# PSEUDOHYPERBOLIC METRIC AND UNIFORMLY DISCRETE SEQUENCES IN THE REAL UNIT BALL＊ 

Guangbin REN（任广斌）<br>Department of Mathematics，University of Science and Technology of China，Hefei 230026，China<br>E－mail：rengb＠ustc．edu．cn<br>Uwe KÄHLER<br>CIDMA－Center of $R \& D$ in Mathematics and Applications，University of Aveiro， P－3810－193 Aveiro，Portugal<br>E－mail：ukaehler＠ua．pt


#### Abstract

We present an overview of the properties of the pseudohyperbolic metric in sev－ eral real dimensions and study uniformly discrete sequences for the real unit ball in $\mathbb{R}^{n}$ ．


Key words Uniformly discrete sequences；harmonic functions；Bergman spaces；pseudohy－ perbolic metric
2010 MR Subject Classification 46E35；47B38

The problem of describing interpolating sequences for analytic and harmonic functions is rather old．L．Carleson［6－8］studied the problem for the complex case since 1959．In［8］L． Carleson and J．Garnett proved for the space of bounded harmonic functions in the upper half space that if a sequence is uniformly discrete and the points fulfil a certain density condition for any Carleson cube，then the sequence can be written as a finite union of interpolating sequences． It is not known if this two sufficient conditions are also necessary．Because of the importance of this problem，this study was extended by other authors［2，4，11，14，15，23，26－28，30］．In［12］ it was even extended to the unit ball in $\mathbb{C}^{n}$ and in［5］to the case of positive harmonic functions in the unit disk．Unfortunately，most of the proofs in the complex case make heavy use of Blaschke products，a tool which is unavailable if one wants to extend this results to the case of the real unit ball in higher dimensions．But there exists another powerful tool in complex analysis，the pseudohyperbolic metric．It is defined in terms of Möbius transformations，which map the unit disk onto the unit disk．Because Möbius transformations can be generalized to higher dimensions（see［1］），we can also consider the pseudohyperbolic metric in this setting

[^0]and show its Möbius invariance. Our aim is to take a closer look into its properties and use them to study uniformly discrete sequences for harmonic Bergman spaces. To this end, we will follow the work of P. Duren and R. Weir [12] for the case of $\mathbb{C}^{n}$ and establish a parallel theory for $\mathbb{R}^{n}$. Hereby, one needs to establish similar arguments, but in the real case the theory is not as "complete" as in the case of several complex variables (see Ahlfors [1]). In our main result, we give necessary and sufficient conditions for uniformly discrete sequences for the case of harmonic Bergman spaces over the unit ball in $\mathbb{R}^{n}$.

## 1 Möbius Transformations and Pseudohyperbolic Metric

Let $B_{n}$ be the open unit ball in $\mathbb{R}^{n}$, that is, the set of points $\{x:|x|<1\}$.
We shall be using the following notation: we will write $x, y \in \mathbb{R}^{n}$ in polar coordinates by $x=|x| x^{\prime}$ and $y=|y| y^{\prime}$.

For any $a \in B$, denote by $\varphi_{a}$ the Möbius transformation in $B$. It is an involution automorphism of $B$ such that $\varphi_{a}(0)=a$ and $\varphi_{a}(a)=0$, which is of the form (see [1])

$$
\begin{equation*}
\varphi_{a}(x)=\frac{|x-a|^{2} a-\left(1-|a|^{2}\right)(x-a)}{\| x\left|a-x^{\prime}\right|^{2}}, \quad a, x \in \mathbb{B} \tag{1.1}
\end{equation*}
$$

For the denominator, we will also use the abbreviation [1],

$$
h(x, y)=\left||x| y-x^{\prime}\right|
$$

Furthermore, $h(x, y)=h(y, x)$ by the symmetry lemma, and

$$
h(x, y)^{2}=\left(1-|x|^{2}\right)\left(1-|y|^{2}\right)+|x-y|^{2}
$$

It is well known that

$$
\begin{gather*}
\left|\varphi_{a}(x)\right|=\frac{|x-a|}{\| a\left|x-a^{\prime}\right|}  \tag{1.2}\\
1-\left|\varphi_{a}(x)\right|^{2}=\frac{\left(1-|x|^{2}\right)\left(1-|a|^{2}\right)}{\| a\left|x-a^{\prime}\right|^{2}} \tag{1.3}
\end{gather*}
$$

Theorem 1.1 For the Möbius transformation $\varphi_{a}(x), a, x, y \in B_{n}$, we have

$$
\begin{equation*}
h\left(\varphi_{a}(x), \varphi_{a}(y)\right)=\frac{h(a, a) h(x, y)}{h(a, y) h(x, a)} \tag{1.4}
\end{equation*}
$$

