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Fueter's theorem and its generalizations in Dunkl–Clifford analysis

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Abstract
In this paper, we give a construction of Dunkl monogenic and Dunkl harmonic functions starting from holomorphic functions in the plane. This construction has the advantage of not needing Dunkl's intertwining operator or Dunkl spherical harmonics. To this end we study Vekua-type systems and prove a version of Fueter's theorem in the case of finite reflection groups. Important examples, such as a Dunkl monogenic Gaussian distribution or a Cauchy kernel, will be given at the end.

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1. Introduction

It is well known that there are many problems in physics which can be treated by methods using harmonic or holomorphic functions. It is maybe not so well known that these methods usually require that the problem itself shows symmetries under a rotation group $SO(n)$. This is due to the fact that the Laplace or Dirac operator is invariant under rotations. In fact, often one is using the correspondence between monogenic functions, i.e. null solution of the Dirac operator and irreducible representations of spin groups. But there are a lot of applications where it would be advantageous to have methods which are based on symmetries given by reflection groups, particularly finite reflection groups, instead of rotation groups. However, there exists one major obstacle. While partial derivatives are closely linked to rotations this is not the case for reflections. The way out seems to be to consider differential-difference operators [7, 9], also called Dunkl operators in the literature. Due to their connection with Coxeter groups these operators are used in many fields of mathematics and physics. They provide a useful tool in the study of special functions with root systems [5, 8, 12]. Moreover,

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the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero–Moser–Sutherland models, which describe quantum mechanical system of N identical particles on a circle or line which interact pairwise through the long-range potentials of inverse square type [6, 13, 14, 16, 24]. Furthermore, since finite reflection groups or Coxeter groups correspond to so-called point groups in crystallography, which describe the symmetries of a crystal, also in this field the application of operators linked to such groups instead of methods using classic harmonic function theory is required. This is the main reason for the importance of Dunkl operators in the study of the crystallographic Radon and x-ray transforms.

One of the most important properties of Dunkl operators is that they are mutually commute. This allowed the authors in [1, 3, 17] to introduce a Dirac operator, called the Dunkl–Dirac operator, based on differential-difference operators which are invariant under reflection groups and also factorize the Dunkl–Laplacian (see section 2 for details). However, the construction of specific Dunkl monogenic functions is a very difficult task. The main reason is that the two major tools used in the literature, the Dunkl intertwining operator and the classes of spherical harmonics associated with root systems are not really adequate for the explicit construction of Dunkl monogenic or Dunkl harmonic functions. While Dunkl's intertwining operator V_κ provides a link with classical partial derivatives, i.e. $T_i V_\kappa = V_\kappa \partial_i$, it is explicitly known only in some special cases. The application of spherical Dunkl harmonics needs the knowledge of the invariant measure and the special orthogonal polynomials related to it. Additionally, it only results in approximation by power series. It would be much better if one has a method which only requires the explicit knowledge of the involved operators alone to construct Dunkl monogenic functions. But, such a tool exists in classical hypercomplex function theory, in the form of the so-called Fueter's theorem.

To explain the idea we take a look into the classical case. If $f(z)$ is a holomorphic function in an open set B in the upper half complex plane and

$$f(z) = u(s,t) + \mathbf{i}v(s,t), \qquad z = s + \mathbf{i}t,$$

then Fueter's theorem [10, 21, 25] asserts that in the set $\vec{B} = \left\{x = x_0 + \underline{x} \in \mathbb{R}_1^d : (x_0, |\underline{x}|) \in B\right\}$ there holds

$$\partial_x \Delta_x^{(d-1)/2} \left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) = 0,$$

where ∂_x and Δ_x denotes the classical generalized Cauchy–Riemann operator and Laplace operator, respectively. In other words, one only needs to apply the Laplacian often enough to a given holomorphic function to obtain a monogenic one.

This theme was further developed and found to play a crucial role in the study of Fourier multipliers and singular integrals on the unit sphere of $\mathbb{R}_1^d$ and its Lipschitz perturbations [22, 23]. As examples, their study shows that by means of Fueter's machinery, some problems on the unit sphere may be reduced to the corresponding ones on the unit circle in the complex plane.

For simplicity, we will denote for $x \in \vec{B}$,

$$\vec{f}(x_0 + \underline{x}) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|).$$

$\vec{B}$ is said to be the *induced set* from B and $\vec{f}(x_0 + \underline{x})$ the *induced function* from f.

In a recent work, Kou, Peña–Peña, Qian and Sommen [15, 18, 26] proved the following generalization of Fueter's theorem: if $P_n(\underline{x})$ is a left monogenic, homogeneous function of degree n in $\mathbb{R}^d$, i.e. $\partial_{\underline{x}} P_n(\underline{x}) = 0$, then

$$\partial_x \left(\Delta_x^{n+(d-1)/2} (\vec{f}(x_0 + \underline{x}) P_n(\underline{x})) \right) = 0,$$

whenever $n + (d - 1)/2$ is a non-negative integer. If the space dimension $d + 1$ is odd, then the above result also holds for n being the non-negative integer but in this case $P_n(\underline{x})$ needs to be a homogeneous left monogenic polynomial of degree n.

When $n = 0$, this reduces to the standard Fueter's theorem. We would like to remark that some other generalizations of Fueter's theorem have been studied in [18, 19, 20].

Since this theorem provides a method to construct the higher dimensional monogenic functions or harmonic functions explicitly, we would like to use it in the case of finite reflection groups. To this end, we will prove a version of Fueter's theorem for Dunkl monogenic functions, as given in the following theorem.

Theorem 1.1. *Let W be a finite reflection group which leaves x_0-axis invariant and f be a holomorphic function in an open set B in the upper half complex plane given by*

$$f = u(s,t) + \mathbf{i}v(s,t), \qquad z = s + \mathbf{i}t.$$

Then in the induced set $\vec{B}$ the function

$$\Delta_h^{\gamma_\kappa + \frac{d-1}{2}} \vec{f}(x_0 + \underline{x})$$

is Dunkl monogenic whenever $\gamma_\kappa + (d-1)/2$ is a positive integer. Hereby Δ_h denotes the Dunkl–Laplacian.

