KREIN REPRODUCING KERNEL MODULES IN CLIFFORD ANALYSIS

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Abstract. Classic hypercomplex analysis is intimately linked with elliptic operators, such as the Laplacian or the Dirac operator, and positive quadratic forms. But there are many applications like the crystallographic X-ray transform or the ultrahyperbolic Dirac operator which are closely connected with indefinite quadratic forms. Although appearing in many papers in such cases Hilbert modules are not the right choice as function spaces since they do not reflect the induced geometry. In this paper we are going to show that Clifford-Krein modules are naturally appearing in this context. Even taking into account the difficulties, e.g. the existence of different inner products for duality and topology, we are going to demonstrate how one can work with them. Taking into account possible applications and the nature of hypercomplex analysis special attention will be given to the study of Clifford-Krein modules with reproducing kernels. In the end we will discuss the interpolation problem in Clifford-Krein modules with reproducing kernel.

1. Introduction

Classic hypercomplex function theory is intimately connected with the factorization of the Laplacian and an elliptic Dirac operator. This leads to the point that Hilbert modules are the appropriate function spaces. They are based on a interplay between two inner products, one which provides the link with Riesz representation theorem and one which leads to norm and, therefore, provides the topological structure. But there are many applications where one uses either Clifford algebras with signatures different of $(0, n)$ or operators which are not elliptic. Besides the obvious applications in Minkowski space, etc, there are even practical problems in which such Clifford algebras and operators naturally arise. For instance, the second order scalar differential operator connected with a Clifford algebra of signature (p, q) which is factorized by the Dirac operator is the ultra-hyperbolic Laplacian $\Delta_p - \Delta_q$. This operator does not only appear in problems of PDE's, but also characterizes the target space of the spherical (or crystallographic) Radon transform. In such a case it turns out that Hilbert modules are not the correct spaces, i.e. the underlying sesqui-linear form of the Hilbert module and the underlying bilinear form of the Clifford algebra are not compatible. As will be seen in this paper in such a setting the correct notion of a function space is the notion of a Krein module where the underlying space of coefficients forms a Pontryagin module. The Hermitean form associated to a Pontryagin module has a finite number of negative squares, but the number of coefficients is infinite, and hence the Krein space structure. To be more explicit, let us give an example from classical function theory. Let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The space $\mathbb{C}^2$ endowed with the Hermitean form

$$[u, v]_J = u^* J v$$

is a two dimensional Pontryagin space. The set $\mathbf{H}_{2,J}$ of $\mathbb{C}^2$-valued functions of the form $f(z) = \sum_{n=0}^{\infty} z^n u_n$ where the vectors $u_n = (u_n^1, u_n^2) \in \mathbb{C}^2$ satisfy $\sum_{n=0}^{\infty} \left(|u_n^1|^2 + |u_n^2|^2\right) < \infty$, is a Krein space

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when endowed with the form

$$[f,g] := \sum_{n=0}^{\infty}[u_n, v_n]_J = \sum_{n=0}^{\infty} u_n^* J v_n,$$

for $f(z) = \sum_{n=0}^{\infty} z^n u_n,\ g(z) = \sum_{n=0}^{\infty} z^n v_n$. See e.g. [18], and also [4] for the similar space in the setting of slice-hyperholomorphic functions, and [7] for an example in the setting of split-quaternions.

We are not aware of any works which study Krein modules in the context of Clifford-algebra valued functions, although there are indeed papers studying it in the context of quaternions. Our impression is that this is due to the fact that in the quaternionic context we still have the classic property that the sequilinear form which gives rise to the Riesz representation theorem is also giving rise to a norm. This is not anymore true for Krein modules in the general context of Clifford-algebra valued norms which requires to work with a careful interplay between two different sesqui-linear forms.

Motivated from applications such as crystallographic diffraction tomography and null-solutions of the ultra-hyperbolic Dirac operator we will use our study of Krein spaces to look into the interpolation problem, i.e. the construction of a function from given data. Since classic approaches like Lagrange interpolation are difficult to study in hypercomplex analysis one has to look for a different approach. In fact, there is a natural setting given by using reproducing kernels and reproducing kernel modules. Interestingly enough, there exist a lot of confusion about this approach and the correct notions to be applied like the notion of a positive function. For instance, in [22] the notion of positive function is given by a complicated and cumbersome formula which leaves the reader none-the-wiser. Although correct approaches are given in the case of quaternions (see e.g. [2, 3, 4, 8]) where one still has the preservation of the norm under the multiplication there does not seem to exist a paper specifically looking into the general non-commutative case where the norm is not preserved under multiplication. To remedy this we will provide a short overview on reproducing kernel Hilbert modules and the corresponding interpolation problem. Afterwards, we will take a detailed look into reproducing Krein modules. In the end we will look into the interpolation problem for null-solutions of the hyperbolic Dirac operator.

The paper consists of five sections besides the introduction. In Section 2 we survey part of Clifford analysis (and in particular modules) necessary for the sequel. Positive kernels and the associated reproducing kernel Hilbert modules are considered in Section 3, where one can in particular find the counterpart of the Moore-Aronszajn theorem in the present setting. In Section 4 we consider the case of Pontryagin reproducing kernel modules. Then, the underlying sesquilinear form is not positive anymore, but has a finite dimensional negative part. We give the main theorems, both geometric and analytic, which allow to proceed in this setting. Krein modules are considered in Section 5. The geometry there is much more involved. Applications to Dirac operators and Radon transforms are presented in the last section.

2. Preliminaries

2.1. **Clifford algebras.** Let $\{e_1, \cdots, e_n\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^n$. The associated Clifford algebra $\mathbb{R}_{p,q}$ is the free algebra generated by $\mathbb{R}^n$ modulo $x^2 = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n} x_i^2$, with $p+q=n$. The defining relation induces the multiplication rules $e_i^2 = +1$ for $i = 1, \cdots, p$, $e_i^2 = -1$ for $i = p+1, \cdots, n$, and $e_i e_j + e_j e_i = \delta_{i,j}, i \neq j$, where $\delta_{i,j}$ denotes the Kronecker symbol.

