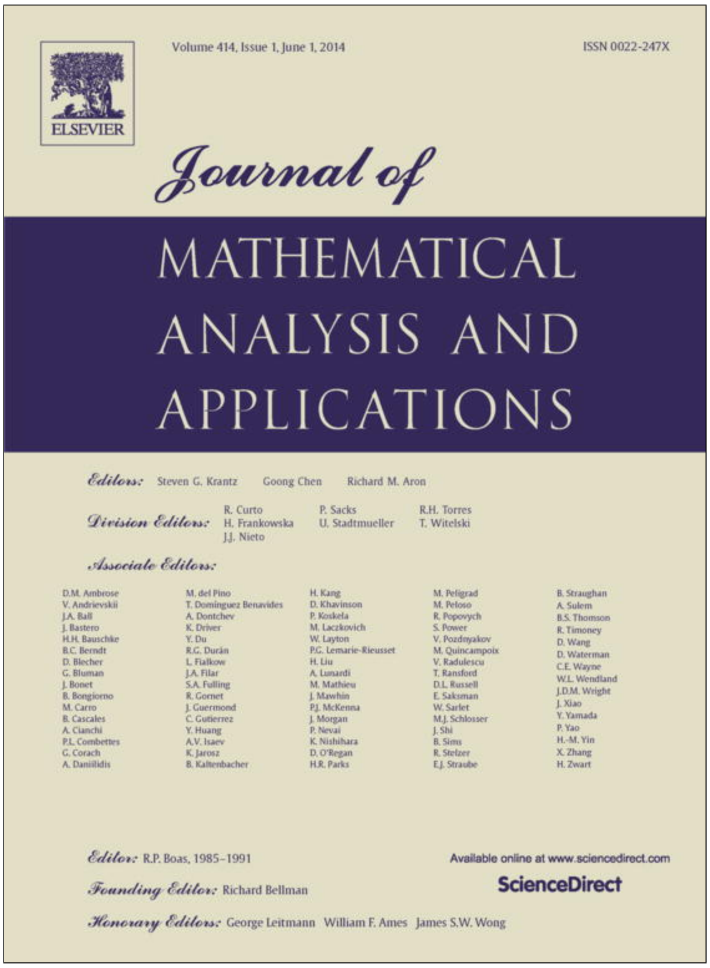
Volume 414, Issue 1, June 1, 2014
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
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Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 414 (2014) 176–187

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Fock space of metaanalytic functions

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ARTICLE INFO

Article history:
Received 12 April 2013
Available online 3 January 2014
Submitted by M. Peloso

Keywords:
Metaanalytic function
Bargmann transform
Fock space
Reproducing kernel

ABSTRACT

We develop the theory on the Fock space of metaanalytic functions, a generalization of some recent results on the Fock space of polyanalytic functions. We show that the metaanalytic Bargmann transform is a unitary mapping between vector-valued Hilbert spaces and metaanalytic Fock spaces. A reproducing kernel of the metaanalytic Fock space is derived in an explicit form. Furthermore, we establish a complete characterization of all lattice sampling and interpolating sequence for the Fock space of metaanalytic functions.

1. Introduction

A function $f(z)$, $z = x + i\omega$, that has continuous partial derivative with respect to x and ω up to order $n \geqslant 1$ on the whole complex plane $\mathbb{C}$ is called a polyanalytic function on $\mathbb{C}$ if it satisfies the generalized Cauchy–Riemann equation

$$\frac{\partial^n f}{\partial \bar{z}^n} = 0, \quad \forall z \in \mathbb{C}, \tag{1.1}$$

where the Cauchy–Riemann operator is defined by

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \omega}\right). \tag{1.2}$$

Polyanalytic functions inherit some of the properties of analytic functions, often in a nontrivial form. However, many of the properties break down once we leave the analytic setting. A clear difference lies in the structure of the zeros. A theory on polyanalytic functions had been investigated thoroughly, notably by the Russian school led by Balk [5], and provided extensions of the classical operators from complex analysis [6].

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.12.058

It is well known that there exists a close connection between the classical Bargman–Fock space of analytic functions and the time-frequency analysis. Specifically, up to a certain weight, the Gabor transform (also called short-time Fourier transform or windowed Fourier transform) with a Gaussian window is tightly related to the Bargmann transform and the corresponding Fock space of analytic functions [10]. Moreover, it had been shown that this is the only choice leading to spaces of analytic functions [4].

Recently, a series of works have been devoted to the theory of the vector-valued Gabor frames (sometimes called Gabor superframe) with Hermite functions [1,9,11,12]. It is worth pointing out that a new connection between polyanalytic functions and time-frequency analysis is established. Actually, the Gabor transform with the kth Hermite function is shown to be a polyanalytic function of order $k+1$, which is corresponding to the true polyanalytic Bargmann transform of order k, a unitary mapping between the Hilbert spaces and true polyanalytic Fock spaces [1–3]. The polyanalytic Fock space of order n can be orthogonally decomposed into a superposition of all true polyanalytic Fock spaces up to order $n-1$. Moreover, the concept of interpolating and sampling sequences corresponds to the case where stable numerical reconstruction of a function from its samples is possible [13]. A complete characterization of all lattice sampling and interpolating sequence was developed for the Fock space of polyanalytic functions, or equivalently, for all vector-valued Gabor frames with Hermite functions [1]. Furthermore, in [2], the authors studied the structure of Gabor and super-Gabor space and obtained an explicit formula of reproducing kernel for the Fock space of polyanalytic functions.

As mentioned briefly in Balk's monography [5], polyanalytic functions can be extended to more general cases, such as metaanalytic functions. Metaanalytic functions are closely connected with many applications in mathematics and physics, some works on them had captured the attention of many researchers [8,14–16].