Furthermore, we also give a generalization of the above theorem as follows:

Theorem 1.2. *If, in addition to the assumptions in theorem 1.1, we assume that $P_n(\underline{x})$ is a homogeneous Dunkl monogenic function of degree n in $\mathbb{R}^d$, where n is any non-negative integer. Then in the induced set $\vec{B}$ the function*

$$\Delta_h^{\gamma_\kappa + n + (d-1)/2}(\vec{f}(x_0 + \underline{x}) P_n(\underline{x}))$$

is Dunkl monogenic whenever $\gamma_\kappa + n + (d-1)/2$ is a positive integer.

The applicability of the method constructed in the present paper is restricted to $\gamma_\kappa + n + (d-1)/2$ being positive integers. That is because in the classical case the applications of Fueter's theorem reduces to the pointwise differentiation. On the other hand, for more general cases where γ_κ is any positive real number, Fueter's theorem will require Fourier multiplier operators. Since we are more interested in the calculation of explicit examples, this case will not be considered here.

The paper is organized as follows. In the next section we collect some basic facts about Clifford algebra and Dunkl operators. The spherical decomposition of Dunkl–Dirac operators will be studied in section 3. Section 4 is devoted to the proofs of our main theorems. In section 5 we will present some examples of Dunkl monogenic functions. Among them is the important example of a Dunkl monogenic Gaussian, i.e. Green's function of the diffusion equation in the Dunkl monogenic case or the monogenic wave function of ground state of the quantum harmonic oscillator in the Dunkl case.

2. Preliminaries

We denote by $\mathbb{R}_{0,d}$ the real Clifford algebra constructed over the orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ of the Euclidean space $\mathbb{R}^d$. The basic axiom of this associative but non-commutative algebra is that the product of a vector with itself equals its squared length up to a minus sign, i.e. for any vector $\underline{x} = \sum_{i=0}^{d} x_i \mathbf{e}_i$ in $\mathbb{R}^d$, we have that

$$\underline{x}^2 = -|\underline{x}|^2 = -\sum_{i=1}^{d} x_i^2.$$

It thus follows that the elements of the basis submit to the multiplication rules

$$\mathbf{e}_i^2 = -1, \qquad i = 1, \ldots, d,$$
$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \qquad 1 \leqslant i \neq j \leqslant d.$$

A basis for the algebra is then given by the elements

$$\mathbf{e}_A = \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_d},$$

where $A = \{i_1, \ldots, i_d\} \subset \{1, \ldots, d\}$ is such that $1 \leqslant i_1 < \cdots < i_d \leqslant d$. For the empty set $\emptyset$, we put $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$, the latter being the identity. It follows that the dimension of $\mathbb{R}_{0,d}$ is 2^d. Hence, each element $a \in \mathbb{R}_{0,d}$ will be represented by

$$a = \sum_A a_A \mathbf{e}_A, \qquad a_A \in \mathbb{R}.$$

An important subspace of the real Clifford algebra $\mathbb{R}_{0,d}$ is the so-called space of paravectors $\mathbb{R}_1^d = \mathbb{R} \bigoplus \mathbb{R}^d$, being sums of scalars and vectors.

In what follows, $sc[x] = x_0$ will denote the scalar part of $x \in \mathbb{R}_{0,d}$, while an element $x = (x_0, x_1, \ldots, x_d)$ of $\mathbb{R}_1^d$ will be identified with $x = x_0 + \underline{x}$, $\underline{x} = \sum_{i=1}^d x_i \mathbf{e}_i$. Also, we need the anti-involution $\bar{\ }$ defined by $\bar{\mathbf{e}}_0 = \mathbf{e}_0$, $\bar{\mathbf{e}}_i = -\mathbf{e}_i$ and $\overline{\mathbf{e}_i\mathbf{e}_j} = \bar{\mathbf{e}}_j\bar{\mathbf{e}}_i$. An important property of algebra $\mathbb{R}_{0,d}$ is that each non-zero vector x in $\mathbb{R}^d$(or in $\mathbb{R}_1^d$) has a multiplicative inverse given by $\frac{\bar{x}}{\|x\|^2}$.

An $\mathbb{R}_{0,d}$-valued function f over $\Omega \subset \mathbb{R}_1^d$ has a representation

$$f = \sum_A \mathbf{e}_A f_A,$$

with component $f_A : \Omega \to \mathbb{R}$.

The reflection $\sigma_\alpha x$ of a given vector $x \in \mathbb{R}_1^d$ on the hyperplane orthogonal to $\alpha \neq 0$ is given, in Clifford notation, by

$$\sigma_\alpha x := \alpha x \alpha^{-1}.$$

We remark that it corresponds (up to a factor of -1) to the standard notation of a reflection given by

$$\sigma_\alpha x := x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha.$$

A finite set $R \subset \mathbb{R}_1^d \backslash \{0\}$ is called a root system if $R \bigcap \mathbb{R}_1^d \cdot \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the finite reflection group (or Coxeter group) associated with R. All reflections in W correspond to the suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R | \langle \alpha, \beta\rangle > 0\}$, i.e. for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

Sometimes we will only consider reflections which only act in $\mathbb{R}^d$. In this case we denote α or β by $\underline{\alpha}$ or $\underline{\beta}$.

A function $\kappa : R \to \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards κ as a function on the corresponding reflections, this means that κ is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index $\gamma_\kappa = \sum_{\alpha\in R_+} \kappa(\alpha)$.