Proof Let $J \varphi_{a}(x)$ denote the Jacobian matrix of $\varphi_{a}$ at $x$ and denote $\left|J \varphi_{a}(x)\right|=$ $\left|\operatorname{det} J \varphi_{a}(x)\right|^{1 / n}$. Then (see [1]),

$$
\frac{J \varphi_{a}(x)}{\left|J \varphi_{a}(x)\right|} \in O(n)
$$

$O(n)$ being the orthogonal group; moreover,

$$
\begin{gathered}
\left|J \varphi_{a}(x)\right|=\frac{1-|a|^{2}}{\left(1-|x|^{2}\right)\left(1-|a|^{2}\right)+|x-a|^{2}}=\frac{h(a, a)}{h(x, a)^{2}} \\
\left|\varphi_{a}(x)-\varphi_{a}(y)\right|=\left|J \varphi_{a}(x)\right|^{1 / 2}\left|J \varphi_{a}(y)\right|^{1 / 2}|x-y|
\end{gathered}
$$

From these, we get

$$
\left|\varphi_{a}(x)-\varphi_{a}(y)\right|=\frac{\sqrt{h(a, a)}}{h(a, x)} \frac{\sqrt{h(a, a)}}{h(a, y)}|x-y|
$$

Thus, for the term $h\left(\varphi_{a}(x), \varphi_{a}(y)\right)$, we have

$$
h\left(\varphi_{a}(x), \varphi_{a}(y)\right)^{2}=\left(1-\left|\varphi_{a}(x)\right|^{2}\right)\left(1-\left|\varphi_{a}(y)\right|^{2}\right)+\left|\varphi_{a}(x)-\varphi_{a}(y)\right|^{2}
$$

$$
\begin{aligned}
& =\frac{h(a, a) h(x, x)}{h(a, x)^{2}} \frac{h(a, a) h(y, y)}{h(a, y)^{2}}+\frac{h(a, a)^{2}}{h(a, x)^{2} h(y, a)^{2}}|x-y|^{2} \\
& =\frac{h(a, a)^{2}}{h(a, x)^{2} h(a, y)^{2}}\left(h(x, x) h(y, y)+|x-y|^{2}\right) \\
& =\frac{h(a, a)^{2}}{h(a, x)^{2} h(a, y)^{2}} h(x, y)^{2} .
\end{aligned}
$$

Furthermore, $\hat{\mathcal{M}}$ denotes the group generated by all similarities of $\mathbb{R}^{n}$ together with the reflection in the unit sphere and $\mathcal{M}$ the subgroup which keeps the unit ball invariant. Let us remark that any $\varphi \in \mathcal{M}$ can be written as the composition of an orthogonal transformation in the orthogonal group $O(n)$ with a Möbius transformation, that is, $\mathcal{M}=\left\{K \varphi_{a}: K \in O(n), a \in\right.$ $B\}$.

The pseudohyperbolic metric for the unit ball is defined by

$$
\rho(x, y)=\left|\varphi_{y}(x)\right|, x, y \in B_{n} .
$$

For this metric, we have the following well-known properties:

1. $\rho(x, y) \geq 0$ and $\rho(x, y)=0 \Leftrightarrow x=y$,
2. $\rho(x, y)=\rho(y, x)$,
3. $\rho(x, y) \leq \rho(x, z)+\rho(z, y)$,
4. $\rho(K x, K y)=\rho(x, y), \quad \forall K \in O(n)$.

The first three points show that $\rho(\cdot, \cdot)$ defines indeed a metric. These assertions except the triangle inequality come easily from the identity (1.3). Indeed, for the first point we remark that if $\rho(x, y)=0$, then we have $\left|\varphi_{x}(y)\right|=0$, so that

$$
1=1-\left|\varphi_{x}(y)\right|^{2}=\frac{\left(1-|x|^{2}\right)\left(1-|y|^{2}\right)}{\left(1-|x|^{2}\right)\left(1-|y|^{2}\right)+|x-y|^{2}},
$$

which implies $x=y$. Again from (1.3), it is easy to see that

$$
1-\left|\varphi_{x}(y)\right|^{2}=1-\left|\varphi_{y}(x)\right|^{2}=1-\left|\varphi_{K y}(K x)\right|^{2}
$$

and it yields the second and fourth statements. Their proof can be found in [22] for the real unit ball and in [29] (p.100) for the complex unit ball. We shall need the Möbius invariant property and the strengthened triangle inequality.

Theorem 1.2 For any $a, x, y \in B_{n}$, we have

$$
\rho\left(\varphi_{a}(x), \varphi_{a}(y)\right)=\rho(x, y) ;
$$

and

$$
\frac{|\rho(x, a)-\rho(a, y)|}{1-\rho(x, a) \rho(a, y)} \leq \rho(x, y) \leq \frac{\rho(x, a)+\rho(a, y)}{1+\rho(x, a) \rho(a, y)} .
$$

Proof For the Möbius invariance, we apply the identity

$$
1-\left|\varphi_{a}(x)\right|^{2}=\frac{h(a, a) h(x, x)}{h(a, x)^{2}},
$$

so that

$$
\left(1-\left|\varphi_{a}(x)\right|^{2}\right)\left(1-\left|\varphi_{a}(y)\right|^{2}\right)=\frac{h(a, a) h(x, x)}{h(a, x)^{2}} \frac{h(a, a) h(y, y)}{h(a, y)^{2}} .
$$