In this paper, our original goal is to deal with the Fock space of metaanalytic functions, which is a generalization of the Fock space of polyanalytic functions, recently proposed by Abreu [1,2]. We establish the definition of the metaanalytic Bargmann transform and prove it to be a unitary mapping between vector-valued Hilbert spaces and metaanalytic Fock spaces. Moreover, we obtain an explicit formula for the reproducing kernel of the metaanalytic Fock space from which growth estimates can be derived. As for sampling and interpolating in the Fock space of metaanalytic functions, we show them to be equivalent to the cases in the Fock space of polyanalytic functions.

This paper is organized as follows: Section 2 is devoted to reviewing some definitions and basic properties of metaanalytic functions and the Bargmann transform. In Section 3, we provide the definitions of the true metaanalytic Bargmann transform and metaanalytic Bargmann transform, and show some properties of them. In Section 4, we proceed with the study of the reproducing kernel in the metaanalytic Fock space. Furthermore, in Section 5, we establish a complete characterization of all lattice sampling and interpolating sequence for the Fock space of metaanalytic functions.

2. Preliminaries

2.1. Metaanalytic functions

To generalize the definition of polyanalytic function given by (1.1), let

$$\mathcal{M}_n := \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{\partial^k}{\partial \bar{z}^k} = \left(\frac{\partial}{\partial \bar{z}} - \lambda \right)^n \tag{2.1}$$

be a polynomial of Cauchy–Riemann operator $\frac{\partial}{\partial \bar{z}}$ defined by (1.2), where λ is a complex constant.

Definition 2.1. A function $f(z)$ that has continuous partial derivatives with respect to x and ω up to order $n \geqslant 1$ is called a metaanalytic function of order n on $\mathbb{C}$ if it satisfies $\mathcal{M}_n f = 0$ on $\mathbb{C}$, denoted by $f \in \mathcal{M}_n(\mathbb{C})$, where the operator $\mathcal{M}_n$ is defined by (2.1).

In particular, if we take $\lambda = 0$ in (2.1), then $\mathcal{M}_n(\mathbb{C})$ is just the class of polyanalytic functions of order n on $\mathbb{C}$ denoted by $\mathcal{P}_n(\mathbb{C})$. Obviously, $\mathcal{P}_1(\mathbb{C})$ denotes the class of the entire functions on $\mathbb{C}$. It is well known that we have the following two decomposition theorems (see [5] or [16] for details).

Theorem 2.2 *(Decomposition theorem). If $g(z) \in \mathcal{P}_n(\mathbb{C})$, then there exists uniquely the expression*

$$g(z) = \sum_{k=0}^{n-1} \bar{z}^k \varphi_k(z), \quad \text{with } \varphi_k(z) \in \mathcal{P}_1(\mathbb{C}),\ z \in \mathbb{C},$$

that is to say, it holds $\mathcal{P}_n(\mathbb{C}) = \bigoplus_{k=0}^{n-1} \bar{z}^k \mathcal{P}_1(\mathbb{C})$.

Theorem 2.3 *(Factorization theorem). If $f(z) \in \mathcal{M}_n(\mathbb{C})$, then there exist unique functions $\varphi_1 \in \mathcal{P}_n(\mathbb{C})$ and $\varphi_2 \in \mathcal{P}_n(\mathbb{C})$ such that*

$$f(z) = \varphi_1(z) e^{\lambda \bar{z}} = \varphi_2(z) e^{\lambda \bar{z} - \bar{\lambda} z}, \quad \forall z \in \mathbb{C}. \tag{2.2}$$

Corollary 2.1. *If functions $\varphi \in \mathcal{P}_n(\mathbb{C})$, then the associated functions*

$$f(z) = \varphi(z) e^{\lambda \bar{z} - \bar{\lambda} z}, \quad \forall z \in \mathbb{C} \tag{2.3}$$

are metaanalytic functions of order n, i.e., $f \in \mathcal{M}_n(\mathbb{C})$.

It is easy to check that the factor $e^{\lambda \bar{z} - \bar{\lambda} z}$ in (2.2) satisfies the following properties

$$\left| e^{\lambda \bar{z} - \bar{\lambda} z} \right| = 1 \tag{2.4}$$

and

$$e^{\lambda \bar{z} - \bar{\lambda} z} \overline{e^{\lambda \bar{z} - \bar{\lambda} z}} = 1, \tag{2.5}$$

which will be fundamental together with Theorems 2.2 and 2.3 in what follows.

2.2. The Bargmann transform

The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to a non-zero window function $g \in L^2(\mathbb{R})$ is defined, for every $x, \omega \in \mathbb{R}$, by

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i t \omega}\, dt = \langle f, M_\omega T_x g \rangle_{L^2(\mathbb{R})}, \tag{2.6}$$

where the translation operator T_x and modulation operator M_ω are defined by $T_x g(t) = g(t-x)$, $M_\omega g(t) = e^{2\pi i \omega t} g(t)$, respectively. The following relation on inner products of the Gabor transform, called orthogonal relations and corresponding to Parseval's identity, will be used. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^2)$ for $j = 1, 2$, and it holds

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R})} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R})}. \tag{2.7}$$

In particular, if we choose the Gaussian window $h_0(t) = 2^{1/4} e^{-\pi t^2}$ as a window function in (2.6), then a simple calculation leads to

$$V_{h_0} f(x, -\omega) = e^{\pi i \omega x - \pi \frac{|z|^2}{2}} Bf(z), \quad z = x + i\omega, \tag{2.8}$$

where $Bf(z)$ denotes the Bargmann transform of a function $f \in L^2(\mathbb{R})$, defined by

$$Bf(z) := 2^{1/4} e^{-\pi z^2/2} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi t z}\, dt. \tag{2.9}$$

The Bargmann transform B is a unitary mapping between the Hilbert space $L^2(\mathbb{R})$ and the Fock space $\mathcal{F}(\mathbb{C})$, consisting of all entire functions $F \in \mathcal{P}_1(\mathbb{C})$ such that

$$\|F\|^2_{\mathcal{F}(\mathbb{C})} := \int_{\mathbb{C}} \left|F(z)\right|^2 e^{-\pi|z|^2}\, dz < \infty.$$