For each fixed positive subsystem R_+ and multiplicity function κ we have, as invariant operators, the differential-difference operators (also called Dunkl operators)

$$T_i f(x) = \frac{\partial}{\partial x_i} f(x) + \sum_{\alpha\in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}\alpha_i, \qquad i = 0, 1, \ldots, d, \quad (1)$$

for $f \in C^1(\mathbb{R}_1^d)$. In the case $\kappa = 0$, T_i, $i = 0, 1, \ldots, d$, reduce to the corresponding partial derivatives. This also give us the justification to think of these differential-difference operators as the equivalent of partial derivatives in the case of finite reflection groups. In this paper, we will assume throughout that $\kappa \geqslant 0$ and $\gamma_\kappa > 0$. More importantly, these operators mutually commute; that is, $T_i T_j = T_j T_i$. This property allows us to define a Dunkl–Dirac operator in $\mathbb{R}^d$ for the corresponding reflection group W given by

$$\underline{D}_h f = \sum_{i=1}^{d} \mathbf{e}_i T_i f. \tag{2}$$

The Dunkl–Laplacian $\underline{\Delta}_h$ in $\mathbb{R}^d$ associated with the finite reflection group W and the multiplicity function κ is defined by

$$\begin{aligned}\underline{\Delta}_h f &= -\underline{D}_h^2 f = \sum_{i=1}^{d} T_i^2 f \\ &= \Delta_{\underline{x}} f + 2\sum_{\underline{\alpha}\in R_+} \kappa(\underline{\alpha}) \frac{\langle \underline{\alpha}, \nabla_{\underline{x}} f(\underline{x})\rangle}{\langle \underline{\alpha}, \underline{x}\rangle} - 2\sum_{\underline{\alpha}\in R_+} \kappa(\underline{\alpha}) \frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}}\underline{x})}{\langle \underline{\alpha}, \underline{x}\rangle^2}\end{aligned} \tag{3}$$

for any $f \in C^2(\mathbb{R}^d)$, where $\triangle_{\underline{x}}$ and $\nabla_{\underline{x}}$ are usual Laplacian and gradient operators in $\mathbb{R}^d$, respectively.

We now introduce the Dunkl–Cauchy–Riemann operator in $\mathbb{R}_1^d$:

$$D_h = T_0 + \underline{D}_h,$$

and Dunkl–Laplacian in $\mathbb{R}_1^d$:

$$\Delta_h = T_0^2 + \underline{\Delta}_h.$$

In this paper, we will assume that our group W will leave the x_0-axis invariant. Since in this case we have $T_0 = \partial_{x_0}$, the Dunkl–Cauchy–Riemann operator and Dunkl–Laplacian in $\mathbb{R}_1^d$ can also be written by

$$D_h = \partial_{x_0} + \underline{D}_h, \tag{4}$$

and

$$\Delta_h = \partial_{x_0}^2 + \underline{\Delta}_h. \tag{5}$$

Functions belonging to the kernel of the Dunkl–Dirac operator $\underline{D}_h$ or the Dunkl–Cauchy–Riemann operator D_h will be called Dunkl monogenic functions. As usual, functions belonging to be the kernel of Dunkl–Laplacian will be called Dunkl harmonic functions.

3. Spherical decomposition of the Dunkl–Dirac operator and Vekua-type systems

The classical Fueter's theorem and its generalizations obtained in [15, 18, 25] provide us with Dunkl monogenic functions of the form

$$A(x_0, r) + \underline{\omega} B(x_0, r), \tag{6}$$

whereby $\underline{x} \in \mathbb{R}^d$, $r = |\underline{x}|$, $\underline{x} = r\underline{\omega}$ and A and B are scalar-valued functions. This means that

$$D_h(A(x_0, r) + \underline{\omega} B(x_0, r)) = 0, \tag{7}$$

where $D_h = \partial_{x_0} + \underline{D}_h$.

To study this system we need the representation of our Dunkl–Dirac operator in terms of spherical coordinates which is given in the next theorem.

Theorem 3.1. *In spherical coordinates the Dunkl–Dirac operator has the form*

$$\underline{D}_h = \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right), \tag{8}$$

with

$$\Gamma_{\underline{\omega}} = \gamma_\kappa + \Phi_{\underline{\omega}} + \Psi,$$

where

$$\Phi_{\underline{\omega}} = -\sum_{i<j} \mathbf{e}_i \mathbf{e}_j \left(x_i \partial_{x_j} - x_j \partial_{x_i}\right),$$

and

$$\Psi f(\underline{x}) = -\sum_{i<j} \mathbf{e}_i \mathbf{e}_j \sum_{\underline{\alpha}\in R^+} \kappa(\underline{\alpha}) \frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}}\underline{x})}{\langle \underline{\alpha}, \underline{x}\rangle}(x_i\alpha_j - x_j\alpha_i) - \sum_{\underline{\alpha}\in R^+} \kappa(\underline{\alpha}) f(\sigma_{\underline{\alpha}}\underline{x}),$$

for any $f \in C^1(\mathbb{R}^d)$.

Remark 3.1. While the operator $\Phi_{\underline{\omega}}$ in the above theorem corresponds to the classical spherical vector derivatives (the classic Gamma operator), the additional operator Ψ and constant γ_κ derive from the difference part.

In order to prove this theorem instead of trying to work with a direct calculation in terms of coordinate functions we will employ a standard technique in higher dimensions whereby we study the commutator and anti-commutator between $\underline{x}$ and $\underline{D}_h$.

