# Riemann Boundary Value Problems for Iterated Dirac Operator on the Ball in Clifford Analysis 

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#### Abstract

In this paper we consider the Riemann boundary value problem for null solutions to the iterated Dirac operator over the ball in Clifford analysis with boundary data given in $\mathbb{L}_{p}(1<p<+\infty)$-space. We will use two different ways to derive its solution, one which is based on the Almansi-type decomposition theorem for null solutions to the iterated Dirac operator and a second one based on the poly-Cauchy type integral operator.


[^0]Keywords Clifford analysis • Riemann boundary value problem . Iterated Dirac operator

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## 1 Introduction

Riemann boundary value problems (RBVP) for polyanalytic functions of one complex variable are of particular interest in solving concrete problems in mathematical physics and engineering. Its higher dimensional equivalent, based on the Dirac equation instead of the complex Cauchy-Riemann equations plays also very important role in pure mathematics and mathematical physics. Its underlying function theory, so-called Clifford analysis, focuses on the study of null solutions to Dirac operator or generalized Cauchy-Riemann operator, i.e., the so-called monogenic functions (see e.g. $[1-3]$ ), which are an elegant generalization of analytic functions from the complex plane to higher dimensions and which refines classic harmonic analysis due to the first-order differential operator factorizing the Laplacian. Making full use of Clifford analysis, a large number of partial differential equations and their related boundary value problems over the various subdomains of $\mathbb{R}^{n}$ and $\mathbb{R}^{n+1}$ were investigated, e.g. in [4-27]. The corresponding solutions were explicitly given in terms of integral representation formulae in [4-18]. In [19-27] a kind of Riemann-Hilbert boundary value problem for monogenic functions over subdomains of $\mathbb{R}^{n}$ and $\mathbb{R}^{n+1}$ was studied. More related results can be found in [28,29]. However, the RBVP for null solutions to iterated Dirac operator on bounded subdomains of $\mathbb{R}^{n}$ is still not considered. In this context, based on ideas contained in [27,30,31], we use two different ways to give the solution to RBVP for null solutions to the iterated Dirac operator on the ball, which generalizes the Riemann boundary value problem for polyanalytic functions of one complex variable(see [30-33] or elsewhere) to the higher-dimensional case. Applying the Almansi decomposition theorem for iterated Dirac operator in Clifford analysis, we first get the solution of RBVP for null solutions to the iterated Dirac operator on the ball. Then by using the poly-Cauchy type integral operator, we obtain another representation for the solution of these RBVP. As a special case we derive solutions to RBVPs for polyanalytic functions on the unit circle in the complex plane (e.g. see Refs. [30-33]). Principally, the first approach is similar to the method used in [24] but with a major difference. In [24] the authors base their method on showing the commutativity between the trace operator and the shifted Euler operator/inverse of the shifted Euler operator. This only works for boundaries which are themselves star-like domains, i.e., the half space. Here we show it in a more direct way which lifts this restriction.

The paper is organized as follows. In Sect. 2 we recall some basic facts about Clifford analysis which will be required in the sequel. In Sect. 3 we provide some technical lemmas which will be needed later. In the last section, we obtain the solution to RBVP for null solutions to iterated Dirac operator on the ball by the proposed methods. First, we give the solution by means of the Almansi-type decomposition theorem for null solutions to iterated Dirac operator in Clifford analysis. Then we
construct the solution to RBVP for null solutions to iterated Dirac operator on the ball by means of the poly-Cauchy type integral operator. As a special case solutions to RBVPs for polyanalytic functions on the unit circle of the complex plane (e.g. see Refs. [30-33]) are obtained.

## 2 Preliminaries and Notations

In this section we recall some basic facts about Clifford analysis which will be needed in the sequel. For more details we refer to Refs. [1-5].

Let $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be an orthogonal basis of the Euclidean space $\mathbb{R}^{n}$. Furthermore, let $\mathbb{R}^{n}$ be endowed with a non-degenerate quadratic form of signature $(0, n)$ and let $\mathbb{R}_{0, n}$ be the $2^{n}$-dimensional real Clifford algebra constructed over $\mathbb{R}^{n}$ with basis

$$
\left\{e_{\mathcal{A}}: \mathcal{A}=\left\{h_{1}, \ldots, h_{r}\right\} \in \mathcal{P N}, 1 \leq h_{1}<h_{r} \leq n\right\}
$$

where $\mathcal{N}$ stands for the set $\{1,2, \ldots, n\}$ and $\mathcal{P N}$ denotes the family of all orderpreserving subsets of $\mathcal{N}$. We denote by $e_{\emptyset}$ the identity element 1 and $e_{\mathcal{A}}$ represents $e_{h_{1} \ldots h_{r}}$ where $\mathcal{A}=\left\{h_{1}, \ldots, h_{r}\right\} \in \mathcal{P N}$. The product in $\mathbb{R}_{0, n}$ is defined by

$$
\begin{cases}e_{\mathcal{A}} e_{\mathcal{B}}=(-1)^{N(\mathcal{A} \cap \mathcal{B})}(-1)^{P(\mathcal{A}, \mathcal{B})} e_{\mathcal{A} \Delta \mathcal{B}}, & \text { if } \mathcal{A}, \mathcal{B} \in \mathcal{P N}, \\ \lambda \mu=\sum_{\mathcal{A}, \mathcal{B} \in \mathcal{P N}} \lambda_{\mathcal{A}} \mu_{\mathcal{B}} e_{\mathcal{A}} e_{\mathcal{B}}, & \text { if } \lambda=\sum_{\mathcal{A} \in \mathcal{P N}} \lambda_{\mathcal{A}} e_{\mathcal{A}}, \mu=\sum_{\mathcal{B} \in \mathcal{P N}} \mu_{\mathcal{B}} e_{\mathcal{B}}\end{cases}
$$

where $N(\mathcal{A})$ denotes the cardinal number of the set $\mathcal{A}$ and $P(\mathcal{A}, \mathcal{B})=\sum_{j \in \mathcal{B}} P(\mathcal{A}, j)$, $P(\mathcal{A}, j)=N(\mathcal{Z})$ and $\mathcal{Z}=\{i: i \in \mathcal{A}, i>j\}$. In particular, we have $e_{i}^{2}=-1$ if $i=1,2, \ldots, n$ and $e_{i} e_{j}+e_{j} e_{i}=0$ if $1 \leq i<j \leq n$. Thus the real Clifford algebra $\mathbb{R}_{0, n}$ is a real linear, associative, but non-commutative algebra.

For arbitrary $a \in \mathbb{R}_{0, n}$ we have $a=\sum_{N(\mathcal{A})=k} a_{\mathcal{A}} e_{\mathcal{A}}=\sum_{N(\mathcal{A})=k}[a]_{k}, a_{\mathcal{A}} \in \mathbb{R}$, where $[a]_{k}=\sum_{N(\mathcal{A})=k} e_{\mathcal{A}}[a]_{\mathcal{A}}$ is the so-called $k$-vector part of a $(k=1,2, \ldots, n)$. The Euclidean space $\mathbb{R}^{n}$ is embedded in $\mathbb{R}_{0, n}$ by identifying ( $x_{1}, x_{2}, \ldots, x_{n}$ ) with the Clifford vector $x$ given by $x=\sum_{j=1}^{n} e_{j} x_{j}$. The conjugation in $\mathbb{R}_{0, n}$ is defined by $\bar{a}=\sum_{\mathcal{A}} a_{\mathcal{A}} \bar{e}_{\mathcal{A}}, \bar{e}_{\mathcal{A}}=(-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}, N(\mathcal{A})=k, a_{\mathcal{A}} \in \mathbb{R}$, and hence $\overline{a b}=\bar{b} \bar{a}$ for arbitrary $a, b \in \mathbb{R}_{0, n}$. Note that $x^{2}=-\langle x, x\rangle=-|x|^{2}$.