The inner product on $\mathcal{F}(\mathbb{C})$ is

$$\langle F, G\rangle_{\mathcal{F}(\mathbb{C})} := \int_{\mathbb{C}} F(z)\overline{G(z)} e^{-\pi|z|^2}\, dz.$$

As we all know, the set $\{h_n\}_{n=0}^{\infty}$ of the Hermite functions, defined by

$$h_n(t) := c_n e^{\pi t^2} \frac{d^n}{dt^n} e^{-2\pi t^2}, \quad n = 0, 1, 2, \ldots, \tag{2.10}$$

forms an orthonormal basis in $L^2(\mathbb{R})$, where the coefficients c_n are chosen in order to have $\|h_n\|_{L^2(\mathbb{R})} = 1$ and $h_0(t)$ corresponds to the Gaussian function mentioned above. The collection of the image of the Hermite functions under the Bargmann transform B is the set $\{e_n\}_{n=0}^{\infty}$, the natural orthonormal basis in the Fock space $\mathcal{F}(\mathbb{C})$,

$$Bh_n(z) = e_n(z), \tag{2.11}$$

where the monomials $e_n(z) = (\frac{\pi^n}{n!})^{1/2} z^n$, $n = 0, 1, 2, \ldots$.

The reproducing kernel of the Fock space $\mathcal{F}(\mathbb{C})$ is the function $e^{\pi \bar{z} w}$, precisely,

$$F(z) = \left\langle F(w), e^{\pi \bar{z} w}\right\rangle_{\mathcal{F}(\mathbb{C})}, \quad \forall F \in \mathcal{F}(\mathbb{C}). \tag{2.12}$$

We introduce the shift $\beta_z : \mathcal{F}(\mathbb{C}) \to \mathcal{F}(\mathbb{C})$ (see [12], change the notation with ζ replaced by $\bar{z}$ in Eq. (26)), defined by

$$\beta_z F(\zeta) := e^{-\pi i \omega x - \pi \frac{|z|^2}{2}} e^{\pi \bar{z} \zeta} F(\zeta - z), \quad z = x + i\omega. \tag{2.13}$$

Then the operator $\beta_{\bar{z}}$ is unitary on the Fock space $\mathcal{F}(\mathbb{C})$, and the Bargmann transform B intertwines the Fock space shift and the time-frequency shift:

$$\beta_z B = BM_{-\omega} T_x, \quad z = x + i\omega. \tag{2.14}$$

3. Metaanalytic Fock space and metaanalytic Bargmann transform

3.1. Metaanalytic Fock space and true metaanalytic Fock space

The so-called metaanalytic Fock space, denoted by $\mathbf{F}^n_\lambda(\mathbb{C})$, is the Hilbert space consisting of all metaanalytic functions $f \in \mathcal{M}_n(\mathbb{C})$ of order n given by Definition 2.1, and such that

$$\|F\|^2_{\mathbf{F}^n_\lambda(\mathbb{C})} := \int_{\mathbb{C}} \left|F(z)\right|^2 e^{-\pi|z|^2}\, dz < \infty.$$

The inner product on $\mathbf{F}^n_\lambda(\mathbb{C})$ is

$$\langle F, G\rangle_{\mathbf{F}^n_\lambda(\mathbb{C})} := \int_{\mathbb{C}} F(z)\overline{G(z)}e^{-\pi|z|^2}\,dz.$$

Observe also that this implies

$$\langle F, G\rangle_{\mathbf{F}^n_\lambda(\mathbb{C})} = \langle F, G\rangle_{L^2(\mathbb{R}^2, e^{-\pi|z|^2})} = \left\langle e^{-\pi\frac{|z|^2}{2}}F, e^{-\pi\frac{|z|^2}{2}}G\right\rangle_{L^2(\mathbb{R}^2)}.$$

We remark that the metaanalytic Fock space $\mathbf{F}^n_\lambda(\mathbb{C})$ reduces to the polyanalytic Fock space $\mathbf{F}^n_0(\mathbb{C})$ if we take $\lambda = 0$ and to the Fock space $\mathcal{F}(\mathbb{C})$ when $\lambda = 0$ and $n = 1$, i.e., $\mathbf{F}^1_0(\mathbb{C}) = \mathcal{F}(\mathbb{C})$, with the same inner product.

Theorems 2.2 and 2.3 and Corollary 2.1 imply that the metaanalytic Fock space $\mathbf{F}^n_\lambda(\mathbb{C})$ admits the following decomposition in terms of the so-called true metaanalytic Fock spaces $\mathcal{F}^k_\lambda(\mathbb{C})$, $k = 0, 1, \dots, n-1$,

$$\mathbf{F}^n_\lambda(\mathbb{C}) = \mathcal{F}^0_\lambda(\mathbb{C}) \oplus \mathcal{F}^1_\lambda(\mathbb{C}) \oplus \cdots \oplus \mathcal{F}^{n-1}_\lambda(\mathbb{C}), \tag{3.1}$$

where the true metaanalytic Fock spaces $\mathcal{F}^k_\lambda(\mathbb{C})$ is also the Hilbert space of all metaanalytic functions $\psi \in \mathcal{M}_{k+1}(\mathbb{C})$ satisfying

$$\psi(z) = e^{\lambda\bar{z} - \bar{\lambda}z}\bar{z}^k\varphi_k(z), \quad \varphi_k(z) \in \mathcal{P}_1(\mathbb{C}),\ k = 0, 1, \dots, n-1,$$

with the same inner product as that in the metaanalytic Fock spaces.

