# Discrete Dirac Operators in Clifford Analysis 

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#### Abstract

We develop a constructive framework to define difference approximations of Dirac operators which factorize the discrete Laplacian. This resulting notion of discrete monogenic functions is compared with the notion of discrete holomorphic functions on quad-graphs. In the end Dirac operators on quad-graphs are constructed.


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## 1. Introduction

Currently, there seems to be much interest in finding discrete counterparts of various structures of the classical (continuous, smooth) mathematics.

As it was shown in [5, 8, 9, 10], developing discrete counterparts of (continuous) function theory is useful in the numerical treatment of problems related to potential theory and boundary values problems.

In the case of classical complex analysis there exist two approaches to this problem. The first one is based on discretizations of the Cauchy-Riemann equations [7], [4], [11], e.g. given by

$$
f_{m, n+1}-f_{m+1, n}=i\left(f_{m+1, n+1}-f_{m, n}\right) .
$$

The second one defines circle patterns to be natural discrete analogues of holomorphic functions [12].

In higher dimensions the situations gets even worse. In [6], the authors introduce discrete versions of Fischer decomposition, Euler and Gamma operators for

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discrete Dirac operators, but Dirac operators which only involve forward/backwarddifferences do not factorize the star Laplacian $\Delta_{h}$, where only the nonzero boundary weights occur on $m h \pm h_{j} \mathbf{b}_{j}$. Hereby $\mathbf{b}_{j}$ denotes the $\mathrm{j}^{\text {th }}$ component of the standard $\mathbb{R}^{n}$ basis.

However, in the quaternionic case, by mixing forward and backward differences, it is possible to construct difference Dirac operators which do factorize $\Delta_{h}[6,10]$. But this is done by a direct construction via $4 \times 4$-matrices, which coincide with the matrix representation of quaternions only asymptotically. It is not clear how this construction can be carried out in higher dimensions.

In this paper we will present a constructive framework to define discrete Dirac operators, which allows to define them in any dimension. This framework also explains the reasons, why the construction represented in [10] works so well in the quaternionic case.

Furthermore, in [2] a notion of discrete holomorphic functions on quad-graphs is presented which is extended to higher dimensions. In the last chapter we will compare our constructive framework with this notion of discrete holomorphic functions on quad-graphs and bricks. This will allow us to define discrete Dirac operators on $n$-rhombohedric embeddings of spatial graphs.

## 2. Preliminaries

Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be an orthonormal basis of $\mathbb{R}^{n}$. The Clifford algebra $C \ell_{0, n}$ is the free algebra over $\mathbb{R}^{n}$ generated modulo the relation

$$
x^{2}=-|x|^{2} \mathbf{e}_{0}
$$

where $\mathbf{e}_{0}$ is the identity of $C \ell_{0, n}$. For the algebra $C \ell_{0, n}$ we have the anti-commutation relationship

$$
\mathbf{e}_{j} \mathbf{e}_{j}+\mathbf{e}_{j} \mathbf{e}_{j}=-2 \delta_{j k} \mathbf{e}_{0},
$$

where $\delta_{j k}$ is the Kronecker symbol. In the following we will identify the Euclidean space $\mathbb{R}^{n}$ with $\bigwedge^{1} C \ell_{0, n}$, the space of all vectors of $C \ell_{0, n}$. This means that each element $x$ of $\mathbb{R}^{n}$ may be represented by

$$
x=\sum_{i=1}^{n} x_{i} \mathbf{e}_{j}
$$

From an analysis viewpoint one extremely crucial property of the algebra $C \ell_{0, n}$ is that each non-zero vector $x \in \mathbb{R}^{n}$ has a multiplicative inverse given by $\frac{-x}{|x|^{2}}$. Up to a sign this inverse corresponds to the Kelvin inverse of a vector in Euclidean space. Moreover, given a general Clifford number $a=\sum_{A} \mathbf{e}_{A} a_{A}, A \subset\{1, \ldots, n\}$ we denote by Sc $a=a_{\emptyset}$ the scalar part and by $\vec{a}=\mathbf{e}_{1} a_{1}+\ldots+\mathbf{e}_{n} a_{n}$ the vector part.

For all what follows let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with a sufficiently smooth boundary $\Gamma=\partial \Omega$. Then any function $f: \Omega \mapsto C \ell_{0, n}$ has a representation $f=\sum_{A} \mathbf{e}_{A} f_{A}$ with $\mathbb{R}$-valued components $f_{A}$. We now introduce the Dirac operator $D=\sum_{i=1}^{n} \mathbf{e}_{j} \frac{\partial}{\partial x_{i}}$. This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In particular we have that $D^{2}=-\Delta$, where $\Delta$ is the Laplacian over $\mathbb{R}^{n}$. A function $f: \Omega \mapsto C \ell_{0, n}$ is said to be left-monogenic if it satisfies the equation $(D f)(x)=0$ for each $x \in \Omega$. A similar definition can be given for right-monogenic functions. Basic properties of the Dirac operator and left-monogenic functions can be found in [1], [3], [8], and [9].

Now, we need some more facts for our discrete setting. To discretize pointwise the partial derivatives $\frac{\partial}{\partial x_{i}}$ in the equidistant lattice with mesh-width $h>0$,

$$
\mathbb{R}_{h}^{n}=\left\{m h=\sum_{j=1}^{n}\left(m_{j} h\right) \mathbf{e}_{j}: m_{j} \in \mathbb{Z}, j=1, \ldots, n\right\}
$$

we introduce for the basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$, the forward/backward differences $\partial_{h}^{ \pm i}$ :

$$
\partial_{h}^{ \pm j} u=\mp h^{-1}\left(I-\sigma_{h}^{ \pm j}\right) u
$$

where $I$ denotes the identity operator and $\sigma_{h}^{ \pm j} u=u\left(\cdot \pm h \mathbf{e}_{j}\right)$ denotes the shift operator on the grid $\mathbb{R}_{h}^{n}$.

## 3. Discrete Versions of Laplace Operators

Definition 3.1. Let $\mathcal{G}$ be a connected graph and let $V(\mathcal{G}), \vec{E}(\mathcal{G})$ and $E(\mathcal{G})$ the sets of vertices, directed and undirected edges of $\mathcal{G}$.

For a vertex $x \in V(\mathcal{G})$ and for the set of all vertices incident to $x, \mathcal{N}(x)$, we define the discrete Laplacian $\Delta_{\mathcal{G}, \nu}$ corresponding to the weight function $\nu$ : $E(\mathcal{G}) \rightarrow \mathbb{C}$ is the operator acting on functions $f: V(\mathcal{G}) \mapsto C \ell_{0, n}$ by

$$
\begin{equation*}
\left(\Delta_{\mathcal{G}, \nu} f\right)(x)=\sum_{y \in \mathcal{N}(x)} \nu(x, y)(f(y)-f(x)) . \tag{3.1}
\end{equation*}
$$

In the continuous case, there is a canonical correspondence between harmonic and holomorphic functions on $\mathbb{C}$ : the real and the imaginary parts of a holomorphic function are harmonic, and any real-valued harmonic function can be considered as a real part of a holomorphic function but these two classes of functions live then on different graphs. Discrete monogenic functions live on quad-graphs.