The complex Clifford algebra $\mathbb{C}_{n}=\mathbb{R}_{0, n} \otimes \mathbb{C}$ can be understood as $\mathbb{C}_{n}=\mathbb{R}_{0, n} \oplus$ $i \mathbb{R}_{0, n}$. Arbitrary $\lambda \in \mathbb{C}_{n}$ may be written as $\lambda=a+i b, a, b \in \mathbb{R}_{0, n}$, leading to the conjugation $\bar{\lambda}=\bar{a}-i \bar{b}$, where the bar denotes the usual Clifford conjugation in $\mathbb{R}_{0, n}$. This leads to the inner product and its associated norm in $\mathbb{C}_{n}$ given by $(\lambda, \mu)=[\bar{\lambda} \mu]_{0}$ and $|\lambda|=\sqrt{[\bar{\lambda} \lambda]_{0}}=\left(\sum_{\mathcal{A}}|\lambda \mathcal{A}|^{2}\right)^{\frac{1}{2}}$.

The vector-valued first-order differential operator $\mathcal{D}=\sum_{j=1}^{n} e_{j} \partial_{x_{j}}$ is called the Dirac operator and it factorizes the Laplacian $\mathcal{D}^{2}=-\Delta$, where $\Delta$ is the Laplace operator in the Euclidean space $\mathbb{R}^{n}$.

Let $\Omega$ be a bounded subdomain of $\mathbb{R}^{n}$. In what follows, we denote the interior of $\Omega$ by $\Omega^{+}$, the exterior of $\Omega$ by $\Omega^{-}$and its smooth boundary by $\partial \Omega . \mathbb{L}_{p}(1<p<+\infty)$ integrability, continuity, continuous differentiability and so on, are defined for a $\mathbb{C}_{n}$ -
valued function $\phi=\sum_{\mathcal{A}} \phi_{\mathcal{A}} e_{\mathcal{A}}: \Omega \rightarrow \mathbb{C}_{n}$ where $\phi_{\mathcal{A}}: \Omega \rightarrow \mathbb{C}$, by being ascribed to each component $\phi_{\mathcal{A}}$. The corresponding spaces are denoted, respectively, by $\mathbb{L}_{p}\left(\Omega, \mathbb{C}_{n}\right)(1<p<+\infty), \mathcal{C}\left(\Omega, \mathbb{C}_{n}\right), \mathcal{C}^{1}\left(\Omega, \mathbb{C}_{n}\right)$ and so on. Null solutions to the Dirac operator $\mathcal{D}$, that is, $\mathcal{D} \phi=0$, are called (left-) monogenic functions, respectively, right-monogenic functions depending whether the Dirac operator acts from the left or right. The set of left-monogenic functions in $\Omega$ forms a right-module, denoted by $\mathbb{M}_{(r)}\left(\Omega, \mathbb{C}_{n}\right)$.

## 3 Several Lemmas

In this section we provide several lemmas with respect to the shifted Euler operator and Almansi-type decomposition which will be needed later.

Definition 3.1 Let $\Omega$ be a star-like subdomain of $\mathbb{R}^{n}$ with center $a \in \mathbb{R}^{n}$. The shifted Euler operator defined on the space $\mathcal{C}^{1}\left(\Omega, \mathbb{C}_{n}\right)$ is given by

$$
E_{s}=s I+\sum_{j=1}^{n}\left(x_{j}-a_{j}\right) \partial_{x_{j}}(s>0),
$$

where $a=\sum_{j=1}^{n} e_{j} a_{j}$ and $I$ denotes the identity operator defined on the space $\mathcal{C}^{1}\left(\Omega, \mathbb{C}_{n}\right)$.

The operator $I_{s}: \mathcal{C}\left(\Omega, \mathbb{C}_{n}\right) \rightarrow \mathcal{C}\left(\Omega, \mathbb{C}_{n}\right)$ is defined by

$$
I_{s} \phi=\int_{0}^{1} \phi(a+t(x-a)) t^{s-1} d t(s>0)
$$

Particularly, when $\Omega$ is the unit ball centred at the origin of $\mathbb{R}^{n}$, we have

$$
I_{s} \phi=\int_{0}^{1} \phi(t x) t^{s-1} d t(s>0), x \in \Omega
$$

Lemma 3.1 Suppose $\Omega$ is a star-like subdomain of $\mathbb{R}^{n}$ with center $a \in \mathbb{R}^{n}$, and operators $\mathcal{D}, E_{s}$ and $I_{s}$ are given as above. Then on the space $\mathcal{C}^{1}\left(\Omega, \mathbb{C}_{n}\right)$ we have
(i) $E_{s} I_{s}=I_{s} E_{s}=I$,
(ii) $\mathcal{D} E_{s} \phi=E_{s+1} \mathcal{D} \phi$ and $E_{S}(x-a) \phi=x E_{s+1} \phi$.
where I denotes the identity operator on the space $\mathcal{C}^{1}\left(\Omega, \mathbb{C}_{n}\right)$. Moreover, if $\phi \in$ $\mathcal{C}^{k}\left(\Omega, \mathbb{C}_{n}\right), k \in \mathbb{N}$, is a solution to $\mathcal{D}^{k} \phi=0$, then $E_{s} \phi$ and $I_{s} \phi$ are both solutions to $\mathcal{D}^{k} \phi=0$, where $\mathcal{D}^{k} \phi \triangleq \mathcal{D}^{k-1}(\mathcal{D} \phi)$. In this context, to avoid a discussion of the well-known case, we always assume that $k \in \mathbb{N}, k \geq 2$ in the following.

Lemma 3.2 [8] Let $\Omega$ be a star-like subdomain of $\mathbb{R}^{n}$ with center a $\in \mathbb{R}^{n}$. If $\phi \in$ $\mathcal{C}^{k}\left(\Omega, \mathbb{C}_{n}\right)$ is a solution to $\mathcal{D}^{k} \phi=0$, then there exist uniquely defined functions $\phi_{j}$ such that

$$
\phi=\sum_{j=1}^{k-1}(x-a)^{j} \phi_{j}, x \in \Omega,
$$

where each $\phi_{j}$ is monogenic in $\Omega(j=0,1,2, \ldots, k-1)$ and given by

$$
\left\{\begin{array}{l}
\phi_{1}=\left(I-x \mathcal{Q}_{1} \mathcal{D}\right)\left(I-x^{2} \mathcal{Q}_{2} \mathcal{D}^{2}\right) \ldots\left(I-x^{k-1} \mathcal{Q}_{k-1} \mathcal{D}^{k-1}\right) \phi \\
\phi_{2}=\mathcal{Q}_{1} \mathcal{D}\left(I-x^{2} \mathcal{Q}_{2} \mathcal{D}^{2}\right) \ldots\left(I-x^{k-1} \mathcal{Q}_{k-1} \mathcal{D}^{k-1}\right) \phi \\
\quad \vdots \\
\vdots \\
\phi_{k-1}=\mathcal{Q}_{k-2} \mathcal{D}^{k-2}\left(I-x^{k-1} \mathcal{Q}_{k-1} \mathcal{D}^{k-1}\right) \phi \\
\phi_{k}=\mathcal{Q}_{k-1} \mathcal{D}^{k-1} \phi
\end{array}\right.
$$

with $\mathcal{Q}_{j}=\frac{1}{a_{j}} I_{\frac{n}{2}} I_{\frac{n}{2}+1} \ldots I_{\frac{n}{2}+\left[\frac{j-1}{2}\right]}, a_{j}=(-2)^{k}\left[\frac{j}{2}\right]!$, for $j=1,2, \ldots, k-1$ and

$$
[s]= \begin{cases}q, & \text { if } q \in \mathbb{N}, \\ q+1, & \text { if } s=q+t, q \in \mathbb{N}, 0<t<1 .\end{cases}
$$

Lemma 3.3 Suppose that $\Omega$ is a subdomain of $\mathbb{R}^{n}$ and $j \in \mathbb{N}$ is arbitrary. If $\phi \in \mathcal{C}^{1}$ $\left(\Omega, \mathbb{C}_{n}\right)$ is monogenic, then

$$
\mathcal{D}\left(x^{j} \phi\right)= \begin{cases}-2 m x^{2 m-1} \phi, & \text { if } j=2 m, \\ -2 x^{2(m-1)} E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi, & \text { if } j=2 m-1,\end{cases}
$$

where $x \in \Omega$ and $m, p \in \mathbb{N}$. Moreover, for $l \in \mathbb{N}$ and $2 \leq l \leq j$, we get

$$
\begin{equation*}
\mathcal{D}^{l} x^{j} \phi=C_{l, j} x^{j-l} E_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi \tag{1}
\end{equation*}
$$

with

$$
C_{l, j}= \begin{cases}2^{l} m(m-1) \ldots(m-p+1), & \text { if } j=2 m, l=2 p, \\ -2^{l} m(m-1) \ldots(m-p), & \text { if } j=2 m, l=2 p+1, \\ 2^{l}(m-1) \ldots(m-p), & \text { if } j=2 m-1, l=2 p, \\ -2^{l}(m-1) \ldots(m-p+1), & \text { if } j=2 m-1, l=2 p-1 .\end{cases}
$$

Especially, for $l=j$, we obtain

$$
\mathcal{D}^{j} x^{j} \phi=C_{j, j} E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+m-1} \phi \text { with } C_{j, j}=\left\{\begin{array}{l}
2^{j} m!, \text { if } j=2 m, \\
-2^{j}(m-1)!, \text { if } j=2 m-1 .
\end{array}\right.
$$

Proof By using Lemma 3.1 the results can be obtained by direct calculation.