A vector space basis for $\mathbb{R}_{p,q}$ is given by the set

$$\{e_\emptyset = 1, e_A = e_{l_1} e_{l_2} \dots e_{l_r} :\ A = \{l_1, l_2, \dots, l_r\}, 1 \le l_1 < \dots < l_r \le n\}. \tag{1}$$

Each $a \in \mathbb{R}_{p,q}$ can be written in the form $a = \sum_A a_A\ e_A$, with $a_A \in \mathbb{R}$. Moreover, each element $a = \sum_A a_A\ e_A$ decomposes into k-blades $[a]_k := \sum_{A:\#A=k} a_A\ e_A$ with $a = \sum_{k=0}^{n} [a]_k$, and we write $\mathbb{R}_{p,q} = \cup_{k=0}^{n} \mathbb{R}_{p,q}^k$, where $\mathbb{R}_{p,q}^k := \{[a]_k, a \in \mathbb{R}_{p,q}\}$.

The conjugation in the Clifford algebra $\mathbb{R}_{p,q}$ is defined as the automorphism $x \mapsto \overline{x} = \sum_A x_A\ \overline{e}_A$, where $\overline{e}_\emptyset = 1, \overline{e}_j = -e_j (j = 1, \dots, n)$, and $\overline{e}_A = \overline{e}_{l_r}\ \overline{e}_{l_{r-1}} \dots \overline{e}_{l_1}$. For a vector $x = \sum_{j=1}^{n} x_j e_j \in \mathbb{R}^{p,q}$

we have $x\overline{x} = |x|^2 := \sum_{j=1}^{p} |x_j|^2 - \sum_{j=p+1}^{q} |x_j|^2$. Hence, each non-zero vector $x = \sum_{j=1}^{n} x_j e_j$ with $|x| \neq 0$ has an unique multiplicative inverse given by $x^{-1} = \frac{\overline{x}}{|x|^2}$.

An $\mathbb{R}_{p,q}$-valued function f over a non-empty domain $\Omega \subset \mathbb{R}^n$ is written as $f = \sum_A f_A e_A$, with components $f_A : \Omega \to \mathbb{R}$. Properties such as continuity are to be understood component-wisely. For example, $f = \sum_A f_A e_A$ is continuous if and only if all components f_A are continuous. Finally, we recall the Dirac operator $D = \sum_{j=1}^{n} e_j\ \partial_{x_j}$, which factorizes the ultra-hyperbolic operator, i.e., $D^2 = \Delta_p - \Delta_q = \sum_{j=1}^{p} \partial^2_{x_j} - \sum_{j=p+1}^{n} \partial^2_{x_j}$. A $\mathbb{R}_{p,q}$-valued function f is said to be *left-monogenic* if it satisfies $Df = 0$ on Ω (resp. *right-monogenic* if it satisfies $fD = 0$ on Ω).

2.2. **Clifford-Hilbert modules.** A right (unitary) module over $\mathbb{R}_{0,n}$ is a vector space V together with an algebra morphism $R : \mathbb{R}_{0,n} \mapsto \mathrm{End}(V)$, or to say it more explicitly, there exists a linear transformation (also called right multiplication) $R(a)$ of V such that

$$R(ab + c) = R(b)R(a) + R(c), \tag{2}$$

for all $a \in \mathbb{R}_{0,n}$, and where $R(1)$ is the identity operator. We consider in V the right multiplication defined by

$$R(a)v = va, \quad v \in V, a \in \mathbb{R}_{0,n}. \tag{3}$$

In particular, if V denotes a function space, the product (3) is defined by point-wise multiplication. We say that V is a right Banach $\mathbb{R}_{0,n}$-module if

- V is a right $\mathbb{R}_{0,n}$-module;
- V is a real Banach space;
- there exists $C > 0$ such that for any $a \in \mathbb{R}_{0,n}$ and $x \in V$ it holds

$$\|xa\|_V \leq C\ |a|\|x\|_V, \quad \text{where } |a|^2 := \sum_A |a_A|^2. \tag{4}$$

In particular, we have $\|xa\|_V = |a|\|x\|_V$ if $a \in \mathbb{R}$. These considerations give rise to the adequate right modules of $\mathbb{R}_{0,n}$-valued functions defined over any suitable subset Ω of $\mathbb{R}^n$. Of course, by similar reasoning one can define adequate left modules of $\mathbb{R}_{0,n}$-valued functions.

Consider $\mathcal{H}$ to be a real Hilbert space. Then $V := \mathcal{H} \otimes \mathbb{R}_{0,n}$ defines a right-Clifford-Hilbert module (rHm for short). Indeed, the inner product $< \cdot, \cdot >$ in $\mathcal{H}$ gives rise to two inner products in V:

$$\langle\!\langle x, y \rangle\!\rangle := \sum_{A,B} < x_A, y_B > \overline{e}_A e_B \quad \text{and} \quad \langle x, y \rangle := \sum_A < x_A, y_A >= [\langle\!\langle x, y \rangle\!\rangle]_0 . \tag{5}$$

We remark that while only the second inner product gives rise to a norm (in the classic sense) the first provides a generalization of Riesz' representation theorem in the sense that a linear functional ϕ is continuous if and only if it can be represented by an element $f_\phi \in V$ such that $\phi(g) = \langle\!\langle f_\phi, g \rangle\!\rangle$. Furthermore, a mapping $K : V \to W$ between two right-Clifford-Hilbert modules V and W is called a $\mathbb{R}_{0,n}$-linear mapping if $K(fa + g) = K(f)a + K(g)$, where $f, g \in V, a \in \mathbb{R}_{0,n}$. For more details we refer to [23, 12].

