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On the Π-operator in Clifford analysis

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A B S T R A C T

In this paper we prove that a generalization of complex Π-operator in Clifford analysis, obtained by the use of two orthogonal bases of a Euclidean space, possesses several mapping and invertibility properties, as studied before for quaternion-valued functions as well as in the standard Clifford analysis setting. We improve and generalize most of those previous results in this direction and additionally other consequent results are presented. In particular, the expression of the jump of the generalized Π-operator across the boundary of the domain is obtained as well as an estimate for the norm of the Π-operator is given. At the end an application of the generalized Π-operator to the solution of Beltrami equations is studied where we give conditions for a solution to realize a local and global homeomorphism.

1. Introduction

In [30], Vekua established the main properties of the most important integral operators in complex analysis. In particular, the complex Π-operator, which plays a mayor role in the theory of generalized analytic functions as well as in the branch of the complex analysis deeply connected with partial differential equations by using functional analytic methods, was introduced and studied in detail. Generalizations of the complex Π-operator to higher-dimensional versions are already considered in recent times [7,9,10,22,23,25,27–29]. Although a thorough treatment is listed, it is nevertheless restricted mainly to the setting of quaternion-valued functions and there is still a strong need for developing further details. The advances achieved in higher dimensions use only the standard orthogonal basic, without achieving a significant progress. The latter is why in this paper we confine attention to the generalized Clifford analysis setting by the help of two arbitrary orthogonal bases.

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Starting from his definition of a generalized Teodorescu transform, see [27,28], Sprössig proposed a generalization of the complex Π-operator in the Clifford analysis setting, which turns out to have most of the useful properties of its complex origin. The definition is stated in terms of Teodorescu transform and Cauchy–Riemann operator arising in standard Clifford analysis of a Euclidean space. In [23,25] the generalization of the complex Π-operator is based now in two different orthonormal bases associated respectively with the Teodorescu transform and Cauchy–Riemann operator.

In the study of the properties of the generalization of Π-operator, one of the qualitative disadvantages of the change of standard orthonormal basic and its conjugate for two different orthonormal bases is the absence of a generalized Borel–Pompeiu representation formula, which allows to extend to the corresponding situations a series of fundamental applications of its standard antecedent. Recently, in [1] there was given such a Borel–Pompeiu representation formula, but in the context of quaternionic analysis. In this paper we offer direct extension of the formula to the Clifford analysis setting.

An important question, which is in the focus of the present paper, is the study of the jump of the generalized Π-operator across the boundary of the domain.

One of the important applications of the complex Π-operator is to the solution of Beltrami systems. Higher-dimensional Beltrami systems in the framework of quaternions were first studied by Shevchenko in 1962 [26], and later on by P. Cerejeiras, K. Gürlebeck, U. Kähler, and others [6,5,9,10,14–16]. Hereby, also the question of local and global homeomorphic solutions was studied. Later on A. Koski [17] studied the solvability of Beltrami equations for VMO-coefficients. For the study of Beltrami equations in the last section we first give estimates for the norm of the generalized Π-operator in Section 4. This will allow us in the last section to apply our Π-operator to the solution of Beltrami equations, and to give conditions for a solution to realize a local and global homeomorphism.

2. Preliminaries

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Consider the 2^n-dimensional real Clifford algebra $\mathbb{R}_{0,n}$ generated by $e_1, e_2, \ldots, e_n$ according to the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker's symbol. The elements $e_A : A \subseteq \mathbb{N}_n := \{1, 2, \ldots, n\}$ define a basis of $\mathbb{R}_{0,n}$, where $e_A = e_{h_1} \cdots e_{h_k}$ if $A = \{h_1, \ldots, h_k\}$ $(1 \le h_1 < \cdots < h_k \le n)$ and $e_\emptyset = e_0 = 1$. Any $a \in \mathbb{R}_{0,n}$ may thus be written as $a = \sum_{A\subseteq\mathbb{N}_n} a_Ae_A$ where $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^{n}[a]_k$, where $[a]_k = \sum_{|A|=k} a_Ae_A$ is a so-called k-vector $(k \in \mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\})$. If we denote the space of k-vectors by $\mathbb{R}_{0,n}^{(k)}$, then $\mathbb{R}_{0,n} = \sum_{k=0}^{n} \oplus\mathbb{R}_{0,n}^{(k)}$.

In such a way, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ will be identified with $\mathbb{R}_{0,n}^{(0)}$ and $\mathbb{R}_{0,n}^{(1)}$ respectively. Moreover, each element $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ can be written as

$$x = x_0 + \sum_{i=1}^{n} x_ie_i \in \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$$

and they are often called *paravectors*. For each $x \in \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ it is remarkable that

$$x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2 = |x|^2. \tag{1}$$

The extension of (1) to a norm of $a \in \mathbb{R}_{0,n}$ is straightforward and leads to

$$|a|^2 = [a\bar{a}]_0 = [\bar{a}a]_0 = \sum_A a_A^2.$$

We will use the *conjugation*, defined by $\bar{a} = \sum_{A\subseteq\mathbb{N}_n} a_A\bar{e}_A$, where

$$\bar{e}_A := (-1)^k e_{h_k} \cdots e_{h_1} = (-1)^{\frac{k(k+1)}{2}} e_A, \quad \text{if } e_A = e_{h_1} \cdots e_{h_k}.$$

The following properties of the norm and conjugation in Clifford algebras are well-known and can be found in many sources, see for instance [8].

Proposition 1. *Let $a, b \in \mathbb{R}_{0,n}$, then*

(i) $\overline{ab} = \bar{b}\,\bar{a}$,
(ii) $|\bar{a}| = |-a| = |a|$,
(iii) $[a\bar{b}]_0 = [\bar{a}b]_0 = \langle a, b\rangle_{\mathbb{R}^{2^n}}$*, where* $\langle\cdot,\cdot\rangle_{\mathbb{R}^m}$ *denotes the standard scalar product in* $\mathbb{R}^m$,
(iv) $|ab| \leq 2^{n/2}|a||b|$,
(v) *if b is such that $b\bar{b} = |b|^2$, $b \neq 0$, then b is invertible and $|ab| = |ba| = |a||b|$.*

Suppose Ω is a bounded domain, with sufficiently smooth boundary Γ, in $\mathbb{R}^{n+1}$. We will be interested in functions $f : \Omega \to \mathbb{R}_{0,n}$, which might be written as $f(x) = \sum_A f_A(x)e_A$ with f_A $\mathbb{R}$-valued. Property, such continuity, differentiability, integrability, and so on, can be ascribed coordinatewisely or directly. A left (unitary) module over $\mathbb{R}_{0,n}$ (left $\mathbb{R}_{0,n}$-module for short) is a vector space V together with an algebra morphism $L : \mathbb{R}_{0,n} \mapsto \mathrm{End}(V)$, or to say it more explicitly, there exists a linear transformation (also called left multiplication) $L(a)$ of V such that

$$L(ab + c) = L(a)L(b) + L(c)$$

for all $a \in \mathbb{R}_{0,n}$, and $L(1)$ is the identity operator. In the same way we have a right (unitary) module if there is a so-called right multiplication $R(a) \in \mathrm{End}(V)$ such that

$$R(ab + c) = R(b)R(a) + R(c).$$

Given either a left or a right multiplication we can always construct a right or a left multiplication by using any anti-automorphism of the algebra, for instance

$$R(a) = L(\overline{a}).$$

A bi-module is a module which is both a left- and a right-module, or with other words, a module where left and right multiplications commute, i.e.

$$L(a)R(b) = R(b)L(a), \text{ for all } a, b \in \mathbb{R}_{0,n}.$$

If V is a vector space of $\mathbb{R}_{0,n}$-valued function we consider the left (right) multiplication defined by point-wise multiplication

$$(L(a)f)(x) = a(f(x)) \text{ and } (R(a)f)(x) = a(f(x)).$$

Also a mapping K between two right modules V and W is called an $\mathbb{R}_{0,n}$-linear mapping if

$$K(fa + g) = K(f)a + K(g).$$

We should also mention what is understanding in this paper by a (left or right) Clifford Banach module (see [18] for example). We say that X is a left Banach $\mathbb{R}_{0,n}$-module if X is a left $\mathbb{R}_{0,n}$-module and X is also a real Banach space such that for any $a \in \mathbb{R}_{0,n}$ and $x \in X$:

$$\|ax\|_X \leq k|a|\|x\|_X, \tag{2}$$

for some $k > 0$. In particular, equality occurs in (2) if $a \in \mathbb{R}$. Similarly one can define a right Banach $\mathbb{R}_{0,n}$-module.

These considerations give rise to the following right modules of $\mathbb{R}_{0,n}$-valued function defined over any suitable subset E of $\mathbb{R}^{n+1}$:

- $C^k(E, \mathbb{R}_{0,n})$, $k \in \mathcal{N} \cup \{0\}$ – the right module of all $\mathbb{R}_{0,n}$-valued functions, k-times continuously differentiable in E. It becomes a right Banach module with the norm

$$\|f\|_{C^k} = \sup_{x \in E} \sum_{|\alpha| \leq k} |D_w^\alpha f(x)|$$

  where α denotes a multi-index and D^α the corresponding partial derivative. In particular, we also have $C^\infty(E, \mathbb{R}_{0,n}) := \bigcap_{k=0}^\infty C^k(E, \mathbb{R}_{0,n})$. Let us remark that the above norm is equivalent to the norm coming from the inductive limit.
- By using the corresponding Hölder-semi-norm we can introduce $C^{0,\mu}(E, \mathbb{R}_{0,n})$, $\mu \in (0, 1]$ as the right Banach module of all μ-Hölder continuous and $\mathbb{R}_{0,n}$-valued functions in E.
- $L_p(E, \mathbb{R}_{0,n})$ $(1 \leq p < \infty)$ denotes the right module of all equivalence classes of Lebesgue measurable functions $f : E \longrightarrow \mathbb{R}_{0,n}$ for which $|f|^p$ is integrable over E. With the norm

$$\|f\|_{L_p(E, \mathbb{R}_{0,n})} := \left(\int_E |f(\xi)|^p \, d\xi \right)^{1/p} < \infty,$$

  $L_p(E, \mathbb{R}_{0,n})$ becomes a right Banach module.
- $C_0^\infty(E, \mathbb{R}_{0,n})$ – the space of all infinitely differentiable functions with compact support in E. An important property of the space $C_0^\infty(E, \mathbb{R}_{0,n})$ is its density in the spaces $C^0(E, \mathbb{R}_{0,n})$ and $L_p(E, \mathbb{R}_{0,n})$.