Lemma 3.4 If $\phi \in \mathcal{C}^{k}\left(\mathbb{R}^{n}, \mathbb{C}_{n}\right)$ is a solution to $\mathcal{D}^{k} \phi=0$, satisfying the condition $\lim \inf _{R \rightarrow+\infty} \frac{M(R, \phi)}{R^{r}}=L<+\infty, r \geq k-1, r \in \mathbb{N}$, where $M(R, \phi)=$ $\max _{|x|=R}|\phi(x)|$, then

$$
\phi(x)=\sum_{j=0}^{k-1} \sum_{m=j}^{r} \sum_{\left(l_{1}, \ldots, l_{m}\right)} x^{j} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)}, A_{l_{1}, \ldots, l_{m}}^{(j)} \in \mathbb{C}_{n}, x \in \mathbb{R}^{n}
$$

i.e., $\phi$ is a polynomial function $P_{r}(x)$ of total degree no greater than $r$ on the variable $x \in \mathbb{R}^{n}$. Hereby, $V_{l_{1}, \ldots, l_{m}}^{(j)}$ denotes the inner spherical monogenic functions of degree $m$. Moreover, we obtain

$$
\phi(x)=\sum_{j=0}^{k-1} x^{j} \phi_{j}(x), x \in \mathbb{R}^{n}
$$

where for $j=0,1,2, \ldots, k-1, \phi_{j}(x)=\sum_{m=j}^{r} \sum_{\left(l_{1}, \ldots, l_{m}\right)} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)}$, $A_{l_{1}, \ldots, l_{m}}^{(j)} \in \mathbb{C}_{n}$ is a polynomial of total degree $r-j$ on the variable $x \in \mathbb{R}^{n}$.
Proof Since $\phi \in \mathcal{C}^{k}\left(\mathbb{R}^{n}, \mathbb{C}_{n}\right)$ is a solution to $\mathcal{D}^{k} \phi=0$, by applying Lemma 3.2, there exist unique monogenic functions $\phi_{j}(j=0,1,2 \ldots, k-1)$ in $\mathbb{R}^{n}$ which satisfy $\phi(x)=\sum_{j=0}^{k-1} x^{j} \phi_{j}(x)$.

As each $\phi_{j}(j=0,1,2 \ldots, k-1)$ is monogenic in $\mathbb{R}^{n}$, by Theorem 11.3.4 in Ref. [1], we have

$$
\begin{aligned}
\phi(x) & =\sum_{j=0}^{k-1} x^{j} \sum_{m=0}^{+\infty} \sum_{\left(l_{1}, \ldots, l_{m}\right)} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)} \\
& =\sum_{m=0}^{+\infty} \sum_{j=0}^{k-1} \sum_{\left(l_{1}, \ldots, l_{m}\right)} x^{j} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)}, x \in \mathbb{R}^{n},
\end{aligned}
$$

where $V_{l_{1}, \ldots, l_{m}}^{(j)}(x)$ is a left inner spherical monogenic polynomial of total degree $m$ on the variable $x \in \mathbb{R}^{n}$ and $A_{l_{1}, \ldots, l_{m}}^{(j)} \in \mathbb{C}_{n}(j=0,1,2 \ldots, k-1)$.

Therefore, by changing to spherical coordinates $x=R \omega \in \mathbb{R}^{n}, R>0, \omega \in S^{n} \subset$ $\mathbb{R}^{n}$, we obtain

$$
\phi(x)=\sum_{m=0}^{+\infty} \sum_{j=0}^{k-1} \sum_{\left(l_{1}, \ldots, l_{m}\right)} R^{j+m} \omega^{j} V_{l_{1}, \ldots, l_{m}}^{(j)}(\omega) A_{l_{1}, \ldots, l_{m}}^{(j)}, \quad A_{l_{1}, \ldots, l_{m}}^{(j)} \in \mathbb{C}_{n}
$$

Now, using the condition $\liminf _{R \rightarrow+\infty} \frac{M(R, \phi)}{R^{r}}=L<+\infty, r \geq k-1$, where $M(R, \phi)=\max _{|x|=R}|\phi(x)|$ and noticing that the set

$$
\left\{x^{j} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)}: j=0,1,2, \ldots, k-1,\left(l_{1}, \ldots, l_{m}\right) \in\{1,2, \ldots, n\}^{m}\right\}
$$

is right $\mathbb{C}_{n}$-free, then for $m, j \in \mathbb{N} \cup\{0\}$, we get

$$
\phi(x)=\sum_{j=0}^{k-1} \sum_{m=j}^{r} \sum_{\left(l_{1}, \ldots, l_{m}\right)} x^{j} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)} \triangleq \sum_{j=0}^{k-1} x^{j} \phi_{j}, \quad A_{l_{1}, \ldots, l_{m}}^{(j)} \in \mathbb{C}_{n},
$$

where each $\phi_{j}(x)=\sum_{m=j}^{r} \sum_{\left(l_{1}, \ldots, l_{m}\right)} V_{l_{1}, \ldots, l_{m}}^{(j)}(x) A_{l_{1}, \ldots, l_{m}}^{(j)}$ is a polynomial of total degree $r-j$ for $j=0,1,2, \ldots, k-1$ on the variable $x \in \mathbb{R}^{n}$.

From here it follows the result.
Remark 1 Similar results about the growth conditions at infinity for null solutions to iterated Dirac operator can also be found in Refs. [18,22]. We would like to point out that Lemma 3.4 presents a much simpler growth condition than the one in [18,22], and refines the corresponding results in [18,22].

## 4 Riemann Boundary Value Problems

In this section, we will consider the RBVP for null solutions to iterated Dirac operator on the ball centered at the origin with boundary values given by Clifford algebra valued $\mathbb{L}_{p}(1<p<+\infty)$-integrable functions. We are going to obtain its unique solution in two different ways. As a special case, we also get the solution to RBVP for polyanalytic functions on the unit circle in the complex plane.

In all what follows we denote the open unit ball centered at the origin by $B(1)$ or for short $B_{+}$whose closure is $\bar{B}(1)$, its boundary given by $S^{n-1}$, and $B_{-}=\mathbb{R}^{n} \backslash \bar{B}(1)$ will denote the exterior. We remark that $\omega \in S^{n-1}$ is the outward pointing unit normal vector of $S^{n-1}$. Functions taking values in $\mathbb{C}_{n}$ defined on $B(1)$ and $S^{n-1}$ will be considered, respectively. Furthermore, without loss of generality, we will only consider the Riemann boundary value problem on the unit ball centered at the origin.