By (1.4), we have

$$
h\left(\varphi_{a}(x), \varphi_{a}(y)\right)=\frac{h(a, a) h(x, y)}{h(a, y) h(x, a)} .
$$

Therefore, we get

$$
\begin{aligned}
1-\left|\varphi_{\varphi_{a}(x)}\left(\varphi_{a}(y)\right)\right|^{2} & =\frac{\left(1-\left|\varphi_{a}(x)\right|^{2}\right)\left(1-\left|\varphi_{a}(y)\right|^{2}\right)}{h\left(\varphi_{a}(x), \varphi_{a}(y)\right)^{2}} \\
& =\frac{h(x, x) h(y, y)}{h(x, y)^{2}}=1-\left|\varphi_{x}(y)\right|^{2} .
\end{aligned}
$$

Regarding the last statement, we can assume $a=0$ in view of the Möbius invariance. In this case, we have $\rho(x, 0)=|x|$, so that the statement can be rewritten as

$$
\frac{\| x|-|y||}{1-|x||y|} \leq \frac{|x-y|}{\| x\left|y-x^{\prime}\right|} \leq \frac{|x|+|y|}{1+|x||y|} .
$$

Notice that

$$
\begin{aligned}
& \frac{|x-y|^{2}}{\| x\left|y-x^{\prime}\right|^{2}} \leq \frac{(|x|+|y|)^{2}}{(1+|x||y|)^{2}} \\
\Leftrightarrow & \frac{|x|^{2}+|y|^{2}-2\langle x, y\rangle}{1+|x|^{2}|y|^{2}-2\langle x, y\rangle} \leq \frac{|x|^{2}+|y|^{2}+2|x||y|}{1+|x|^{2}|y|^{2}+2|x||y|} \\
\Leftrightarrow & \left(1-|x|^{2}\right)\left(1-|y|^{2}\right)(|x||y|+\langle x, y\rangle) \geq 0 .
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
& \frac{\|x|-| y\|^{2}}{(1-|x||y|)^{2}} \leq \frac{|x-y|^{2}}{\| x\left|y-x^{\prime}\right|^{2}} \\
\Leftrightarrow & \left(1-|x|^{2}\right)\left(1-|y|^{2}\right)(|x||y|-\langle x, y\rangle) \geq 0 .
\end{aligned}
$$

which completes the proof.
Let $\nu_{n}$ denote the Lebesgue measure in $\mathbb{R}^{n}$, normalized so that $\nu_{n}\left(B_{n}\right)=1$, and let $\sigma_{n}$ be the corresponding measure on the surface of the unit sphere $\partial B_{n}$, normalized so that $\sigma_{n}\left(\partial B_{n}\right)=1$. Then,

$$
\int_{\mathbb{R}^{n}} f(x) \mathrm{d} \nu_{n}(x)=n \int_{0}^{\infty} r^{n-1} \int_{\partial B_{n}} f(r \xi) \mathrm{d} \sigma_{n}(\xi) \mathrm{d} r
$$

for any function $f \in L^{1}\left(\mathbb{R}^{n}\right)$.
The invariant (Haar) measure on $B_{n}$ is given by $\mathrm{d} \tau_{n}(x)=\frac{\mathrm{d} \nu_{n}(x)}{\left(1-\mid x^{2}\right)^{n}}$. The hyperbolic volume of a measurable set $\Omega \subset B_{n}$ is defined by

$$
\tau_{n}(\Omega)=\int_{\Omega} \frac{\mathrm{d} \nu_{n}(x)}{\left(1-|x|^{2}\right)^{n}} .
$$

Hereby, we obtain the fact that the hyperbolic volume is preserved under the action of Möbius transformations $\tau_{n}\left(\varphi_{a}(\Omega)\right)=\tau_{n}(\Omega), a \in B_{n}$. Let us denote the pseudohyperbolic ball with center $a \in B_{n}$ and radius $r \in(0,1)$ by

$$
\Delta(a, r)=\left\{x \in B_{n}:\left|\varphi_{a}(x)\right|<r\right\} .
$$

As $\varphi_{0}(x)=-x, \Delta(0, r)$ is the true Euclidean ball $|x|<r$. As the Möbius transformation is an involution, $\Delta(a, r)=\varphi_{a}(\Delta(0, r))$. Now, we give an estimate for the hyperbolic volume $\tau_{n}(\Delta(a, r))$ in $\mathbb{R}^{n}$. We like to think of it as the equivalent result to the hyperbolic volume in Duren/Weir [12], but here an explicit calculation is not possible.