If we define the commutator and anti-commutator for two linear operators X and Y as follows:

$$[X, Y] = XY - YX, \qquad \{X, Y\} = XY + YX,$$

then we get the following properties:

Lemma 3.1.

$$\{\underline{x}, \underline{D}_h\} = -2\left(E_{\underline{x}} + \frac{d}{2} + \gamma_\kappa\right), \tag{9}$$

and

$$[\underline{x}, \underline{D}_h] = -2\left(\Phi_{\underline{\omega}} - \frac{d}{2} + \Psi\right), \tag{10}$$

whereby $E_{\underline{x}}$ *is the classical Euler operator and* $\Phi_{\underline{\omega}}$ *and* Ψ *are the operators defined in the previous theorem.*

Proof. For the first relation, we have

$$\underline{x}\underline{D}_h + \underline{D}_h\underline{x} = \sum_{i\neq j} \mathbf{e}_i \mathbf{e}_j (x_i T_j - T_j x_i) - \sum_{i=1}^{d} (x_i T_i + T_i x_i). \tag{11}$$

First we shall verify that the term $\sum_{i\neq j} \mathbf{e}_i \mathbf{e}_j (x_i T_j - T_j x_i)$ vanishes. To this end, for all $f \in C^1(\mathbb{R}^d)$,

$$\begin{aligned} x_i T_j f - T_j(x_i f) &= x_i \left(\sum_{\underline{\alpha}\in R^+} \kappa(\underline{\alpha}) \frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}}\underline{x})}{\langle \underline{\alpha}, \underline{x}\rangle}\alpha_j\right) - \sum_{\underline{\alpha}\in R^+} \kappa(\underline{\alpha}) \frac{x_i f(\underline{x}) - (\sigma_{\underline{\alpha}}\underline{x})_i f(\sigma_{\underline{\alpha}}\underline{x})}{\langle \underline{\alpha}, \underline{x}\rangle}\alpha_j \\ &= -2\sum_{\underline{\alpha}\in R^+} \kappa(\underline{\alpha}) \frac{\alpha_i\alpha_j f(\sigma_{\underline{\alpha}}\underline{x})}{|\underline{\alpha}|^2}, \end{aligned}$$

which is an expression obviously symmetric in i and j, so that summing over the product of the basis elements gives zero. For the second sum on the right-hand side of (11) we obtain

$$\begin{aligned}
&\sum_{i=1}^{d}(x_i T_i f + T_i(x_i f)) \\
&\quad = 2\sum_{i=1}^{d} x_i \partial_{x_i} f(\underline{x}) + \mathrm{d} f(\underline{x}) + \sum_{i=1}^{d}\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha}) x_i\left(\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha})\frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}}\underline{x})}{\langle\underline{\alpha},\underline{x}\rangle}\alpha_i\right) \\
&\qquad + \sum_{i=1}^{d}\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha})\frac{x_i f(\underline{x}) - (\sigma_{\underline{\alpha}}\underline{x})_i f(\sigma_{\underline{\alpha}}\underline{x})}{\langle\underline{\alpha},\underline{x}\rangle}\alpha_i \\
&\quad = 2E_{\underline{x}} f(\underline{x}) + \mathrm{d} f(\underline{x}) + 2f(\underline{x})\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha}) \\
&\quad = 2\left(E_{\underline{x}} + \frac{d}{2} + \gamma_\kappa\right) f(\underline{x}).
\end{aligned}$$

This proves the first relation.

For the second relation, since we have

$$\underline{x}\,\underline{D}_h - \underline{D}_h\underline{x} = \sum_{i\neq j}\mathbf{e}_i\mathbf{e}_j(x_i T_j + T_j x_i) - \sum_{i=1}^{d}(x_i T_i - T_i x_i), \tag{12}$$

similar calculations as for the first relation yield for all $f \in C^1(\mathbb{R}^d)$

$$\begin{aligned}
&[\underline{x}, \underline{D}_h] f(\underline{x}) = d f(\underline{x}) + 2\sum_{i<j}\mathbf{e}_i\mathbf{e}_j(x_i\partial_{x_j} f(\underline{x}) - x_j\partial_{x_i} f(\underline{x})) \\
&\qquad + 2\sum_{i<j}\mathbf{e}_i\mathbf{e}_j\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha})\frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}}\underline{x})}{\langle\underline{\alpha},\underline{x}\rangle}(x_i\alpha_j - x_j\alpha_i) + 2\sum_{\underline{\alpha}\in R^+}\kappa(\underline{\alpha}) f(\sigma_{\underline{\alpha}}\underline{x}) \\
&\quad = -2\left(\Phi_{\underline{\omega}} f(\underline{x}) - \frac{d}{2} f(\underline{x}) + \Psi f(\underline{x})\right) \\
&\quad = -2\left(\Phi_{\underline{\omega}} - \frac{d}{2} + \Psi\right) f(\underline{x}).
\end{aligned}$$

This completes the proof. □

Remark 3.2. The first relation in lemma 3.1 can also be found in [17].

Now we are able to give a proof for theorem 3.1.

Proof of theorem 3.1. From lemma 3.1, if we consider the sum of (9) and (10), for any $f \in C^1(\mathbb{R}^d)$, we get

$$2\underline{x}\,\underline{D}_h f(\underline{x}) = -2E_{\underline{x}} f(\underline{x}) - 2\gamma_\kappa f(\underline{x}) - 2\Phi_{\underline{\omega}} f(\underline{x}) - 2\Psi f(\underline{x}),$$

i.e.

$$\begin{aligned}
\underline{D}_h &= -\frac{\bar{\underline{x}}}{|\underline{x}|^2}(E_{\underline{x}} + \gamma_\kappa + \Phi_{\underline{\omega}} + \Psi) \\
&= \underline{\omega}\left(\partial_r + \frac{1}{r}(\gamma_\kappa + \Phi_{\underline{\omega}} + \Psi)\right) \\
&= \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right).
\end{aligned} \tag{13}$$

This ends the proof. □

Furthermore, we have

Lemma 3.2. *For any radial function* $f(\underline{x}) = f(|\underline{x}|) = f(r)$ *and* $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$, *we have*

(i) $\Psi f(r) = -\gamma_\kappa f(r)$,

(ii) $\Psi(\underline{\omega} f(r)) = \Psi(\underline{\omega}) f(r)$,

(iii) $\Psi(\underline{\omega}) = \gamma_\kappa \underline{\omega}$.