For arbitrary $f \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty)$, we define the Cauchy type integral operator via

$$
\begin{equation*}
\phi(x)=\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f(\omega) \triangleq \mathrm{C} f(x), x \notin S^{n-1} \tag{2}
\end{equation*}
$$

where $E(\omega-x)=\frac{1}{w_{n}} \frac{\frac{\overline{\omega-x}}{|\omega-x|^{n}} \text { and } w_{n} \text { is the area of the unit sphere of } \mathbb{R}^{n} \text {. The following } \quad \left\lvert\, \frac{1}{}\right. \text {. }}{}$ properties in the sense of nontangential limit are well-known:

$$
\begin{aligned}
\mathcal{D} \phi(x)=0, \phi^{ \pm}(t) & \triangleq \lim _{x \rightarrow t \in S^{n-1}} \phi(x)= \pm \frac{1}{2} f(t)+\int_{S^{n-1}} E(\omega-t) d \sigma_{\omega} f(\omega) \\
& \triangleq \pm \frac{1}{2} f(t)+\mathcal{H} f(t)
\end{aligned}
$$

where $x \in B_{ \pm}$and the singular integral exits in the sense of the Cauchy principle value. Next we introduce the integral operators
$\Phi_{1}(x)=\left\{\begin{array}{l}\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f(\omega) \triangleq \mathrm{C} f(x), x \in B_{+}, \\ \frac{1}{2} f(x)+\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f(\omega) \triangleq \frac{1}{2} f(x)+\mathcal{H} f(x), x \in S^{n-1} .\end{array}\right.$
$\Phi_{2}(x)=\left\{\begin{array}{l}\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f(\omega) \triangleq \mathrm{C} f(x), x \in B_{-}, \\ -\frac{1}{2} f(x)+\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f(\omega) \triangleq-\frac{1}{2} f(x)+\mathcal{H} f(x), x \in S^{n-1} .\end{array}\right.$
For these operators we have $\Phi_{1} \in \mathbb{L}_{p}\left(B_{+} \cup S^{n-1}, \mathbb{C}_{n}\right)$ and $\Phi_{2} \in \mathbb{L}_{p}\left(B_{-} \cup S^{n-1}, \mathbb{C}_{n}\right)$ with $1<p<+\infty$.

We also need some technical lemmas for the operator $I_{s}$ and shifted Euler operator $E_{S}$, respectively, which can be obtained by direct application of Lemma 3.2.

Lemma 4.1 If $f \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)$ and $\phi$ as above, then $\left(I_{s} \phi\right)(x), x \notin S^{n-1}$ is welldefined and its boundary values $\left(I_{s} \phi\right)^{ \pm}(t), t \in S^{n-1}$, exist and it holds $\left(I_{s} \phi\right)^{ \pm} \in$ $\mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)$ with $1<p<+\infty$.

Lemma 4.2 (i) If $\phi \in \mathcal{C}^{1}\left(B_{ \pm}, \mathbb{C}_{n}\right)$, then $\left(E_{s} \phi\right)(x), x \notin S^{n-1}$ is well-defined.
(ii) If $\phi \in \mathcal{C}^{k}\left(B_{ \pm}, \mathbb{C}_{n}\right)$ is a solution to $\mathcal{D}^{k} \phi=0$ and $\mathcal{D}^{l} \phi(l=0,1, \ldots, k-1)$ can be continuously extended to the boundary $S^{n-1}$ from $B_{ \pm}$, respectively, then for $1 \leq l \leq j \leq k-1$, the boundary values $\left(E_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi_{j}\right)^{ \pm}$of $E_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi_{j}$ exist on the boundary $S^{n-1}$ from $B_{ \pm}$, respectively, where $\phi_{j}$ is given by the decomposition $\phi=\sum_{j=0}^{k-1} x^{j} \phi_{j}$.
Let us now state the Riemann boundary value problem we will be considering in the sequel.

RBVP Given the boundary data $f_{j} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty, j=0,1$, $2, \ldots, k-1)$, find a function $\phi \in \mathcal{C}^{k}\left(B_{ \pm}, \mathbb{C}_{n}\right)$ such that $\mathcal{D}^{l} \phi(l=1,2, \ldots, k-1)$ and $\phi$ are continuously extendable to $S^{n-1}$ from $B_{ \pm}$, respectively, and it satisfies

$$
(\star)\left\{\begin{array}{l}
\mathcal{D}^{k} \phi(x)=0, x \in B_{ \pm} \\
\phi^{+}(t)=\phi^{-}(t) G+f_{0}(t), t \in S^{n-1}, \\
(\mathcal{D} \phi)^{+}(t)=(\mathcal{D} \phi)^{-}(t) G+f_{1}(t), t \in S^{n-1}, \\
\vdots \\
\left(\mathcal{D}^{l} \phi\right)^{+}(t)=\left(\mathcal{D}^{l} \phi\right)^{-}(t) G+f_{l}(t), t \in S^{n-1}, \\
\vdots \\
\left(\mathcal{D}^{k-1} \phi\right)^{+}(t)=\left(\mathcal{D}^{k-1} \phi\right)^{-}(t) G+f_{k-1}(t), t \in S^{n-1}, \\
\liminf _{R \rightarrow+\infty} \frac{M(R, \phi)}{R^{r}}=L<+\infty,
\end{array}\right.
$$

where $r \geq k-1, r \in \mathbb{N}, M(R, \phi)=\max _{|x|=R}|\phi(x)|$ and $G \in \mathbb{C}_{n}$ is an invertible constant.

A first statement about the solution of this problem is given in the next theorem.
Theorem 4.1 Problem ( $\star$ ) is solvable and its unique solution is given by

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} x^{j} \phi_{j}(x), x \in B_{+}, \\
\sum_{j=0}^{k-1}(x-a)^{j} \psi_{j}(x), x \in B_{-},
\end{array}\right.
$$

where $a \in B_{-}$. For $j=0,1,2, \ldots, k-1$ the functions $\phi_{j}$ and $\psi_{j}$ are given via the function

$$
\begin{aligned}
& \widetilde{\phi}_{j}(x)=\left\{\begin{array}{l}
\phi_{j}(x), x \in B_{+} \\
\psi_{j}(x), x \in B_{-}
\end{array}\right. \text {with } \\
& \widetilde{\phi}_{j}(x)=\left\{\begin{array}{l}
\left\{\begin{array}{l}
\mathrm{C} \widetilde{f}_{0}(x)+g_{l_{0}}(x), x \in B_{+}, \\
\mathrm{C} \widetilde{f}_{0}(x) G^{-1}+g_{l_{0}}(x), x \in B_{-},
\end{array} \quad \text { if } j=0,\right. \\
\left\{\begin{array}{l}
C_{1,1}^{-1} I_{\frac{n+1}{}} \mathrm{C} \widetilde{f}_{1}(x)+g_{l_{1}}, x \in B_{+}, \\
C_{1,1}^{-1} I_{\frac{n+1}{2}} \mathrm{C} \widetilde{f}_{1}(x) G^{-1}+g_{l_{1}}, x \in B_{-},
\end{array} \quad \text { if } j=1,\right. \\
\left\{\begin{array}{l}
C_{l, l}^{-1} I_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \mathrm{C} \widetilde{f_{l}}(x)+g_{l_{l}}, x \in B_{+}, \\
C_{l, l}^{-1} I_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \mathrm{C} \widetilde{f_{l}}(x) G^{-1}+g_{l_{l}}, x \in B_{-},
\end{array} \quad \text { if } 2 \leq l \leq k-1 .\right.
\end{array}\right.
\end{aligned}
$$

Hereby, $\mathbf{C} \widetilde{f}_{j}(x)(j=0,1,2, \ldots, k-1)$ is the Cauchy-type integral of
and, for $j=0,1,2, \ldots, k-1, g_{l_{j}}=\sum_{l_{j}=0}^{r-j} P_{l_{j}}$ where $P_{l_{j}}$ denotes a left inner spherical monogenic polynomial of order $l_{j}$ in the variable $x$.