Lemma 1.3 For any $a \in B_{n}, r \in(0,1)$, and $n \geq 2$, we have

$$
\frac{1}{2} \frac{r^{n}}{\left(1-r^{2}\right)^{n-1}} \leq \tau_{n}(\Delta(a, r)) \leq \frac{r^{n}}{\left(1-r^{2}\right)^{n-1}}
$$

Proof Because of the Möbius invariance, we have for the hyperbolic volume of $\Delta(a, r)$

$$
\begin{aligned}
\tau_{n}(\Delta(a, r)) & =\tau_{n}(\Delta(0, r))=\int_{\Delta(0, r)} \frac{\mathrm{d} \nu_{n}(x)}{\left(1-|x|^{2}\right)^{n}}=n \int_{0}^{r} \frac{t^{n-1}}{\left(1-t^{2}\right)^{n}} \mathrm{~d} t \\
& =n \int_{0}^{r} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!\Gamma(n)} t^{2 k+n-1} \mathrm{~d} t=r^{n} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n-1)} \frac{\Gamma(n+k)}{2 k+n} r^{2 k}
\end{aligned}
$$

We would like to remark that

$$
\frac{\Gamma(n-1+k)}{2} \leq \frac{\Gamma(n+k)}{2 k+n} \leq \Gamma(n-1+k)
$$

Now, $\sum_{k=0}^{\infty} \frac{\Gamma(n-1+k)}{k!\Gamma(n-1)} r^{2 k}=\frac{1}{\left(1-r^{2}\right)^{n-1}}$ gives us the desired result.
We would like to mention that, unlike in the complex case of $\mathbb{C}^{n}[12], \tau_{n}(\Delta(a, r))$ is only equivalent to $r^{n}\left(1-r^{2}\right)^{-n+1}$, but not equal its scale multiple. In fact, otherwise there exists a constant $C$ independent of $r \in(0,1)$ such that

$$
n \int_{0}^{r} \frac{t^{n-1}}{\left(1-t^{2}\right)^{n}} \mathrm{~d} t=C \frac{r^{n}}{\left(1-r^{2}\right)^{n-1}}
$$

so that by derivating both sides, we reach a contradiction.
For $0<p<\infty$, the harmonic Bergman space $L_{h}^{p}\left(B_{n}\right)$ consists of all functions $f$ harmonic in $B_{n}$, that is,

$$
\Delta f(x):=\left(\frac{\partial^{2}}{\partial x_{1}^{2}}+\cdots+\frac{\partial^{2}}{\partial x_{n}^{2}}\right) f(x)=0, \quad \forall x \in B_{n}
$$

and with finite volume integral

$$
\|f\|_{p}^{p}=\int_{B_{n}}|f(x)|^{p} \mathrm{~d} \nu_{n}(x)<\infty
$$

Lemma 1.4 Let $0<p<\infty$ and $0<r<1$, and define $s \in(r, 1)$. Then, for each $a \in B_{n}$, and any $x \in \Delta(a, r)$, the inequality

$$
|f(x)|^{p} \leq \frac{C}{s^{n}} \int_{\Delta(a, s)}|f(\xi)|^{p} \mathrm{~d} \nu_{n}(\xi), \quad \forall f \in L_{h}^{p}\left(B_{n}\right)
$$

holds for some constant $C$ depending only on $n$ and $p$.
One well-known fact is that harmonic functions are not invariant under Möbius transformations in higher dimensions. But if $f$ is harmonic, then $\| x\left|a-x^{\prime}\right|^{2-n} f\left(\varphi_{a}(x)\right)$ is again harmonic [13].

## 2 Uniformly Discrete Sequences in the Ball

A sequence $\Gamma=\left\{x_{k}\right\}_{k=1}^{\infty} \subset B_{n}$ is said to be uniformly discrete if there exists a positive constant $\delta \in(0,1)$ such that

$$
\left|\varphi_{x_{j}}\left(x_{k}\right)\right| \geq \delta>0
$$

for all $j \neq k$. The number $\delta(\Gamma)=\inf _{j \neq k}\left|\varphi_{x_{j}}\left(x_{k}\right)\right|$ is called the separation constant of $\Gamma$.

Lemma 2.1 If $\left\{x_{k}\right\}_{k=1}^{\infty}$ is a uniformly discrete sequence in $B_{n}$ with separation constant $\delta$, then,

$$
\sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n}\left|f\left(x_{k}\right)\right|^{p} \leq C\left(\frac{1}{\delta}\right)^{n} \int_{B_{n}}|f(x)|^{p} \mathrm{~d} \nu_{n}(x)
$$

for any $f \in L_{h}^{p}\left(B_{n}\right)$, where $C$ is a constant independent of $f$ and $\delta$.
Proof By the triangle inequality, the pseudohyperbolic balls $\Delta\left(x_{k}, \frac{\delta}{2}\right)$ are pairwise disjoint. Moreover, one obstacle for our proof is the fact that harmonic functions are not invariant under Möbius transformations in higher dimensions. But we obtain the result that if $f$ is harmonic, then $\left||x| a-x^{\prime}\right|^{2-n} f\left(\varphi_{a}(x)\right)$ is again harmonic [13]. Now,

$$
\begin{aligned}
\int_{B_{n}}|f(x)|^{p} \mathrm{~d} \nu_{n}(x) & \geq \sum_{k=1}^{\infty} \int_{\Delta\left(x_{k}, \frac{\delta}{2}\right)}|f(x)|^{p}\left(1-|x|^{2}\right)^{n} \mathrm{~d} \tau_{n}(x) \\
& =\sum_{k=1}^{\infty} \int_{\Delta\left(0, \frac{\delta}{2}\right)}\left|f\left(\varphi_{x_{k}}(\xi)\right)\right|^{p}\left(1-\left|\varphi_{x_{k}}(\xi)\right|^{2}\right)^{n} \mathrm{~d} \tau_{n}(\xi) \\
& =\sum_{k=1}^{\infty} \int_{\Delta\left(0, \frac{\delta}{2}\right)}\left|f\left(\varphi_{x_{k}}(\xi)\right)\right|^{p}\left(\frac{1-\left|x_{k}\right|^{2}}{\| x_{k}\left|\xi-x_{k}^{\prime}\right|^{2}}\right)^{n} \mathrm{~d} \nu_{n}(\xi)
\end{aligned}
$$