Definition 3.2. A cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n}$ is called a quad-graph, if the graph is regular and the boundary of each cell consists of 4-cycles.

We will denote by $F(\mathcal{G})$ the set of faces of $\mathcal{G}$. To any such $\mathcal{G}$ there corresponds canonically a combinatorial/geometric quad-graph called its double (or diamond) constructed from $\mathcal{G}$ and its dual $\mathcal{G}^{*}$. Recall that, in general, a dual cell decomposition $\mathcal{G}^{*}$ is only defined up to isotopy, but it can be fixed uniquely with the help
of Voronoi/Delaunay construction. The dual $\mathcal{G}^{*}$ is characterized as follows. Vertices of $\mathcal{G}^{*}$ are in a one-to-one correspondence to faces of $\mathcal{G}$. Each edge $\mathfrak{e} \in E(\mathcal{G})$ separates two faces of $\mathcal{G}$, which in turn correspond to two vertices of $\mathcal{G}^{*}$. It is declared that these two vertices are connected by the edge $\mathfrak{e}^{*} \in E\left(\mathcal{G}^{*}\right)$ dual to $\mathfrak{e}$. Finally, the faces of $\mathcal{G}^{*}$ are in a one-to-one correspondence with the vertices of $\mathcal{G}$ : if $x_{0} \in V(\mathcal{G})$, and $x_{1}, \ldots, x_{d} \in V(\mathcal{G})$ are neighbors connected with $x_{0}$ by the edges $\mathfrak{e}_{1}=\left(x_{0}, x_{1}\right), \ldots, \mathfrak{e}_{d}=\left(x_{0}, x_{d}\right)$, then the face of $\mathcal{G}^{*}$ corresponding to $x_{0}$ is defined by its boundary $\mathfrak{e}_{1}^{*} \cup \ldots \cup \mathfrak{e}_{n}^{*}$. If one assigns a direction to an edge $\mathfrak{e} \in E(\mathcal{G})$, then it will be assumed that the dual edge $\mathfrak{e}^{*} \in E(\mathcal{G})$ is also directed, in a way consistent with the orientation of the underlying surface, namely so that the pair ( $\left.\mathfrak{e}, \mathfrak{e}^{*}\right)$ is oriented directly at this crossing point. This orientation convention implies that $\mathfrak{e}^{* *}=-\mathfrak{e}$.

Now the double is constructed from $\mathcal{G}, \mathcal{G}^{*}$ as follows. The set of vertices of the double $\mathcal{D}$ is $V(\mathcal{D})=V(\mathcal{G}) \cup V\left(\mathcal{G}^{*}\right)$. Each pair of dual edges, say $\mathfrak{e}=\left(x_{0}, x_{1}\right) \in$ $E(\mathcal{G})$ and $\mathfrak{e}^{*}=\left(y_{0}, y_{1}\right) \in E\left(\mathcal{G}^{*}\right)$, defines a quadrilateral $\left(x_{0}, x_{1}, y_{0}, y_{1}\right)$. These quadrilaterals constitutes the faces of the cell decomposition (quad-graph) $\mathcal{D}$. The edges of $\mathcal{D}$ belong neither to $E(\mathcal{G})$ nor to $E\left(\mathcal{G}^{*}\right)$. A star of a vertex $x_{0} \in V(\mathcal{G})$ produces a flower of adjacent quadrilaterals from $F(\mathcal{D})$ around the common vertex $x_{0}$.

Observe that the double $\mathcal{D}$ is automatically bipartite since its vertices $V(\mathcal{D})$ are decomposed in two halves $V(\mathcal{D})=V(\mathcal{G}) \cup V\left(\mathcal{G}^{*}\right)$ ("black" and "white" vertices), such that the ends of each edge from $E(\mathcal{D})$ are of different colors.

Without loss of generality we will restrict our study to lattice functions with equidistant mesh-width $h>0$ with respect to a non-negative weight function $\nu: E(\mathcal{G}) \rightarrow[0, \infty)$. In this case, the cell decomposition of $\mathbb{R}^{n}$, say $\mathcal{D}$, can be identified by $\mathbb{R}_{h}^{n}$ and for $m h \in \mathbb{R}_{h}^{n}$, the set of all vertices incident to $m h$, can be described by $\mathcal{N}(m h)=m h+h \mathcal{K}$ with

$$
\mathcal{K}=\bigcup_{j=1}^{n} \bigcup_{k \leq j}\left\{\alpha_{j} \mathbf{e}_{j}+\beta_{k} \mathbf{e}_{k}: \alpha_{j}^{2}, \beta_{k}^{2} \in \mathbb{Z}_{2}, 1 \leq \alpha_{j}^{2}+\beta_{k}^{2} \leq 2-\delta_{j k}\right\}
$$

and for a small mesh-size $h$, the weight function has the same behaviour of $\frac{1}{h^{2}}$. Hence,

$$
\begin{align*}
\Delta_{\mathcal{G}, \nu} & =\sum_{r \in \mathcal{K}} \nu(\cdot, \cdot+r h)\left(\sigma_{h}^{r}-I\right)  \tag{3.2}\\
& \sim \sum_{r \in \mathcal{K}} \frac{1}{h^{2}}\left(\sigma_{h}^{r}-I\right) \tag{3.3}
\end{align*}
$$

where $\sigma_{h}^{r}:=\left(\sigma_{h}^{j \alpha_{j}}\right)^{\alpha_{j}^{2}}\left(\sigma_{h}^{j \beta_{j}}\right)^{\beta_{j}^{2}}$ for a certain $r=\alpha_{j} \mathbf{e}_{j}+\beta_{k} \mathbf{e}_{k} \in \mathcal{K}$.
One example of a discrete Laplacian is

- the star Laplacian

$$
\begin{aligned}
\Delta_{h} & =\sum_{j=1}^{n} \frac{\sigma_{h}^{+j}+\sigma_{h}^{-j}-2 I}{h^{2}} \\
& =\sum_{j=1}^{n} \partial_{h}^{-j} \partial_{h}^{+j}
\end{aligned}
$$

In this case, only $m h \in \mathbb{R}_{h}^{n}$ and its incident vertices $m h+h \mathcal{K}^{*}$, with $\mathcal{K}^{*}=\bigcup_{j=1}^{n}\left\{-\mathbf{e}_{j}, \mathbf{e}_{j}\right\}$, are considered for the discrete approximation of the Laplace operator.