Proof Since $\mathcal{D}^{k} \phi=0, x \in B_{ \pm}$, by applying Lemma 3.2 we know that there exist unique functions $\phi_{j}$, defined in $B_{+}$, and $\psi_{j}$, defined in $B_{-}$, satisfying $\mathcal{D} \phi_{j}=0$, $\mathcal{D} \psi_{j}=0(j=0,1,2, \ldots, k-1)$, such that for $a \in B_{-}$we have

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} x^{j} \phi_{j}(x), x \in B_{+}, \\
\sum_{j=0}^{k-1}(x-a)^{j} \psi_{j}(x), x \in B_{-}
\end{array}\right.
$$

Using Lemma 3.3, for $l \in \mathbb{N}$ and $l \leq j$ we obtain

$$
\mathcal{D}^{l} \phi=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} \mathcal{D}^{l}\left(x^{j} \phi_{j}\right)=\sum_{j=l}^{k-1} C_{l, j} x^{j-l} E_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi_{j}, x \in B_{+}, \\
\sum_{j=0}^{k-1} \mathcal{D}^{l}\left((x-a)^{j} \psi_{j}\right)=\sum_{j=1}^{k-1} C_{l, j}(x-a)^{j-l} E_{\frac{n+1}{2}+\left[\frac{j-1}{2}\right]} \cdots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \psi_{j}, x \in B_{-} .
\end{array}\right.
$$

Now, applying Lemma 4.2 we get that Problem ( $\star$ ) is equivalent to the problem

$$
(*)\left\{\begin{array}{l}
\mathcal{D} \phi_{j}(x)=0(j=0,1,2, \ldots, k-1), x \in B_{ \pm}, \\
\sum_{j=0}^{k-1} t^{j} \phi_{j}^{+}(t)=\sum_{j=0}^{k-1}(t-a)^{j} \psi_{j}^{-}(t) G+f_{0}(t), t \in S^{n-1}, \\
\sum_{j=1}^{k-1} C_{1, j} t^{j-1}\left(E_{\left.\frac{n+1}{2}+\left[\frac{j-1}{2}\right] \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi_{j}\right)^{+}(t)}=\sum_{j=1}^{k-1} C_{1, j}(t-a)^{j-1}\left(E_{\left.\frac{n+1}{2}+\frac{j-1}{2}\right]} \ldots E_{\left.\frac{n+1}{2}+\frac{j}{2}\right]-1} \psi_{j}\right)^{-}(t) G+f_{1}(t), t \in S^{n-1},\right. \\
\vdots \\
\vdots \\
\sum_{j=l}^{k-1} C_{l, j} t^{j-l}\left(E_{\left.\frac{n+1}{2}+\left[\frac{j-l}{2}\right] \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \phi_{j}\right)^{+}(t)}\right. \\
=\sum_{j=l}^{k-1} C_{l, j}(t-a)^{j-l}\left(E_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots E_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \psi_{j}\right)^{-}(t) G+f_{l}(t), t \in S^{n-1}, \\
\vdots \\
\vdots \\
C_{k-1, k-1}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \phi_{k-1}\right)^{+}(t) \\
=C_{k-1, k-1}\left(E_{\frac{n+1}{2}} \cdots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \psi_{k-1}\right)^{-}(t) G+f_{k-1}(t), t \in S^{n-1},
\end{array}\right.
$$

Next, applying Lemma 3.4, we consider the above problem in the case $k-1$ :
$(*)\left\{\begin{array}{l}\mathcal{D} \widetilde{\phi}_{k-1}(x)=0, x \in B_{ \pm}, \\ C_{k-1, k-1}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \widetilde{\phi}_{k-1}\right)^{+}(t) \\ =C_{k-1, k-1}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \widetilde{\phi}_{k-1}\right)^{-}(t) G+f_{k-1}(t), t \in S^{n-1}, \\ \liminf _{R \rightarrow+\infty} \frac{M\left(R, E_{\frac{n+1}{2}} \cdots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \widetilde{\phi}_{k-1}\right)}{R^{r-k+1}}=L_{k-1}<+\infty,\end{array}\right.$
where

$$
\widetilde{\phi}_{k-1}(x)=\left\{\begin{array}{l}
\phi_{k-1}(x), x \in B_{+} \\
\psi_{k-1}(x), x \in B_{-}
\end{array}\right.
$$

Here, by applying Lemma 3.1 and the same argument as in Lemma 4.1 in Ref. [25] we get that Problem $(*)$ has the unique solution

$$
\tilde{\phi}_{k-1}(x)=\left\{\begin{array}{l}
C_{k-1, k-1}^{-1} I_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right\}-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f_{k-1}(\omega)+g_{k-1}(x), x \in B_{+}, \\
C_{k-1, k-1}^{-1} I_{\frac{n+1}{2}+\left\{\frac{k-1}{2}\right\}-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f_{k-1}(\omega) G^{-1}+g_{k-1}(x), x \in B_{-},
\end{array}\right.
$$

where $\widetilde{f}_{k-1}(\omega)=f_{k-1}(\omega), \omega \in S^{n-1}, g_{k-1}(x)=\sum_{l_{k-1}=0}^{r-k+1} P_{l_{k-1}}(x)$, and $P_{l_{k-1}}$ is a left inner spherical monogenic polynomial of order $l_{k-1}$ on the variable $x$.

By using Lemma 4.1 for arbitrary $x \in B_{ \pm}$we get

$$
\begin{aligned}
& I_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \mathrm{C} f_{k-1}(t) \\
& \triangleq \lim _{x \rightarrow t} I_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} f_{k-1}(\omega) \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)
\end{aligned}
$$

That is, we have $\widetilde{\phi}_{k-1}^{ \pm},\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} g_{k-1}\right)^{ \pm} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)$ with $1<$ $p<+\infty$.

Now we can proceed in an inductive way. Applying Lemmas 3.1, 3.4, 4.1, 4.2, we continue by considering the next boundary value problem (for case $k-2$ ).

$$
(* *)\left\{\begin{array}{l}
\mathcal{D} \widetilde{\phi}_{k-2}(x)=0, x \in B_{ \pm}, \\
C_{k-2, k-2}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-2}{2}\right]-1} \widetilde{\phi}_{k-2}\right)^{+}(t) \\
=C_{k-2, k-2}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{k-2}{2}\right]-1} \widetilde{\phi}_{k-2}\right)^{-}(t) G+\widetilde{f}_{k-2}(t), t \in S^{n-1}, \\
\liminf _{R \rightarrow+\infty} \frac{M\left(R, E_{\frac{n+1}{2}} \cdots E_{\frac{n+1}{2}+\left[\frac{k-2}{2}\right]-1} \widetilde{\phi}_{k-2}\right)}{R^{r-k+2}}=L_{k-2}<+\infty
\end{array}\right.
$$

where

$$
\widetilde{\phi}_{k-2}(x)=\left\{\begin{array}{l}
\phi_{k-2}(x), x \in B_{+} \\
\psi_{k-2}(x), x \in B_{-}
\end{array}\right.
$$

and

$$
\begin{aligned}
\widetilde{f}_{k-2}(t)= & f_{k-2}(t)-C_{k-1, k-2} C_{k-1, k-1}^{-1} t I_{\frac{n+1}{2}} f_{k-1}(t)-a C_{k-1, k-2} C_{k-1, k-1}^{-1} I_{\frac{n+1}{2}} \mathbf{C} f_{k-1}(t) \\
& -a\left(E_{\frac{n+1}{2}} \cdots E_{\frac{n+1}{2}+\left[\frac{k-1}{2}\right]-1} g_{k-1}\right)^{-}(t) G
\end{aligned}
$$

In a similar way as above we prove that Problem ( $* *$ ) has the solution

$$
\widetilde{\phi}_{k-2}(x)=\left\{\begin{array}{l}
C_{k-2, k-2}^{-1} I_{\frac{n+1}{2}+\left[\frac{k-2}{2}\right\rfloor-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{k-2}(\omega)+g_{k-2}(x), x \in B_{+}, \\
C_{k-2, k-2}^{-1} I_{\frac{n+1}{2}+\left[\frac{k-2}{2}\right\rfloor-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{k-2}(\omega) G^{-1}+g_{k-2}(x), x \in B_{-},
\end{array}\right.
$$

where $g_{k-2}(x)=\sum_{l_{k-2}=0}^{r-k+2} P_{l_{k-2}}(x), P_{l_{k-2}}$ is a left inner spherical monogenic polynomial of order $l_{k-2}$ on the variable $x$.