Notice that

$$
\begin{equation*}
\left||x| \xi-x^{\prime}\right| \simeq 1, \quad \forall \xi \in \Delta\left(0, \frac{\delta}{2}\right) \tag{2.1}
\end{equation*}
$$

for any $x \in B_{n}$. Indeed, $2>\left||x| \xi-x^{\prime}\right| \geq 1-|x||\xi| \geq 1-|\xi| \geq 1-\delta / 2>1 / 2$. Therefore, from (2.1) and Lemma 1.4, the preceding summation can be further estimated from below by

$$
\begin{aligned}
& \left.C_{1} \sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n} \int_{\Delta\left(0, \frac{\delta}{2}\right)}| | x_{k}\left|\xi-x_{k}^{\prime}\right|^{2-n} f\left(\varphi_{x_{k}}(\xi)\right)\right|^{p} \mathrm{~d} \nu_{n}(\xi) \\
\geq & C_{2} \delta^{n} \sum_{k=0}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n}\left|f\left(x_{k}\right)\right|^{p} .
\end{aligned}
$$

By taking $f \equiv 1$, we have the following result.
Lemma 2.2 If $\left\{x_{k}\right\}_{k=1}^{\infty}$ is a uniformly discrete sequence in $B_{n}$ with separation constant $\delta$, then,

$$
\sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n} \leq C\left(\frac{1}{\delta}\right)^{n}
$$

For $a \in B_{n}, 0<r<1$, and $\Gamma$ a sequence in $B_{n}$, we define the counting function

$$
N(\Gamma, a, r)=\sum_{x \in \Gamma} \chi_{\Delta(a, r)}(x)
$$

where $\chi_{A}(x)$ denotes the characteristic function of the set $A$. Namely, $N(\Gamma, a, r)$ is the number of points in $\Gamma$ that lie in the pseudohyperbolic ball $\Delta(a, r)$. Clearly,

$$
\begin{equation*}
N(\Gamma, a, r)=N\left(\varphi_{a}(\Gamma), 0, r\right) \tag{2.2}
\end{equation*}
$$

In fact,

$$
N(\Gamma, a, r)=\sum_{\varphi_{a}(y) \in \Gamma} \chi_{\Delta(a, r)}\left(\varphi_{a}(y)\right)=\sum_{y \in \varphi_{a}(\Gamma)} \chi_{\Delta(0, r)}(y)=N\left(\varphi_{a}(\Gamma), 0, r\right)
$$

Lemma 2.3 If $\left\{x_{k}\right\}_{k=1}^{\infty}$ is a uniformly discrete sequence in $B_{n}$ with separation constant $\delta$, then its counting function satisfies

$$
N(\Gamma, a, r)<2\left(\frac{3}{2}\right)^{n-2}\left(1+\frac{2}{\delta}\right)^{n} \frac{1}{(1-r)^{n-1}}
$$

In particular,

$$
N(\Gamma, a, r)=O\left(\frac{1}{(1-r)^{n-1}}\right), \quad r \rightarrow 1
$$

Proof As the pseudohyperbolic metric is Möbius invariant, we have the sequence $\varphi_{a}(\Gamma)=$ $\left\{\varphi_{a}\left(x_{k}\right)\right\}_{k=1}^{\infty}$ that is again a uniformly discrete sequence with separation constant $\delta$. So, by (2.2) there is no loss of generality in taking $a=0$.

We claim that for $\Gamma=\left\{x_{k}\right\}_{k=1}^{\infty}$, we have

$$
\bigcup_{x_{k} \in \Delta(0, r)} \Delta\left(x_{k}, \frac{\delta}{2}\right) \subset \Delta(0, R), \quad \text { and } \quad R:=\frac{r+\frac{\delta}{2}}{1+\frac{r \delta}{2}}
$$

Indeed, $\left\{\Delta\left(x_{k}, \frac{\delta}{2}\right)\right\}$ are pairwise disjoint. If $x \in \Delta\left(x_{k}, \frac{\delta}{2}\right)$ and $x_{k} \in \Delta(0, r)$, then,

$$
|x|=\rho(x, 0) \leq \frac{\rho\left(0, x_{k}\right)+\rho\left(x_{k}, x\right)}{1+\rho\left(0, x_{k}\right) \rho\left(x_{k}, x\right)} \leq \frac{r+\frac{\delta}{2}}{1+r \frac{\delta}{2}} .
$$