We end this section with some important inequalities involving the sesquilinear form and the norm coming from the complex-valued inner product:

- Hölder inequality: $|\langle f, g \rangle| \leq 2^{n/2} \|f\|_{L_p(\Omega,\mathbb{R}_{0,n})} \|g\|_{L_q(\Omega,\mathbb{R}_{0,n})}$ with $\frac{1}{p} + \frac{1}{q} = 1$
- $\|af\|_{L_2(\Omega,\mathbb{R}_{0,n})} \leq 2^{n/2} |a| \|f\|_{L_2(\Omega,\mathbb{R}_{0,n})}$ for all $a \in \mathbb{R}_{0,n}$, but $\|af\|_{L_2(\Omega,\mathbb{R}_{0,n})} = |a| \|f\|_{L_2(\Omega,\mathbb{R}_{0,n})}$ whenever a is a scalar or a vector.
- $\|f\|_{L_2(\Omega,\mathbb{R}_{0,n})} \leq |\langle f, f \rangle| \leq 2^{n/2} \|f\|_{L_2(\Omega,\mathbb{R}_{0,n})}$
- $\|f\|_{L_2(\Omega,\mathbb{R}_{0,n})} \leq \sup_{\|g\|_{L_2(\Omega,\mathbb{R}_{0,n})} \leq 1} |\langle f, g \rangle| \leq 2^{n/2} \|f\|_{L_2(\Omega,\mathbb{R}_{0,n})}$

Many facts from classic Hilbert spaces carry over to the notion of a Clifford Hilbert module. Here too we refer to [23] for more details.

3. Positive kernels

In this section we present some important results about reproducing right-Clifford-Hilbert modules. While we believe that experts in the field of Clifford analysis are familiar with these results we could not find them anywhere in the literature.

Consider a right-Clifford-Hilbert module $V = \mathcal{H} \otimes \mathbb{R}_{0,n}$ where $\mathcal{H}$ denotes a Hilbert space of real-valued functions on an open set $\Omega \subset \mathbb{R}^n$, with a reproducing kernel $k_x = k(x, \cdot), \ x \in \Omega \subset \mathbb{R}^n$, i.e.

$$f(y) = \langle\!\langle k(y,\cdot), f(\cdot)\rangle\!\rangle, \ y \in \Omega, \qquad \forall f \in V. \tag{6}$$

Consider now $f = \sum_{l=1}^{M} k(x_l,\cdot)c_l$, where $c_l \in \mathbb{R}_{0,n}$. We have

$$\begin{aligned}
\langle\!\langle f, f\rangle\!\rangle &= \langle\!\langle \sum_{i=1}^{M} k(x_i,\cdot)c_i, \sum_{j=1}^{M} k(x_j,\cdot)c_j\rangle\!\rangle \\
&= \sum_{i,j=1}^{M} \overline{c_i}\langle\!\langle k(x_i,\cdot), k(x_j,\cdot)\rangle\!\rangle c_j \\
&= \sum_{i,j=1}^{M} \overline{c_i} k(x_j, x_i) c_j = \underline{c}^* K \underline{c}
\end{aligned}$$

with $\underline{c} = (c_1, \cdots, c_M)^T, \underline{c}^* = (\overline{c}_1, \cdots, \overline{c}_M)$, and $K = (k(x_j, x_i))_{i,j=1}^{M}$. Now, in general we have $\langle\!\langle f, f\rangle\!\rangle$ being Clifford-valued and, therefore, it does not corresponding to a classic inner product. But, as in (5), we have that

$$[\underline{c}^* K \underline{c}]_0 = \langle f, f\rangle \geq 0. \tag{7}$$

This corresponds to the following notion of positivity of a matrix with Clifford valued entrances.

Definition 1. A matrix $A \in (\mathbb{R}_{0,n})^{M\times M}$ is said to be positive (positive semi-definite) if and only if

$$[\underline{c}^* A \underline{c}]_0 \geq 0 \tag{8}$$

for all $\underline{c} \in (\mathbb{R}_{0,n})^M$.

Based on this notion of positivity we introduce the notion of a positive kernel.

Definition 2. A kernel $k : \Omega \times \Omega \to \mathbb{R}_{0,n}$, where $\Omega \subset \mathbb{R}^n$ is an open set, is said to be positive if and only if for every $M \in \mathbb{N}$ the matrix $K = (k(x_j, x_i))_{i,j=1}^{M}$ satisfies

$$[\underline{c}^* K \underline{c}]_0 \geq 0 \tag{9}$$

for all $(x_i, x_j) \in \Omega \times \Omega$ and all $\underline{c} \in (\mathbb{R}_{0,n})^M$.

Let us remark that the above notion of positivity also corresponds to the established notion of positivity in the case of quaternionic analysis [2, 3, 4, 8]. The reason is that in the quaternionic case the inner products $\langle\cdot,\cdot\rangle$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ while not coinciding do give rise to the same norm.

The above considerations lead to the following theorem.

Theorem 1. *A reproducing kernel $k : \Omega \times \Omega \mapsto \mathbb{R}_{0,n}$ associated to a reproducing right-Clifford-Hilbert module is positive.*

We are in conditions to establish in the Clifford setting a version of Moore-Aronszajn Theorem (see [9, 25]).

Theorem 2. *Let $k : \Omega \times \Omega \mapsto \mathbb{R}_{0,n}$ be a Hermitean positive kernel. Then there exists a unique right-reproducing kernel Hilbert module (rRKHM) which has k as its reproducing kernel.*