Proof. The results (i) and (ii) can be easily proved since radial functions are invariant under reflections.

The key result is the last one. By straightforward, but careful calculations, we obtain

$$\underline{D}_h \underline{\omega} = -\frac{d-1}{r} - \frac{2\gamma_\kappa}{r}.$$

In addition, it is well known that the operator $\Phi_{\underline{\omega}}$ has the following property:

$$\Phi_{\underline{\omega}}(\underline{\omega}) = (d-1)\underline{\omega}.$$

So, invoking relation (13), we get

$$\Psi(\underline{\omega}) = \gamma_\kappa \underline{\omega},$$

and the lemma follows. □

Summarily, we have for the Gamma operator in the Dunkl case:

Theorem 3.2. *For any radial function* $f(\underline{x}) = f(|\underline{x}|) = f(r)$ *and* $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$, *it holds*

$$\text{(i)} \quad \Gamma_{\underline{\omega}} f(r) = 0, \tag{14}$$

$$\text{(ii)} \quad \Gamma_{\underline{\omega}}(\underline{\omega} f(r)) = \Gamma_{\underline{\omega}}(\underline{\omega}) f(r), \tag{15}$$

$$\text{(iii)} \quad \Gamma_{\underline{\omega}}(\underline{\omega}) = (2\gamma_\kappa + d - 1)\underline{\omega}. \tag{16}$$

Using the obtained properties in theorems 3.1 and 3.2 we find that the assumed Dunkl monogenicity of (6) requires that our functions A and B satisfy the following Vekua-type system:

$$\begin{cases} \partial_{x_0} A - \partial_r B = \frac{2\gamma_\kappa + d - 1}{r} B, \\ \partial_{x_0} B + \partial_r A = 0. \end{cases} \tag{17}$$

For a more general approach, one may consider Dunkl monogenic functions of the form

$$(A(x_0, r) + \underline{\omega} B(x_0, r)) P_n(\underline{x}), \tag{18}$$

whereby

$$P_n(\underline{x}) = r^n P_n(\underline{\omega}),$$

is Dunkl monogenic in $\mathbb{R}^d$, i.e. $\underline{D}_h P_n(\underline{x}) = 0$, and homogeneous of degree $n \in \mathbf{Z}^+, \mathbf{Z}^+ = \mathcal{N} \cup \{0\}$.

In order to get the corresponding Vekua-type system for function of type (18), we need the following theorem:

Theorem 3.3. *If* $P_n(\underline{x})$ *is a homogeneous Dunkl monogenic function of degree* n *in* $\mathbb{R}^d$, *then we have*

$$\Gamma_{\underline{\omega}} P_n(\underline{\omega}) = -n P_n(\underline{\omega}), \tag{19}$$

and

$$\Gamma_{\underline{\omega}}(\underline{\omega} P_n(\underline{\omega})) = (2\gamma_\kappa + n + d - 1)(\underline{\omega} P_n(\underline{\omega})). \tag{20}$$

The classical idea to prove the above theorem consists in the the following result.

If $f(\underline{x})$ is any monogenic function in an open ball $B(c) \subset \mathbb{R}^d$ centred at the origin with radius c, then its Kelvin inversion

$$\mathbf{I} f(\underline{x}) \equiv G(\underline{x}) f\left(\frac{\bar{\underline{x}}}{|\underline{x}|^2}\right)$$

is monogenic in $\mathbb{R}^d \backslash \bar{B}(1/c)$, where $G(\underline{x}) = \frac{\bar{\underline{x}}}{|\underline{x}|^d}$ denotes the Cauchy kernel.

But this idea does not work for the proof of the above theorem since at present we do not have a similar result about Kelvin inversion in the Dunkl case. Therefore, we are going to consider a different approach.

Proof. In view of our representation of the Dunkl–Dirac operator in spherical coordinates the property $\underline{D}_h(r^n P_n(\underline{\omega})) = 0$ implies (19) easily. To prove (20), we can apply the anti-commutator $\{\underline{x}, \underline{D}_h\}$ on the term $\frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x})$, i.e. using (9) we get

$$\{\underline{x}, \underline{D}_h\} \frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x}) = -2\left(E_{\underline{x}} + \frac{d}{2} + \gamma_\kappa\right) \frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x}). \tag{21}$$

On the one hand, the left-hand side of the above equality yields

$$\begin{aligned} \{\underline{x}, \underline{D}_h\} \frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x}) &= (\underline{x}\,\underline{D}_h + \underline{D}_h \underline{x}) \frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x}) \\ &= r\underline{\omega}\left(\underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right)\right)(r^{n-1}\underline{\omega} P_n(\underline{\omega})) - \underline{D}_h P_n(\underline{x}) \\ &= -((n-1)r^{n-1}\underline{\omega} P_n(\underline{\omega}) + r^{n-1}\Gamma_{\underline{\omega}}(\underline{\omega} P_n(\underline{\omega}))), \end{aligned}$$

since $\underline{D}_h P_n(\underline{x}) = 0$.

On the other hand, the right-hand side of equality (21) gives

$$\begin{aligned} -2\left(E_{\underline{x}} + \frac{d}{2} + \gamma_\kappa\right) \frac{\underline{x}}{|\underline{x}|^2} P_n(\underline{x}) &= -2\left(r\partial_r + \frac{d}{2} + \gamma_\kappa\right)(r^{n-1}\underline{\omega} P_n(\underline{\omega})) \\ &= -2\left((n-1)r^{n-1}\underline{\omega} P_n(\underline{\omega}) + \frac{d}{2} r^{n-1}\underline{\omega} P_n(\underline{\omega}) + \gamma_\kappa r^{n-1}\underline{\omega} P_n(\underline{\omega})\right). \end{aligned}$$