By induction for $2 \leq l \leq k-1$, the boundary value problem

$$
\left\{\begin{array}{l}
\mathcal{D} \widetilde{\phi}_{l}(x)=0, x \in B_{ \pm}, \\
C_{l, l}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\frac{l}{2}-1} \widetilde{\phi}_{l}\right)^{+}(t)=C_{l, l}\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\frac{l}{2}-1} \widetilde{\phi}_{l}\right)^{-}(t) G+\widetilde{f}_{l}(t), t \in S^{n-1}, \\
\liminf _{R \rightarrow+\infty} \frac{M\left(R, E_{\frac{n+1}{2} \cdots E_{\left.\frac{n+1}{2}+l \frac{l}{2}\right]-1} \widetilde{\phi}_{l}}\right)}{R^{r-l}}=L_{l}<+\infty,
\end{array}\right.
$$

where

$$
\widetilde{\phi}_{l}(x)=\left\{\begin{array}{l}
\phi_{l}(x), x \in B_{+} \\
\psi_{l}(x), x \in B_{-}
\end{array}\right.
$$

and

$$
\begin{aligned}
\widetilde{f}_{l}(t)= & f_{l}(t)-\sum_{j=l+1}^{k-1} C_{k-1, j} C_{j, j}^{-1} t^{j-l} I_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots I_{\frac{n+1}{2}} \widetilde{f}_{j}(t) \\
& +\sum_{j=l+1}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-l}-t^{j-l}\right) I_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots I_{\frac{n+1}{2}} \mathbf{C} \widetilde{f}_{j}(t) \\
& +\sum_{j=l+1}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-l}-t^{j-l}\right)\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} g_{l_{j}}\right)^{-}(t) G
\end{aligned}
$$

with $\left(E_{\frac{n+1}{2}} \ldots E_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} g_{l_{j}}\right)^{ \pm} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)$ for arbitrary $j=l+1, \ldots, k-1$, has the unique solution

$$
\widetilde{\phi}_{l}(x)=\left\{\begin{array}{l}
C_{l, l}^{-1} I_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{l}(\omega)+g_{l_{l}}(x), x \in B_{+}, \\
C_{l, l}^{-1} I_{\frac{n+1}{2}+\left[\frac{l}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{l}(\omega) G^{-1}+g_{l}(x), x \in B_{-},
\end{array}\right.
$$

where $g_{l_{l}}(x)=\sum_{l_{l}=0}^{r-l} P_{l_{l}}(x)$ with $P_{l_{l}}$ being a left inner spherical monogenic polynomial of order $l_{l}$ on the variable $x$.

For $l=1$, again applying Lemmas 3.1, 3.4, 4.1, 4.2, the following boundary value problem

$$
\left\{\begin{array}{l}
\mathcal{D} \widetilde{\phi}_{1}(x)=0, x \in B_{ \pm} \\
C_{1,1}\left(E_{\frac{n+1}{2}} \widetilde{\phi}_{1}\right)^{+}(t)=C_{1,1}\left(E_{\frac{n+1}{2}} \widetilde{\phi}_{1}\right)^{-}(t)+\widetilde{f}_{1}(t), t \in S^{n-1}, \\
\liminf _{R \rightarrow+\infty} \frac{M\left(R, E_{\frac{n+1}{2}} \widetilde{\phi}_{1}\right)}{R^{r-1}}=L_{1}<+\infty
\end{array}\right.
$$

where

$$
\widetilde{\phi}_{1}(x)=\left\{\begin{array}{l}
\phi_{1}(x), x \in B_{+}, \\
\psi_{1}(x), x \in B_{-},
\end{array}\right.
$$

and

$$
\widetilde{f}_{1}(t)=\left\{\begin{array}{l}
f_{1}(t)-C_{1,2} C_{2,2}^{-1} t I_{\frac{n+1}{2}}^{2} \tilde{f}_{2}(t)-\cdots-C_{1, k-1} C_{k-1, k-1}^{-1} t^{k-2} I_{\left.\frac{n+1}{2}+\frac{k-1}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \widetilde{f}_{k-1}(t) \\
+\sum_{j=2}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-1}-t^{j-1}\right) I_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right] \cdots I_{\frac{n+1}{2}} \mathbf{C} \tilde{f}_{j}(t)}+\sum_{j=2}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-1}-t^{j-1}\right)\left(E_{\frac{n+1}{2}} g_{l_{j}}\right)^{-}(t) G, \text { if } k \text { odd, } \\
f_{1}(t)-C_{1,2} C_{2,2}^{-1} t I_{\frac{n+1}{2}} \widetilde{f}_{2}(t)-\cdots-C_{1, k-1} C_{k-1, k-1}^{-1} t^{k-2} I_{\left.\frac{n+1}{2}+\frac{k-1}{2}\right]} \ldots I_{\frac{n+1}{2}} \widetilde{f}_{k-1}(t) \\
+\sum_{j=2}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-1}-t^{j-1}\right) I_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \cdots I_{\frac{n+1}{2}} \mathbf{C} \widetilde{f}_{j}(t) \\
+\sum_{j=2}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j-1}-t^{j-1}\right)\left(E_{\frac{n+1}{2}} g_{l_{j}}\right)^{-}(t) G, \text { if } k \text { even, }
\end{array}\right.
$$

with $\left(E_{\frac{n+1}{2}} g_{l_{j}}\right)^{-} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(j=2,3, \ldots, k-1)$, has the unique solution

$$
\widetilde{\phi}_{1}(x)=\left\{\begin{array}{l}
C_{1,1}^{-1} I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{1}(\omega)+g_{l_{1}}(x), x \in B_{+}, \\
C_{1,1}^{-1} I_{\frac{n+1}{2}} \int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{1}(\omega)+g_{l_{1}}(x), x \in B_{-},
\end{array}\right.
$$

where $g_{l_{1}}(x)=\sum_{l_{1}=0}^{r-1} P_{l_{1}}(x), P_{l_{1}}$ being a left inner spherical monogenic polynomial of order $l_{1}$ on the variable $x$.

Finally, for $l=0$, in the same way we can show that the boundary value problem

$$
\left\{\begin{array}{l}
\mathcal{D} \widetilde{\phi}_{0}(x)=0, x \in B_{ \pm} \\
\widetilde{\phi}_{0}^{+}(t)=\widetilde{\phi}_{0}^{-}(t)+\widetilde{f}_{0}(t), t \in S^{n-1} \\
\liminf _{R \rightarrow+\infty} \frac{M\left(R, \widetilde{\phi}_{0}\right)}{R^{r}}=L_{0}<+\infty
\end{array}\right.
$$

where

$$
\widetilde{\phi}_{0}(x)=\left\{\begin{array}{l}
\phi_{0}(x), x \in B_{+}, \\
\psi_{0}(x), x \in B_{-},
\end{array}\right.
$$

and

$$
\begin{aligned}
\widetilde{f}_{0}(t)= & f_{0}(t)-\sum_{j=1}^{k-1} t^{j} C_{j, j}^{-1} t^{j} I_{\frac{n+1}{2}+\left[\frac{j}{2}\right]-1} \ldots I_{\frac{n+1}{2}} \widetilde{f}_{j}(t) \\
& +\sum_{j=1}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j}-t^{j}\right) I_{\frac{n+1}{2}+\left[\frac{j-l}{2}\right]} \ldots I_{\frac{n+1}{2}} \mathbf{C} \widetilde{f}_{j}(t) \\
& +\sum_{j=1}^{k-1} C_{k-1, j} C_{j, j}^{-1}\left((t-a)^{j}-t^{j}\right) g_{l_{j}}^{-}(t) G
\end{aligned}
$$

with $t^{j} \widetilde{\phi}_{j}^{-}(t) \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(j=1,2, \ldots, k-1)$, has the unique solution

$$
\widetilde{\phi}_{0}(x)=\left\{\begin{array}{l}
\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{0}(\omega)+g_{l_{0}}(x), x \in B_{+}, \\
\int_{S^{n-1}} E(\omega-x) d \sigma_{\omega} \widetilde{f}_{0}(\omega) G^{-1}+g_{l_{0}}(x), x \in B_{-},
\end{array}\right.
$$

where $g_{l_{0}}(x)=\sum_{l_{0}=0}^{r} P_{l_{0}}(x), P_{l_{0}}$ being a left inner spherical monogenic polynomial of order $l_{0}$ on the variable $x$.