Furthermore, the above proposition allows us to introduce the following sesquilinear form (also called symmetric inner product in the literature) for two functions $f, g : \Omega \to \mathbb{R}_{0,n}$

$$(f, g) := \overline{f} g.$$

In particular, this sesquilinear form is a real-bilinear mapping and a Clifford-linear mapping in the second argument. Furthermore, this sesquilinear form gives rise to an $\mathbb{R}_{0,n}$-valued sesquilinear form

$$\langle f, g \rangle_2 = \int_E \overline{f(\xi)} g(\xi) \, d\xi.$$

This form satisfies the following properties:

1. $\langle \cdot, \cdot \rangle_2$ is sesquilinear
2. $\langle f, gb \rangle_2 = \langle f, g \rangle_2 \, b$ with $b \in \mathbb{R}_{0,n}$
3. $\overline{\langle f, g \rangle_2} = \langle g, f \rangle_2$

Furthermore, the right $\mathbb{R}_{0,n}$-module $L_2(E, \mathbb{R}_{0,n})$ equipped with this sesquilinear form is complete under the norm

$$\|f\| = \sqrt{\int_E [\overline{f(\xi)} g(\xi)]_0 d\xi}, \tag{3}$$

which makes it a Clifford–Hilbert module. Many facts from classic Hilbert spaces carry over to the notion of a Hilbert module. In particular, a generalization of Riesz' representation theorem is valid in the sense that a linear functional ϕ is continuous if and only if it can be represented by an element $f_\phi \in V$ such that

$$\phi(g) = \langle f_\phi, g\rangle_2.$$

Additionally, there are some important inequalities involving the sesquilinear form and the norm coming from the real-valued inner product:

- Cauchy–Schwarz inequality: $|\langle f,g\rangle_2| \leq 2^{n/2}\|f\|_{L_p(E,\mathbb{R}_{0,n})}\|g\|_{L_q(E,\mathbb{R}_{0,n})}$ with $\frac{1}{p}+\frac{1}{q}=1$
- $\|af\|_{L_2(E,\mathbb{R}_{0,n})} \leq 2^{n/2}|a|\|f\|_{L_2(E,\mathbb{R}_{0,n})}$ for all $a \in \mathbb{R}_{0,n}$, but $\|af\|_{L_2(E,\mathbb{R}_{0,n})} = |a|\|f\|_{L_2(E,\mathbb{R}_{0,n})}$ whenever a is a paravector or belongs to the paravector group
- $\|f\|_{L_2(E,\mathbb{R}_{0,n})} \leq |\langle f,f\rangle_2| \leq 2^{n/2}\|f\|_{L_2(E,\mathbb{R}_{0,n})}$
- $\|f\|_{L_2(E,\mathbb{R}_{0,n})} \leq \sup_{\|g\|_{L_2(E,\mathbb{R}_{0,n})}\leq 1} |\langle f,g\rangle_2| \leq 2^{n/2}\|f\|_{L_2(E,\mathbb{R}_{0,n})}$

The same statements are also true for any other Clifford Hilbert module coming from a tensor product between the elements of the Clifford basis and elements of a real or complex Hilbert space [12].

In the same way we can introduce $\mathcal{S}$ as the corresponding Schwartz space of rapidly decaying functions. Its dual space $\mathcal{S}'$ given by the continuous linear functionals is the space of tempered distributions. Again, this can be either defined componentwisely or via the sesquilinear form $\langle f,g\rangle_2$ but the space $\mathcal{S}'$ is again considered as a right module. Let us remark that strictly speaking if we consider $\mathcal{S}$ as a Fréchet right module its algebraic dual $\mathcal{S}'$ is the space of all left-$\mathbb{R}_{0,n}$-linear functionals over $\mathcal{S}$, but it can be identified with elements of a right-linear module by means of the standard anti-automorphism in the above mentioned way.

This allows us to introduce the *Sobolev* space $W_p^k(E,\mathbb{R}_{0,n})$, $k \in \mathbb{N}\cup\{0\}$, $1 \leq p < \infty$, as the right module of all functionals $f \in \mathcal{S}'$ whose derivatives[1] $D_w^\alpha f$ for $|\alpha| \leq k$ belong to $L_p(E,\mathbb{R}_{0,n})$, with norm

$$\|f\|_{W_p(E,\mathbb{R}_{0,n})} := \left(\sum_{\|\alpha\|\leq k} \|D_w^\alpha f\|^p_{L_p(E,\mathbb{R}_{0,n})}\right)^{1/p}.$$

Let us remark that we only consider Sobolev spaces with non-negative exponent, which means that duality discussions between Sobolev spaces are not required for this paper, but can be dealt with similarly as to the above argument. For more details we refer to the classic book [4].

Let $\psi := \{\psi^0,\psi^1,\ldots,\psi^n\} \subset \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$. By abuse of notation, we will write $\overline{\psi} := \{\overline{\psi^0},\overline{\psi^1},\ldots,\overline{\psi^n}\}$. On the set $C^1(\Omega,\mathbb{R}_{0,n})$ we define respectively the left and the right Cauchy–Riemann operators by:

$$^{\psi}D[f] := \sum_{i=0}^{n} \psi^i \frac{\partial f}{\partial x_i}, \qquad D^{\psi}[f] := \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\psi^i. \tag{4}$$

For simplicity of notation we continue to write ∂_i instead of $\dfrac{\partial}{\partial x_i}$.

Let Δ_{n+1} be the $(n+1)$-dimensional Laplace operator. It is easy to prove that the equalities

$$^{\psi}D \cdot {}^{\overline{\psi}}D = {}^{\overline{\psi}}D \cdot {}^{\psi}D = D^{\psi}\cdot D^{\overline{\psi}} = D^{\overline{\psi}}\cdot D^{\psi} = \Delta_{n+1}, \tag{5}$$

[1] $D_w^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_0^{\alpha_0}\cdots\partial x_n^{\alpha_n}}$ where $\alpha = (\alpha_0,\ldots,\alpha_n) \in (\mathbb{N}\cup\{0\})^{n+1}$ is a multi-index and $|\alpha| = \alpha_0+\cdots+\alpha_n$.

hold, if and only if

$$\psi^i \cdot \overline{\psi^j} + \psi^j \cdot \overline{\psi^i} = 2\delta_{i,j}, \quad i, j \in \mathbb{N}_n^0.$$

Note that last equality yields

$$2\delta_{i,j} = \psi^i \cdot \overline{\psi^j} + \psi^j \cdot \overline{\psi^i} = \psi^i \cdot \overline{\psi^j} + \overline{\psi^i \cdot \overline{\psi^j}} = 2\left[\psi^i \cdot \overline{\psi^j}\right]_0 = 2\left\langle \psi^i, \psi^j \right\rangle_{\mathbb{R}^{n+1}}, \tag{6}$$

thus, factorization (5) holds if and only if ψ represents an orthonormal basis of $\mathbb{R}^{n+1} \cong \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$.

Any set ψ with the property (6) is called *structural set*. It is evident that ψ and $\overline{\psi}$ are structural sets simultaneously. One can find basic properties of the structural sets in [23,24] and the references given therein.

Remark 2.1. In the case $n = 1$ ($\mathbb{R}_{0,1} \cong \mathbb{C}$), the only structural sets that we can find have the form $\psi = \{e^{i\theta}, \pm ie^{i\theta}\}$ where $0 \leq \theta \leq 2\pi$. In other words, in complex analysis, all structural sets are *equivalent* to one of the sets $\{1, i\}$ or $\{1, -i\}$.

For fixed ψ and Ω we introduce the sets

$$\begin{aligned} {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n}) &:= \ker {}^{\psi}D = \{f \in \mathbb{C}^1(\Omega, \mathbb{R}_{0,n}) : {}^{\psi}D[f] = 0_\Omega\}, \\ \mathfrak{M}^{\psi}(\Omega, \mathbb{R}_{0,n}) &:= \ker D^{\psi} = \{f \in \mathbb{C}^1(\Omega, \mathbb{R}_{0,n}) : D^{\psi}[f] = 0_\Omega\}. \end{aligned}$$

Sometimes, elements of these sets are called ψ-hyperholomorphic functions (left or right respectively). Denote by Θ_{n+1} the fundamental solution of the Laplace operator Δ_{n+1}

$$\Theta_{n+1}(x) = \frac{1}{(1-n)|\mathbb{S}^n|}|x|^{1-n}.$$

Of central importance is the fundamental solution (= the Cauchy kernel for the corresponding theory) of the operators ${}^{\psi}D$ and D^{ψ}, which is given, thanks to the relation (5), by:

$$K_\psi(x) := {}^{\overline{\psi}}D[\Theta_{n+1}](x) = D^{\overline{\psi}}[\Theta_{n+1}](x) = \frac{x_{\overline{\psi}}}{|\mathbb{S}^n| \cdot |x|^{n+1}}.$$

Here $x_{\overline{\psi}} := \sum_{i=0}^n x_i \overline{\psi^i}$ if $x = \sum_{i=0}^n x_i e_i$ and $|\mathbb{S}^n|$ is the area of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The kernel K_ψ has the following important properties:

1. $K_\psi \in C^\infty(\mathbb{R}^{n+1} \backslash \{0\}, \mathbb{R}_{0,n})$.
2. $K_\psi \in {}^{\psi}\mathfrak{M}(\mathbb{R}^{n+1} \backslash \{0\}, \mathbb{R}_{0,n}) \cap \mathfrak{M}^{\psi}(\mathbb{R}^{n+1} \backslash \{0\}, \mathbb{R}_{0,n})$.

2.1. Stokes formula

One of the most crucial facts in standard Clifford analysis is the existence of a Stokes formula. Here we present this formula for an arbitrary structural set ψ, see for instance [23] and elsewhere.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with a sufficiently smooth boundary Γ. The following theorem is standard in classical analysis.