Hereby, we used the fact that $g(x)=\frac{x+y}{1+x y}$ is an increasing function of $x \geq 0$ for any fixed $y \in[0,1]$. This proves the claim. From our claim, we obtain

$$
\sum_{x_{k} \in \Delta(0, r)} \tau_{n}\left(\Delta\left(x_{k}, \frac{\delta}{2}\right)\right) \leq \tau_{n}(\Delta(0, R))
$$

Therefore, from Lemma 1.3,

$$
N(\Gamma, 0, r) \frac{1}{2} \frac{\left(\frac{\delta}{2}\right)^{n}}{\left(1-\left(\frac{\delta}{2}\right)^{2}\right)^{n-1}} \leq \frac{R^{n}}{\left(1-R^{2}\right)^{n-1}}
$$

As

$$
1-R^{2}=\frac{1-\left(\frac{\delta}{2}\right)^{2}}{\left(1+\frac{r \delta}{2}\right)^{2}}\left(1-r^{2}\right)
$$

we have

$$
\begin{aligned}
N(\Gamma, 0, r) & \leq 2\left(\frac{\delta}{2}\right)^{-n}\left(r+\frac{\delta}{2}\right)^{n}\left(1+\frac{r \delta}{2}\right)^{n-2} \frac{1}{(1-r)^{n-1}} \\
& \leq 2\left(\frac{3}{2}\right)^{n-2}\left(1+\frac{2}{\delta}\right)^{n} \frac{1}{(1-r)^{n-1}}
\end{aligned}
$$

As in the complex case, we have the following result.
Lemma 2.4 Let $\Gamma=\left\{x_{k}\right\}_{k=1}^{\infty}$ be a sequence of points in the unit ball $B_{n}$ such that for some fixed radius $r>0$, each pseudohyperbolic ball $\Delta(a, r)$ contains at most $N$ points. Then, $\Gamma$ is the disjoint union of at most $N$ uniformly discrete sequences.

The proof of this lemma is exactly the same as that of the corresponding lemma in [11]. For the sake of completeness, we will give it here.

Proof Consider first the disk $\Delta\left(x_{1}, r\right)$. By hypothesis, it contains at most $N$ points in the sequence $\Gamma$, including $x_{1}$. Let those points be assigned to $M$ different subsets, $\Gamma_{1}, \Gamma_{2}, \cdots, \Gamma_{M}$ with $M \leq N$. Let $x_{k}$ be the first point of $\Gamma$ not already assigned. Then, $\rho\left(x_{k_{1}}, x_{1}\right) \geq r$, so $x_{k_{1}}$ is placed into the set $\Gamma_{j}$ containing $x_{1}$.

Now, we proceed inductively. Suppose that a finite number of points have been assigned to subsets $\Gamma_{1}, \Gamma_{2}, \cdots, \Gamma_{m}$ with $m \leq N$ and that $\rho(x, y) \geq r$ for all points $x, y \in \Gamma_{j}, j=1, \cdots, m$. Let $x^{*}$ be the first point of $\Gamma$ not already assigned to a subset $\Gamma_{j}$. By hypothesis, the disk $\Delta\left(x^{*}, r\right)$ contains at most $N-1$ points of $\Gamma$ that have already been assigned, and they represent at most $N-1$ different subsets $\Gamma_{j}$, so that the point $x^{*}$ can be assigned to some subset $\Gamma_{k_{0}}$ not represented in this list. It is clear by construction that $\Delta\left(x^{*}, r\right) \cap \Gamma_{k_{0}}=\phi$; namely, $\rho\left(x^{*}, \xi\right) \geq r$ for all points $\xi \in \Gamma$ already assigned to $\Gamma_{k_{0}}$. This inductive process therefore divides the given set $\Gamma$ into disjoint subsets $\Gamma_{1}, \cdots, \Gamma_{m}$ with $m \leq N$ and $\rho(x, y) \geq r$ for all $x, y \in \Gamma_{j}, j=1, \cdots, m$.

Theorem 2.5 For a sequence $\Gamma=\left\{x_{k}\right\}_{k=1}^{\infty}$ of distinct points in $B_{n}$, the following six statements are equivalent.

1. $\Gamma$ is a finite union of uniformly discrete sequences;
2. $\sup N(\Gamma, a, r)<\infty$ for some $r \in(0,1)$;
$a \in B_{n}$
3. $\sup N(\Gamma, a, r)<\infty$ for all $r \in(0,1)$;
$a \in B_{n}$
4. For some $p \in(0, \infty)$, there exists a constant $c$ such that

$$
\sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n}\left|f\left(x_{k}\right)\right|^{p} \leq c\|f\|_{p}^{p}, f \in L_{h}^{p}\left(B_{n}\right)
$$

5. For each $p \in(0, \infty)$, there exists a constant $c$ such that

$$
\sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n}\left|f\left(x_{k}\right)\right|^{p} \leq c\|f\|_{p}^{p}, f \in L_{h}^{p}\left(B_{n}\right)
$$

6. $\sup _{a \in B_{n}} \sum_{k=1}^{\infty}\left(1-\left|\varphi_{a}\left(x_{k}\right)\right|^{2}\right)^{n}<\infty$.