3.2. *True metaanalytic Bargmann transform*

Consider general Hermite functions h_n defined by (2.10) as a window functions for the Gabor transform in (2.6). Using the shift operator β_z defined by (2.13), recall that an easy computation (see [12] or [1] for details) leads to

$$e^{\pi\frac{|z|^2}{2} - \pi i\omega x}V_{h_k}f(x, -\omega) = B^k f(z), \tag{3.2}$$

where $B^k f(z)$ is the so-called true polyanalytic Bargmann transform of order k, defined by

$$\begin{aligned} B^k f(z) &:= \left(\pi^k k!\right)^{-1/2}\sum_{j=0}^{k}\binom{k}{j}(-\pi\bar{z})^j\left(\frac{d}{dz}\right)^{k-j}Bf(z) \\ &= \left(\pi^k k!\right)^{-1/2}e^{\pi|z|^2}\frac{\partial^k}{\partial z^k}\left[e^{-\pi|z|^2}Bf(z)\right]. \end{aligned} \tag{3.3}$$

It is clear that (3.2) is a generalization of (2.8). By (3.3) and Theorem 1.1 we see that $B^k f(z) \in \mathcal{P}_{k+1}(\mathbb{C})$. Associated with Corollary 2.3, we give the following definition for the metaanalytic case.

Definition 3.1. The true metaanalytic Bargmann transform of order k, of a function on $\mathbb{R}$, is defined for $k = 0, 1, \dots, n-1$ by the formula

$$B^k_\lambda f(z) := \left(\pi^k k!\right)^{-1/2}e^{\lambda\bar{z} - \bar{\lambda}z}e^{\pi|z|^2}\frac{\partial^k}{\partial z^k}\left[e^{-\pi|z|^2}Bf(z)\right], \tag{3.4}$$

where the Bargmann transform $Bf(z)$ is defined as in (2.9).

Note that B_0^k is the true polyanalytic Bargmann transform B^k given by (3.3) and B_0^0 is the Bargmann transform B defined by (2.9).

We will provide the fundamental properties of the true metaanalytic Bargmann transform B_λ^k following the line of [1]. Based on (3.2), (2.4) and Corollary 2.1, the proof of the following theorem is straightforward. For simplicity, we omit it here (see [1, Proposition 1] for details).

Theorem 3.2.

(1) *If $f(t)$ is a function on $\mathbb{R}$ with polynomial growth, then its true metaanalytic Bargmann transform $B_\lambda^k f(z)$, given by (3.4), is a metaanalytic function of order $k+1$ on $\mathbb{C}$, i.e., $B_\lambda^k f(z) \in \mathcal{M}_{k+1}(\mathbb{C})$.*

(2) *If we write $z = x + i\omega$, then B_λ^k is related to the Gabor transform with Hermite windows $h_k(t)$ in the following way*

$$V_{h_k} f(x, -\omega) = e^{\pi i\omega x - \pi\frac{|z|^2}{2}} e^{\bar{\lambda} z - \lambda\bar{z}} B_\lambda^k f(z), \tag{3.5}$$

where the Hermite functions h_k are defined as in (2.10).

(3) *If $f \in L^2(\mathbb{R})$, then $B_\lambda^k f(z) \in \mathcal{F}_\lambda^k(\mathbb{C})$ and*

$$\left\| B_\lambda^k f \right\|_{\mathcal{F}_\lambda^k(\mathbb{C})} = \|f\|_{L^2(\mathbb{R})}, \quad k = 0, 1, \ldots, n-1. \tag{3.6}$$

The true metaanalytic Bargmann transform B_λ^k keeps the unitary property as the Bargmann transform B and true polyanalytic Bargmann transform B^k.

Theorem 3.3. *The true metaanalytic Bargmann transform B_λ^k, defined as in (3.4), is an isometric isomorphism*

$$B_\lambda^k : L^2(\mathbb{R}) \to \mathcal{F}_\lambda^k(\mathbb{C}), \quad k = 0, 1, \ldots, n-1. \tag{3.7}$$

Proof. By (3.6) we know that B_λ^k is isometry. Now we only need to show that B_λ^k maps an orthonormal basis of $L^2(\mathbb{R})$ to an orthonormal basis of $\mathcal{F}_\lambda^k(\mathbb{C})$. Since the sets $\{h_m\}_{m=0}^\infty$ of the Hermite functions, defined by (2.10), forms an orthonormal basis in $L^2(\mathbb{R})$, for a fixed k, $0 \leqslant k \leqslant n-1$, by Theorem 3.2 we consider the image of the Hermite functions h_m under the true metaanalytic transform B_λ^k

$$e_{k,m}(z) = B_\lambda^k h_m(z), \quad m = 0, 1, 2, \ldots. \tag{3.8}$$

On the one hand, for a fixed k, the set $\{e_{k,m}\}_{m=0}^\infty$ is an orthonormal system of the true metaanalytic Fock spaces $\mathcal{F}_\lambda^k(\mathbb{C})$. In fact, by (2.4), (2.7) and (3.5) a simple calculation leads to

$$\begin{aligned}
\langle e_{k,m}, e_{k,j}\rangle_{\mathcal{F}_\lambda^k(\mathbb{C})} &= \left\langle B_\lambda^k h_m, B_\lambda^k h_j \right\rangle_{\mathcal{F}_\lambda^k(\mathbb{C})} \\
&= \left\langle e^{\pi\frac{|z|^2}{2} - \pi i\omega x} e^{\bar{\lambda}\bar{z} - \lambda z} V_{h_k} h_m, e^{\pi\frac{|z|^2}{2} - \pi i\omega x} e^{\bar{\lambda}\bar{z} - \lambda z} V_{h_k} h_j \right\rangle_{\mathcal{F}_\lambda^k(\mathbb{C})} \\
&= \langle V_{h_k} h_m, V_{h_k} h_j\rangle_{L^2(\mathbb{R}^2)} \\
&= \langle h_m, h_j\rangle_{L^2(\mathbb{R})} \\
&= \delta_{m,j}.
\end{aligned}$$