Proof. Let k be a Hermitean positive kernel, that is to say, for all $M \in \mathbb{N}$ the matrix

$$K = (k(x_j, x_i))_{i,j=1}^{M} := (\langle\!\langle k(x_i, \cdot), k(x_j, \cdot)\rangle\!\rangle)_{i,j=1}^{M}$$

is Hermitean and satisfy to $[\underline{c}^* K \underline{c}]_0 \geq 0$, for all $(x_i, x_j) \in \Omega \times \Omega$ and all $\underline{c} \in (\mathbb{R}_{0,n})^M$. We can consider now the functions $k_x = k(x, \cdot)$ and the associated right-linear module $V = \operatorname{span}\{k_x, x \in \Omega\}$. On this right-linear module V we introduce the sesquilinear forms $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ and $\langle \cdot, \cdot \rangle_V$ defined by

$$\langle\!\langle \sum_{i=1}^{M} k_{x_i} c_i, \sum_{j=1}^{M} k_{x_j} d_j \rangle\!\rangle_V := \sum_{i,j=1}^{M} \overline{c_i} \langle\!\langle k(x_i, \cdot), k(x_j, \cdot)\rangle\!\rangle d_j = \sum_{i,j=1}^{M} \overline{c_i} k(x_j, x_i) d_j, \tag{10}$$

$$\left\langle \sum_{i=1}^{M} k_{x_i} c_i, \sum_{j=1}^{M} k_{x_j} d_j \right\rangle_V := \left[\sum_{i,j=1}^{M} \overline{c_i} \langle\!\langle k(x_i, \cdot), k(x_j, \cdot)\rangle\!\rangle d_j \right]_0 = \left[\sum_{i,j=1}^{M} \overline{c_i} k(x_j, x_i) d_j \right]_0 \tag{11}$$

for every finite linear combination in V. By construction the sesquilinear form is both Hermitean and positive, i.e.

$$\left\langle \sum_{i=1}^{M} k_{x_i} c_i, \sum_{j=1}^{M} k_{x_j} c_j \right\rangle_V \geq 0.$$

Subsequently, (10) defines a Clifford-valued inner product and V is a right-linear pre-Hilbert Clifford module. Next, we consider the closure $\overline{V}$ of V and define a sesquilinear Hermitean and positive definite form in $\overline{V}$ as an extension of the sesquilinear form in V. Given $f, g \in \overline{V}$ there exist two Cauchy sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in V, such that $f = \lim_n f_n$ and $g = \lim_n g_n$. We define the desired sesquilinear form as

$$\langle\!\langle f, g\rangle\!\rangle_{\overline{V}} := \lim_{n\to\infty} \langle\!\langle f_n, g_n \rangle\!\rangle_V. \tag{12}$$

It is easy to see that this form is well-defined, possesses all desired properties, and it does not dependent on the Cauchy sequences taken. Let us call equivalent two Cauchy sequences whose difference converges to 0. The space $\overline{V}$ is the space of equivalence classes of Cauchy sequences, endowed with the inner product (12). We associate to every element in $\overline{V}$ a function f in Ω in a unique way by noting that a Cauchy sequence converges also weakly, and in particular pointwise. Therefore, for a Cauchy sequence (f_n) we define

$$F(y) = \lim_{n\to\infty} \langle\!\langle k_y, f_n \rangle\!\rangle_{\overline{V}}. \tag{13}$$

The limit is the same for two equivalent Cauchy sequences, and will be identically equal to 0 if and only if the Cauchy sequence is equivalent to the zero-sequence. We associate to an equivalent class of Cauchy sequences the limit (13). The space of such functions, with the inner product (12) is the required reproducing kernel Hilbert module. □

Remark 1. We remark that Theorem 1 and Theorem 2 ensure a one-to-one correspondence between positive definite kernels and right-reproducing kernel Hilbert modules (rRKHM).

Furthermore, let us assume that V is a rRKHM and let $\{\varphi_j,\ j \in J\}$ be a basis in V, orthonormal w.r.t. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$. Then we know that $\langle\!\langle f, f \rangle\!\rangle_V = \sum_{j\in J} \overline{\langle\!\langle \varphi_j, f\rangle\!\rangle_V} \langle\!\langle \varphi_j, f \rangle\!\rangle_V$ for all $f \in V$. For $f = k(x, \cdot)$ we get

$$k(x,x) = \langle\!\langle k(x,\cdot), k(x,\cdot)\rangle\!\rangle_V = \sum_{j\in J} \overline{\langle\!\langle \varphi_j(\cdot), k(x,\cdot)\rangle\!\rangle_V} \langle\!\langle \varphi_j(\cdot), k(x,\cdot)\rangle\!\rangle_V = \sum_{j\in J} \overline{\varphi_j(x)} \varphi_j(x) \in \mathbb{R}_{0,n},$$

and, in particular, $\left| \sum_{j\in J} \overline{\varphi_j(x)} \varphi_j(x) \right|^2 < \infty$ for all $x \in \Omega$. This allows us to write

$$k(x,y) = \sum_{j\in J} \overline{\varphi_j(y)} \varphi_j(x).$$

One application of reproducing kernel Hilbert modules in Clifford analysis is interpolation. Given evaluation mappings

$$E(x_i) : f \in V \to f(x_i) \in \mathbb{R}_{0,n},$$

we are interested in finding a function f such that

$$z_i = E(x_i)f$$

for all i.

Given the functions $k_i = k(x_i, \cdot)$ associated with N observations $f(x_i)$ we can consider the subspace V_N of V spanned by the $k_i's$. Therefore, we have for each $g \in V_N$

$$g = \sum_{i=1}^{N} k_i c_i$$

Since V is a Hilbert module there exists a unique best approximation f_N in V_N for any $f \in V$. Since $f_N - f \perp V_N$ we have

$$0 = \langle\!\langle k_i, f - f_N \rangle\!\rangle_V = \langle\!\langle k_i, f - \sum_{i=1}^{N} k_i c_i \rangle\!\rangle_V$$

This leads to the system $Kc = z$ where K is the Gram matrix with entries $K_{ij} = \langle\!\langle k_i, k_j \rangle\!\rangle_V = k(x_j, x_i)$ and $z_i = E(x_i)f = \langle\!\langle f, k_i \rangle\!\rangle_V$. Note that the Gram matrix K is hermitean. From the condition that

$$[\sum_{i,j=1}^{M} \overline{c_i} K_{ij} c_j]_0 > 0,$$

for all choices of $c = (c_i)_{i=1}^M$, with $c_i \in R_{0,n}$, we also have the positivity of all sub-matrices $K_N = (K_{ij})_{i,j=1}^N$ with $N \leq M$. Furthermore, the positivity condition for the matrices K_N implies that the corresponding Schur complements are positive and, consequently, the quasideterminants of K_N since a quasideterminant is built from the Schur complements and satisfies the heredity principle [20, 16]. This implies that the quasideterminant of the matrix K is invertible and, therefore, the system $Kc = z$ has a unique solution.