Combining the above terms, Problem ( $*$ ) has the unique solution

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} x^{j} \phi_{j}(x), x \in B_{+}, \\
\sum_{j=0}^{k-1}(x-a)^{j} \psi_{j}(x), x \in B_{-},
\end{array}\right.
$$

where $\phi_{j}$ and $\psi_{j}(j=0,1,2, \ldots, k-1)$ are given explicitly as above.
The proof of the result is completed.
Remark 2 In the proof of Theorem 4.1, Problem ( $\star$ ) is solved by means of the Almansitype decomposition theorem and the inverse operators of the shifted Euler operators. This way can be also applied to solve the Dirchlet problem for null solutions to the iterated Dirac operator. A similar argument is used to discuss the Riemann-Hilbert boundary value problem on half space with boundary data in a Hölder space in [24], but this is only adapted to the case of half space. However, the method we used here can be applied to any arbitrary bounded star-like domain with Lyapunov boundary, which is different from the mentioned method in Ref. [24] due to Lemmas 4.1, 4.2.

In the above considerations the Almansi decomposition plays a key point. Next we will show that one can also use an integral representation in terms of the poly-Cauchy integral operator. To this end we will establish the corresponding integral kernels.

For arbitrary $x \in \mathbb{R}^{n}$, by Lemma 3.3, we get

$$
\mathcal{D} x^{j}=\left\{\begin{array}{l}
-2 m x^{2 m-1}, \quad \text { if } j=2 m, m \in \mathbb{N}, \\
-(n+2 m) x^{2 m}, \quad \text { if } j=2 m+1, m \in \mathbb{N} .
\end{array}\right.
$$

Hence, for arbitrary $x \in \mathbb{R}^{n} \backslash\{0\}$, we have

$$
\mathcal{D} \frac{\bar{x}^{j}}{|x|^{n}}=\left\{\begin{array}{l}
(2 m-n) \frac{\bar{x}^{2 m-1}}{|x|^{n}}, \quad \text { if } j=2 m, m \in \mathbb{N}, \\
2 m \frac{\bar{x}^{2 m}}{|x|^{n}}, \quad \text { if } j=2 m+1, m \in \mathbb{N} .
\end{array}\right.
$$

Next, for arbitrary $j \in \mathbb{N}, x \in \mathbb{R}^{n} \backslash\{0\}$, we introduce the function

$$
H_{j}(x)=\left\{\begin{array}{l}
\frac{c_{j}}{\omega_{n}} \frac{\bar{x}^{j}}{|x| n}, \quad \text { if } n \text { odd or } j<n, n \text { even, } \\
\frac{c_{n-1}}{(-1)^{\frac{n}{2}} \omega_{n}} \ln |x|, \quad \text { if } j=n, n \text { even, } \\
\frac{c_{n-1} b_{l}}{(-1)^{\frac{n}{2}} \omega_{n}} x^{l}\left(\ln |x|+\sum_{i=0}^{l-1} \frac{b_{i+1}}{b_{i}}\right), \quad \text { if } j>n, l=j-n, n \text { even, }
\end{array}\right.
$$

where $\omega_{n}$ denotes the area of the unit sphere in $\mathbb{R}^{n}$. The coefficients are given by

$$
c_{j}=\left\{\begin{array}{l}
\frac{1, j=1}{\frac{1}{2^{m-1}(m-1)!\prod_{l=1}^{m}(2 l-n)}}, j=2 m, m=1,2,3, \ldots, \\
\frac{1}{2^{m} m!\prod_{l=1}^{m}(2 l-n)}, j=2 m+1, m=1,2,3, \ldots,
\end{array}\right.
$$

and for $l=j-n$ with $j>n$,

$$
b_{j}=\left\{\begin{array}{l}
1, j=0 \\
\frac{1}{2^{m} m!\prod_{s=0}^{m-1}(2 s+n)}, l=2 m, m=1,2,3, \ldots, \\
-\frac{1}{2^{m} m!\prod_{s=1}^{m}(2 s+n)}, j=2 m+1, m=1,2,3, \ldots
\end{array}\right.
$$

It is easy to check that

$$
\left\{\begin{array}{l}
\mathcal{D} H_{1}(x)=H_{1}(x) \mathcal{D}=0, x \in \mathbb{R}^{n} \backslash\{0\} \\
\mathcal{D} H_{j+1}(x)=H_{j+1}(x) \mathcal{D}=H_{j}(x), x \in \mathbb{R}^{n} \backslash\{0\}, j \geq 2, j \in \mathbb{N}
\end{array}\right.
$$

Now, we will give another way to solve Riemann boundary value problem ( $\star$ ). For arbitrary $k \geq 2, k \in \mathbb{N}$, we first introduce the poly-Cauchy type integral

$$
\begin{equation*}
\Phi(x)=\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y), x \notin S^{n-1} \tag{3}
\end{equation*}
$$

with $f_{j} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty)(j=0,1,2, \ldots, k-1)$.
Theorem 4.2 If $f_{j} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty, j=0,1,2, \ldots, k-1)$, then for arbitrary $x \in \mathbb{R}^{n} \backslash S^{n-1}$, the above given function $\Phi($ see (3)) is well defined, and we have

$$
\begin{equation*}
\left(\mathcal{D}^{l} \Phi\right)(x)=\sum_{j=0}^{k-l-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j+l}(y), l=0,1,2, \ldots, k-1 \tag{4}
\end{equation*}
$$

In particular, we have $\left(\mathcal{D}^{k} \Phi\right)(x)=0$, i.e., $\Phi$ is a solution to $\mathcal{D}^{k} \phi=0$.
Proof Applying the properties of the function $H_{j}(j=0,1,2, \ldots, k-1)$, the result is immediate.

Theorem 4.3 If $f_{j} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty, j=0,1,2, \ldots, k-1$, $)$, then for arbitrary $x \in \mathbb{R}^{n} \backslash S^{n-1}$,

$$
\begin{align*}
\left(\mathcal{D}^{l} \Phi\right)^{ \pm}(t) & \triangleq \lim _{x \rightarrow t \in S^{n-1}} \mathcal{D}^{l} \Phi(x) \\
& = \pm \frac{1}{2} f_{l}(t)+\sum_{j=0}^{k-l-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j+l}(y) \tag{5}
\end{align*}
$$

for $l=0,1,2, \ldots, k-1$.
Moreover, $\left(\mathcal{D}^{l} \Phi\right)^{ \pm} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty)$ for $l=0,1,2, \ldots, k-1$.
Proof When $l=0$, it is necessary to prove that for arbitrary $x \in \mathbb{R}^{n} \backslash S^{n-1}$, we obtain

$$
\Phi^{ \pm}(t)=\lim _{x \rightarrow t \in S^{n-1}} \Phi(x)= \pm \frac{1}{2} f_{0}(t)+\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j}(y)
$$

For arbitrary $x \notin S^{n-1}$ we have

$$
\begin{aligned}
\Phi(x) & =\int_{S^{n-1}} H_{1}(y-x) d \sigma_{y} f_{0}(y)+\sum_{j=1}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y) \\
& =\Phi_{0}(x)+\widehat{\Phi}(x)
\end{aligned}
$$

On the one hand, for $f_{0} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty)$, we have

$$
\Phi_{0}^{ \pm}(t)=\lim _{x \rightarrow t \in S^{n-1}} \Phi_{0}(x)= \pm \frac{1}{2} f_{0}(t)+\int_{S^{n-1}} H_{1}(y-t) d \sigma_{y} f_{0}(y)
$$

On the other hand, for $f_{j} \in \mathbb{L}_{p}\left(S^{n-1}, \mathbb{C}_{n}\right)(1<p<+\infty)$, we remark that for $k<n$,

$$
\sum_{j=1}^{k-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j}(y) \text { has a weak singularity. }
$$

By the Lebesgue dominated convergence theorem, noticing that $\ln |x|$ has also only a weak singularity for $k \geq n$, we get

$$
\lim _{x \rightarrow t \in S^{n-1}} \widehat{\Phi}(x)=\sum_{j=1}^{k-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j}(y)
$$

Hence, we obtain

$$
\Phi^{ \pm}(t)= \pm \frac{1}{2} f_{0}(t)+\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j}(y)
$$