Theorem 1 *(Gauss' formula). For $f \in C^1(\overline{\Omega}, \mathbb{R})$ and any $i \in \mathbb{N}_n^0$ we have*

$$\int_\Gamma n_i(\xi) \cdot f(\xi)\, d\Gamma_\xi = \int_\Omega \partial_i[f](\xi)\, d\xi$$

where $n_i(\xi)$ is the i-th component of the outward unit normal vector on Γ at the point $\xi \in \Gamma$.

As a consequence a Stokes formula for $\mathbb{R}_{0,n}$-valued functions yields the following theorem.

Theorem 2 *(Stokes formula). Let $f, g \in C^1(\overline{\Omega}, \mathbb{R}_{0,n})$ and ψ be a structural set. Then*

$$\int_\Gamma g(\xi) \cdot n_\psi(\xi) \cdot f(\xi)\, d\Gamma_\xi = \int_\Omega \big(D^\psi[g](\xi) \cdot f(\xi) + g(\xi) \cdot {}^\psi D[f](\xi)\big)\, d\xi, \tag{7}$$

where $n_\psi(\xi) = \sum_{i=0}^n n_i(\xi)\psi^i$.

Proof. Using the Gauss's formula coordinatewisely directly we have:

$$\begin{aligned}\int_\Omega \big(D^\psi[g]f + g\,{}^\psi D[f]\big)\, d\xi &= \int_\Omega \sum_{i=0}^n \big(\partial_i[g]\psi^i f + g\,\psi^i \partial_i[f]\big)\, d\xi \\ &= \int_\Omega \sum_{i=0}^n \partial_i\left[g\,\psi^i f\right] d\xi = \sum_{i=0}^n \int_\Omega \partial_i\left[g\,\psi^i f\right] d\xi \\ &= \sum_{i=0}^n \int_\Gamma n_i\left(g\,\psi^i f\right) d\Gamma_\xi = \int_\Gamma g\left(\sum_{i=0}^n n_i\psi^i\right) f\, d\Gamma_\xi \\ &= \int_\Gamma g\, n_\psi\, f\, d\Gamma_\xi. \qquad \square\end{aligned}$$

Remark 2.2. In the case $n = 1$, for the structural sets $\{1, i\}$ and $\{1, -i\}$, and taking into account the commutativity in $\mathbb{C}$, one can obtain the complex versions of (7), see for example [30, p. 24].

2.2. Integral operators

The Cauchy kernel generates the following two important integrals:

$${}^\psi T_\Omega[f](x) := -\int_\Omega K_\psi(\xi - x) \cdot f(\xi)\, d\xi, \quad x \in \mathbb{R}^{n+1},$$

and

$${}^{\varphi,\psi} K_\Gamma[f](x) := \int_\Gamma K_\varphi(\xi - x) \cdot n_\psi(\xi) \cdot f(\xi)\, d\Gamma_\xi, \quad x \notin \Gamma.$$

While the first is a generalization of the usual Teodorescu transform, the second represents a boundary *exotic* operator which connects two arbitrary structural sets φ and ψ. When $\psi = \varphi$, ${}^{\varphi,\psi} K_\Gamma$ reduces to the usual Cauchy type integral

$$ {}^{\psi}K_{\Gamma}[f](x) := \int_{\Gamma} K_{\psi}(\xi - x) \cdot n_{\psi}(\xi) \cdot f(\xi)\, d\Gamma_{\xi}. $$

The singular version of ${}^{\varphi,\psi}K_{\Gamma}[f]$ on Γ, denoted by ${}^{\varphi,\psi}S_{\Gamma}[f]$, is given, as usual, by

$$ {}^{\varphi,\psi}S_{\Gamma}[f] := 2\,{}^{\varphi,\psi}K_{\Gamma}[f]. $$

The integral which defines the operator ${}^{\varphi,\psi}S_{\Gamma}[f]$ is to be taken in the sense of the Cauchy principal value.

In the case of $\varphi = \psi$, it is known (see [23]) the exact relation between the boundary value of ${}^{\psi}K_{\Gamma}[f]$ and ${}^{\psi}S_{\Gamma}[f] := {}^{\psi,\psi}S_{\Gamma}[f]$.

Let us introduce the notations $\Omega^{+} := \Omega$ and $\Omega^{-} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$, then the following result holds.

Theorem 3 *(Plemelj–Sokhotski formulas). Let $f \in C^{0,\mu}(\Gamma, \mathbb{R}_{0,n})$, $\mu \in (0, 1]$. Then we have:*

$$ {}^{\psi}K^{\pm}_{\Gamma}[f](t) := \lim_{\Omega^{\pm} \ni x \to t \in \Gamma} {}^{\psi}K_{\Gamma}[f](x) = \frac{1}{2}\left[{}^{\psi}S_{\Gamma}[f](t) \pm f(t)\right]. $$

The Stokes formula leads immediately to three important consequences, which are widely known and can be found in many sources.

Theorem 4 *(Cauchy integral theorem). Let $f \in {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n}) \cap C^{0}(\overline{\Omega}, \mathbb{R}_{0,n})$ and $g \in \mathfrak{M}^{\psi}(\Omega, \mathbb{R}_{0,n}) \cap C^{0}(\overline{\Omega}, \mathbb{R}_{0,n})$. Then*

$$ \int_{\Gamma} g(\xi) \cdot n_{\psi}(\xi) \cdot f(\xi)\, d\Gamma_{\xi} = 0. \tag{8} $$

Theorem 5 *(Borel–Pompeiu (Cauchy–Green) formula). Let $f \in C^{1}(\overline{\Omega}, \mathbb{R}_{0,n})$. Then*

$$ \int_{\Gamma} K_{\psi}(\xi - x) \cdot n_{\psi}(\xi) \cdot f(\xi)\, d\Gamma_{\xi} - \int_{\Omega} K_{\psi}(\xi - x) \cdot {}^{\psi}D[f](\xi) d\xi = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}. \end{cases} $$

Theorem 6 *(Cauchy integral formula). Let $f \in {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n}) \cap C^{0}(\overline{\Omega}, \mathbb{R}_{0,n})$. Then*

$$ \int_{\Gamma} K_{\psi}(\xi - x) \cdot n_{\psi}(\xi) \cdot f(\xi)\, d\Gamma_{\xi} = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}. \end{cases} \tag{9} $$

Remark 2.3. Particularly, for $\varphi = \psi = \psi_{st} := \{1, e_1, \ldots, e_n\}$, Theorems 4, 5 and 6 have been known for a long time, see for instance [7,9,11].

3. ${}^{\varphi,\psi}\Pi_{\Omega}$-operator and its integral representation

From [11] it is known that in the case of the standard structural set ψ_{st} of $\mathbb{R}_{0,n}$ the operator

$$ {}^{\psi_{st}}T_{\Omega} : W^{k}_{p}(\Omega, \mathbb{R}_{0,n}) \mapsto W^{k+1}_{p}(\Omega, \mathbb{R}_{0,n}) $$

is a continuous mapping for $1 < p < \infty$, $k \in \mathbb{N}$. Using an orthogonal transformation of coordinates we obtain the same property for general structural sets. Therefore, the operator ${}^{\varphi}D^{\psi}T_{\Omega}$ is well defined on $L_p(\Omega, \mathbb{R}_{0,n})$ and on $C^{0,\mu}(\Omega, \mathbb{R}_{0,n})$. On these spaces we have that

$$^{\psi_{st}}D^{\psi_{st}}T_{\Omega} = I,$$

where I stands for the identity operator.

Definition 1. For a pair of structural sets φ, ψ we define the operator $^{\varphi,\psi}\Pi_{\Omega}$ by

$$^{\varphi,\psi}\Pi_{\Omega} := {}^{\varphi}D^{\psi}T_{\Omega}.$$

Applying the differential theory of singular integral operators we get an integral representation formula for the $^{\varphi,\psi}\Pi_{\Omega}$-operator, which generalizes those obtained in [7, Theorem 4] and [9, Theorem 2] as well as [10, Theorem 3.2] for the quaternionic case.

Theorem 7. *Let $\Omega \in \mathbb{R}^{n+1}$ and $f \in L_p(\Omega, \mathbb{R}_{0,n})$, $p \in (1, \infty)$, then for all $x \in \Omega$ we have*

$$\begin{aligned} ^{\varphi,\psi}\Pi_{\Omega}[f](x) &= \int_{\Omega} \left(\frac{\sum_{i=0}^{n} \varphi^i \overline{\psi^i}}{|\mathbb{S}^n||\xi - x|^{n+1}} - \frac{(n+1)(\xi - x)_{\varphi}(\xi - x)_{\overline{\psi}}}{|\mathbb{S}^n||\xi - x|^{n+3}} \right) f(\xi)\, d\xi \\ &\quad + \frac{\sum_{i=0}^{n} \varphi^i \overline{\psi^i}}{n+1} f(x) \\ &= \int_{\Omega} {}_{\varphi,\psi}\Lambda(\xi - x) f(\xi)\, d\xi + \frac{\sum_{i=0}^{n} \varphi^i \overline{\psi^i}}{n+1} f(x), \end{aligned} \tag{10}$$

where ${}_{\varphi,\psi}\Lambda(\xi - x) := -{}^{\varphi}D_x {}^{\overline{\psi}}D_{\xi}\left[\Theta_{n+1}(\xi - x)\right]$.

Proof. From [19, Theorem 11.1, Chapter XI] we get for $f \in L_p(\Omega, \mathbb{R}_{0,n})$, $i \in \mathbb{N}_n^0$, that $\partial_i \left[{}^{\psi}T_{\Omega}[f]\right] \in L_p(\Omega, \mathbb{R}_{0,n})$. Moreover, we have

$$\partial_i \left[{}^{\psi}T_{\Omega}[f]\right](x) = \frac{-1}{|\mathbb{S}^n|} \left\{ \int_{\Omega} \partial_i \left[\frac{(\xi - x)_{\overline{\psi}}}{|\xi - x|^{n+1}} \right] f(\xi)\, d\xi - \int_{\partial B[0,1]} u(x, \theta) \cos(\vec{r}, x_i) dS_{\theta}\, f(x) \right\},$$

where $r = |\xi - x|$, $\vec{r} = \xi - x$, $\theta = \dfrac{\xi - x}{|\xi - x|}$ and $u(x, \theta) = \dfrac{(\xi - x)_{\overline{\psi}}}{|\xi - x|} = \theta_{\overline{\psi}}$.