Summarily, one has

$$\Gamma_{\underline{\omega}}(\underline{\omega} P_n(\underline{\omega})) = (2\gamma_\kappa + n + d - 1)(\underline{\omega} P_n(\underline{\omega})),$$

which is exactly relation (20). □

While we stated before that we do not have the corresponding result for the Kelvin inverse in the Dunkl case, we are now able to obtain such a result, but only for homogeneous functions. By (20), we can easily prove the following result:

Corollary 3.1. *If $f(\underline{x})$ is a homogeneous Dunkl monogenic function in an open ball $B(c) \subset \mathbb{R}^d$ centred at the origin with radius c, then its corresponding Kelvin inversion*

$$\mathbf{I}_h f(\underline{x}) \equiv G_h(\underline{x}) f\left(\frac{\bar{\underline{x}}}{|\underline{x}|^2}\right)$$

is Dunkl monogenic in $\mathbb{R}^d \backslash \bar{B}(1/c)$, where $G_h(\underline{x}) = \frac{\bar{\underline{x}}}{|\underline{x}|^{2\gamma_\kappa + d}}$ is the Cauchy kernel in Dunkl–Clifford analysis.

We want to emphasize that we do not have a direct proof for the above statement, only as a consequence of theorem 3.3.

Based on theorems 3.1 and 3.3, we get that the Dunkl monogenicity of (18) requires that the functions A and B satisfy the Vekua-type system

$$\begin{cases} \partial_{x_0} A - \partial_r B = \frac{2\gamma_\kappa + 2n + d - 1}{r} B, \\ \partial_{x_0} B + \partial_r A = 0. \end{cases} \tag{22}$$

4. Proof of the main theorems

With the work of the previous section we are now able to provide a proof of theorems 1.1 and 1.2. Since the proof will be given in a constructive way, it will allow us to compute some examples in the next section. Additionally, since theorem 1.1 is a special case ($n = 0$) of theorem 1.2, we will only prove theorem 1.2.

Now let us outline our proof. First, we would like to remark that this version of Fueter's theorem provides us with axial monogenic functions of degree n, i.e.

$$\Delta_h^{\gamma_\kappa + n + (d-1)/2}(u(x_0, |\underline{x}|) + \underline{\omega} v(x_0, |\underline{x}|) P_n(\underline{x})) = (A(x_0, r) + \underline{\omega} B(x_0, r)) P_n(\underline{x})$$

for some scalar-valued and continuously differentiable functions A and B. Hence, the proof consists in showing that A and B satisfy our Vekua-type system (22). To this end, we start with the following lemma from [18–20] which we only state the special case that we will use in this paper.

Lemma 4.1. *Suppose that $f(x_0, r)$ and $g(x_0, r)$ are scalar-valued infinitely differentiable functions in $\mathbb{R}^2$ and that D_r and D^r are differential operators defined by $D_r(0)\{f\} = D^r(0)\{f\} = f$ and*

$$D_r(m)\{f\} = \left(\frac{1}{r}\partial_r\right)^m \{f\},$$

$$D^r(m)\{f\} = \partial_r \left(\frac{D^r(m-1)\{f\}}{r}\right)$$

for $m \geqslant 1$. Then one has

(i) $\partial_r^2 D_r(m)\{f\} = D_r(m)\{\partial_r^2 f\} - 2m D_r(m+1)\{f\}$,

(ii) $\partial_r D_r(m-1)\{f/r\} = D^r(m)\{f\}$,

(iii) $D^r(m)\{\partial_r f\} = \partial_r D_r(m)\{f\}$,

(iv) $D_r(m)\{\partial_r f\} - \partial_r D^r(m)\{f\} = 2m/r\, D^r(m)\{f\}$,

(v) $\partial_r^2 D^r(m)\{f\} = D^r(m)\left\{\partial_r^2 f\right\} - 2m D^r(m+1)\{f\}$,

(vi) $D_r(m)\{fg\} = \sum_{j=0}^{m} \binom{m}{j} D_r(m-j)\{f\} D_r(j)\{g\}$,

(vii) $D^r(m)\{fg\} = \sum_{j=0}^{m} \binom{m}{j} D_r(m-j)\{f\} D^r(j)\{g\}$.

Furthermore, we need the following lemma which shows that the iterated Dunkl–Laplacian Δ_h^m, for any positive integer m, keeps functions of the form $(A(x_0, r) + \underline{\omega} B(x_0, r)) P_n(\underline{x})$ invariant whenever A and B are scalar-valued harmonic functions in $\mathbb{R}^2$.

Lemma 4.2. *Let $h(x_0, r)$ be a scalar-valued harmonic function in $\mathbb{R}^2$, i.e.*

$$\partial_{x_0}^2 h + \partial_r^2 h = 0, \tag{23}$$

and $P_n(\underline{x})$ be a homogeneous Dunkl monogenic function of degree n in $\mathbb{R}^d$. Then we have

$$\Delta_h^m(h(x_0, r)P_n(\underline{x})) = \prod_{i=1}^{m}(2\gamma_\kappa + 2n + d - (2i-1))D_r(m)\{h(x_0, r)\}P_n(\underline{x})$$

and

$$\Delta_h^m(h(x_0, r)\underline{\omega}P_n(\underline{x})) = \prod_{i=1}^{m}(2\gamma_\kappa + 2n + d - (2i-1))D^r(m)\{h(x_0, r)\}\underline{\omega}P_n(\underline{x}),$$

with m being a positive integer.