Another example is given by

- the cross Laplacian

$$
\begin{aligned}
\Delta_{h}^{\times} & =\sum_{j=1}^{n} \sum_{k<j} \frac{\sigma_{h}^{+j} \sigma_{h}^{+k}+\sigma_{h}^{-j} \sigma_{h}^{-k}+\sigma_{h}^{-j} \sigma_{h}^{+k}+\sigma_{h}^{+j} \sigma_{h}^{-k}-4 I}{h^{2}} \\
& =\sum_{j=1}^{n} \sum_{k<j} \tilde{\Delta}_{h}^{j, k},
\end{aligned}
$$

where $\tilde{\Delta}_{h}^{j, k}:=\partial_{h}^{+j} \partial_{h}^{+k}+\partial_{h}^{-j} \partial_{h}^{-k}-\partial_{h}^{+j} \partial_{h}^{-k}-\partial_{h}^{-j} \partial_{h}^{+k}$ denotes the cross Laplacian on the plane $\mathbf{e}_{j} \mathbf{e}_{k}$.

In this case, $m h \in \mathbb{R}_{h}^{n}$ and its incident vertices $m h+h \mathcal{K}^{\times}$, with

$$
\mathcal{K}^{\times}=\bigcup_{j=1}^{n} \bigcup_{k<j}\left\{-\mathbf{e}_{j}+\mathbf{e}_{k}, \mathbf{e}_{j}+\mathbf{e}_{k}, \mathbf{e}_{j}-\mathbf{e}_{k},-\mathbf{e}_{j}-\mathbf{e}_{k}\right\}
$$

are considered for the discrete approximation of the Laplace operator.
Let us remark that $\mathcal{K}=\mathcal{K}^{*} \cup \mathcal{K}^{\times}$. Thus approximations of the continuous Laplace operator which take all the incident vertices of a lattice point $m h \in \mathbb{R}_{h}^{n}$ into account can be obtained if we take linear combinations of the star Laplacian and the cross Laplacian. This is the case when both representatives ( $m h, m h \pm r h$ ) of any edge carry the same value $\nu(m h, m h+r h)=\nu(m h, m h-r h)$ as the underlying undirected one.


Figure 1. Example of a star Laplacian and a mixed Laplacian in $\mathbb{R}^{3}$.

## 4. Factorization of Discrete Laplacians

### 4.1. A first look

One of the main goals in the discrete case is to approximate the classical Dirac operator

$$
\begin{equation*}
D=\sum_{i=1}^{n} \mathbf{e}_{j} \frac{\partial}{\partial x_{i}} \tag{4.1}
\end{equation*}
$$

by its discrete analogues such that they factorize the discrete Laplacian.
It is clear that one has to replace the partial derivatives $\frac{\partial}{\partial x_{i}}$ by difference operators on the right hand side of (4.1). Some canonical choices are

1. The forward/backward differences $\partial_{h}^{ \pm i}$.
2. The bi-directional difference $\frac{1}{2}\left(\partial_{h}^{+i}+\partial_{h}^{-i}\right)$.

These canonical choices induce the forward/backward discretizations of the Dirac operator, $D_{h}^{ \pm}$, defined in [8], and the bi-directional discretization $\frac{1}{2}\left(D_{h}^{+}+D_{h}^{-}\right)$, respectively.

More general, we can obtain a difference approximation as linear combinations of forward and backward differences, that is,

$$
\begin{array}{rlr}
\partial_{h, \theta}^{j} & = & \theta \partial_{h}^{+j}+(1-\theta) \partial_{h}^{-j} \\
& = & h^{-1}\left(\theta \sigma_{h}^{+j}+(\theta-1) \sigma_{h}^{-j}-(2 \theta-1) I\right) \quad, \theta \in \mathbb{R} \tag{4.2}
\end{array}
$$

and, therefore, for a sequence $\Theta_{n}=\left(\theta_{j}\right)_{j=1}^{n}$, we can define a difference approximation of (4.1) by

$$
\begin{equation*}
D_{h, \Theta_{n}}=\sum_{j=1}^{n} \mathbf{e}_{j} \partial_{h, \theta_{j}}^{j} \tag{4.3}
\end{equation*}
$$

Theorem 4.1. For all sequences $\Theta_{n}$ and $\Gamma_{n}$, the difference Dirac operators $D_{h, \Theta_{n}}$ and $D_{h, \Gamma_{n}}$ do not factorize $-\Delta_{h} \mathbf{e}_{0}$.

Proof. Suppose that $D_{h, \Theta_{n}}$ and $D_{h, \Gamma_{n}}$ factorize $-\Delta_{h} \mathbf{e}_{0}$.
Starting from the definition, we can split $D_{h, \Theta_{n}} D_{h, \Gamma_{n}}$ into the sum

$$
D_{h, \Theta_{n}} D_{h, \Gamma_{n}}=\sum_{j, k=1}^{n} \mathbf{e}_{j} \mathbf{e}_{k}\left(I_{h}^{-}(j, k)+I_{h}^{+}(j, k)+I_{h}^{+-}(j, k)+I_{h}^{-+}(j, k)\right)
$$

with

$$
\begin{aligned}
I_{h}^{-}(j, k) & =\left(1-\theta_{j}\right)\left(1-\gamma_{k}\right) \partial_{h}^{-j} \partial_{h}^{-k} \\
I_{h}^{+}(j, k) & =\theta_{i} \gamma_{j} \partial_{h}^{+j} \partial_{h}^{+k} \\
I_{h}^{+-}(j, k) & =\theta_{j}\left(1-\gamma_{k}\right) \partial_{h}^{+j} \partial_{h}^{-k} \\
I_{h}^{-+}(j, k) & =\left(1-\theta_{i}\right) \gamma_{j} \partial_{h}^{-j} \partial_{h}^{+k} .
\end{aligned}
$$

Because $D_{h, \Theta_{n}} D_{h, \Gamma_{n}}=-\Delta_{h}$, we have

$$
\begin{aligned}
I_{h}^{-}(j, k) & =0 \\
I_{h}^{+}(j, k) & =0 \\
I_{h}^{-+}(j, k) & =0, \quad j \neq k \\
I_{h}^{+-}(j, k) & =0, \quad j \neq k .
\end{aligned}
$$

That is, the sequences $\Theta_{n}$ and $\Gamma_{n}$ satisfy the conditions

$$
\begin{aligned}
\theta_{j} \gamma_{j} & =0 \\
\theta_{j}+\gamma_{j} & =1 \\
\theta_{j} & =\theta_{j} \gamma_{k} \quad j \neq k \\
\gamma_{k} & =\theta_{j} \gamma_{k} \quad j \neq k,
\end{aligned}
$$

which is a contradiction.