It is easily seen that the outward unit normal vector on $\partial B[0, 1]$ at the point θ is precisely θ. Moreover, taking into account that $\cos(\vec{r}, x_i) = \dfrac{\xi_i - x_i}{|\xi - x|}$ is the i-th component of the unit vector parallel to $\vec{r}$, we have that $\cos(\vec{r}, x_i) = n_i(\theta)$. Then, using the Gauss's theorem we get the equality

$$\begin{aligned} \int_{\partial B[0,1]} u(x, \theta) \cos(\vec{r}, x_i) dS_{\theta} &= \int_{\partial B[0,1]} n_i(\theta) \theta_{\overline{\psi}}\ dS_{\theta} = \int_{B[0,1]} \partial_{\theta_i} \left[\theta_{\overline{\psi}}\right] d\theta \\ &= \overline{\psi^i} \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2} + 1\right)}. \end{aligned}$$

On the other hand,

$$\partial_i \left[\frac{(\xi - x)_{\overline{\psi}}}{|\xi - x|^{n+1}} \right] = \frac{-\overline{\psi^i}|\xi - x|^2 + (n+1)(\xi_i - x_i)(\xi - x)_{\overline{\psi}}}{|\xi - x|^{n+3}}.$$

From this we obtain our representation formula by summation over i. $\square$

From the mapping properties of ${}^{\psi}T_{\Omega}$ it is easy to see that

$$ {}^{\varphi,\psi}\Pi_{\Omega} : W_p^k(\Omega, \mathbb{R}_{0,n}) \mapsto W_p^k(\Omega, \mathbb{R}_{0,n}), $$

for $1 < p < \infty$ and $k \in \mathbb{N} \cup \{0\}$. Applying the theorem of Calderon and Zygmund (see [19, Chapter XI, 3.1]) for $k = 0$ we obtain the following result:

Theorem 8. *Suppose $1 < p < \infty$. Then for $\Omega \subseteq \mathbb{R}^{n+1}$ we have that*

$$ {}^{\varphi,\psi}\Pi_{\Omega} : L_p(\Omega, \mathbb{R}_{0,n}) \mapsto L_p(\Omega, \mathbb{R}_{0,n}) $$

is a continuous mapping.

Particularly, for $\varphi = \psi$, an important consequence of Theorem 7 follows.

Corollary 1. *Let ψ be a structural set and $\Omega \subset \mathbb{R}^{n+1}$ a bounded domain. Then, for all $f \in L_p(\Omega, \mathbb{R}_{0,n})$, $p \in (1, \infty)$, we have*

$$ {}^{\psi,\psi}\Pi_{\Omega}[f](x) = {}^{\psi}D^{\psi}T_{\Omega}[f](x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}. \end{cases} $$

This property can be extended to the case $\Omega = \mathbb{R}^{n+1}$.

Corollary 2. *Let ψ be a structural set. Then, for all $f \in L_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, $p \in (1, \infty)$, we have*

$$ {}^{\psi,\psi}\Pi_{\mathbb{R}^{n+1}}[f](x) = {}^{\psi}D^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = f(x). $$

Proof. Suppose that $f \in C_0^{\infty}(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, then there exists a suitable compact subset $K \subset \mathbb{R}^{n+1}$ such that $f(x) = 0$ for all $x \notin K$. Using Corollary 1

$$ {}^{\psi,\psi}\Pi_{\mathbb{R}^{n+1}}[f](x) = {}^{\psi}D^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = {}^{\psi}D^{\psi}T_{K}[f](x) = \begin{cases} f(x) & \text{if } x \in K, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus K. \end{cases} $$

Then we get that ${}^{\psi}D^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = f(x)$ for all $x \in \mathbb{R}^{n+1}$. Finally, taking into account the density of $C_0^{\infty}(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ in $L_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ and the continuity of the operator ${}^{\psi,\psi}\Pi_{\mathbb{R}^{n+1}}$ in the space $L_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, we obtain the desired result. □

3.1. Generalized Borel–Pompeiu formula

We are now proceeded to derive an essential integral formula, to be called generalized Borel–Pompeiu formula, which expresses a very profound relation among the operators ${}^{\varphi}D$, ${}^{\psi}T_{\Omega}$ and ${}^{\varphi,\psi}K_{\Gamma}$.

Theorem 9 *(Generalized Borel–Pompeiu formula). Let $f \in C^1(\overline{\Omega}, \mathbb{R}_{0,n})$. Then:*

$$ {}^{\varphi,\psi}\Pi_{\Omega}[f](x) = \int_{\Gamma} K_{\overline{\varphi}}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_{\xi} - \int_{\Omega} K_{\overline{\varphi}}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi, \quad x \notin \Gamma. \tag{11} $$

Remark 3.1. The above formula may be written in a shorter way as:

$$ {}^{\varphi,\psi}\Pi_{\Omega}[f](x) = {}^{\overline{\varphi},\overline{\psi}}K_{\Gamma}[f](x) + {}^{\overline{\varphi}}T_{\Omega}{}^{\overline{\psi}}D[f](x), \quad x \notin \Gamma. $$

Proof. Let $x \in \Omega$. Using the Stokes formula (7) for the structural set $\overline{\psi}$ and the functions $g(\xi) = \Theta_{n+1}(\xi - x)$ and $f(\xi)$, in the multiply connected domain $\Omega_{x,\epsilon} := \Omega \setminus B[x, \epsilon]$ ($\epsilon > 0$ small) we have the equality

$$\int_{\Gamma} \Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi - \int_{\partial B(x,\epsilon)} \Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi$$
$$= \int_{\Omega_{x,\epsilon}} K_\psi(\xi - x) \cdot f(\xi)\, d\xi + \int_{\Omega_{x,\epsilon}} \Theta_{n+1}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi. \tag{12}$$

Further, for every $\xi \in \partial B(x, \epsilon)$ we have

$$\left|\Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\right| = \frac{|f(\xi)|}{(n-1)|\mathbb{S}^n|\epsilon^{n-1}}.$$

By hypothesis f is a continuous function in the compact set $\overline{\Omega}$ and then, there exists $M > 0$ such that $|f(\xi)| < M$, $\forall\, \xi \in \Omega \cup \Gamma$. Consequently,

$$\left|\int_{\partial B(x,\epsilon)} \Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi\right| < \frac{M}{(n-1)|\mathbb{S}^n|\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} d\Gamma_\xi = \frac{M}{(n-1)}\epsilon,$$

which implies that

$$\lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon)} \Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi = 0. \tag{13}$$

On the other hand,

$$-{}^{\psi}T[f](x) = \int_{\Omega} K_\psi(\xi - x) \cdot f(\xi)\, d\xi := \lim_{\epsilon \to 0} \int_{\Omega_{x,\epsilon}} K_\psi(\xi - x) \cdot f(\xi)\, d\xi, \tag{14}$$

and

$$\int_{\Omega} \Theta_{n+1}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi := \lim_{\epsilon \to 0} \int_{\Omega_{x,\epsilon}} \Theta_{n+1}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi. \tag{15}$$

Letting $\epsilon \to 0$ in (12) and using (13), (14) and (15) we see that

$${}^{\psi}T_\Omega[f](x) = \int_{\Omega} \Theta_{n+1}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi - \int_{\Gamma} \Theta_{n+1}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi.$$

Applying the operator ${}^{\varphi}D$ in both sides of the above equality we obtain:

$${}^{\varphi}D^{\psi}T_\Omega[f](x) = \int_{\Gamma} K_{\overline{\varphi}}(\xi - x) \cdot n_{\overline{\psi}}(\xi) \cdot f(\xi)\, d\Gamma_\xi - \int_{\Omega} K_{\overline{\varphi}}(\xi - x) \cdot {}^{\overline{\psi}}D[f](\xi)\, d\xi,$$

and (11) is proved.

The proof for the case $x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}$ is similar. First, we use again formula (7) but now in Ω and then the task is only to apply the operator ${}^{\varphi}D$ in both sides of the resulting equality. □

Remark 3.2. In the quaternionic case, formula (11) is the one presented in [1].

4. Properties of the ${}^{\varphi,\psi}\Pi_\Omega$-operator

The ideas presented in [8,11] allow us to establish the following results about the decomposition of $L_2(\Omega, \mathbb{R}_{0,n})$.

Theorem 10. *Let ψ be a structural set. Then, the Hilbert space $L_2(\Omega, \mathbb{R}_{0,n})$ permits the following orthogonal decomposition with respect to the inner product (3)*

$$L_2(\Omega, \mathbb{R}_{0,n}) = \big(L_2(\Omega, \mathbb{R}_{0,n}) \cap {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n})\big) \oplus {}^{\overline{\psi}}D(\overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}_{0,n})). \tag{16}$$

There $\overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}_{0,n})$ denotes the subspace of all functions of $W_2^1(\Omega, \mathbb{R}_{0,n})$ whose traces in Γ vanish.

The above orthogonal decomposition generates a pair of mutually complementary orthoprojections

$$\begin{aligned} {}^{\psi}\mathbf{P} &: L_2(\Omega, \mathbb{R}_{0,n}) \longrightarrow L_2(\Omega, \mathbb{R}_{0,n}) \cap {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n}), \\ {}^{\psi}\mathbf{Q} &: L_2(\Omega, \mathbb{R}_{0,n}) \longrightarrow {}^{\overline{\psi}}D(\overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}_{0,n})). \end{aligned}$$

In the following we state and prove a collection of properties of the ${}^{\varphi,\psi}\Pi_\Omega$-operator extending the results given in [7,9,14,31] for the particular case of $\varphi = \overline{\psi}_{st}$ and $\psi = \vartheta = \psi_{st}$.