Proof By Lemma 2.3, statement (3) follows from statement (1). Obviously, (3) implies (2). By Lemma 2.4 from (2) we obtain (1). That (5) implies (4) is trivial and statement (5) can be obtained from (1) using Lemma 2.1.

Now, because assertion (1) as a property is invariant under Möbius transformations, it follows from Lemma 2.2 that

$$
\sum_{k=1}^{\infty}\left(1-\left|\varphi_{a}\left(z_{k}\right)\right|^{2}\right)^{n} \leq C\left(\frac{1}{\delta}\right)^{n}
$$

This means that $(1) \Rightarrow(6)$. Moreover, to show that (6) implies (3), we fix any $r \in(0,1)$ and let $a \in B_{n}$. As $\left|\varphi_{a}\left(x_{k}\right)\right|<r$ for $N(\Gamma, a, r)$ points $x_{k}$, we infer from (6) that

$$
N(\Gamma, a, r)\left(1-r^{2}\right)^{n} \leq \sum_{x_{k} \in \Delta(a, r)}\left(1-\left|\varphi_{a}\left(x_{k}\right)\right|^{2}\right)^{n} \leq C
$$

Now, we will show that (5) follows from (4). Statement (4) holds precisely when the measure $\sum_{k=0}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n} \delta_{x_{k}}$ is a Carleson measure for $L_{h}^{p}\left(B_{n}\right)$. As Carleson measures are known to be independent of $p$, this proves the implication [10].

The only part which is still left is to prove that assertion (5) implies (1) which we will do by contradiction following the ideas of the proof of Lemma 3.3 in [25]. Hereby, we remark that if $\Gamma$ is not a finite union of uniformly discrete sequences, then there exists a sequence of points $\left\{y_{k}\right\}_{k=1}^{\infty}$ in $B_{n}$, such that $N\left(\Gamma, y_{k}, \frac{1}{2}\right)$ is unbounded as $k$ becomes large. Take

$$
f_{k}(x)=\frac{K\left(x, y_{k}\right)}{\sqrt{K\left(y_{k}, y_{k}\right)}} .
$$

Here, $K$ is the harmonic Bergman kernel in $B_{n}$. It is known [16] that

$$
K\left(x, y_{k}\right) \simeq \frac{1}{\left(1-|x|^{2}\right)^{n}} \text { for any } x \in \Delta\left(y_{k}, r\right)
$$

Notice that in $\Delta\left(y_{k}, r\right)$, we have (see [22])

$$
1-|x|^{2} \simeq 1-\left|y_{k}\right|^{2} \simeq| | x\left|y_{k}-x^{\prime}\right|, \quad \forall x \in \Delta\left(y_{k}, r\right)
$$

Now, we take $p=2$. From the reproducing property of the Bergman kernel, we get $\|f\|_{L^{2}}=1$. From this, we obtain

$$
\begin{aligned}
\|f\|_{L^{2}}^{2} & \geq C \sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n}\left|f\left(x_{k}\right)\right|^{2}=C \sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n} \frac{K\left(x_{k}, y_{k}\right)^{2}}{K\left(y_{k}, y_{k}\right)} \\
& \geq C \sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|^{2}\right)^{n} K\left(x_{k}, y_{k}\right) \\
& \geq C \sum_{x_{k} \in \Delta\left(y_{k}, 1 / 2\right)}\left(1-\left|x_{k}\right|^{2}\right)^{n} \frac{\left(1-\left|y_{k}\right|^{2}\right)^{n}}{\| x_{k}\left|y_{k}-x_{k}^{\prime}\right|^{2 n}} \\
& =C \sum_{x_{k} \in \Delta\left(y_{k}, 1 / 2\right)}\left(1-\left|\varphi_{x_{k}}\left(y_{k}\right)\right|^{2}\right)^{n} \\
& \geq C\left(\frac{3}{4}\right)^{n} N\left(\Gamma, y_{k}, \frac{1}{2}\right),
\end{aligned}
$$

which is a contradiction.
As in the complex case in [12], we have the following result.
Theorem 2.6 If a sequence $\left\{x_{k}\right\}_{k=1}^{\infty}$ in $B_{n}$ is uniformly discrete and $x_{k} \neq 0$ for all $k$, then,

$$
\sum_{k=1}^{\infty}\left(1-\left|x_{k}\right|\right)^{n-1}\left(\log \frac{1}{1-\left|x_{k}\right|}\right)^{-(1+\epsilon)}<\infty
$$

for each $\epsilon>0$.
Remark 2.7 The main theorem remains true if the Bergman space $L_{h}^{p}\left(B_{n}\right)$ is replaced by the harmonic Hardy space.

Remark 2.8 Although the results are given for the case of the unit ball in $\mathbb{R}^{n}$, they can be easily transferred to the case of the upper half plane by means of the Cayley transform.