### 4.2. Factorization of the star Laplacian

According to Theorem 4.1, we cannot define discrete Dirac operators which factorize $-\Delta_{h}$ if we only consider the Clifford basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ and hence, we cannot identify the discrete Dirac operator as a pure Clifford number as it is the case with the continuous Dirac operator (c.f. [10]) On the other hand, the last proof tells us that to factorize the discrete Laplacian, we must choose two sequences $\Theta_{n}=\left(\theta_{j}\right)_{j=1}^{n}$ and $\Gamma_{n}=\left(1-\theta_{j}\right)_{j=1}^{n}$ with $\theta_{j} \in \mathbb{Z}_{2}$, for all $j=1, \ldots, n$. For simplification we will write $\gamma_{j}=1-\theta_{j}$.

For this reasons, we will introduce generalizations of the discrete Dirac operators introduced in (4.3) using the following approach:

Splitting the basis vectors $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ in $\mathbf{e}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}$and defining the operators $D_{h, \Theta_{n}}^{ \pm}$by

$$
\begin{equation*}
D_{h, \Theta_{n}}^{ \pm}=\sum_{j=1}^{n} \mathbf{e}_{j}^{ \pm} \partial_{h, \theta_{j}}^{j}, \tag{4.4}
\end{equation*}
$$

we can approximate the Dirac operator by $D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}$and $D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}$.
To apply our formalism, let us consider the following example:

Example 4.2. The discrete Dirac operators introduced by Gürlebeck and Hommel in [10] have the matrix representation

$$
\begin{align*}
D_{h}^{-+} & =\left(\begin{array}{cccc}
0 & -\partial_{h}^{-1} & -\partial_{h}^{-2} & -\partial_{h}^{-3} \\
\partial_{h}^{-1} & 0 & -\partial_{h}^{3} & \partial_{h}^{2} \\
\partial_{h}^{-2} & \partial_{h}^{3} & 0 & -\partial_{h}^{1} \\
\partial_{h}^{-3} & -\partial_{h}^{2} & \partial_{h}^{1} & 0
\end{array}\right)  \tag{4.5}\\
D_{h}^{+-} & =\left(\begin{array}{cccc}
0 & -\partial_{h}^{1} & -\partial_{h}^{2} & -\partial_{h}^{3} \\
\partial_{h}^{1} & 0 & -\partial_{h}^{-3} & \partial_{h}^{-2} \\
\partial_{h}^{2} & \partial_{h}^{-3} & 0 & -\partial_{h}^{-1} \\
\partial_{h}^{3} & -\partial_{h}^{-2} & \partial_{h}^{-1} & 0
\end{array}\right) . \tag{4.6}
\end{align*}
$$

For $f_{h}=f_{h}^{0}+\vec{f}_{h}$, these operators factorize $-\Delta_{h}$ in the sense that

$$
D_{h}^{+-} D_{h}^{-+} f_{h}=-\Delta_{h} f_{h}=D_{h}^{-+} D_{h}^{+-} f_{h}
$$

Considering the sequences $\Theta_{n}=(1)_{j=1}^{n}$ and $\Gamma_{n}=(0)_{j=1}^{n}$ for $n=3$ and the matrix elements

$$
\begin{array}{ll}
\mathbf{e}_{1}^{-}=\left(\begin{array}{cccc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) & \mathbf{e}_{1}^{+}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right) \\
\mathbf{e}_{2}^{-}=\left(\begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) & \mathbf{e}_{2}^{+}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right) \\
\mathbf{e}_{3}^{-}=\left(\begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right) & \mathbf{e}_{3}^{+}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

we can write $D_{h}^{-+}=D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}$and $D_{h}^{+-}=D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}$.
The matrix elements $\mathbf{e}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}, j=1,2,3$ satisfy the Clifford algebra property $\mathbf{e}_{j} \mathbf{e}_{k}+\mathbf{e}_{k} \mathbf{e}_{j}=-2 \delta_{j k} \mathbf{e}_{0}$ and the matrix elements $\mathbf{e}_{j}^{ \pm}, i=1,2,3$ satisfy the conditions

$$
\begin{array}{r}
\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{+}=-\delta_{j k} \mathbf{e}_{0} \\
\mathbf{e}_{j}^{+} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{-}=0 \\
\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{+}+\mathbf{e}_{k}^{-} \mathbf{e}_{j}^{+}=0 \\
\mathbf{e}_{j}^{+} \mathbf{e}_{j}^{-}=0=\mathbf{e}_{j}^{-} \mathbf{e}_{j}^{+} . \tag{4.10}
\end{array}
$$

More generally, in $\mathbb{R}^{0, n}$ we can construct a Clifford basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ by taking $\mathbf{e}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}$such that $\mathbf{e}_{j}^{ \pm}, j=1, \ldots, n$ satisfy the conditions (4.7)-(4.9).

Furthermore, we obtain the following theorem.
Theorem 4.3. If the elements $\mathbf{e}_{j}^{ \pm}$satisfy the conditions (4.7)-(4.10), the discrete Dirac operators $D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}$and $D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}$factorize $-\Delta_{h}$.

Proof. Starting from the definition, we can split the product $\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)$ $\left(D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}\right)$into the sum
$\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)\left(D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}\right)=J_{h}\left(\Gamma_{n}, \Theta_{n}\right)+L_{h}\left(\Gamma_{n}, \Theta_{n}\right)+M_{h}\left(\Gamma_{n}, \Theta_{n}\right)$
with

$$
\begin{aligned}
& J_{h}\left(\Gamma_{n}, \Theta_{n}\right) \\
& \quad=\sum_{j=1}^{n}\left(\mathbf{e}_{j}^{+} \mathbf{e}_{j}^{-} \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{j}}^{j}+\mathbf{e}_{j}^{-} \mathbf{e}_{j}^{+} \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{j}}^{j}+\left(\left(\mathbf{e}_{j}^{+}\right)^{2}+\left(\mathbf{e}_{j}^{-}\right)^{2}\right) \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{j}}^{j}\right) \\
& L_{h}\left(\Gamma_{n}, \Theta_{n}\right) \\
& \quad=\sum_{j=1}^{n} \sum_{k<j}\left(\mathbf{e}_{j}^{+} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{-}\right) \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{k}}^{j}+\sum_{j=1}^{n} \sum_{k<j}\left(\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{+}+\mathbf{e}_{k}^{-} \mathbf{e}_{j}^{+}\right) \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{k}}^{j} \\
& M_{h}\left(\Gamma_{n}, \Theta_{n}\right) \\
& \quad=\sum_{j=1}^{n} \sum_{k<j}\left(\mathbf{e}_{j}^{+} \mathbf{e}_{k}^{+}+\mathbf{e}_{k}^{-} \mathbf{e}_{j}^{-}\right) \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{j}+\sum_{j=1}^{n} \sum_{k<j}\left(\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{+}\right) \partial_{h, \theta_{j}}^{j} \partial_{h, \gamma_{k}}^{j} .
\end{aligned}
$$