Theorem 11. *Let φ, ψ, ϑ be three arbitrary structural sets and $f \in W_p^1(\Omega, \mathbb{R}_{0,n})$, $1 < p < \infty$. Then, the following equalities[2] holds in Ω*

$$\begin{aligned} {}^{\overline{\varphi}}D\,{}^{\varphi,\psi}\Pi[f] &= {}^{\overline{\psi}}D[f], && (17)\\ {}^{\varphi,\psi}\Pi\,{}^{\vartheta}D[f] &= {}^{\varphi}D\,{}^{\overline{\psi},\overline{\vartheta}}\Pi[f] - {}^{\varphi}D\,{}^{\psi,\vartheta}K_\Gamma[f], && (18)\\ {}^{\overline{\varphi}}K_\Gamma\,{}^{\varphi,\psi}\Pi[f] &= ({}^{\varphi,\psi}\Pi - {}^{\overline{\varphi}}T_\Omega\,{}^{\overline{\psi}}D)[f], && (19)\\ ({}^{\overline{\varphi}}D\,{}^{\varphi,\psi}\Pi - {}^{\overline{\psi},\vartheta}\,\Pi\,{}^{\vartheta}D)[f] &= {}^{\overline{\psi}}D\,{}^{\vartheta}K_\Gamma[f], && (20)\\ {}^{\overline{\varphi}}K_\Gamma\,{}^{\varphi,\psi}\Pi\,{}^{\overline{\psi}}K_\Gamma[f] &= {}^{\varphi,\psi}\Pi\,{}^{\overline{\psi}}K_\Gamma[f]. && (21) \end{aligned}$$

Proof. The proofs of these equalities are based in the use of the classical or generalized version of the Borel–Pompeiu formula.

(17): ${}^{\overline{\varphi}}D\,{}^{\varphi,\psi}\Pi[f] = {}^{\overline{\varphi}}D({}^{\overline{\varphi},\overline{\psi}}K_\Gamma + {}^{\overline{\varphi}}T_\Omega\,{}^{\overline{\psi}}D)[f] = {}^{\overline{\psi}}D[f]$,
(18): ${}^{\varphi,\psi}\Pi\,{}^{\vartheta}D[f] = {}^{\varphi}D\,{}^{\psi}T\,{}^{\vartheta}D[f] = {}^{\varphi}D({}^{\overline{\psi},\overline{\vartheta}}\Pi - {}^{\psi,\vartheta}K_\Gamma)[f]$,
(19): ${}^{\overline{\varphi}}K_\Gamma\,{}^{\varphi,\psi}\Pi[f] = ({}^{\varphi,\psi}\Pi - {}^{\overline{\varphi}}T_\Omega\,{}^{\overline{\varphi}}D\,{}^{\varphi}D\,{}^{\psi}T_\Omega)[f] = ({}^{\varphi,\psi}\Pi - {}^{\overline{\varphi}}T\,{}^{\overline{\psi}}D\,{}^{\psi}D\,{}^{\psi}T)[f] = ({}^{\varphi,\psi}\Pi - {}^{\overline{\varphi}}T\,{}^{\overline{\psi}}D)[f]$,
(20): $({}^{\overline{\varphi}}D\,{}^{\varphi,\psi}\Pi - {}^{\overline{\psi},\vartheta}\,\Pi\,{}^{\vartheta}D)[f] = ({}^{\overline{\psi}}D - {}^{\overline{\psi}}D + {}^{\overline{\psi}}D\,{}^{\vartheta}K_\Gamma)[f] = {}^{\overline{\psi}}D\,{}^{\vartheta}K_\Gamma[f]$ (see (17) and (18)),
(21): ${}^{\overline{\varphi}}K_\Gamma\,{}^{\varphi,\psi}\Pi\,{}^{\overline{\psi}}K_\Gamma[f] = ({}^{\varphi,\psi}\Pi - {}^{\overline{\varphi}}T_\Omega\,{}^{\overline{\psi}}D)(I - {}^{\overline{\psi}}T_\Omega\,{}^{\overline{\psi}}D)[f] = {}^{\varphi,\psi}\Pi(I - {}^{\overline{\psi}}T_\Omega\,{}^{\overline{\psi}}D)[f] = {}^{\varphi,\psi}\Pi\,{}^{\overline{\psi}}K_\Gamma[f]$. □

Remark 4.1. Some particular choices of ϑ allow to different versions of the above formulas. Indeed, making $\vartheta = \psi$ and $\vartheta = \overline{\varphi}$ in the equalities (18) and (20) respectively we have

$$\begin{aligned} {}^{\varphi,\psi}\Pi\,{}^{\psi}D[f] &= {}^{\varphi}D[f] - {}^{\varphi}D\,{}^{\psi}K_\Gamma[f], \\ ({}^{\overline{\varphi}}D\,{}^{\varphi,\psi}\Pi - {}^{\overline{\psi},\overline{\varphi}}\,\Pi\,{}^{\overline{\varphi}}D)[f] &= {}^{\overline{\psi}}D\,{}^{\overline{\varphi}}K_\Gamma[f]. && (22) \end{aligned}$$

[2] Equalities (17) and (20) extend those obtained in [30, 8.21 and 8.22].

Moreover, if we compare (19) and (21) with the generalized Borel–Pompeiu formula we get

Corollary 3. *Let $f \in W_p^1(\Omega, \mathbb{R}_{0,n})$. Then, we have in Ω that*

$$ {}^{\overline{\varphi}}K_\Gamma {}^{\varphi,\psi}\Pi[f] = {}^{\bar{\varphi},\bar{\psi}}K_\Gamma[f](x). \tag{23}$$

The following result extends that proved in [7,9,14,31]. Also generalize the particular case given in [10].

Proposition 2. *For any pair φ, ψ of structural sets we have*

$$ {}^{\varphi,\psi}\Pi : \operatorname{im} {}^{\overline{\psi}}\mathbf{Q} \mapsto \operatorname{im} {}^{\overline{\varphi}}\mathbf{Q}, \tag{24}$$

$$ {}^{\varphi,\psi}\Pi : \operatorname{im} {}^{\overline{\psi}}\mathbf{P} \mapsto \operatorname{im} {}^{\overline{\varphi}}\mathbf{P}. \tag{25}$$

Proof. It is sufficient to use (22) and (17). □

In addition, note that property (25) can be written in a more complete form using the equality (17) as follows.

Proposition 3. ${}^{\varphi,\psi}\Pi[f] \in \operatorname{im} {}^{\overline{\varphi}}\mathbf{P} \Leftrightarrow f \in \operatorname{im} {}^{\overline{\psi}}\mathbf{P}$.

Our next results again generalize those proved in [7,9,10].

Theorem 12. *Let $f \in W_p^k(\Omega, \mathbb{R}_{0,n})$, $1 < p < \infty$, $k \in \mathbb{N}$, then*

$$ {}^{\psi,\varphi}\Pi {}^{\varphi,\psi}\Pi[f] = {}^{\psi}D {}^{\varphi}T_\Omega {}^{\varphi}D {}^{\psi}T_\Omega[f] = {}^{\psi}D(I - {}^{\varphi}K_\Gamma) {}^{\psi}T_\Omega[f] = f - {}^{\psi}D {}^{\varphi}K_\Gamma {}^{\psi}T_\Omega[f].$$

Observe that interchanging φ and ψ in the above equality we get

$$ {}^{\varphi,\psi}\Pi {}^{\psi,\varphi}\Pi[f] = f - {}^{\varphi}D {}^{\psi}K_\Gamma {}^{\varphi}T_\Omega[f].$$

Therefore, as a consequence of the previous theorem, we obtain immediately the one-sided invertibility of the ${}^{\varphi,\psi}\Pi$ operator in those spaces of functions where $\mathbf{tr}{}^{\psi}T_\Omega$ or $\mathbf{tr}{}^{\varphi}T_\Omega$ vanishes.

Proposition 4. *For an arbitrary structural set ψ, we have*

$$ f \in \operatorname{im} {}^{\psi}\mathbf{Q} \Leftrightarrow \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega[f] = 0_\Gamma,$$

where 0_Γ stands for the function $x \in \Gamma \to 0 \in \mathbb{R}$.

Proof. ($\Rightarrow$) Let $f \in \operatorname{im} {}^{\psi}\mathbf{Q}$, then there exists $u \in \overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}_{0,n})$ such as $f = {}^{\overline{\psi}}D[u]$. Then,

$$ {}^{\overline{\psi}}T_\Omega[f] = {}^{\overline{\psi}}T_\Omega {}^{\overline{\psi}}D[u] = u - {}^{\overline{\psi}}K_\Gamma[u] = u \Rightarrow \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega[f] = 0_\Gamma.$$

($\Leftarrow$) Let $\mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega[f] = 0_\Gamma$. We have

$$ 0_\Gamma = \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega[f] = \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega {}^{\psi}\mathbf{P}[f] + \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega {}^{\psi}\mathbf{Q}[f] \Rightarrow \mathbf{tr}_\Gamma {}^{\overline{\psi}}T_\Omega {}^{\psi}\mathbf{P}[f] = 0_\Gamma.$$

Then, using that the function ${}^{\overline{\psi}}T_\Omega {}^{\psi}\mathbf{P}[f]$ is harmonic in Ω, we can conclude that it is identically zero in Ω. Therefore, ${}^{\overline{\psi}}D {}^{\overline{\psi}}T_\Omega {}^{\psi}\mathbf{P}[f] = 0_\Omega \Rightarrow {}^{\psi}\mathbf{P}[f] = 0_\Omega \Rightarrow f \in \operatorname{im} {}^{\psi}\mathbf{Q}$. □

From Theorem 12 and Proposition 4 we have the following result.

Proposition 5 *(One-sided invertibility of the ${}^{\varphi,\psi}\Pi$ operator).*

$$
\begin{aligned}
{}^{\varphi,\psi}\Pi\,{}^{\psi,\varphi}\Pi &= I_{\operatorname{im}{}^{\overline{\varphi}}\mathbf{Q}} \quad \textit{in } \operatorname{im}{}^{\overline{\varphi}}\mathbf{Q},\\
{}^{\psi,\varphi}\Pi\,{}^{\varphi,\psi}\Pi &= I_{\operatorname{im}{}^{\overline{\psi}}\mathbf{Q}} \quad \textit{in } \operatorname{im}{}^{\overline{\psi}}\mathbf{Q}.
\end{aligned}
$$

In the remainder of this section we concentrate in the case $\Omega = \mathbb{R}^{n+1}$.