Proof. We will prove this lemma by induction. When $m = 1$, we need to show that the following identities hold:

$$\Delta_h(hP_n) = (2\gamma_\kappa + 2n + d - 1)D_r(1)\{h\}P_n \tag{24}$$

and

$$\Delta_h(h\underline{\omega}P_n) = (2\gamma_\kappa + 2n + d - 1)D^r(1)\{h\}\underline{\omega}P_n. \tag{25}$$

To prove (24), we start from

$$\Delta_h = \partial_{x_0}^2 - \underline{D}_h\underline{D}_h, \qquad \underline{D}_h = \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right).$$

Then using (19) and (20) in theorem 3.3 we get

$$\begin{aligned}\underline{D}_h(hP_n) &= \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right)(hr^nP_n(\underline{\omega}))\\ &= (\partial_r h)r^n\underline{\omega}P_n(\underline{\omega})\end{aligned}$$

and

$$\begin{aligned}\underline{D}_h\underline{D}_h(hP_n) &= \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right)((\partial_r h)r^n\underline{\omega}P_n(\underline{\omega}))\\ &= -\left(\partial_r^2 h\right)r^nP_n(\underline{\omega}) - n(\partial_r h)r^{n-1}P_n(\underline{\omega})\\ &\quad - (2\gamma_\kappa + d + n - 1)(\partial_r h)r^{n-1}P_n(\underline{\omega})\\ &= -\left(\partial_r^2 h + \frac{2\gamma_\kappa + 2n + d - 1}{r}(\partial_r h)\right)r^nP_n(\underline{\omega}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta_h(hP_n) &= \left(\partial_{x_0}^2 h + \partial_r^2 h + \frac{2\gamma_\kappa + 2n + d - 1}{r}(\partial_r h)\right)r^nP_n(\underline{\omega})\\ &= (2\gamma_\kappa + 2n + d - 1)D_r(1)\{h\}P_n.\end{aligned}$$

To prove (25), again applying (19) and (20) from theorem 3.3 we obtain

$$\begin{aligned}\underline{D}_h(h\underline{\omega}P_n) &= \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right)(hr^n\underline{\omega}P_n(\underline{\omega}))\\ &= \underline{\omega}((\partial_r h)r^n\underline{\omega}P_n(\underline{\omega}) + nhr^{n-1}\underline{\omega}P_n(\underline{\omega}) + hr^{n-1}\Gamma_{\underline{\omega}}(\underline{\omega}P_n(\underline{\omega})))\\ &= -(\partial_r h)r^nP_n(\underline{\omega}) - (2\gamma_\kappa + 2n + d - 1)hr^{n-1}P_n(\underline{\omega})\end{aligned}$$

and

$$\begin{aligned}
\underline{D}_h\underline{D}_h(h\underline{\omega}P_n) &= -\underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}\right)((\partial_r h)r^n P_n(\underline{\omega}) + (2\gamma_\kappa + 2n + d - 1)hr^{n-1}P_n(\underline{\omega}))\\
&= -\underline{\omega}\big(\big((\partial_r^2 h)r^n + n(\partial_r h)r^{n-1}\big)P_n(\underline{\omega}) + (\partial_r h)r^{n-1}\Gamma_{\underline{\omega}}(P_n(\underline{\omega}))\big)\\
&\quad + (2\gamma_\kappa + 2n + d - 1)((\partial_r h)r^{n-1} + (n-1)hr^{n-2})P_n(\underline{\omega})\\
&\quad + (2\gamma_\kappa + 2n + d - 1)hr^{n-2}\Gamma_{\underline{\omega}}(P_n(\underline{\omega})))\\
&= -\left(\partial_r^2 h + \frac{n}{r}(\partial_r h) - \frac{n}{r}(\partial_r h) + \frac{2\gamma_\kappa + 2n + d - 1}{r}(\partial_r h)\right.\\
&\quad \left. + \frac{(2\gamma_\kappa + 2n + d - 1)(n-1)}{r^2}h - \frac{(2\gamma_\kappa + 2n + d - 1)n}{r^2}h\right)\underline{\omega}r^n P_n(\underline{\omega})\\
&= -\left(\partial_r^2 h + (2\gamma_\kappa + 2n + d - 1)\left(\frac{\partial_r h}{r} - \frac{h}{r^2}\right)\right)\underline{\omega}P_n.
\end{aligned}$$

This leads to

$$\begin{aligned}
\Delta_h(h\underline{\omega}P_n) &= \left(\partial_{x_0}^2 h + \partial_r^2 h + (2\gamma_\kappa + 2n + d - 1)\left(\frac{\partial_r h}{r} - \frac{h}{r^2}\right)\right)\underline{\omega}P_n\\
&= (2\gamma_\kappa + 2n + d - 1)D^r(1)\{h\}\underline{\omega}P_n.
\end{aligned}$$

Summarizing we have that the lemma is true in the case $m = 1$. Assume that our formulae hold for a positive integer m, we have to show them for $m + 1$.

We thus get

$$\begin{aligned}
\Delta_h^{m+1}(hP_n) &= \prod_{i=1}^{m}(2\gamma_\kappa + 2n + d - (2i-1))\Delta_h D_r(m)\{h\}P_n,\\
&= \prod_{i=1}^{m}(2\gamma_\kappa + 2n + d - (2i-1))\\
&\quad \cdot \big(\partial_{x_0}^2 D_r(m)\{h\} + \partial_r^2 D_r(m)\{h\} + (2\gamma_\kappa + 2n + d - 1)D_r(m+1)\{h\}\big)P_n\\
&= \prod_{i=1}^{m}(2\gamma_\kappa + 2n + d - (2i-1))\\
&\quad \cdot \big(D_r(m)\big\{\partial_{x_0}^2 h + \partial_r^2 h\big\} + (2\gamma_\kappa + 2n + d - (2m+1))D_r(m+1)\{h\}\big)P_n\\
&= \prod_{i=1}^{m+1}(2\gamma_\kappa + 2n + d - (2i-1))D_r(m+1)\{h\}P_n,
\end{aligned}$$

which establishes the first formula. The other one may be proved in a similar way. □

We are now ready to present our proof of theorem 1.2.

Proof. By lemma 4.2, we get that

$$\begin{aligned}
&\Delta_h^{\gamma_\kappa + n + (d-1)/2}\left((u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|))P_n(\underline{x})\right)\\
&\quad = (2\gamma_\kappa + 2n + d - 1)!!(A(x_0, r) + \underline{\omega}B(x_0, r))P_n(\underline{x}),
\end{aligned}$$

with

$$\begin{aligned}
A &= D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{u\},\\
B &= D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{v\}.
\end{aligned}$$

The task is now to show that A and B satisfy the Vekua-type system (22). In order to do that, it will be necessary to use the assumptions on u and v and statements (iii) and (iv) of lemma 4.1.