Using the properties (4.7)-(4.10), we obtain

$$
\begin{aligned}
J_{h}\left(\Gamma_{n}, \Theta_{n}\right) & =-\sum_{j=1}^{n} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{j}}^{j} \mathbf{e}_{0} \\
L_{h}\left(\Gamma_{n}, \Theta_{n}\right) & =0 \\
M_{h}\left(\Gamma_{n}, \Theta_{n}\right) & =0 .
\end{aligned}
$$

Therefore, by remembering $\gamma_{j}=1-\theta_{j}$ we conclude our proof.
With the last proof, we generalize the discrete Dirac operators introduced by Gürlebeck and Hommel in [10]. Moreover, with this result we prove that there exists more than one pair of discrete Dirac operators which factorize $-\Delta_{h}$. From now on, we will denote by

$$
\begin{align*}
D_{h, \Theta_{n}}^{+-} & =\sum_{j=1}^{n} \mathbf{e}_{j}^{+} \partial_{h, \theta_{j}}^{j}+\mathbf{e}_{j}^{-} \partial_{h, 1-\theta_{j}}^{j},  \tag{4.11}\\
D_{h, \Theta_{n}}^{-+} & =\sum_{j=1}^{n} \mathbf{e}_{j}^{-} \partial_{h, \theta_{j}}^{j}+\mathbf{e}_{j}^{+} \partial_{h, 1-\theta_{j}}^{j}, \tag{4.12}
\end{align*}
$$

for all $\Theta_{n}=\left(\theta_{j}\right)_{j=1}^{n} \in\left(\mathbb{Z}_{2}\right)^{n}$, our discrete Dirac operators.

Let us remark that using the last approach, when $h \rightarrow 0$, we approximate the factorization property $D^{2}=-\Delta$, where $D$ denotes the standard Dirac operator. However, contrary to the continuous case, we need two discrete Dirac operators to split the star Laplacian.

### 4.3. Factorization of the star Laplacian: Another approach

In the previous section we could see how one can obtain discrete Dirac operators using forward/backward differences. In this section we will present a different approach using the bi-directional differences $\partial_{h}^{i}=\frac{1}{2}\left(\partial_{h}^{+i}+\partial_{h}^{-i}\right)$.

This will allow us afterwards to develop a unified framework for the construction of discrete Laplacians using the power of Clifford algebras on a $3 * \ldots * 3$-grid.

The basic idea is to consider two Clifford basis $e_{j}^{+}, j=1, \ldots, m$ and $e_{j}^{-}, j=$ $1, \ldots, m$, assuming that the whole collection $\left\{e_{j}^{+}, e_{j}^{-}\right\}$would be a Clifford basis of dimension $2 m$ for some suitable metric. This means we introduce two symmetric matrices $g_{j k}^{+}, g_{j k}^{-}$and one general matrix $M_{j k}$ for which we assume the relations

$$
\begin{aligned}
e_{j}^{+} e_{k}^{+}+e_{k}^{+} e_{j}^{+} & =-2 g_{j k}^{+}, \quad e_{j}^{-} e_{k}^{-}+e_{k}^{-} e_{j}^{-}=-2 g_{j k}^{-} \\
e_{j}^{+} e_{k}^{-}+e_{k}^{-} e_{k}^{+} & =-2 M_{j k} .
\end{aligned}
$$

The idea now is to define the discrete Dirac operator by

$$
D_{h}=\frac{1}{2}\left(D_{h}^{+}+D_{h}^{-}\right)
$$

with $D_{h}^{ \pm}=\sum_{k=1}^{m} e_{j}^{ \pm} \partial_{h}^{ \pm i}$.
Of course we need $D_{h}$ to approximate the Euclidean Dirac operator for which we make the following assumption.

Assumption 1. The basis $e_{j}=e_{j}^{+}+e_{j}^{-}$is the standard Clifford basis, i.e. $e_{j} e_{k}+$ $e_{k} e_{j}=-2 \delta_{j k}$.

This immediately gives rise to the constraint

$$
g_{j k}^{+}+g_{j k}^{-}+M_{j k}+M_{k j}=\delta_{j k} .
$$

There are many possibilities still. Next, for a rectangular grid it is natural to assume that all coordinates are equally important.

Assumption 2. (Dimensional democracy) For the diagonal entities of the matrices we assume $g_{j j}^{+}=\lambda^{+}, g_{j j}^{-}=\lambda^{-}, M_{j j}=\mu$ is independent from $j$.

Hence, we get $\lambda^{+} \lambda^{-}+2 \mu=1$.
Also $g_{j k}^{+}, g_{j k}^{-}, M_{j k}$ for $j \neq k$ should not depend on $j$ and $k$ so that in particular $M_{j k}=M_{k j}=M$ and $g_{j k}^{ \pm}=g^{ \pm}$, whereby thus $g^{+}+g^{-}+2 M=0$.

Finally, another assumption would be
Assumption 3. The two directions "+" and "-" are equally important, leading to $\lambda^{+}=\lambda^{-}=\lambda$ and $2 \lambda+2 \mu=1$ and also $g^{+}=g^{-}=g$ and $g+M=0$, i.e. $M=-g$.

So now we have the relations

$$
\begin{array}{ll}
e_{j}^{+} e_{j}^{+}=e_{j}^{-} e_{j}^{-}=-\lambda, & e_{j}^{+} e_{j}^{-}+e_{j}^{-} e_{j}^{+}=-2 \mu \\
e_{j}^{ \pm} e_{k}^{ \pm}+e_{k}^{ \pm} e_{j}^{ \pm}=-2 g, & e_{j}^{+} e_{k}^{-}+e_{k}^{-} e_{j}^{+}=+2 g
\end{array}
$$

If we now choose $\lambda=0$ and $\mu=1 / 2$ we end up with the relation $\left(e_{j}^{+}\right)^{2}=$ $\left(e_{j}^{-}\right)^{2}=0$, as well as

$$
e_{j}^{+} e_{j}^{-}+e_{j}^{-} e_{j}^{+}=-1
$$

This results in

$$
\begin{align*}
D_{h}^{2} & =\frac{1}{4}\left(D_{h}^{+}+D_{h}^{-}\right)^{2}  \tag{4.13}\\
& =-\sum_{j=1}^{m} \partial_{h}^{+j} \partial_{h}^{-j}+g \sum_{j \neq l}\left(-\partial_{h}^{+j} \partial_{h}^{+l}+2 \partial_{h}^{+j} \partial_{h}^{-l}-\partial_{h}^{-j} \partial_{h}^{-l}\right), \tag{4.14}
\end{align*}
$$

so that choosing $g=0$ we obtain the usual discrete (star-)Laplacian.

### 4.4. Factorization of discrete Laplacians: A constructive framework

The main idea is to approximate the standard Dirac operator using the formulae (4.11) and (4.12). This means that the elements $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ given by $\mathbf{e}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}$ should be generators of a Clifford basis.

