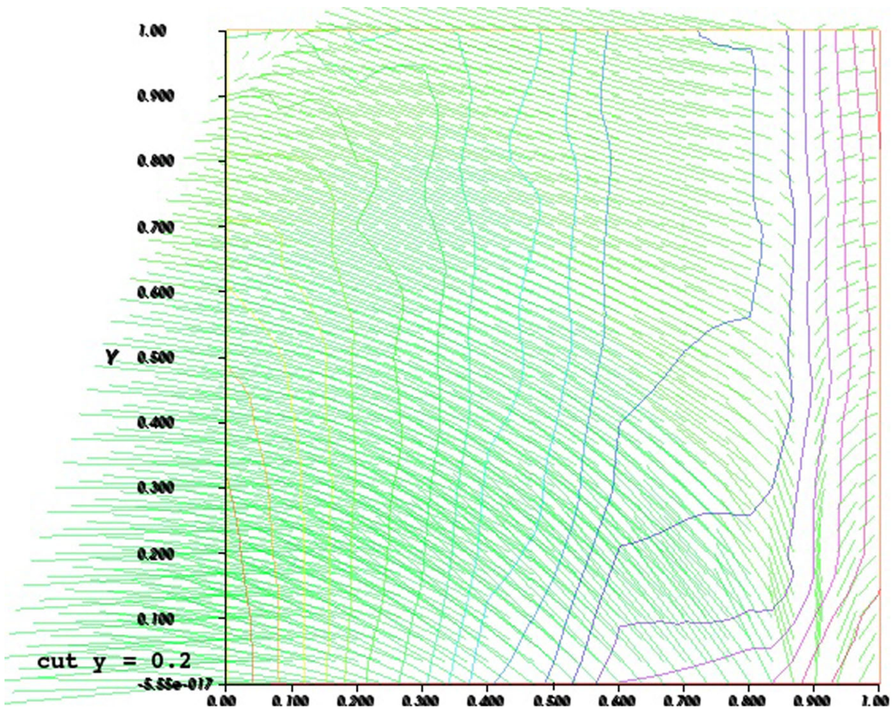

Fig. 4 2D cross-section for y = 0.2 with h = 0.1

7 Discussion and Outlook

The Almansi-type decomposition theorem in Clifford analysis gives the characterization of the conservation relationship between the space of poly-monogenic functions and the space of monogenic functions (cf. [20] or elsewhere). Moreover, by using it many boundary value problems and growth orders for null-solutions of iterated Dirac operators are solved by transferring them to those for null-solutions to the Dirac operator (cf. [19,24]). Very recently, in the setting of discrete Clifford analysis, a discrete Hilbert boundary value problem has been considered (cf. [16]). However, to the authors' knowledge, discrete Hilbert value problems for null-solutions to iterated discrete Dirac operators have not been looked at, yet. This will be not only a purely theoretical question, since such problems are closely linked to discrete physical applications like Ising models or problems in computational mechanics (cf. [16]). In our idea, a discrete version of the Almansi-type decomposition theorem might provide a possible tool to study discrete Hilbert boundary value problems for null-solutions to iterated discrete Dirac operators. Therefore, as a first step in this context we introduced the discrete Euler operator and its inverse based on the observation that the discrete Taylor series expansion of a discrete monogenic function defined in a bounded domains is always convergent, and then derive the Almansi-type decomposition theorem for the space of discrete poly-monogenic functions defined in a bounded domain of $h\mathbb{Z}^l (h > 0)$. As a special case when discrete monogenic functions are

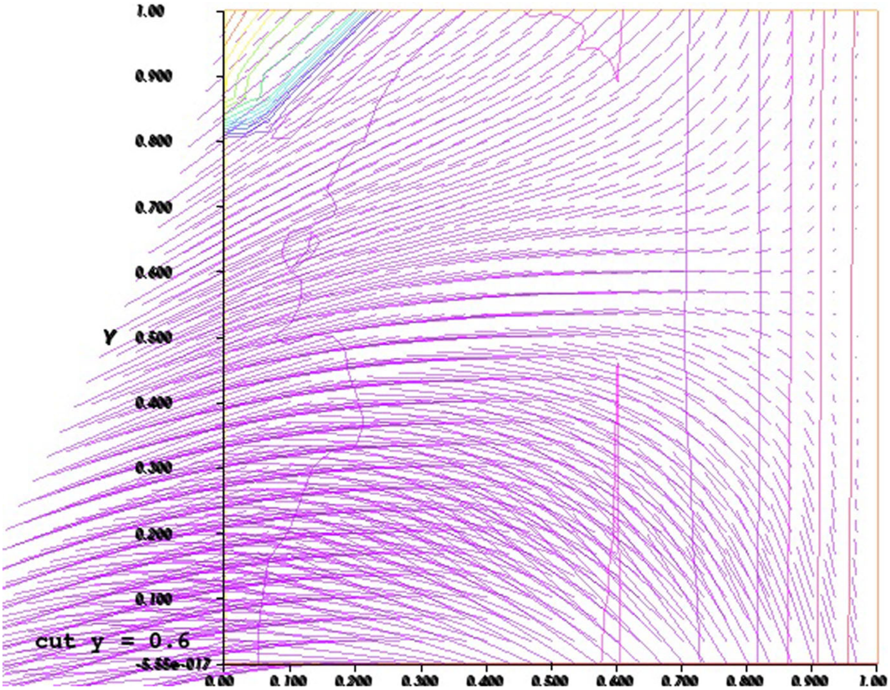

Fig. 5 2D cross-section for y = 0.6 with h = 0.1

restricted to discrete monogenic polynomials our results reduce to those in [13] while they are similar to those in [5,6,11] due to the fact that different discrete Dirac operators were considered. Furthermore, following our proof of the discrete Almansi-type decomposition theorem, we obtain a numerical method to determine null-solutions to iterated Dirac equations in bounded domains and show error estimates between the discrete solutions and the continuous ones. Moreover, our basic idea in this context can be extended to null-solutions to iterated discrete heat operators and iterated discrete wave operators. For the next step, by applying the obtained discrete Almansi-type decomposition theorem, we will discuss discrete Hilbert boundary value problems for null-solutions to iterated discrete Dirac operators in a forthcoming paper.

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