On the other hand, for a fixed k, to prove the completeness of the set $\{e_{k,m}\}_{m=0}^\infty$ in the true metaanalytic Fock spaces $\mathcal{F}_\lambda^k(\mathbb{C})$, we start by supposing that $F \in \mathcal{F}_\lambda^k(\mathbb{C})$ is such that

$$\langle F, e_{k,m}\rangle_{\mathcal{F}_\lambda^k(\mathbb{C})} = 0, \quad m = 0, 1, 2, \ldots,$$

that is to say, by (2.5), (3.4) and (3.8) we have

$$\left\langle e^{\lambda\bar{z}-\bar{\lambda}z}F, B^k h_m\right\rangle_{L^2(\mathbb{C},e^{-\pi|z|^2})} = 0,$$

where the true polyanalytic Bargmann transform B^k is defined by (3.3) and $e^{\lambda\bar{z}-\bar{\lambda}z}F$ is a metaanalytic function in terms of Corollary 2.1. Consequently, due to the fact that the set $\{B^k h_m(z)\}_{m=0}^{\infty}$ is an orthonormal basis in the true polyanalytic Fock space (see [1, Proposition 2]), we obtain $e^{\lambda\bar{z}-\bar{\lambda}z}F = 0$, $\forall z \in \mathbb{C}$, which implies $F = 0$. $\square$

Based on the proof of Theorem 3.6, it follows that $\{e_{k,m}\}_{m=0}^{\infty}$ is an orthonormal basis of the true metaanalytic Fock spaces $\mathcal{F}_\lambda^k(\mathbb{C})$, thus, for $0 \leqslant k \leqslant n-1$ we have

$$\mathcal{F}_\lambda^k(\mathbb{C}) = \mathrm{Span}\left[\{e_{k,m}\}_{m=0}^{\infty}\right].$$

Furthermore, on account of (3.1), we can also check the fact that the set $\{\{e_{k,m}\}_{m=0}^{\infty}\}_{k=0}^{n-1}$ forms an orthonormal basis of the metaanalytic Fock spaces $\mathbf{F}_\lambda^n(\mathbb{C})$.

3.3. Metaanalytic Bargmann transform

Suppose $f_k \in L^2(\mathbb{R})$, $k = 0, 1, \ldots, n-1$, consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$ with the inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\mathcal{H}} := \sum_{k=0}^{n-1} \langle f_k, g_k\rangle_{L^2(\mathbb{R})}. \tag{3.9}$$

Moreover, associated with the true metaanalytic Bargmann transform B_λ^k defined as in (3.4), the metaanalytic Bargmann transform of a vector-valued function $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$ is defined by

$$\mathbf{B}_\lambda^n \mathbf{f}(z) := \sum_{k=0}^{n-1} B_\lambda^k f_k(z). \tag{3.10}$$

The principle significance of the following theorem is that it allows us to discuss the main result regarding sampling in the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$.

Theorem 3.4. *The metaanalytic Bargmann transform $\mathbf{B}_\lambda^n \mathbf{f}$, defined as in (3.10), is an isometric isomorphism*

$$\mathbf{B}_\lambda^n : \mathcal{H} \to \mathbf{F}_\lambda^n(\mathbb{C}).$$

Proof. At first, we prove the isometry. For the true metaanalytic Bargmann transform B_λ^k defined as in (3.4), by Theorem 3.2 we have $B_\lambda^k f_k \in \mathcal{F}_\lambda^k(\mathbb{C})$. Since $\mathcal{F}_\lambda^k(\mathbb{C}) \subset \mathbf{F}_\lambda^n(\mathbb{C})$, $k = 0, \ldots, n-1$, we consider the inner product

$$\begin{aligned}
\left\langle B_\lambda^k f_k, B_\lambda^j f_j\right\rangle_{\mathbf{F}_\lambda^n(\mathbb{C})} &= \left\langle e^{\pi\frac{|z|^2}{2}-\pi i\omega x} e^{\bar{\lambda}\bar{z}-\lambda z} V_{h_k} f_k, e^{\pi\frac{|z|^2}{2}-\pi i\omega x} e^{\bar{\lambda}\bar{z}-\lambda z} V_{h_j} f_j\right\rangle_{\mathbf{F}_\lambda^n(\mathbb{C})} \\
&= \langle V_{h_k} f_k, V_{h_j} f_j\rangle_{L^2(\mathbb{R}^2)} \\
&= \overline{\langle h_k, h_j\rangle}_{L^2(\mathbb{R})} \langle f_k, f_j\rangle_{L^2(\mathbb{R})} \\
&= \delta_{k,j} \langle f_k, f_j\rangle_{L^2(\mathbb{R})},
\end{aligned} \tag{3.11}$$

where we use (2.5), (2.7) and (3.5). Then, by (3.11) we obtain

$$\left\|\mathbf{B}_\lambda^n\mathbf{f}\right\|^2_{\mathbf{F}_\lambda^n(\mathbb{C})} = \left\langle \mathbf{B}_\lambda^n\mathbf{f}, \mathbf{B}_\lambda^n\mathbf{f}\right\rangle_{\mathbf{F}_\lambda^n(\mathbb{C})} = \left\langle \sum_{k=0}^{n-1} B_\lambda^k f_k, \sum_{j=0}^{n-1} B_\lambda^j f_j \right\rangle_{\mathbf{F}_\lambda^n(\mathbb{C})}$$
$$= \sum_{k=0}^{n-1} \langle f_k, f_k\rangle_{L^2(\mathbb{R})} = \|f\|_{\mathcal{H}}.$$