Theorem 13. *Let φ, ψ be two arbitrary structural sets and $f, g \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, then*

$$\left\langle {}^{\varphi,\psi}\Pi f, g\right\rangle_2 = \left\langle f, {}^{\psi,\varphi}\Pi g\right\rangle_2. \tag{26}$$

Proof. If $g \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, then, there exists a compact subset $K \subset \mathbb{R}^{n+1}$ such that $g(x) = 0$ for all $x \notin K$. It is obvious that, if we take a ball B such that $K \subset B$ strictly, we have ${}^{\overline{\psi}}T_{\mathbb{R}^{n+1}}{}^{\overline{\varphi}}D[g] = {}^{\overline{\psi}}T_B{}^{\overline{\varphi}}D[g]$. Then, according to the formula (11),

$$
\begin{aligned}
{}^{\overline{\psi}}T_{\mathbb{R}^{n+1}}{}^{\overline{\varphi}}D[g] &= {}^{\overline{\psi}}T_B{}^{\overline{\varphi}}D[g] = {}^{\psi}D^{\varphi}T_B[g] - {}^{\overline{\psi},\overline{\varphi}}K_{\partial B}[g] = {}^{\psi}D^{\varphi}T_B[g]\\
&= {}^{\psi}D^{\varphi}T_{\mathbb{R}^{n+1}}[g] = {}^{\psi,\varphi}\Pi[g].
\end{aligned}
$$

It is easy to check for all pairs $u, v \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ that

$$\left\langle {}^{\psi}Tu, v\right\rangle_2 = -\langle u, {}^{\overline{\psi}}Tv\rangle_2,$$

and with the additional supposition of $v \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ we get

$$\langle {}^{\varphi}Du, v\rangle_2 = -\left\langle u, {}^{\overline{\varphi}}Dv\right\rangle_2.$$

Then, for all pairs of functions $f \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ and $g \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ we obtain

$$\left\langle {}^{\varphi,\psi}\Pi f, g\right\rangle_2 = \left\langle {}^{\varphi}D^{\psi}Tf, g\right\rangle_2 = -\left\langle {}^{\psi}Tf, {}^{\overline{\varphi}}Dg\right\rangle_2 = \langle f, {}^{\overline{\psi}}T^{\overline{\varphi}}Dg\rangle_2 = \left\langle f, {}^{\psi,\varphi}\Pi g\right\rangle_2.$$

Finally, by density of the space $C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ in $L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ and using the continuity of the operator ${}^{\psi,\psi}\Pi_{\mathbb{R}^{n+1}}$ in this space, the desired result fails. □

The formula (26) shows that ${}^{\varphi,\psi}\Pi$ and ${}^{\psi,\varphi}\Pi$ are adjoint operators in the space $L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$.

Theorem 14. *Let ϑ, φ, ψ be three arbitrary structural sets and $f \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$. Then*

$${}^{\vartheta,\varphi}\Pi\,{}^{\varphi,\psi}\Pi[f] = {}^{\vartheta,\psi}\Pi[f]. \tag{27}$$

Proof. Suppose that $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, then there exists a compact subset $K \subset \mathbb{R}^{n+1}$ such that $f \equiv 0$ in $\mathbb{R}^{n+1} \setminus K$. Let $B_r := B(0, r)$, where $r > 0$ is a fixed number such that $K \subset B_r$. Let $x \in \mathbb{R}^{n+1}$ be an arbitrary point and $R > \text{máx}\{|x|, r\}$. Then, by virtue of the classical Borel–Pompeiu formula inside the ball $B_R := B(0, R)$

$${}^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = {}^{\varphi}K_{\partial B_R}{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x) + {}^{\varphi}T_{B_R}{}^{\varphi}D^{\psi}T_{\mathbb{R}^{n+1}}[f](x).$$

But

$$\left|{}^{\varphi}K_{\partial B_R}{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x)\right| \leq \frac{1}{|\mathbb{S}^n|}\int\limits_{\partial B_R} \frac{1}{|\xi - x|^n} \cdot \left|{}^{\psi}T[f](\xi)\right| dS_\xi, \tag{28}$$

and for any $\xi \in \partial B_R$, $|\xi - x| \geq \operatorname{dist}(x, \partial B_R) = R - |x|$, then

$$\frac{1}{|\xi - x|^n} \leq \frac{1}{(R - |x|)^n}. \tag{29}$$

On the other hand

$$\left|{}^{\psi}T_{\mathbb{R}^{n+1}}[f](\xi)\right| = \left|{}^{\psi}T_{B_r}[f](\xi)\right| \leq \frac{\|f\|_1}{|\mathbb{S}^n|}\operatorname{dist}(\xi, B_r)^{-n} = \frac{\|f\|_1}{|\mathbb{S}^n|(R-r)^n}. \tag{30}$$

Using (29) and (30) in (28) we obtain

$$\begin{aligned}
&\left|{}^{\varphi}K_{\partial B_R}{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x)\right| \leq C\,\|f\|_1 \frac{R^n}{(R-|x|)^n\,(R-r)^n} \\
&\Rightarrow \lim_{R\to\infty} {}^{\varphi}K_{\partial B_R}{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = 0.
\end{aligned}$$

Then,

$${}^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = {}^{\varphi}T_{\mathbb{R}^{n+1}}{}^{\varphi}D{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x).$$

Applying in both sides of this relation the operator ${}^{\vartheta}D$ we obtain for $f \in C_0^{\infty}(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ that

$${}^{\vartheta,\psi}\Pi[f] = {}^{\vartheta}D{}^{\psi}T_{\mathbb{R}^{n+1}}[f](x) = {}^{\vartheta,\varphi}\Pi{}^{\varphi,\psi}\Pi[f](x).$$

We now apply again the density argument of $C_0^{\infty}(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ in $L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ and the continuity of ${}^{\varphi,\psi}\Pi_{\mathbb{R}^{n+1}}$ in this space and the proof is complete. $\square$

Remark 4.2. A more general result can be proved. Let ς, ϑ, φ, ψ be four arbitrary structural sets. Then the following equality holds

$${}^{\varsigma,\vartheta}\Pi{}^{\varphi,\psi}\Pi[f] = {}^{\varsigma}D{}^{\overline{\vartheta},\overline{\varphi}}\Pi{}^{\psi}T[f].$$

For the particular case of $\vartheta = \psi$ we obtain from (27) the both-side invertibility of the operator ${}^{\varphi,\psi}\Pi$.

Corollary 4. *Let φ, ψ be two arbitrary structural sets and $f \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$. Then*

$${}^{\psi,\varphi}\Pi{}^{\varphi,\psi}\Pi[f] = f. \tag{31}$$

Remark 4.3. The relation (31) proves that in the space $L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ the operator ${}^{\varphi,\psi}\Pi$ has a both-side inverse ${}^{\varphi,\psi}\Pi^{-1}$ which coincides with its adjoint ${}^{\psi,\varphi}\Pi$. Indeed, from (31) we obtain that ${}^{\psi,\varphi}\Pi$ is the left inverse of ${}^{\varphi,\psi}\Pi$, but interchanging φ and ψ we get that ${}^{\psi,\varphi}\Pi$ is also the right inverse.

By virtue of Theorem 13 and Theorem 14 we obtain the following important result.

Theorem 15. *Let ϑ, φ, ψ be three arbitrary structural sets and $f, g \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$. Then,*

$$\left\langle {}^{\varphi,\vartheta}\Pi f, {}^{\varphi,\psi}\Pi g\right\rangle_2 = \left\langle f, {}^{\vartheta,\psi}\Pi g\right\rangle_2. \tag{32}$$

As a consequence, for the particular case of $\vartheta = \psi$ we get that

$${}^{\varphi,\psi}\Pi : L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}) \to L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}),$$

is a unitary operator.

Proposition 6. *Let φ, ψ be two arbitrary structural sets and $f, g \in L_2(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$. Then*

$$\left\langle {}^{\varphi,\psi}\Pi f, {}^{\varphi,\psi}\Pi g \right\rangle_2 = \langle f, g\rangle_2 ,$$
$$\left\| {}^{\varphi,\psi}\Pi f \right\|_{L_2(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} = \| f\|_{L_2(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} .$$

By a simple extension argument we can also obtain that $\left\| {}^{\varphi,\psi}\Pi f \right\|_{L_2(\Omega,\mathbb{R}_{0,n})} = 1$ for any bounded smooth domain Ω.

One of the most interesting conjectures with respect to the complex Π-operator is the famous conjecture by Iwaniec [13] about the norm of the complex Π-operator:

$$\|\Pi\|_{L_p} = \begin{cases} \frac{1}{p-1} & 1 < p \leq 2 \\ p-1 & 2 < p < \infty \end{cases}$$

Applying the Fourier transform $\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{-\mathbf{i}\langle x,\xi\rangle} f(x)dx$ to our ${}^{\varphi,\psi}\Pi$-operator we get for the Fourier symbol of our operator

$$\mathcal{F}\left({}^{\varphi,\psi}\Pi f\right)(\xi) = \sum_{i=0}^{n}\sum_{j=0}^{n} \varphi^i \psi^j \frac{\xi_i\xi_j}{|\xi|^2}(\mathcal{F}f)(\xi)$$

From the observation of the Fourier symbol we get the following connection between the ${}^{\varphi,\psi}\Pi$ and the Riesz transforms $R_j f(y) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(x)dx$:

$${}^{\varphi,\psi}\Pi f = \sum_{i=0}^{n}\sum_{j=0}^{n} \varphi^i \psi^j R_i R_j f.$$