The corresponding bi-orthogonal basis is given by

$$k^i = \sum_{j=1}^{M} K^{ij} k_j$$

where $K^{ij} = (K^{-1})_{ij}$ and, therefore,

$$f_M = \sum_{i=1}^{M} z_i k^i.$$

For the last step we recall that $\langle k^i, k_j \rangle_V = \delta_{ij}$.

The above statements mean that the solution to the interpolation problem in rKHM corresponds to the orthogonal projection into the (finite-) dimensional subspace spanned by the functions $k_i, i = 1, \ldots, M$, or, equivalently, to the solution of the problem

$$E_M^* E_M f = E_M^* z$$

where $E_M = \sum_{i=1}^{M} e_i E(x_i)$ and $E_M^* = \sum_{i=1}^{M} e_i k_i$. Furthermore, the operator

$$E_M E_M^* = \sum_{i=1}^{M} e_i k_{ij} \overline{e_j}$$

is equivalent to the kernel matrix K. Its solution is given by the Moore-Penrose (or generalized) inverse

$$f_M = E_M^\dagger z = E_M^* (E_M E_M^*)^\dagger z. \tag{14}$$

4. Pontryagin reproducing kernel modules

Let us consider a right-linear module V with a sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ denoted by $(V, \langle\!\langle \cdot, \cdot \rangle\!\rangle_V)$. We call $(V, -\langle\!\langle \cdot, \cdot \rangle\!\rangle_V)$ its anti-module. Let us just point out the obvious fact that if $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ is a non-negative form, i.e. $\langle x, x \rangle_V \geq 0$, for all $x \in V$, then obviously the anti-module of $(V, \langle\!\langle \cdot, \cdot \rangle\!\rangle_V)$ is endowed with a non-positive form.

Definition 3 (adapted from [11])**.** A right-linear module V endowed with a sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ is called a Krein module if

(i) it admits a decomposition

$$V = V_+ \oplus V_-, \tag{15}$$

where both, V_+ and the anti-module of V_-, are Hilbert modules;

(ii) the decomposition is orthogonal with respect to the sesquilinear form, i.e.

$$\langle\!\langle v_+, v_- \rangle\!\rangle_V = 0, \tag{16}$$

for each pair $(v_+, v_-) \in V_+ \times V_-$.

The decomposition (15) is called a fundamental decomposition. Some remarks must be made.

Remark 2. A Krein module is a inner product space which is non-degenerate, decomposable, and complete. In general V_+ and the anti-module of V_- are infinite-dimensional Hilbert modules which are orthogonal to each other with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$. Moreover, the decomposition is not unique. More important, and in difference to the classic case of Krein spaces, the sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ induces a secondary linear form $\langle \cdot, \cdot \rangle_V := [\langle\!\langle \cdot, \cdot \rangle\!\rangle_V]_0$ which determines positivity.

To characterize the Krein module in a unique way we introduce the associated signature operator, or canonical symmetry, J_V of V via

$$J_V(v_+ + v_-) := v_+ - v_-, \tag{17}$$

where $v = v_+ + v_- \in V$, with $v_\pm \in V_\pm$. Moreover, J_V is a self-adjoint, involutory and unitary operator. Remark now that V_+ (resp. V_-) is a maximal strictly positive (resp. maximal strictly negative) submodule of V. This implies that the dimensions of V_+ and of V_- are independent of the fundamental decomposition for V. Hence, we define the *positive and negative indices* of V as $\mathrm{ind}_\pm V := \mathrm{dim} V_\pm \in \mathbb{N}_0 \cup \{\infty\}$. Since $\mathrm{ind}_\pm V$ are independent of the decomposition they characterize the module $(V, \langle\!\langle \cdot, \cdot \rangle\!\rangle_V)$ in a unique way. Indeed, consider the associated right-linear Hilbert module $|V| := V_+ \oplus |V_-|$ where V_- is replaced by its anti-module $|V_-|$. This new Hilbert module has a norm induced by the linear form $\langle \cdot, \cdot \rangle_V := [\langle\!\langle \cdot, \cdot \rangle\!\rangle_V]_0$, that is,

$$\|v\|_V^2 := \langle J_V v, v \rangle_V = \langle v_+, v_+ \rangle_V - \langle v_-, v_- \rangle_V \,,$$

and since two norms arising from different fundamental decompositions are equivalent the induced norm topology is unique (see also [11, Theorem 7.19]). Since in our case the norms are arising from the corresponding Clifford-valued inner products the justification still holds.

Definition 4. A Krein module $V = V_+ \oplus V_-$ is a Pontryagin module if

$$\min\{\mathrm{ind}_+ V, \mathrm{ind}_- V\} < \infty.$$

Example 1. The Clifford module $V = \mathbb{R}_{p,q}$ endowed with the sesquilinear form

$$\langle\!\langle x, y \rangle\!\rangle_V := \sum_{A,B} < x_A, y_B > \overline{e}_A e_B, \tag{18}$$

and

$$\langle x, y \rangle_V := [\langle\!\langle x, y \rangle\!\rangle_V]_0 = \sum_{A: \overline{e}_A e_A = +1} < x_A, y_A > - \sum_{A: \overline{e}_A e_A = -1} < x_A, y_A > . \tag{19}$$

is a Pontryagin module.

In a similar way, the Clifford module $L^2(\Omega;\mathbb{R}_{p,q}) := L^2(\Omega)\otimes_{\mathbb{R}}\mathbb{R}_{p,q}$, where $\Omega\subset\mathbb{R}^n$, endowed with the sesquilinear form

$$\langle\!\langle f,g\rangle\!\rangle_2 := \sum_{A,B}\langle f_A,g_B\rangle_{L^2(\Omega)}\,\overline{e}_A e_B, \tag{20}$$

and

$$\langle f,g\rangle_2 := [\langle\!\langle f,g\rangle\!\rangle_2]_0 = \sum_{A:\overline{e}_A e_A=+1}\langle f_A,g_A\rangle_{L^2(\Omega)} - \sum_{A:\overline{e}_A e_A=-1}\langle f_A,g_A\rangle_{L^2(\Omega)}\,. \tag{21}$$

is a Krein module but not a Pontryagin module.