Moreover, $\mathbf{B}_\lambda^n[\mathcal{H}]$ is dense in $\mathbf{F}_\lambda^n(\mathbb{C})$. In fact, by the decomposition (3.1), every element $F \in \mathbf{F}_\lambda^n(\mathbb{C})$ can be written as $F = \sum_{k=0}^{n-1} F_k$, where $F_k \in \mathcal{F}_\lambda^k(\mathbb{C})$, $k = 0, 1, \ldots, n-1$. Since Theorem 3.4 tells us that B_λ^k is a unitary mapping between $L^2(\mathbb{R})$ and $\mathcal{F}_\lambda^k(\mathbb{C})$, there exists $f_k \in L^2(\mathbb{R})$ such that $F_k = B_\lambda^k f_k$ for every $0 \leqslant k \leqslant n-1$. It follows that $F = \mathbf{B}_\lambda^n\mathbf{f}$ with $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$. $\square$

4. Reproducing kernel in metaanalytic Fock space

Denote the subspace of $L^2(\mathbb{R}^2)$, which is the image of $L^2(\mathbb{R})$ under the Gabor transform with window functions $g \in L^2(\mathbb{R})$, by $\mathcal{S}_g$ such that

$$\mathcal{S}_g := \left\{V_g f(x, -\omega)\colon\ f \in L^2(\mathbb{R})\right\}.$$

It is well known (see [7]) that the Gabor spaces $\mathcal{S}_g$ have a reproducing kernel given by

$$K_g(u, \eta, x, \omega) = \langle M_{-\omega}T_x g, M_{-\eta}T_u g\rangle_{L^2(\mathbb{R})}. \tag{4.1}$$

We remark that we use the time-frequency shift $M_{-\omega}T_x$ in (4.1) to get the reproducing kernel here because we use the Gabor transform $V_g f(x, -\omega)$ to establish the relation to the metaanalytic Bargmann transform $B_\lambda^k f(z)$ in (3.5).

Analogously to the proof of Theorem 2 in [2], a reproducing kernel of Gabor spaces with Hermite windows h_k, defined by (2.10), can be provided by

$$K_{h_k}(w, z) = \frac{1}{k!} e^{i\pi(u\eta - x\omega) + \pi\frac{|w|^2 - |z|^2}{2}} \frac{\partial^k}{\partial w^k}\left[e^{\pi\bar{z}w - \pi|w|^2}(w - z)^k\right] \tag{4.2}$$

with $z = x + i\omega$ and $w = u + i\eta$. Note that we slightly modify the result given by Theorem 2 in [2] due to the reason that we use intertwining relation (2.13) and (2.14), which are a little different from the cases introduced in [2].

Since the Fock space $\mathcal{F}(\mathbb{C})$ is related to the Gabor transform with the Gaussian window h_0, setting $k = 0$, this reproducing kernel can be related with the reproducing kernel of the Fock space in the following way

$$K_{h_0}(w, z) = e^{i\pi(u\eta - x\omega) - \pi\frac{|w|^2 + |z|^2}{2}} e^{\pi\bar{z}w},$$

where $\mathcal{K}_0(w, z) = e^{\pi\bar{z}w}$ is the reproducing kernel of the Fock space $\mathcal{F}(\mathbb{C})$ given in (2.12).

Furthermore, by (3.5) we see that the true metaanalytic Fock space $\mathcal{F}_\lambda^k(\mathbb{C})$ is related to the Gabor space with Hermite functions h_k, $0 \leqslant k \leqslant n-1$. Then, we show that the true metaanalytic Fock space $\mathcal{F}_\lambda^k(\mathbb{C})$ has a reproducing kernel and compute it explicitly associated with (4.2).

Theorem 4.1. *Let $K_{h_k}(w, z)$ be the reproducing kernel of the Gabor space $\mathcal{S}_{h_k}$. Then, the true metaanalytic Fock space $\mathcal{F}_\lambda^k(\mathbb{C})$ is a Hilbert space with a reproducing kernel, $\mathcal{K}_k(z, w)$, satisfying*

$$\mathcal{K}_k(w, z) = e^{\pi i(\omega x - \eta u) + \pi\frac{|w|^2 + |z|^2}{2}} e^{\bar{\lambda}(z - w) - \lambda(\bar{z} - \bar{w})} K_{h_k}(w, z). \tag{4.3}$$

Proof. Given $F \in \mathcal{F}_\lambda^k(\mathbb{C})$, from (3.5) it follows that there exists $f \in \mathcal{S}_{h_k}$ such that

$$f(z) = e^{\pi i \omega x - \pi \frac{|z|^2}{2}} e^{\bar{\lambda} z - \lambda \bar{z}} F(z).$$

Since $\mathcal{S}_{h_k}$ is a Hilbert space with the reproducing kernel $K_{h_k}(w,z)$, the reproducing property yields

$$f(z) = \big\langle f(w), K_{h_k}(w,z) \big\rangle_{L^2(\mathbb{R})},$$

or

$$\begin{aligned} e^{\pi i \omega x - \pi \frac{|z|^2}{2}} e^{\bar{\lambda} z - \lambda \bar{z}} F(z) &= \big\langle e^{\pi i \eta u - \pi \frac{|w|^2}{2}} e^{\bar{\lambda} w - \lambda \overline{w}} F(w), K_{h_k}(w,z) \big\rangle_{L^2(\mathbb{R})} \\ &= \big\langle e^{\pi i \eta u + \pi \frac{|w|^2}{2}} e^{\bar{\lambda} w - \lambda \overline{w}} F(w), K_{h_k}(w,z) \big\rangle_{\mathcal{F}_\lambda^k(\mathbb{R})}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} F(z) &= \big\langle e^{\pi i (\eta u - \omega x) + \pi \frac{|w|^2 + |z|^2}{2}} e^{\lambda(\bar{z} - \overline{w}) - \bar{\lambda}(z - w)} F(w), K_{h_k}(w,z) \big\rangle_{\mathcal{F}_\lambda^k(\mathbb{R})} \\ &= \big\langle F(w), e^{\pi i (\omega x - \eta u) + \pi \frac{|w|^2 + |z|^2}{2}} e^{\bar{\lambda}(z - w) - \lambda(\bar{z} - \overline{w})} K_{h_k}(w,z) \big\rangle_{\mathcal{F}_\lambda^k(\mathbb{R})} \\ &= \big\langle F(w), \mathcal{K}_k(w,z) \big\rangle_{\mathcal{F}_\lambda^k(\mathbb{R})}. \qquad \square \end{aligned}$$