Let us suppose that one of the following two sets of assumptions hold for $\mathbf{e}_{1}^{ \pm}, \ldots, \mathbf{e}_{n}^{ \pm}$:

Assumption 4. There exist two symmetric matrices $\mathfrak{f}_{j k}^{+-}, \mathfrak{f}_{j k}^{-+}$and one general ma$\operatorname{trix} L_{j k}$ for which we assume the relations

1. $\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{+}=-2 L_{j k}$,
2. $\mathbf{e}_{j}^{+} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{+} \mathbf{e}_{j}^{-}=-2 \mathfrak{f}_{j k}^{+-}$,
3. $\mathbf{e}_{j}^{-} \mathbf{e}_{k}^{+}+\mathbf{e}_{k}^{-} \mathbf{e}_{j}^{+}=-2 \mathfrak{f}_{j k}^{-}$,
4. $\mathbf{e}_{j}^{ \pm} \mathbf{e}_{j}^{\mp}=0$,
5. $\mathbf{e}_{j}^{+}$and $\mathbf{e}_{j}^{-}$have the same direction, that is, $\mathfrak{f}_{j k}^{+-}=\mathfrak{f}_{j k}^{-+}$and $L_{j k}=L_{k j}$,
6. $\mathfrak{f}_{j k}^{+-}+\mathfrak{f}_{j k}^{-+}+L_{j k}+L_{k j}=\delta_{j k} \mathbf{e}_{0}$.

Assumption 5. There exist two symmetric matrices $\mathfrak{g}_{j k}^{+}, \mathfrak{g}_{j k}^{-}$and one general matrix $M_{j k}$ for which we assume the relations

1. $\mathbf{e}_{j}^{ \pm} \mathbf{e}_{k}^{ \pm}+\mathbf{e}_{k}^{ \pm} \mathbf{e}_{j}^{ \pm}=-2 \mathfrak{g}_{j k}^{ \pm}$
2. $\mathbf{e}_{j}^{+} \mathbf{e}_{k}^{-}+\mathbf{e}_{k}^{-} \mathbf{e}_{j}^{+}=-2 M_{j k}$
3. $\left(\mathbf{e}_{j}^{ \pm}\right)^{2}=0$
4. $\mathbf{e}_{j}^{+}$and $\mathbf{e}_{j}^{-}$have the same direction, that is, $\mathfrak{g}_{j k}^{+}=\mathfrak{g}_{j k}^{-}$and $M_{j k}=M_{k j}$.
5. $\mathfrak{g}_{j k}^{+}+\mathfrak{g}_{j k}^{-}+M_{j k}+M_{k j}=\delta_{j k} \mathbf{e}_{0}$.

In the set of assumptions 4 by the constraints 4 and 6 we have the relation $L_{j j}=\frac{1}{2} \mathbf{e}_{0}$. Moreover, we get

$$
\begin{array}{r}
\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)\left(D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}\right)=-\mathbf{e}_{0} \sum_{j=1}^{n} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{j}}^{j} \\
-2 \sum_{j=1}^{n} \sum_{k<j}\left(\mathfrak{f}_{j k}^{+-} \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{k}}^{k}+\mathfrak{f}_{j k}^{-+} \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{k}}^{k}+L_{j k} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}+L_{k j} \partial_{h, \gamma_{k}}^{k} \partial_{h, \theta_{j}}^{j}\right) \cdot( \tag{4.15}
\end{array}
$$

In case of the assumptions 5 from the constraints 5 and 3 we obtain the relation $M_{j j}=\frac{1}{2} \mathbf{e}_{0}$. This leads to

$$
\begin{array}{r}
\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)^{2}=-\mathbf{e}_{0} \sum_{j=1}^{n} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{j}}^{j}- \\
-2 \sum_{j=1}^{n} \sum_{k<j}\left(\mathfrak{g}_{j k}^{+} \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{k}}^{k}+\mathfrak{g}_{j k}^{-} \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{k}}^{k}+M_{j k} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}+M_{k j} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}\right) . \tag{4.16}
\end{array}
$$

Hence, we obtain the following theorems, respectively:
Theorem 4.4. Under the assumptions 4, we have

1. If $L_{j k}=0$ for all $j<k$, then $D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}=-\Delta_{h} \mathbf{e}_{0}=D_{h, \Theta_{n}}^{-+} D_{h, \Theta_{n}}^{+-}$.
2. If for all $j<k, L_{j k}=c_{j k},\left(c_{j k} \geq 0\right)$ then $-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}=-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}$ is the discrete Laplacian according to (3.2) only if $\Theta_{n} \in\left(\mathbb{Z}_{2}\right)^{n}$ is a constant sequence.

Moreover,

$$
-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}=\mathbf{e}_{0}\left(\Delta_{h}+2 \sum_{j=1}^{n} \sum_{k<j} c_{j k} \tilde{\Delta}_{h}^{j, k}\right)
$$

with $\tilde{\Delta}_{h}^{j, k}:=\partial_{h}^{+j} \partial_{h}^{+k}+\partial_{h}^{-j} \partial_{h}^{-k}-\partial_{h}^{+j} \partial_{h}^{-k}-\partial_{h}^{-j} \partial_{h}^{+k}$ being the cross Laplacian on the plane $\mathbf{e}_{j} \mathbf{e}_{k}$.

Furthermore, if all the directions are equally important, i.e. $c_{j k}=c$ : $k=1, \ldots, n, j<k$, the right-hand side of $-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}$is equal to $\mathbf{e}_{0}\left(\Delta_{h}+\right.$ $\left.2 c \Delta_{h}^{\times}\right)$.
Theorem 4.5. Under assumptions 5, we get

1. If $M_{j k}=0$ for all $j<k$, then $\left(D_{h, \Theta_{n}}^{+-}\right)^{2}=-\Delta_{h} \mathbf{e}_{0}=\left(D_{h, \Theta_{n}}^{-+}\right)^{2}$.
2. If for all $j<k, M_{j k}=c_{j k},\left(c_{j k} \geq 0\right)$ then $-\left(D_{h, \Theta_{n}}^{+-}\right)^{2}=-\left(D_{h, \Theta_{n}}^{-+}\right)^{2}$ is a discrete Laplacian according to (3.2).

Moreover,

$$
-\left(D_{h, \Theta_{n}}^{-+}\right)^{2}=\mathbf{e}_{0}\left(\Delta_{h}+2 \sum_{j=1}^{n} \sum_{k<j} c_{j k} \tilde{\Delta}_{h}^{j, k}\right)
$$

and if all the directions are equally important (i.e. $c_{j k}=c: j<k, k=$ $1, \ldots, n$, ) the right-hand side of $-\left(D_{h, \Theta_{n}}^{-+}\right)^{2}$ is equal to $\mathbf{e}_{0}\left(\Delta_{h}+2 c \Delta_{h}^{\times}\right)$.

Under the conditions of assumptions 4 and 5 we obtain factorizations of the star Laplacian, a sum between a star Laplacian and a "weighted" cross Laplacian. However, we never obtain factorizations of "weighted" cross Laplacians.