For Pontryagin modules we have the following very important property.

Theorem 3 (Pontryagin's Theorem (see [24]))**.** *Let D be a dense linear submodule of a Pontryagin module $\mathcal{P}$. Then D contains the negative submodule $\mathcal{P}_-$ in some fundamental decomposition $\mathcal{P}=\mathcal{P}_+\oplus\mathcal{P}_-$*

Now we consider a Pontryagin module $\mathcal{P}$ consisting of functions from a certain domain Ω which take values in a Krein space $\mathcal{K}$. Then $\mathcal{K}$ is called its coefficient space. Furthermore, assume that for each $x\in\Omega$ the evaluation mappings

$$E(x): f\in\mathcal{P}\to f(x)\in\mathcal{K},$$

are linear mappings from $\mathcal{P}$ to $\mathcal{K}$. In other words, they are linear operators on $\mathcal{P}$ which are defined pointwise.

A reproducing kernel for $\mathcal{P}$ is a kernel $k=k(x,y)$ defined on $\Omega\times\Omega$ with values in $\mathcal{K}$ such that for every $x\in\Omega$ and $f\in\mathcal{P}$ we have

(1) $k_x(y)=k(x,y)$ belongs to $\mathcal{P}$ as a function of y;
(2) $\langle\!\langle k(x,\cdot),f\rangle\!\rangle_{\mathcal{P}}=f(x)$.

Theorem 4. *Let $\mathcal{P}$ be a Pontryagin module over the space of functions defined on a set Ω and taking values in a Krein space $\mathcal{K}$. Then $\mathcal{P}$ has a reproducing kernel if and only if all evaluation mappings $E(x), x\in\Omega$, act continuously from $\mathcal{P}$ to $\mathcal{K}$. The reproducing kernel is unique with*

$$k(x,y)=E(x)E(y)^*,\qquad x,y\in\Omega, \tag{22}$$

where $E(x)^*$ denotes the adoint of the evaluation mapping operator, and it admits a decomposition $k=k_+-k_-$ where the rRKHM generated by $-k_-$ has minimal finite dimension κ, (here, κ being the negative index of $\mathcal{P}$).

Furthermore, if Ω is an open set in $\mathbb{R}^n$ and the elements of $\mathcal{P}$ are monogenic functions then $k(x,y)$ is monogenic at right in x and is adjoint-monogenic at left in y, that is to say

$$k(x,y)D_x=0,\qquad \overline{D}_y k(x,y)=0. \tag{23}$$

Proof. If $\mathcal{P}$ has a reproducing kernel, then the closed graph theorem implies continuity of the evaluation mappings $E(\cdot)$, with $E(x)E(y)^*$ being a reproducing kernel of $\mathcal{P}$. Conversely, if the evaluation mappings are continuous then $E(x)E(y)^*$ is a reproducing kernel of $\mathcal{P}$. The uniqueness of the kernel ensures $k(x,y)=E(x)E(y)^*$.

Since $k(x,\cdot)\in\mathcal{K}=\mathcal{P}_+\oplus\mathcal{P}_-$ it admits a fundamental decomposition $k(x,\cdot)=k_+(x,\cdot)-k_-(x,\cdot)$, with k_+ and $-k_-$ reproducing kernels which span the Hilbert modules $\mathcal{P}_+$ and $\mathcal{P}_-$. But $\mathcal{P}_-$ cannot have dimension smaller than κ otherwise $\mathcal{P}_+$ would not be a reproducing kernel Hilbert module. Finally, if the elements of $\mathcal{P}$ are monogenic functions then $E(\cdot)$ is a monogenic operator-valued function, that is $x\mapsto f(x)=E(x)f$ is monogenic for all $f\in\mathcal{P}$. In consequence, the evaluation mappings $E(\cdot)$ are monogenic and, therefore, the reproducing kernel satisfy the monogenicity equations (23). ☐

We denote by $\mathcal{L}(\mathcal{K})$ the space of linear bounded operators from a Krein module $\mathcal{K}$ into itself. A very important theorem for rRKPM is the following.

Theorem 5. *Let $k = k(x, y)$ be a Hermitean kernel on $\Omega \times \Omega$, satisfying to (23) and taking values in $\mathcal{L}(\mathcal{K})$ for some Krein module $\mathcal{K}$ and region Ω. If the restriction $k_0(x, y)$ to $\Omega_0 \times \Omega_0$, $(\Omega_0 \subset \Omega)$, can be written in the form $k_0 = k_{0,+} - k_{0,-}$ where the rRKHM generated by $k_{0,-}$ has dimension κ then $k = k_+ - k_-$ where the rRKHM generated by k_- has also dimension κ.*

Theorem 5 basically states that reducing the domain to a smaller domain does not change the negative index of the rRKPM. For the proof we will need following two lemmas.

Lemma 1. *Let $k(x, y)$ be a monogenic Hermitean kernel defined for $|x| < R$ and $|y| < R$ with values in $\mathcal{L}(\mathcal{K})$ for some Krein module $\mathcal{K}$ and positive number R. If the restriction of $k(x, y)$ to $|x| < r, |y| < r$ is nonnegative for some $r \in (0, R)$ then $k(x, y)$ is nonnegative for $|x| < R, |y| < R$.*