Consider now the Cauchy problem for the heat equation $U_t + \Delta U = 0$ with $U(0, x) = u(x)$. Its solution is given by $U(t, x) = (\mathcal{F}^{-1} e^{-|\xi|^2 t}\mathcal{F}u)(x)$. Also, we will denote by V the corresponding solution with initial data $V(0, x) = v(x)$. Considering the sesquilinear form

$$\langle U, V\rangle = \int_0^\infty \int_{\mathbb{R}^{n+1}} \overline{U(x,t)}V(x,t)dxdt$$

we obtain

$$\begin{aligned}
\langle \psi^i \partial_i U, \overline{\varphi^j}\partial_j V\rangle &= \int_0^\infty \int_{\mathbb{R}^{n+1}} \overline{\psi^i (\mathcal{F}^{-1}(-\mathbf{i}\xi_i) e^{-|\xi|^2 t}\mathcal{F}u)(x)}\,\overline{\varphi^j}(\mathcal{F}^{-1}(-\mathbf{i}\xi_j)e^{-|\xi|^2 t}\mathcal{F}v)(x)dxdt \\
&= \int_{\mathbb{R}^{n+1}} \overline{((-\mathbf{i}\xi_i)\mathcal{F}u)(\xi)}\,\overline{\psi^i}\,\overline{\varphi^j}(-\mathbf{i}\xi_j)\mathcal{F}v)(\xi)\int_0^\infty e^{-2|\xi|^2 t}dtd\xi \\
&= \int_{\mathbb{R}^{n+1}} \overline{(\xi_i\mathcal{F}u)(\xi)}\,\overline{\psi^i}\,\overline{\varphi^j}(\xi_j\mathcal{F}v)(\xi)\frac{1}{2|\xi|^2}d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^{n+1}} \overline{\psi^i R_i u}\,\overline{\varphi^j} R_j v dx.
\end{aligned}$$

Now, since we have

$$\begin{aligned}\langle {}^{\varphi,\psi}\Pi u, v\rangle_2 &= \langle \sum_{i=0}^{n}\sum_{j=0}^{n} \varphi^i \psi^j R_i R_j u, v\rangle_2 \\ &= \langle \sum_{j=0}^{n} \psi^j R_j u, \sum_{i=0}^{n} \overline{\varphi^i} R_i v\rangle_2 \\ &= \langle \sum_{j=0}^{n} \psi^j \partial_j U, \sum_{i=0}^{n} \overline{\varphi^i} \partial_i V\rangle \\ &= \langle {}^{\psi}DU, \overline{{}^{\varphi}D}V\rangle\end{aligned}$$

we get

$$|\langle {}^{\varphi,\psi}\Pi u, v\rangle_2| = \left| \int_0^\infty \int_{\mathbb{R}^{n+1}} {}^{\psi}DU\overline{{}^{\varphi}D}V\, dxdt \right|. \tag{33}$$

Here we could use the idea of estimates via Martingales (see [3] and references therein). But for the purpose of this paper it is enough to apply the idea of Petermichl, Slavin and Wick [21] of using the Bellman function technique. The proof of the following theorem is a straightforward application of the corresponding proofs in [21] and [20].

Theorem 16. *For the norm of the operator*

$${}^{\varphi,\psi}\Pi : L_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}) \to L_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}),\ 1 < p < \infty,$$

we have the estimate

$$\|{}^{\varphi,\psi}\Pi\|_{L_p} \le 2^{n/2}(n+1)(p^* - 1)$$

with $p^* = \max\left\{p, \frac{p}{p-1}\right\}$.

The principal point for the proof is that for each partial derivative Petermichl, Slavin and Wick [21] provide the following theorem (adapted to our case).

Theorem 17. *Let* $u, v \in C_0^\infty(\mathbb{R}^m, H)$ *with* H *being a Hilbert module and* $p \ge 2$. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1$ *then*

$$2\sum_{i,j=1}^{m} \int \int_{R_+^{m+1}} \|\partial_i U(x,t)\|_H \|\partial_i V(x,t)\|_H dxdt \le (p^* - 1)\|u\|_{L_p(\mathbb{R}^m, H)}\|v\|_{L_q(\mathbb{R}^m, H)}.$$

Strictly speaking Petermichl, Slavin and Wick [21] proved the above theorem for the case of H being a Hilbert space over a Grassmann algebra, but the proof can be easily adapted to the case of a Clifford–Hilbert module with the obvious modifications.

Starting with formula (33) we can now write

$$|\langle {}^{\varphi,\psi}\Pi u, v\rangle_2| \le 2^{n/2}\left(\sum_{i=0}^{n} \int_{\mathbb{R}_+^{n+2}} |\partial_i U||\partial_i V| dxdt + \sum_{i\neq j} \int_{\mathbb{R}_+^{n+2}} |\partial_i U||\partial_j V| dxdt \right)$$

and apply the above theorem.

Since the ${}^{\varphi,\psi}\Pi$-operator commutes with the partial derivatives we obtain the following theorem.

Theorem 18. *For the norm of the operator*

$$ {}^{\varphi,\psi}\Pi : W_p^k(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}) \to W_p^k(\mathbb{R}^{n+1}, \mathbb{R}_{0,n}) $$

with $1 < p < \infty$, $k \in \mathbb{N}$, *we get*

$$ \|{}^{\varphi,\psi}\Pi\|_{W_p^k} \leq 2^{n/2}(n+1)(p^* - 1). $$

Since it is well-known that $\|{}^{\varphi,\psi}\Pi\| \geq (p^* - 1)$ we get the following estimate

$$ p^* - 1 \leq \|{}^{\varphi,\psi}\Pi\|_{W_p^k} \leq 2^{n/2}(n+1)(p^* - 1) \tag{34} $$

with $1 < p < \infty$ and $k \in \mathbb{N}$.

In the last section during our study of the application to higher-dimensional Beltrami equations this result will be quite helpful.

5. Jump of ${}^{\varphi,\psi}\Pi$ operator

The aim of this section is to find an expression for the jump of the function ${}^{\varphi,\psi}\Pi_\Omega[f]$ across the boundary Γ of the domain Ω. To this end, we will need to assume the boundary Γ to be a Lyapunov surface and $f \in C^{0,\mu}(\Gamma, \mathbb{R}_{0,n})$. Without loss of generality we may regard f to be continuous out of Γ and the class being conserved. Therefore the representation (11) holds for $x \notin \Gamma$.

We claim that, without difficulties Plemelj–Sokhotski formulas remain true also for the *exotic* Cauchy type integral ${}^{\varphi,\psi}K_\Gamma$. The success of our method will depend on using the Hölder continuity of the outward unit normal vector on a Lyapunov surface. Indeed:

Theorem 19. *Let* $f \in C^{0,\mu}(\Gamma, \mathbb{R}_{0,n})$, $\mu \in (0, 1]$. *Then, we have*

$$ {}^{\varphi,\psi}K_\Gamma^{\pm}[f](t) := \lim_{\Omega^{\pm} \ni x \to t \in \Gamma} {}^{\varphi,\psi}K_\Gamma[f](x) = \frac{1}{2}\left[{}^{\varphi,\psi}S_\Gamma[f](t) \pm n_{\overline{\varphi}}(t) n_\psi(t) f(t)\right]. $$

Proof. Note that, as a consequence of $n_\varphi(\xi)n_{\overline{\varphi}}(\xi) = 1$, we have

$$ {}^{\varphi,\psi}K_\Gamma[f](x) = \int_\Gamma K_\varphi(\xi - x)\left[n_\varphi(\xi)n_{\overline{\varphi}}(\xi)\right] n_\psi(\xi) f(\xi)\, d\Gamma_\xi = {}^{\varphi}K_\Gamma[n_{\overline{\varphi}} n_\psi f](x). $$

Next, the Plemelj–Sokhotski formulas applied to ${}^{\varphi}K_\Gamma[n_{\overline{\varphi}} n_\psi f]$ show that

$$ {}^{\varphi,\psi}K_\Gamma^{\pm}[f](t) = \frac{1}{2}\left[{}^{\varphi,\psi}S_\Gamma[f](t) \pm n_{\overline{\varphi}}(t) n_\psi(t) f(t)\right]. \qquad \square $$

To proceed with, it is worth noting that the function ${}^{\overline{\varphi}}T_\Omega[f]$ is continuous on the whole $\mathbb{R}^{n+1}$ if $f \in L_p(\Omega, \mathbb{R}_{0,n})$, $p > n+1$. This result is obtained in [8, Proposition 8.1] in the case of standard structural set $\varphi = \varphi_{st}$. The proof for an arbitrary φ is similar.

After the above preparations, we are now ready to give the expression for the jump of the function ${}^{\varphi,\psi}\Pi_\Omega[f]$ on the contour Γ, using the previous Plemelj–Sokhotski formulas. Precisely

$$ \left({}^{\varphi,\psi}\Pi_\Omega[f]\right)^+(t) - \left({}^{\varphi,\psi}\Pi_\Omega[f]\right)^-(t) = n_\varphi(t) n_{\overline{\psi}}(t) f(t), \quad t \in \Gamma. \tag{35} $$

Remark 5.1. In the case of $\varphi = \psi = \psi_{st}$, formula (35) can be found in [14, Deduction 3.9, p. 22].

Remark 5.2. Let $n = 1$, and Γ (being in this case a curve) consist of points $t(s) = x(s) + iy(s)$, $0 \le s \le l$, where l is the length of Γ and s is the length of the arc counted from a fixed point on Γ to t. For the structural sets $\psi_{st} = \{1, i\}$ and $\overline{\psi}_{st} = \{1, -i\}$, we have

$$n_{\psi_{st}} = -i\frac{dt}{ds} \quad \text{and} \quad n_{\overline{\psi}_{st}} = i\frac{d\bar{t}}{ds}.$$

Then, applying (35) the jump of ${}^{\overline{\psi}_{st},\psi_{st}}\Pi$ is given by

$$\left({}^{\overline{\psi}_{st},\psi_{st}}\Pi[f]\right)^{+}(t) - \left({}^{\overline{\psi}_{st},\psi_{st}}\Pi[f]\right)^{-}(t) = \left(n_{\overline{\psi}_{st}}(t)\right)^{2} f(t) = -\left(\frac{d\bar{t}}{ds}\right)^{2} f(t).$$

This relation is exactly that obtained in [30, p. 62] for the jump of the classical complex Π-operator.