Combining (4.3) and (4.2) leads to an explicit representation for the reproducing kernel of the true metaanalytic Fock space $\mathcal{F}_\lambda^k(\mathbb{C})$ given by

$$\mathcal{K}_k(w,z) = \frac{1}{k!} e^{\pi |w|^2} e^{\bar{\lambda}(z-w) - \lambda(\bar{z} - \overline{w})} \frac{\partial^k}{\partial w^k} \big[e^{\pi \bar{z} w - \pi |w|^2} (w-z)^k \big]. \tag{4.4}$$

Note that $\mathcal{F}_0^k(\mathbb{C})$ is the true polyanalytic Fock space. If we take $\lambda = 0$ in (4.4), we recover the reproducing kernel of the true polyanalytic Fock space $\mathcal{F}_0^k(\mathbb{C})$ (see [2, Corollary 5]). If we take $\lambda = 0$ and $k = 0$, $\mathcal{F}_0^0(\mathbb{C}) = \mathcal{F}(\mathbb{C})$ accordingly, then (4.4) reduces to the reproducing kernel in the Fock space given by (2.12).

Recall the fact that, valid in any Hilbert space H with a reproducing kernel $K(w,z)$, if we can estimate the diagonal function $K(z,z)$, we automatically have estimates for the growth of an arbitrary function $F \in \mathrm{H}$ according to

$$|F| = \big| \big\langle F, K(\cdot, z) \big\rangle_{\mathrm{H}} \big| \leqslant \|F\|_{\mathrm{H}} \big\| K(\cdot, z) \big\|_{\mathrm{H}} = \|F\|_H \sqrt{K(z,z)}. \tag{4.5}$$

Therefore, based on (4.4) and (4.5), an estimate can be derived for the modulus of the metaanalytic functions $F \in \mathcal{F}_\lambda^k(\mathbb{C})$, $k = 0, 1, \ldots, n-1$,

$$|F| \leqslant \|F\|_{\mathcal{F}_\lambda^k(\mathbb{C})} \sqrt{K(z,z)} = \left(\frac{1}{k!} \right)^{1/2} e^{\pi \frac{|z|^2}{2}} \|F\|_{\mathcal{F}_\lambda^k(\mathbb{C})}.$$

Applying (3.1), we also see that reproducing kernel of the metaanalytic Fock space $\mathbf{F}_\lambda^k(\mathbb{C})$ is given as

$$\mathbf{K}_n(w,z) = e^{\pi |w|^2} e^{\bar{\lambda}(z-w) - \lambda(\bar{z} - \overline{w})} \sum_{k=0}^{n-1} \frac{1}{k!} \frac{\partial^k}{\partial w^k} \big[e^{\pi \bar{z} w - \pi |w|^2} (w-z)^k \big],$$

and that every function $F \in \mathbf{F}_\lambda^k(\mathbb{C})$ satisfies

$$|F| \leqslant \left(\sum_{k=0}^{n-1} \frac{1}{k!} \right)^{1/2} e^{\pi \frac{|z|^2}{2}} \|F\|_{\mathcal{F}_\lambda^k(\mathbb{C})}.$$

5. Sampling and interpolating in metaanalytic Fock space

5.1. Sampling in metaanalytic Fock space

Let $\Lambda = A\mathbb{Z}^2$ be a lattice in $\mathbb{R}^2$, where A is a non-singular real 2×2-matrix. Let $s(\Lambda) = |\det A|$ be the volume of a fundamental domain of Λ. The density of Λ is defined by $d(\Lambda) = s(\Lambda)^{-1}$ so that $d(\Lambda)$ coincides with the usual notions of density. In what follows, we will use the notion $\Gamma = \{z = x + i\omega\}$ to indicate the complex sequence associated with the sequence $\Lambda = (x, \omega)$.

First of all, let us recall the definition of the Gabor superframe for the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$.

Definition 5.1. Let $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$ and $\mathbf{g} = (g_0, g_1, \ldots, g_{n-1})$. The vector-valued system $\mathcal{G}(\mathbf{g}, \Lambda) = \{M_\omega T_x \mathbf{g}\}_{(x,\omega)\in\Lambda}$ is a Gabor superframe for $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ if there exist positive constants C and D such that, for arbitrary $\mathbf{f} \in \mathcal{H}$,

$$C\|\mathbf{f}\|_{\mathcal{H}}^2 \leqslant \sum_{(x,\omega)\in\Lambda} \left|\langle \mathbf{f}, M_\omega T_x \mathbf{g}\rangle_{\mathcal{H}}\right|^2 \leqslant D\|\mathbf{f}\|_{\mathcal{H}}^2, \tag{5.1}$$

where the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is defined as in (3.9).

On the other hand, we give the definition of sampling sequence in the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$.

Definition 5.2. Γ is a sampling sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if there exist positive constants C and D such that, for arbitrary $F \in \mathbf{F}_\lambda^n(\mathbb{C})$,

$$C\|F\|_{\mathbf{F}_\lambda^n(\mathbb{C})}^2 \leqslant \sum_{z\in\Gamma} \left|F(z)\right|^2 e^{-\pi|z|^2} \leqslant D\|F\|_{\mathbf{F}_\lambda^n(\mathbb{C})}^2. \tag{5.2}$$

The following lemma is a key step of the argument in [1] where the unitary of the polyanalytic Bargmann transform is essential. For more details, we refer the reader to Lemma 2 in [1].