## References

[1] Ahlfors L. Möbius transformations in several dimensions. Ordway Professorship Lectures in Mathematics. Minneapolis, Minn: University of Minnesota, School of Mathematics, 1981
[2] Amar E. Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^{n}$. Canad J Math, 1978, 30: 711-737
[3] Axler S, Bourdon P, Ramey W. Harmonic Function Theory. Graduate Texts in Math 137. New York: Springer-Verlag, 1992
[4] Berndtsson B, Ortega Cerdà J. On interpolation and sampling in Hilbert spaces of analytic functions. J Reine Angew Math, 1995, 464: 109-128
[5] Blasi D, Nicolau A. Interpolation by positive harmonic functions. Journal of the London Mathematical Society, 2007, 76(1): 253-271
[6] Carleson L. An interpolation problem for bounded analytic functions. Amer J Math, 1958, 80: 921-930
[7] Carleson L. A moment problem and harmonic interpolation. Preprint. Institut Mittag-Leffler, 1972
[8] Carleson L, Garnett J. Interpolating sequences and separation properties. J Anal Math, 1975, 28: 273-299
[9] Chen Z, Ouyang W. A Littlewood-Paley type theorem for Bergman spaces, Acta Mathematica Scientia, 2013, 33B(1): 150-154
[10] Choe B L, Lee Y J, Na K. Toeplitz operators on harmonic Bergman spaces. Nagoya Math J, 2004, 174: 165-186
[11] Duren P, Schuster A, Vukotić D. On uniformly discrete sequences in the disk//Ebenfelt, Peter et al. Quadrature domains and their applications. The Harold S. Shapiro anniversary volume. Expanded version of talks and papers presented at a conference on the occasion of the 75 th birthday of Harold S. Shapiro, Santa Barbara, CA, USA, March 2003. Basel: Birkhäuser. Operator Theory: Advances and Applications, 2005, 156: 131-150
[12] Duren P, Weir R. The Pseudohyperbolic metric and Bergman spaces in the ball. Trans Amer Math Soc, 2007, 359(1): 63-76
[13] Hua L K. Starting with the unit circle. New York, Heidelberg, Berlin: Springer, 1981
[14] Luecking D H. Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives. Amer J Math, 1985, 107: 85-111
[15] Massaneda X. $A^{-p}$ interpolation in the unit ball. J London Math Soc, 1995, 52: 391-401
[16] Miao J. Reproducing kernels for harmonic Bergman spaces of the unit ball. Mh Math, 1998, 125: 25-35
[17] Pavlovič M. Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball. Indag Math (NS), 1991, 2(1): 89-98
[18] Peng R, Ouyang C. Carleson Measures for Besov-Sobolev Spaces with Applications in the Unit Ball of $C^{n}$. Acta Mathematica Scientia, 2013, 33B(5): 1219-1230
[19] Ren G, Kähler U. Radial Derivative on Bounded Symmetric Domains. Studia Mathematica, 2003, 157(1): 57-70
[20] Ren G, Kähler U. Boundary behavior of Gleason's problem in hyperbolic harmonic Bergman space. Science in China A, 2005, 48(2): 145-154
[21] Ren G, Shi J. Bergman type operator on mixed norm spaces with applications. Chin Ann of Math, 1997, 18B: 265-278
[22] Ren G, Kähler U, Shi J, Liu C. Hardy-Littlewood inequalities on the space of invariant harmonic functions. Complex Analysis and Operator Theory, 2012, 6(2): 373-396
[23] Rochberg R. Interpolation by functions in Bergman spaces. Michigan Math J, 1982, 29: 229-236
[24] Rudin W. Function Theory in the Unit Ball of $\mathbb{C}^{n}$. New York: Springer-Verlag, 1980
[25] Schuster A. On Seip's Description of Sampling Sequences for Bergman spaces. Complex Variables, 2000, 42(4): 347-367
[26] Seip K. Interpolating and sampling in spaces of analytic functions. University Lecture Series 33. Providence, RI: AMS, 2004
[27] Seip K. Density theorems for sampling and interpolation in the Bargmann-Fock space I. J Reine Angew Math, 1992, 429: 91-106
[28] Seip K. Density theorems for sampling and interpolation in the Bargmann-Fock space II. J Reine Angew Math, 1992,429: 107-113
[29] Stoll M. Invariant potential theory in the unit ball of $\mathbb{C}^{n}$. London Mathematical Society Lecture Note Series, 199. Cambridge: Cambridge University Press, 1994
[30] Xiao J. Carleson measure, atomic decomposition and free interpolation from Bloch space. Annal Acad Scient Fenn Series A, 1994, 19: 35-46
[31] Zhang Y, Deng G, Kou K I. On the lower bound for a class of harmonic functions in the half space. Acta Mathematica Scientia, 2012, 32B(4): 1487-1494


[^0]:    ＊Received June 14，2012；revised September 2，2013．The first author is partially supported by the NNSF of China（11071230，11371337），RFDP（20123402110068）．The second author is supported by FEDER funds through COMPETE－Operational Programme Factors of Competitiveness（Programa Operacional Factores de Competitividade），and by Portuguese funds through the Center for Research and Development in Mathematics and Applications（University of Aveiro）and the Portuguese Foundation for Science and Technology（FCT－ Fundação para a Ciência e a Tecnologia），within project PEst－C／MAT／UI4106／2011 with COMPETE number FCOMP－01－0124－FEDER－022690．