6. Application of the ${}^{\varphi,\psi}\Pi_\Omega$-operator to the solution of Beltrami equations

Let us now consider the following Beltrami equation ${}^{\psi}Dw(z) = q(z)\,{}^{\varphi}Dw(z)$, $z \in \mathbb{R}^{n+1}$. This type of Beltrami equations was first studied in [16] in the case of quaternionic analysis, where it was shown that these equations have a solution of the form $w(z) = \Phi(z) + Th(z)$, where $\Phi(z)$ is a monogenic function and $h(z)$ a solution of the singular integral equation $h = q^{\varphi}D\Phi + q^{\varphi,\psi}\Pi h$, if only $\|q^{\varphi,\psi}\Pi\| < 1$ in the appropriate norm. This system includes the case of D and $\overline{D}$ as well as the case of Shevchenko [26]. Here and in the following we consider the case of q taking values in the paravector group $\mathcal{P}$. This includes the most important cases for applications. To obtain a global solution w we require $q \in W_p^1(\mathbb{R}^{n+1}, \mathcal{P})$, $p > n+1$, $\|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \le q_c < 1/\|{}^{\varphi,\psi}\Pi\|$. We remark that in [16] the condition on the structural sets is $\sum_{k=0}^{n} \varphi^k \overline{\psi^k}$ which is not satisfied by the standard choice $\psi^k = \overline{\varphi^k}$, $k = 0, \ldots, n$.

One of the principal questions is if and when the solution of a Beltrami equation is a local or global quasi-conformal homeomorphism. The basis for such a discussion is the following theorem [5] adapted to our case:

Theorem 20. *(See [6].) Let f be monogenic in z_0, i.e. ${}^{\varphi}Df(z) = 0$ in a neighborhood of z_0, then we have*

$$\operatorname{rk} J_f(z_0) < n+1 \Leftrightarrow \exists p : |p| = 1 \wedge \overline{{}^{\varphi}D} f(\overline{p}z)|_{z=z_0} = 0,$$

where $\operatorname{rk} J_f(z_0)$ denotes the rank of the Jacobian at the point z_0.

We fix now the monogenic function $\Phi(z)$ such that ${}^{\psi}D\Phi = 0$ and ${}^{\varphi}D\Phi = 1$. Using the affine transformation $\xi = (z - z_0)$ we can reduce our study to the point $z_0 = 0$. For each neighborhood $U_\delta(0) = \{\xi \in \mathbb{R}^{n+1},\ |\xi| < \delta\}$ we consider the function $q_\delta(\xi) = q(\xi)b(\xi)$ with

$$b(\xi) = \begin{cases} 1 & |\xi| < \frac{1}{2}\delta, \\ 2\left(\frac{|\xi|}{\delta}\right)^3 - \frac{9}{2}\left(\frac{|\xi|}{\delta}\right)^2 + 3\left(\frac{|\xi|}{\delta}\right) - \frac{1}{2} & \frac{1}{2}\delta \le |\xi| \le \delta, \\ 0 & |\xi| > \delta. \end{cases}$$

It holds $\|q_\delta\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \le \|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})}\|b\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \le C_\varphi \delta^n \|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})}$, $C_\varphi = \left(\frac{2^n+15}{2^{n+1}(n+1)} + \frac{2}{n+4} + \frac{3}{2(n+3)} - \frac{6}{n+2}\right)|\mathbb{S}^n|$.

Let us denote by $W^1_{p,0}(U_\delta(0), \mathbb{R}_{0,n})$ the space of all $W^1_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$-functions with support in $U_\delta(0)$. Then we have for the operator ${}^{\varphi,\psi}\Pi_\delta$, defined by ${}^{\varphi,\psi}\Pi_\delta h = q_\delta {}^{\varphi,\psi}\Pi h$, the mapping property ${}^{\varphi,\psi}\Pi_\delta : W^1_{p,0,}(U_\delta(0), \mathbb{R}_{0,n}) \mapsto W^1_{p,0}(U_\delta(0), \mathbb{R}_{0,n})$, $1 < p < \infty$. Moreover, from $\|q\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq q_c < 1/\|{}^{\varphi,\psi}\Pi\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})}$, $p > n+1$, and $\|q_\delta\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq C_\varphi \delta^n \|q\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})}$ it follows that for all $\delta \leq \frac{1}{\sqrt[n]{C_\varphi}}$ the operator ${}^{\varphi,\psi}\Pi_\delta$ is a contraction over $W^1_p(U_\delta(0), \mathbb{R}_{0,n})$, $p > n+1$.

This implies that $w = \Phi + {}^{\psi}Th_\delta$, where h_δ is a solution of $h_\delta = q_\delta + {}^{\varphi,\psi}\Pi_\delta h_\delta$, is a solution of the Beltrami-type equation ${}^{\psi}Dw(\xi) = q_\delta(\xi)\, {}^{\varphi}Dw(\xi)$ over $\mathbb{R}^{n+1}$ and, moreover, also a solution of the Beltrami-type equation ${}^{\psi}Dw(\xi) = q(\xi)\, {}^{\varphi}Dw(\xi)$ over $U_\delta(0)$. Finally, after applying the inverse transformation of our affine transformation we get a solution of our original Beltrami-type equation ${}^{\psi}Dw(z) = q(z)\, {}^{\varphi}Dw(z)$ over $U_\delta(z_0)$. Additionally, by Banach's fixed-point theorem and $\delta < \frac{1}{\sqrt[n]{C_\varphi}}$ we obtain the estimate

$$\|h_\delta\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} < 2C_\varphi \|q\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \delta^n < 2\frac{C_\varphi}{\|{}^{\varphi,\psi}\Pi\|_{W^1_p}} \delta^n.$$

Therefore, we get that for all $\epsilon > 0$ there exists a δ such that

$$\|h_\delta\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} < \frac{\epsilon}{C_p \|{}^{\varphi,\psi}\Pi\|_{W^1_p}}, \tag{36}$$

where C_p denotes the embedding constant of $W^1_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, $p > n+1$, in $C(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, and we can use Theorem 20.

From this theorem we have the condition

$$\mathrm{rk}\, J_w|_{\xi=0} < n+1 \Leftrightarrow \exists p : |p| = 1 \wedge \overline{{}^{\varphi}D}_\xi w(p\xi)|_{\xi=0} = 0.$$

We consider now the terms $\overline{{}^{\varphi}D}\xi w(p\xi)$ and $\overline{\overline{{}^{\varphi}D}_\xi w(p\xi)}$ at zero and transforming them into $\sum_A \psi_A(P^T Q_A)$, where Q_A are real symmetric matrices and P denotes our rotation written as a vector. Since it is enough that one of the matrices Q_A is positive or negative definite we can restrict ourselves to the scalar part. This leads to the following matrix:

$$\begin{pmatrix} 3 + C_{00} & C_{01} & C_{02} & \dots & C_{0n} \\ C_{10} & 3 + C_{11} & C_{12} & \dots & C_{13} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n0} & C_{n1} & C_{n2} & \dots & 3 + C_{nn} \end{pmatrix} \tag{37}$$

Hereby, we refer to [6] for the coefficients and details. We only remark that the coefficients C_{ij} of system (37) fulfil $|C_{ij}| \leq C_p \|{}^{\overline{\psi},\psi}\Pi\|_{W^1_p} \|h^\delta\|_{W^1_p}$ for all indices i, j, where C_p denotes again the embedding constant of W^1_p in $C(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ [9].

Moreover, from (36) we obtain that $|C_{ij}| < \frac{\|{}^{\overline{\psi},\psi}\Pi\|}{\|{}^{\varphi,\psi}\Pi\|}\epsilon$ holds. Choosing now $\frac{\|{}^{\overline{\psi},\psi}\Pi\|}{\|{}^{\varphi,\psi}\Pi\|}\epsilon < 1/3$ the elements q_{ij} of our matrix (37) satisfy $q_{ii} > \sum_{j \neq i} |q_{ij}|$.

This allows us to establish the following theorem.

Theorem 21. *Suppose that $q \in W^1_p(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ for a certain $p > n+1$ and $\|q\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq q_c < 1/\|{}^{\varphi,\psi}\Pi\|_{W^1_p(\mathbb{R}^{n+1},\mathbb{R}_{0,n})}$. Suppose also that $q+1$ is not a zero divisor for any point z_0 in $\mathbb{R}^{n+1}$. Then the Beltrami-type equation ${}^{\psi}Dw(z) = q(z)\, {}^{\varphi}Dw(z)$ has in each point z_0 a solution, which realizes a local quasi-conformal mapping.*

Using our norm estimate for the ${}^{\varphi,\psi}\Pi$-operator (34) we can state the following corollary.

Corollary 5. *A sufficient condition for the above theorem is* $\|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq q_c < \frac{1}{2^{n/2}(n-1)(p^*-1)}$.

Now, we assume $q \in W_p^1(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$ with

$$\|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq q_c < \frac{1}{(n-1)\min\left(\|^{\varphi,\psi}\Pi\|_{W_p^1}, \|^{\overline{\psi},\psi}\Pi\|_{W_p^1}\right)}$$

for some $p > n+1$. This means that we have $\|h\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} < \frac{q_c}{1-q_c\|^{\varphi,\psi}\Pi\|_{W_p^1}} < \frac{1}{3\|\Pi\|_{W_p^1}}$ and the solution $w = \Phi + Th$ belongs to the space $C^1(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$. Moreover, our linear system (37) again turns out to be diagonal dominant.

The well-known theorem of Hadamard [2], which states, that a function $w(z)$, where the Jacobian determinant is not equal to zero in any point and $w(z) \to \infty$ if $|z| \to \infty$, realizes a global homeomorphism, together with our norm estimate of the ${}^{\varphi,\psi}\Pi$-operator (34) allows us to establish the following theorem.

Theorem 22. *If* $q \in W_p^1(\mathbb{R}^{n+1}, \mathbb{R}_{0,n})$, $\|q\|_{W_p^1(\mathbb{R}^{n+1},\mathbb{R}_{0,n})} \leq q_c < \frac{1}{2^{n/2}(n-1)(p^*-1)}$, $p^* = \max\left\{p, \frac{p}{p-1}\right\}$, $q+1$ *not being a zero divisor at any point in* $\mathbb{R}^{n+1}$*, for some* $p > n+1$ *then the function*

$$w = \Phi + Th,$$

with h *being a solution of the corresponding singular integral equation, is a solution of the Beltrami-type equation* ${}^{\psi}Dw(z) = q(z)\,{}^{\varphi}Dw(z)$ *which realizes a global homeomorphism.*

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