Lemma 5.3. *Let $\mathbf{h}_n = (h_0, h_1, \ldots, h_{n-1})$, where the Hermite functions $h_k(t)$ are defined by (2.10). The vector-valued system $\mathcal{G}(\mathbf{h}_n, \Lambda)$ is a Gabor superframe for $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ if and only if the associated complex sequence Γ is a sampling sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$.*

Observe that Theorem 3.4 shows that the metaanalytic Bargmann transform $\mathbf{B}_\lambda^n\mathbf{f}$, defined as in (3.10), is a unitary mapping between $\mathcal{H}$ and $\mathbf{F}_\lambda^n(\mathbb{C})$. The following result may be proved in much the same way as Lemma 5.3.

Lemma 5.4. *Let $\mathbf{h}_n = (h_0, h_1, \ldots, h_{n-1})$, where the Hermite functions $h_k(t)$ are defined by (2.10). The vector-valued system $\mathcal{G}(\mathbf{h}_n, \Lambda)$ is a Gabor superframe for $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ if and only if the associated complex sequence Γ is a sampling sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$.*

Proof. Based on (3.9), (3.10) and (3.5), a simple calculation leads to

$$\begin{aligned}
\langle \mathbf{f}, M_\omega T_x \mathbf{h}_n\rangle_{\mathcal{H}} &= \sum_{k=0}^{n-1} \langle f_k, M_\omega T_x h_k\rangle_{L^2(\mathbb{R})} \\
&= \sum_{k=0}^{n-1} e^{\pi i\omega x - \pi\frac{|z|^2}{2}} e^{\bar{\lambda}z - \lambda\bar{z}} B_\lambda^k f(z) \\
&= e^{\pi i\omega x - \pi\frac{|z|^2}{2}} e^{\bar{\lambda}z - \lambda\bar{z}} \mathbf{B}_\lambda^n \mathbf{f}(z).
\end{aligned}$$

Therefore, applying (2.4) and setting $F = \mathbf{B}_\lambda^n \mathbf{f}(z)$, the unitary of the metaanalytic Bargmann transform $\mathbf{B}_\lambda^n \mathbf{f}(z)$ shows that the inequality (5.2) is equivalent to the inequality (5.1), which completes the proof. $\square$

Combining Lemmas 5.3 and 5.4 yields the following theorem which provides the relationship of sampling sequence between the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ and polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$.

Theorem 5.5. *Γ is a sampling sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if and only if Γ is a sampling sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$.*

Moreover, we recall the characterization of sampling lattice in the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$ (see [1, Theorem 4]).

Lemma 5.6. *The associated complex sequence Γ is a sampling sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$ if and only if the density $d(\Gamma) > n$.*

Thus, combining Theorem 5.5 and Lemma 5.6, we obtain the following theorem which gives a characterization of sampling lattice in the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$.

Theorem 5.7. *Γ is a sampling sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if and only if the density $d(\Gamma) > n$.*

5.2. Interpolating in metaanalytic Fock space

Definition 5.8. The sequence Γ is an interpolating sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if, for every sequence $\{\alpha_{m,j}\} \in l^2$, there exists a function $F(z) \in \mathbf{F}_\lambda^n(\mathbb{C})$ such that

$$e^{\pi i \omega x - \pi \frac{|z|^2}{2}} e^{\bar{\lambda} z - \lambda \bar{z}} F(z) = \alpha_{m,j}$$

holds for every $z \in \Gamma$.

As mentioned before, when $\lambda = 0$, the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ reduces to the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$. We remark here that Definition 5.8 is a generalization of the definition of interpolating sequence for the polyanalytic Fock space (see [1, Definition 7]).

Moreover, we proceed to establish the relation of interpolating sequence between the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ and polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$, which allows us to adapt the result of the interpolating sequence in the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$ to our case.

Theorem 5.9. *The sequence Γ is an interpolating sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if and only if the sequence Γ is an interpolating sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$.*

Proof. Suppose that the sequence Γ is an interpolating sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$. Then, by Definition 5.8 we have, for every sequence $\{\alpha_{m,j}\} \in l^2$, there exists a function $F(z) \in \mathbf{F}_\lambda^n(\mathbb{C})$ such that

$$e^{\pi i \omega x - \pi \frac{|z|^2}{2}} e^{\bar{\lambda} z - \lambda \bar{z}} F(z) = \alpha_{m,j}$$

holds for every $z \in \Gamma$. Set $G(z) := e^{\bar{\lambda} z - \lambda \bar{z}} F(z)$. According to Corollary 2.1 and (2.4), we see that $G(z) \in \mathbf{F}_0^n(\mathbb{C})$. Therefore, for every sequence $\{\alpha_{m,j}\} \in l^2$, there exists a function $G(z) \in \mathbf{F}_0^n(\mathbb{C})$ such that $e^{\pi i\omega x-\pi\frac{|z|^2}{2}}G(z)=\alpha_{m,j}$ holds for every $z\in\Gamma$, which means that the sequence Γ is an interpolating sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$, and vice versa. $\square$

As before, we recall the characterization of interpolating lattice in the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$ (see [1, Theorem 6]).

Lemma 5.10. *The lattice Γ is an interpolating sequence for the polyanalytic Fock space $\mathbf{F}_0^n(\mathbb{C})$ if and only if the density $d(\Gamma)<n$.*

Thus, combining Theorem 5.9 and Lemma 5.10 leads to a characterization of interpolating lattice in the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$.

Theorem 5.11. *The lattice Γ is an interpolating sequence for the metaanalytic Fock space $\mathbf{F}_\lambda^n(\mathbb{C})$ if and only if the density $d(\Gamma)<n$.*

Acknowledgments

The first author is the recipient of a postdoctoral grant from *Fundação para a Ciência e a Tecnologia*, No. SFRH/BPD/46250/2008. This work was supported by *FEDER* funds through *COMPETE* – Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT – Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. The work was also supported by the Foundation of Hubei Educational Committee (No. Q20091004) and the NSFC (No. 11371007).

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