Proof. The proof follows the following lines: a monogenic Hermitean kernel k can be written as $k(x, y) = \sum_{|\alpha|,|\beta|=0}^{\infty} \overline{V_\beta(y)} C_{\alpha,\beta} V_\alpha(x)$, $|x| < R, |y| < R$, where V_α denotes the inner spherical monogenic of degree $\alpha \in \mathbb{N}^d$. It will be shown that $k(x, y)$ is nonnegative if and only if the matrix $(C_{\alpha,\beta})_{\alpha,\beta}$ is positive for $|\alpha|, |\beta| \le N$, and for all $N \ge 0$. Since this condition does not depend on R our lemma will follow.
First, we assume $k(x, y)$ is non-negative for $|x| < R, |y| < R$. Then,

$$k(\rho x, \rho y) = \sum_{|\alpha|,|\beta|=0}^{\infty} \rho^{|\alpha|+|\beta|} \overline{V_\beta(y)} C_{\alpha,\beta} V_\alpha(x), \quad \text{for } 0 < \rho < R.$$

This allows us to reduce the problem to the case of $R > 1$. By Cauchy's integral theorem we have

$$k(x, y) = \int_\Gamma \int_\Gamma \frac{\overline{u - x}}{|u - x|^n} n(u) k(u, v) \overline{n(v)} \frac{v - y}{|v - y|^n} dS_v dS_u, \qquad |x| < 1, |y| < 1,$$

where dS denotes the surface measure, $n(u)$ the outward pointing normal vector at $u \in \Gamma$, and Γ the surface of the unit ball in $\mathbb{R}^n$. For arbitrary points $x_1, \dots, x_M$ of modulus less than one and $f_1, \dots, f_M \in \mathcal{K}$ the non-negativity of k means that

$$\begin{aligned} 0 &\le \sum_{i,j=1}^{M} [\langle\!\langle k(x_i, x_j) f_j, f_i \rangle\!\rangle_{\mathcal{K}}]_0 \\ &= \left[\int_\Gamma \int_\Gamma \sum_{i,j=1}^{M} \overline{f_j} \frac{\overline{v - x_j}}{|v - x_j|^n} n(v) k(u, v) \overline{n(u)} \frac{u - x_i}{|u - x_i|^n} f_i dS_v dS_u \right]_0 \\ &= \int_\Gamma \int_\Gamma [\langle\!\langle k(u, v)\varphi(u), \varphi(v) \rangle\!\rangle_{\mathcal{K}}]_0 \, dS_v dS_u \end{aligned}$$

with $\varphi(u) = \sum_{i=1}^{M} \overline{n(u)} \frac{u - x_i}{|u - x_i|^n} f_i$ for every u. Now an approximation argument gives us

$$\int_\Gamma \int_\Gamma [\langle\!\langle k(u, v) p(u), p(v) \rangle\!\rangle_{\mathcal{K}}]_0 dS_v dS_u \ge 0$$

for every monogenic polynomial $p(u) = \sum_{|\alpha|=0}^{N} \overline{V_\alpha}(u) g_\alpha$ with coefficients in $\mathcal{K}$. Then

$$0 \le \sum_{|\alpha|,|\beta|=1}^{N} \int_\Gamma \int_\Gamma [\langle\!\langle k(u, v) g_\alpha, g_\beta \rangle\!\rangle_{\mathcal{K}}]_0 dS_v dS_u = [\sum_{|\alpha|,|\beta|=1}^{N} \langle\!\langle C_{\alpha,\beta} g_\alpha, g_\beta \rangle\!\rangle_{\mathcal{K}}]_0$$

Therefore $(C_{\alpha,\beta})_{\alpha,\beta}$ is positive for all $N \ge 0$.

Now, assume that $(C_{\alpha,\beta})_{\alpha,\beta}$ with $|\alpha|, |\beta| \le N$ is nonnegative. Choose $f_1, \dots, f_M \in \mathcal{K}$ and restrict to the case $R > 1$ and $x_1, \dots, x_M$ with modulus less than one (without loss of generality). Let φ_N be the N-th degree approximant for the function φ above. Reversing the steps and passing to the limit as $N \to \infty$ we obtain the result. □

Lemma 2. *Let $k = k(x,y)$ be a monogenic Hermitean kernel on $\Omega \times \Omega$ in $\mathcal{L}(\mathcal{K})$ for some Krein module $\mathcal{K}$ and domain Ω. Suppose that $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are domains and the restrictions $k_1(x,y)$ and $k_2(x,y)$ to $\Omega_1 \times \Omega_1$ and $\Omega_2 \times \Omega_2$ are nonnegative. Then $k(x,y)$ is nonnegative.*

Proof. Since $\Omega = \Omega_1 \cup \Omega_2$ is a domain (and therefore connected) the intersection $\Omega_0 = \Omega_1 \cap \Omega_2$ is nonempty. We have that the restriction $k_0(x,y)$ of $k(x,y)$ to $\Omega_0 \times \Omega_0$ is nonnegative. Denote by V_0, V_1, V_2 the Hilbert modules of monogenic functions on $\Omega_0, \Omega_1, \Omega_2$ with reproducing kernels $k_0(x,y), k_1(x,y), k_2(x,y)$, respectively. For $j = 1,2$ we consider the restriction mapping

$$R_j \varphi_j = \varphi_j\big|_{\Omega_0} =: \varphi_0, \qquad \varphi_j \in V_j,\ j = 1,2,$$

which represents a Hilbert module isomorphism from V_j onto V_0. We now define a new Hilbert module in the following way. Each $\varphi_0 \in V_0$ satisfies

$$\varphi_0 = \varphi_1\Big|_{\Omega_0} = \varphi_2\Big|_{\Omega_0}$$

with $\varphi_1 \in V_1$ and $\varphi_2 \in V_2$. Consequently, $\varphi_0 = \varphi\big|_{\Omega_0}$ where φ is a monogenic function on Ω. Let $\mathcal{M}$ be the space of all such functions φ. Then $\mathcal{M}$ is a Hilbert module with a unique sesquilinear form such that the mapping $\varphi_0 \to \varphi$ is an isomorphism from V_0 onto $\mathcal{M}$. Moreover, evaluation mappings on $\mathcal{M}$ are continuous since if $x \in \Omega$ we have $x \in \Omega_1$ or $x \in \Omega_2$. For simplicity sake let us assume that $x \in \Omega_1$.