To solve this drawback, instead of last relation in the set of assumptions 4 and 5 , we will suppose:

$$
\mathfrak{f}_{j k}^{+-}+\mathfrak{f}_{j k}^{-+}+L_{j k}+L_{k j}=0
$$

and

$$
\mathfrak{g}_{j k}^{+}+\mathfrak{g}_{j k}^{-}+M_{j k}+M_{k j}=0,
$$

respectively.
Under the above conditions, the elements $\tilde{\mathbf{e}}_{1}, \ldots, \tilde{\mathbf{e}}_{n}$ given by $\tilde{\mathbf{e}}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}$ satisfy the relation $\tilde{\mathbf{e}}_{j} \tilde{\mathbf{e}}_{k}+\tilde{\mathbf{e}}_{k} \tilde{\mathbf{e}}_{j}=0$, i.e. form a Grassmann basis.

Therefore, we get

$$
\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)\left(D_{h, \Gamma_{n}}^{-}+D_{h, \Theta_{n}}^{+}\right)
$$

$$
\begin{equation*}
=-2 \sum_{j=1}^{n} \sum_{k<j}\left(\mathfrak{f}_{j k}^{+-} \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{k}}^{k}+\mathfrak{f}_{j k}^{-+} \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{k}}^{k}+L_{j k} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}+L_{k j} \partial_{h, \gamma_{k}}^{k} \partial_{h, \theta_{j}}^{j}\right) \tag{4.17}
\end{equation*}
$$

as well as

$$
\left(D_{h, \Gamma_{n}}^{+}+D_{h, \Theta_{n}}^{-}\right)^{2}
$$

$=-2 \sum_{j=1}^{n} \sum_{k<j}\left(\mathfrak{g}_{j k}^{+} \partial_{h, \gamma_{j}}^{j} \partial_{h, \gamma_{k}}^{k}+\mathfrak{g}_{j k}^{-} \partial_{h, \theta_{j}}^{j} \partial_{h, \theta_{k}}^{k}+M_{j k} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}+M_{k j} \partial_{h, \gamma_{j}}^{j} \partial_{h, \theta_{k}}^{k}\right)$.
which leads to the following theorems.
Theorem 4.6. If assumptions 4 with $\mathfrak{f}_{j k}^{+-}+\mathfrak{f}_{j k}^{-+}+L_{j k}+L_{k j}=0$ are valid, we obtain:
If for all $j<k, L_{j k}=c_{j k}, c_{j k}>0$, then $-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}=-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}$is a discrete Laplacian according to (3.2).

Moreover,

$$
-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}=\mathbf{e}_{0} \sum_{j=1}^{n} \sum_{k<j} 2 c_{j k} \tilde{\Delta}_{h}^{j, k},
$$

and if all the directions are equally important, i.e. $c_{j k}=c: j=1, \ldots, n, j<k$, the right-hand side of $-D_{h, \Theta_{n}}^{+-} D_{h, \Theta_{n}}^{-+}$is equal to $2 c \mathbf{e}_{0} \Delta_{h}^{\times}$.

Theorem 4.7. If assumptions 4 with $\mathfrak{g}_{j k}^{+}+\mathfrak{g}_{j k}^{-}+M_{j k}+M_{k j}=0$, are valid, we get:
If for all $j<k, M_{j k}=c_{j k}, c_{j k}>0$, then $-\left(D_{h, \Theta_{n}}^{+-}\right)^{2}=-\left(D_{h, \stackrel{\Theta}{\Theta}_{n}}^{-+}\right)^{2}$ is the discrete Laplacian according to (3.2).

Moreover,

$$
-\left(D_{h, \Theta_{n}}^{-+}\right)^{2}=2 \mathbf{e}_{0} \sum_{j=1}^{n} \sum_{k<j} c_{j k} \tilde{\Delta}_{h}^{j, k}
$$

and if all the directions are equally important, i.e. $c_{j k}=c: j=1, \ldots, n, j<k$, the right-hand side of $-\left(D_{h, \Theta_{n}}^{-+}\right)^{2}$ is equal to $2 c \mathbf{e}_{0} \Delta_{h}^{\times}$.

## 5. Discrete Dirac Operators and Discrete Holomorphic Functions on Graphs

In [2] a notion of discrete analytic functions on quad-graphs is studied. To this end quasicristallic rhombic embeddings $\mathcal{D}$ with set of labels $\left\{ \pm \alpha_{1}, \ldots, \pm \alpha_{n}\right\}$ are considered. Extending the labelling $\alpha: \vec{E}(\mathcal{D}) \mapsto \mathbb{C}$ to all edges $\mathbb{Z}^{n}$, assuming that all edges parallel to (and directed as) $\mathbf{e}_{k}$ carry the label $\alpha_{k}$, leads then to the following definition.

Definition 5.1. [2] A function $f: \mathbb{Z}^{n} \mapsto \mathbb{C}$ is called discrete holomorphic, if it satisfies, on each elementary square of $\mathbb{Z}^{n}$, the equations

$$
\frac{f\left(m+\mathbf{e}_{j}+\mathbf{e}_{k}\right)-f(m)}{f\left(m+\mathbf{e}_{j}\right)-f\left(m+\mathbf{e}_{k}\right)}=\frac{\alpha_{j}+\alpha_{k}}{\alpha_{j}-\alpha_{k}}
$$

for all $j$ and $k$.
In other words, the quotient of the diagonals of the $f$-image on each elementary quadrilateral $\left(m, m+\mathbf{e}_{k}, m+\mathbf{e}_{j}+\mathbf{e}_{k}, m+\mathbf{e}_{j}\right) \in \mathcal{F}(\mathcal{D})$ is equal to the quotient of diagonals of the corresponding parallelogram.


Figure 2. Elementary square of $\mathbb{Z}^{n}$

To extend this ideas to our setting, we introduce for a set of labels $\mathcal{A}_{\Theta_{n}}=$ $\left\{\alpha_{j}: j=1, \ldots, n\right\} \subset \mathbb{R}^{0, n}$ and consider the discrete Dirac operator

$$
\begin{equation*}
\tilde{D}_{h, \Theta_{n}}^{j, k}=\left(\alpha_{j}-\alpha_{k}\right) \frac{\sigma_{h, \theta_{j}}^{j} \sigma_{h, \theta_{k}}^{k}-I}{2 h}-\left(\alpha_{j}+\alpha_{k}\right) \frac{\sigma_{h, \theta_{j}}^{j}-\sigma_{h, \theta_{k}}^{k}}{2 h}, \tag{5.1}
\end{equation*}
$$

with $\sigma_{h, \theta_{j}}^{j}=\theta \sigma_{h}^{+j}+(1-\theta) \sigma_{h}^{-j}$, and we define discrete holomorphic functions as null solutions of the Dirac operator (5.1). Therefore, considering the sequence $\Gamma_{n}=\left(\gamma_{l}\right)_{l=1}^{n}$ with $\gamma_{l}=1-\theta_{l} \in \mathbb{Z}_{2}$ we obtain