Similarly, for $l=1,2, \ldots, k-1$, by using Theorem 4.2 , we get

$$
\begin{aligned}
\left(\mathcal{D}^{l} \Phi\right)^{ \pm}(t) & =\lim _{x \rightarrow t \in S^{n-1}} \mathcal{D}^{l} \Phi(x) \\
& = \pm \frac{1}{2} f_{l}(t)+\sum_{j=0}^{k-l-1} \int_{S^{n-1}} H_{j+1}(y-t) d \sigma_{y} f_{j+l}(y)
\end{aligned}
$$

From this our result follows.
Now let us consider the following Riemann boundary value problem:

$$
(\star \star)\left\{\begin{array}{l}
\phi^{+}(t)-\phi^{+}(t)=f_{0}(t) \\
(\mathcal{D} \phi)^{+}(t)-(\mathcal{D} \phi)^{-}(t)=f_{1}(t) \\
\vdots \\
\left(\mathcal{D}^{l} \phi\right)^{+}(t)-\left(\mathcal{D}^{l} \phi\right)^{-}(t)=f_{l}(t) \\
\vdots \\
\left(\mathcal{D}^{k-1} \phi\right)^{+}(t)-\left(\mathcal{D}^{k-1} \phi\right)^{-}(t)=f_{k-1}(t)
\end{array}\right.
$$

Especially, when $f_{j}(t) \equiv 0, t \in S^{n-1}(j=0,1,2, \ldots, k-1)$, it further reduces to the case

Theorem 4.4 Problem ( $\star \star$ ) is solvable and its unique solution is expressed by

$$
\phi(x)=\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+\sum_{l=0}^{+\infty} P_{l}(x), x \in B_{ \pm}
$$

where $P_{l}$ is again a left inner spherical monogenic polynomial of order $l$ on the variable $x$.

Proof Let $\psi=\phi-\Phi$ where $\Phi$ is defined as the term (3). Applying Theorems 4.2,4.3, and the same argument as in Lemma 4.1[25], it follows the result.

By using Theorem 4.4 and Lemma 3.4, we get
Corollary 4.1 Consider boundary value problem ( $\star \star$ ). If

$$
\liminf _{R \rightarrow+\infty} \frac{M(R, \phi)}{R^{r}}=L<+\infty, r \geq k-1, r \in \mathbb{N}
$$

where $M(R, \phi)=\max _{|x|=R}|\phi(x)|$, then the solution to Problem $(\star \star)$ is given by

$$
\phi(x)=\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+P_{r}(x), x \in B_{ \pm},
$$

where $P_{r}(x)$ is a polynomial function of total degree no greater than $r$ on the variable $x \in \mathbb{R}^{n}$. Moreover, when $r=0$, the solution to boundary value problem ( $\star \star$ ) is simply given by

$$
\phi(x)=\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+d, x \in B_{ \pm}
$$

where $d \in \mathbb{C}_{n}$ is a constant.
Now, we are at the point to study the general case of Problem ( $\star$ ).
Theorem 4.5 Problem ( $\star$ ) is solvable and its unique solution is given by

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+\sum_{l=0}^{+\infty} P_{l}(x), x \in B_{+}, \\
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y) G^{-1}+\sum_{l=0}^{+\infty} P_{l}(x), x \in B_{-},
\end{array}\right.
$$

where for arbitrary $l \in \mathbb{N}, P_{l}$ is a left inner spherical monogenic polynomial of order $l$ on the variable $x$.

Proof Let

$$
\widetilde{\psi}(x)=\left\{\begin{array}{l}
\phi(x)-\Phi(x), x \in B_{+}, \\
\phi(x)-\Phi(x) G^{-1}, x \in B_{-},
\end{array}\right.
$$

where $\Phi$ is given by (3). By applying Theorem 4.4 we obtain the desired result.
Using Theorem 4.5 and Lemma 3.4, we obtain the following corollary.

Corollary 4.2 Consider boundary value problem (*). If

$$
\liminf _{R \rightarrow+\infty} \frac{M(R, \phi)}{R^{r}}=L<+\infty, r \geq k-1, r \in \mathbb{N}
$$

where $M(R, \phi)=\max _{|x|=R}|\phi(x)|$, then the solution to Problem $(\star)$ is given by

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+P_{r}(x), x \in B_{+} \\
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y) G^{-1}+P_{r}(x), x \in B_{-}
\end{array}\right.
$$

where $P_{r}(x)$ is a polynomial function of total degree no greater than $r$ on the variable $x \in \mathbb{R}^{n}$. Moreover, when $r=0$, the solution to boundary value problem ( $\star$ ) is simply given by

$$
\phi(x)=\left\{\begin{array}{l}
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y)+d, x \in B_{+}, \\
\sum_{j=0}^{k-1} \int_{S^{n-1}} H_{j+1}(y-x) d \sigma_{y} f_{j}(y) G^{-1}+d, x \in B_{-},
\end{array}\right.
$$

where $d \in \mathbb{C}_{n}$ is a constant.
Remark 3 By Theorem 4.5, Problem ( $\star$ ) is solved using the poly-Cauchy type integral operator. In [22] the analogous higher-order Cauchy type integral operator is used to solve the Riemann boundary value problem with boundary data in a Hölder space for bi-harmonic functions. Here, we construct the poly-Cauchy type integral operator to solve the corresponding Riemann boundary value problem for null solutions to iterated Dirac operator $\mathcal{D}^{k}$ with boundary data given in $\mathbb{L}_{p}(1<p<+\infty)$-space for arbitrary $k \geq 2(k \in \mathbb{N})$.

Remark 4 When $k=2 m, m \in \mathbb{N}$, Problem ( $\star$ ) is equivalent to the case

$$
(\star)\left\{\begin{array}{l}
\Delta^{m} \phi(x)=0, x \in B_{ \pm} \\
\phi^{+}(t)=\phi^{-}(t) G+f_{0}(t), t \in S^{n-1}, \\
(\mathcal{D} \phi)^{+}(t)=(\mathcal{D} \phi)^{-}(t) G+f_{1}(t), t \in S^{n-1}, \\
\vdots \\
\left(\mathcal{D}^{l} \phi\right)^{+}(t)=\left(\mathcal{D}^{l} \phi\right)^{-}(t) G+f_{l}(t), t \in S^{n-1}, \\
\vdots \\
\left(\mathcal{D}^{2 m-1} \phi\right)^{+}(t)=\left(\mathcal{D}^{2 m-1} \phi\right)^{-}(t) G+f_{2 m-1}(t), t \in S^{n-1},
\end{array}\right.
$$

where $G \in \mathbb{C}_{n}$ is a constant with an inverse denoted by $G^{-1}$. This means that the solution to RBVP for poly-harmonic functions can be given by means of Clifford analysis.

When $n=2$, Problem ( $\star$ ) reduces to the complex case

$$
(\star)\left\{\begin{array}{l}
\partial_{\bar{z}}^{k} \phi(x)=0, x \in B_{ \pm} \\
\phi^{+}(t)=\phi^{-}(t) G+f_{0}(t), t \in S^{n-1}, \\
\left(\partial_{\bar{z}} \phi\right)^{+}(t)=\left(\partial_{\bar{z}} \phi\right)^{-}(t) G+f_{1}(t), t \in S^{n-1}, \\
\vdots \\
\left(\partial_{\bar{z}}^{l} \phi\right)^{+}(t)=\left(\partial_{\bar{z}}^{l} \phi\right)^{-}(t) G+f_{l}(t), t \in S^{n-1}, \\
\vdots \\
\left(\partial_{\bar{z}}^{k-1} \phi\right)^{+}(t)=\left(\partial_{\bar{z}}^{k-1} \phi\right)^{-}(t) G+f_{k-1}(t), t \in S^{n-1}
\end{array}\right.
$$

where $\partial_{\bar{z}}=e_{1} \partial_{x_{1}}+e_{2} \partial_{x_{2}}, z=e_{1} x_{1}+e_{2} x_{2} \in \mathbb{R}^{2}(\cong \mathbb{C}), G$ is a constant with an inverse denoted by $G^{-1}$. This implies the RBVP for poly-analytic functions in the complex plane is a special case of our problem in Clifford analysis.

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