Indeed, we obtain

$$\begin{aligned}
\partial_{x_0} A - \partial_r B &= D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_{x_0} u\} - \partial_r D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{v\} \\
&= D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_r v\} - \partial_r D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{v\} \\
&= \frac{2\gamma_\kappa + 2n + d - 1}{r} D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{v\} \\
&= \frac{2\gamma_\kappa + 2n + d - 1}{r} B
\end{aligned}$$

and

$$\begin{aligned}
\partial_{x_0} B + \partial_r A &= D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_{x_0} v\} - \partial_r D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{u\} \\
&= D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_{x_0} v\} + D^r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_r u\} \\
&= D_r\left(\gamma_\kappa + n + \frac{d-1}{2}\right)\{\partial_{x_0} v + \partial_r u\} \\
&= 0,
\end{aligned}$$

which completes the proof. □

5. Examples of Dunkl monogenic functions

In this section, we will give some examples of Dunkl monogenic functions by using our main theorems. As it was stated, to construct our function we just need to choose a holomorphic function $f(z)$. To simplify our work instead of directly applying our theorem we will use the method of our proof.

Example 5.1. Let $f(z) = iz$. It easily follows that

$$D_r(m)\{-r\} = (-1)^m \frac{(2m-3)!!}{r^{2m-1}},$$

and

$$D^r(m)\{x_0\} = (-1)^m \frac{(2m-1)!!}{r^{2m}} x_0.$$

We thus get the Dunkl monogenic function

$$\left(\frac{1}{r^{2\gamma_\kappa+2n+d-2}} + \frac{(2\gamma_\kappa + 2n + d - 2)}{r^{2\gamma_\kappa+2n+d}} x_0 \underline{x}\right) P_n(\underline{x}).$$

Example 5.2. Consider $f(z) = 1/z$. It is easy to check that

$$D_r(m)\left\{\frac{x_0}{x_0^2 + r^2}\right\} = (-1)^m \frac{2^m m! x_0}{\left(x_0^2 + r^2\right)^{m+1}},$$

and

$$D^r(m)\left\{\frac{r}{x_0^2+r^2}\right\} = (-1)^m \frac{2^m m! r}{\left(x_0^2+r^2\right)^{m+1}}.$$

With this choice of initial function, we obtain the Dunkl monogenic function

$$\left(\frac{\bar{x}}{|x|^{2\gamma_\kappa+2n+d+1}}\right) P_n(\underline{x}).$$

We would like to remark that for $n = 0$ the above Dunkl monogenic function is just the Cauchy kernel for the Dunkl–Dirac operator.

For the last example, we will construct a Dunkl monogenic version of the Gaussian distribution. To this end it is enough to consider the complex Gaussian, $f(z) = \exp(z^2/2)$. Our construction method will yield an n-dimensional function with the desired properties.

Example 5.3. Choose $f(z) = \exp(z^2/2)$, we use the fact that

$$D_r(m)\left\{\exp\left(\frac{x_0^2-r^2}{2}\right)\right\} = (-1)^m \exp\left(\frac{x_0^2-r^2}{2}\right).$$

This leads to

$$D_r(m)\{\cos(x_0 r)\} = \sum_{j=1}^{m} a_j^{(m)} \frac{x_0^j}{r^{2m-j}} \cos(x_0 r + j\pi/2),$$

and

$$D^r(m)\{\sin(x_0 r)\} = \sum_{j=1}^{m} a_{j+1}^{(m+1)} \frac{x_0^j}{r^{2m-j}} \sin(x_0 r + j\pi/2),$$

where

$$\begin{aligned} a_1^{(m)} &= (-1)^{m+1}(2m-3)!!, \\ a_j^{(m+1)} &= -(2m-j)a_j^{(m)} + a_{j-1}^{(m)}, \qquad j = 2, \ldots, m, \\ a_m^{(m)} &= 1. \end{aligned}$$

By statements (vi) and (vii) of lemma 4.2, we see that

$$D_r(m)\left\{\exp\left(\frac{x_0^2-r^2}{2}\right)\cos(x_0 r)\right\} = \exp\left(\frac{x_0^2-r^2}{2}\right)\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j} D_r(j)\{\cos(x_0 r)\},$$

and

$$D^r(m)\left\{\exp\left(\frac{x_0^2-r^2}{2}\right)\sin(x_0 r)\right\} = \exp\left(\frac{x_0^2-r^2}{2}\right)\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j} D^r(j)\{\sin(x_0 r)\}.$$

Hence, we have that

$$\begin{aligned} \exp\left(\frac{x_0^2-r^2}{2}\right) & \left(\sum_{j=0}^{\gamma_\kappa+n+\frac{d-1}{2}} \binom{\gamma_\kappa+n+\frac{d-1}{2}}{j} (-1)^{\gamma_\kappa+n+\frac{d-1}{2}-j} D_r(j)\{\cos(x_0 r)\}\right. \\ & \left. + \underline{\omega} \sum_{j=0}^{\gamma_\kappa+n+\frac{d-1}{2}} \binom{\gamma_\kappa+n+\frac{d-1}{2}}{j} (-1)^{\gamma_\kappa+n+\frac{d-1}{2}-j} D_r(j)\{\sin(x_0 r)\}\right) P_n(\underline{x}) \end{aligned}$$

is a Dunkl monogenic function, which is the equivalent of the Gauss distribution in classical hypercomplex function theory analysis.

Note that if $\gamma_\kappa = 0$, which implies $\kappa = 0$, i.e. the classical case of partial derivatives, all the above examples are reduced to the classical cases [18].

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