$$
\begin{align*}
\tilde{D}_{h, \Gamma_{n}}^{j, k} \tilde{D}_{h, \Theta_{n}}^{j, k}= & \left(\alpha_{j}-\alpha_{k}\right)\left(\alpha_{j}-\alpha_{k}\right) \frac{2 I-\sigma_{h, \theta_{j}}^{j} \sigma_{h, \theta_{k}}^{k}-\sigma_{h, \gamma_{j}}^{j} \sigma_{h, \gamma_{k}}^{k}}{h^{2}} \\
+ & \left(\alpha_{j}+\alpha_{k}\right)\left(\alpha_{j}+\alpha_{k}\right) \frac{2 I-\sigma_{h, \theta_{j}}^{j} \sigma_{h, \gamma_{k}}^{k}-\sigma_{h, \gamma_{j}}^{j} \sigma_{h, \theta_{k}}^{k}}{h^{2}} \\
& +2\left(\alpha_{j} \alpha_{j}-\alpha_{k} \alpha_{k}\right) \frac{\sigma_{h, \theta_{j}}^{j}-\sigma_{h, \theta_{k}}^{k}+\sigma_{h, \gamma_{j}}^{j}-\sigma_{h, \gamma_{k}}^{k}}{h^{2}} . \tag{5.2}
\end{align*}
$$

Therefore, $-\tilde{D}_{h, \Gamma_{n}}^{j, k} \tilde{D}_{h, \Theta_{n}}^{j, k}$ is a discrete Laplacian in the sense of (3.2), that is $-\tilde{D}_{h, \Gamma_{n}}^{j, k} \tilde{D}_{h, \Theta_{n}}^{j, k} \sim \mathbf{e}_{0} \tilde{\Delta}_{h}^{j, k}$, if and only if $\alpha_{j}^{2}=\alpha_{k}^{2}$ and $\alpha_{j} \alpha_{k}+\alpha_{k} \alpha_{j}=-\delta_{j k} \mathbf{e}_{0}$. Moreover, the formula (5.2) becomes then

$$
\begin{align*}
\tilde{D}_{h, \Gamma_{n}}^{j, k} & \tilde{D}_{h, \Theta_{n}}^{j, k}= \\
& =\left(\alpha_{j} \alpha_{k}+\alpha_{k} \alpha_{j}\right) \frac{4 I-\sigma_{h, \theta_{j}}^{j} \sigma_{h, \theta_{k}}^{k}-\sigma_{h, \gamma_{j}}^{j} \sigma_{h, \gamma_{k}}^{k}-\sigma_{h, \theta_{j}}^{j} \sigma_{h, \gamma_{k}}^{k}-\sigma_{h, \gamma_{j}}^{j} \sigma_{h, \theta_{k}}^{k}}{4 h^{2}}  \tag{5.3}\\
& =\frac{\alpha_{j} \alpha_{k}+\alpha_{k} \alpha_{j}}{4}\left(\partial_{h}^{+j} \partial_{h}^{+k}+\partial_{h}^{-j} \partial_{h}^{-k}-\partial_{h}^{+j} \partial_{h}^{-k}-\partial_{h}^{-j} \partial_{h}^{+k}\right) \\
& =-\frac{1}{4} \mathbf{e}_{0} \tilde{\Delta}_{h}^{j, k} .
\end{align*}
$$

Therefore, choosing $\mathbf{e}_{k}=\alpha_{k}-\alpha_{j}$ and $\mathbf{e}_{j}=\alpha_{k}+\alpha_{j}$ we obtain $\tilde{D}_{h, \Gamma_{n}}^{j, k}=D_{h, \Theta_{n}}^{+-}$ (satisfying assumptions 4 or assumptions 5) restricted to the plane $e_{k} e_{j}$. Hereby, we used the fact that $\alpha_{k}-\alpha_{j}$ and $\alpha_{k}+\alpha_{j}$ are always orthogonal to each other in a rhombic embedding.

Moreover, we obtain as an immediate consequence, that a discrete monogenic function, i.e. a function $f$ with $D_{h, \Theta_{n}}^{+-} f=0$, which is additionally complex-valued, is also a discrete holomorphic function, i.e. satisfies Definition 5.1.

An interesting consequence of the above considerations is that they allow us to define discrete Dirac operators on spatial graphs. To this end we will consider $n$-rhombohedric embeddings $\mathcal{D}$ with set of labels $\left\{ \pm \alpha_{1}, \ldots, \pm \alpha_{n}\right\}$. Extending the labelling $\alpha: \vec{E}(\mathcal{D}) \mapsto \mathbb{C}$ to all edges of $\mathbb{R}_{h}^{n}$, we create a new basis (locally for each rhombohedron) via

$$
\begin{aligned}
\mathbf{e}_{1} & =\alpha_{1}+\ldots+\alpha_{n-2}+\alpha_{n-1}+\alpha_{n} \\
\mathbf{e}_{2} & =\alpha_{1}+\ldots+\alpha_{n-2}+\alpha_{n-1}-\alpha_{n} \\
\mathbf{e}_{3} & =\alpha_{1}+\ldots+\alpha_{n-2}-\alpha_{n-1}-\alpha_{n} \\
\mathbf{e}_{4} & =\alpha_{1}+\ldots-\alpha_{n-2}-\alpha_{n-1}-\alpha_{n} \\
& \vdots
\end{aligned}
$$

and, afterwards, we decompose it into $\mathbf{e}_{j}=\mathbf{e}_{j}^{+}+\mathbf{e}_{j}^{-}$. Therefore, joining the center points of the rhombohedrons we create two graphs, $\mathcal{G}$ ("white" vertices) and $\mathcal{G}^{*}$
("black" vertices), which lead to the following definition of a discrete monogenic function on the graph $\mathcal{D}$, constructed from $\mathcal{G}$ and $\mathcal{G}^{*}\left(V(\mathcal{D})=V(\mathcal{G}) \cup V\left(\mathcal{G}^{*}\right)\right)$.


Figure 3. Local construction in a rhombohedron.

Definition 5.2. A function $f: V(\mathcal{D}) \mapsto C \ell_{0, n}$ is called discrete monogenic, if it satisfies one of the following equations:

$$
D_{h, \Theta_{n}}^{+-} f=0 \text { or } D_{h, \Theta_{n}}^{-+} f=0 .
$$

for a certain sequence $\Theta_{n} \in\left(\mathbb{Z}_{2}\right)^{n}$.
An immediate consequence of our definition of a discrete monogenic function on the graph $\mathcal{D}$ is that the projection into the plane $\mathbf{e}_{k j}=\mathbf{e}_{k} \mathbf{e}_{j}$ will be discrete holomorphic for all $j$ and $k$.

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