Consider a norm $\|\cdot\|_{\mathcal{K}}$ which determines the strong topology of $\mathcal{K}$. Let $\varphi \in \mathcal{K}$ and take $\varphi_0 = \varphi\Big|_{\Omega_0}$ and $\varphi_1 = \varphi\Big|_{\Omega_1}$. Then, we have $\varphi_0 \in V_0$ and $\varphi_1 \in V_1$. For each $f \in \mathcal{K}$, and due to the Hilbert module isomorphisms, it holds

$$\begin{aligned} \big|[\langle\!\langle \varphi(x), f \rangle\!\rangle_{\mathcal{K}}]_0\big| &= \big|[\langle\!\langle \varphi_1(x), f \rangle\!\rangle_{\mathcal{K}}]_0\big| \\ &\le M_x \|f\|_{\mathcal{K}} \|\varphi_1\|_{V_1} = M_x \|f\|_{\mathcal{K}} \|\varphi_0\|_{V_0} = M_x \|f\|_{\mathcal{K}} \|\varphi\|_{\mathcal{M}} \end{aligned}$$

with M_x being a constant. Therefore, $\mathcal{M}$ has a reproducing kernel $l(x,y)$ which is nonnegative since $\mathcal{M}$ is a Hilbert module. Since the restriction to Ω_0 is an isomorphism from $\mathcal{M}$ to V_0 the restriction of $l(x,y)$ to $\Omega_0 \times \Omega_0$ is a reproducing kernel for V_0. Since $k(x,y)$ and $l(x,y)$ are monogenic Hermitean kernels which coincide on $\Omega_0 \times \Omega_0$ we have that they are identical. Therefore, $k(x,y)$ is nonnegative. □

These two lemmas allow us to prove now Theorem 5.

Proof. Let $k(x,y)$ be nonnegative on Ω_0. Denote by $B_0, \ldots, B_n$ balls in Ω such that the center of B_0 is in Ω_0 and the center of B_j is in $B_{j-1}, j = 1, \ldots, n$. By our previous lemmas $k(x,y)$ is nonnegative on B_0, B_1, and $B_0 \cup B_1$. In the same way, $k(x,y)$ is nonnegative on B_2 and $(B_0 \cup B_1) \cup B_2$. Continuing in thus way we have that $k(x,y)$ is nonnegative on B_n and $B_0 \cup B_1 \cup \ldots \cup B_n$. In case of $\kappa = 0$ the result follows immediately because any finite set of points in Ω are contained in the union of a system of such balls.

In the general case we use the fundamental decomposition

$$k_0(x,y) = k_{0,+}(x,y) - k_{0,-}(x,y), \qquad x, y \in \Omega_0,$$

where $k_{0,+}(x,y)$ is a reproducing kernel and $k_{0,-}(x,y)$ is the reproducing kernel in the κ-dimensional anti-space. The kernel $k_{0,-}(x,y)$ can be written as

$$\langle\!\langle k_{0,-}(x,\cdot), f \rangle\!\rangle_{\mathcal{K}} = -\sum_{j=1}^{\kappa} u_j(x) \langle\!\langle u_j(\cdot), f \rangle\!\rangle_{\mathcal{K}}, \qquad f \in \mathcal{K},$$

on $\Omega \times \Omega$ and $k_{0,+}$ is the restriction to $\Omega_0 \times \Omega_0$ of the monogenic Hermitean kernel $k_+(x,y) = k(x,y) + k_-(x,y)$ on $\Omega \times \Omega$. Using now the case of $\kappa = 0$, we have that $k_+(w,z)$ is nonnegative and, hence, the reproducing kernel for a Hilbert module $\mathcal{K}_+$ of monogenic functions on Ω and $k_-(w,z)$ is the reproducing kernel for a κ-dimensional module $\mathcal{K}_-$ which is the anti-space of a Hilbert module. It is easy to see that there is a Pontryagin module $\mathcal{K}$ which contains $\mathcal{K}_+$ and $\mathcal{K}_-$ isometrically as orthogonal submodules and $k(x,y)$ is a reproducing kernel for $\mathcal{K}$. □

One of the important concepts in Pontryagin modules is the possiblity to extend densely defined contractions to continuous contractions.

Theorem 6. *Let T be a densely defined contraction, i.e. $[\langle\langle Tx, Tx\rangle\rangle_{\mathcal{W}}]_0 \leq [\langle\langle x, x\rangle\rangle_{\mathcal{V}}]_0$ for all x in the domain of T, between two Pontryagin modules $\mathcal{V}$ and $\mathcal{W}$ of the same finite index. Then T can be extended to a continuous contraction and the adjoint of T is again a contraction.*

Proof. Let us start with choosing a fundamental decomposition $\mathcal{V} = \mathcal{V}_+ + \mathcal{V}_-$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ such that $\mathcal{V}_- \subset D(T)$ and $T\mathcal{V}_- = \mathcal{W}_-$. Such a choice of $\mathcal{V}_-$ is always possible since every dense linear submodule contains a negative submodule of maximum dimension κ and $T\mathcal{V}_- = \mathcal{W}_-$ is justified by the contraction property.

Let now $Q_\pm$ denote the projections of $\mathcal{W}$ into $\mathcal{W}_\pm$ and let $\mathcal{M}$ be any κ-dimensional uniformly negative submodule of $D(T)$. Let $Q_-Tf = 0$ for some $f \in \mathcal{M}$. Then,

$$\begin{aligned} 0 &\leq [\langle\langle Q_+Tf, Q_+Tf\rangle\rangle_{\mathcal{W}}]_0 \\ &= [\langle\langle Q_+Tf, Q_+Tf\rangle\rangle_{\mathcal{W}}]_0 + [\langle\langle Q_-Tf, Q_-Tf\rangle\rangle_{\mathcal{W}}]_0 \\ &= [\langle\langle Tf, Tf\rangle\rangle_{\mathcal{W}}]_0 \leq [\langle\langle f, f\rangle\rangle_{\mathcal{V}}]_0 \leq 0. \end{aligned}$$

Therefore, $[\langle\langle f, f\rangle\rangle_{\mathcal{V}}]_0 = 0$ and, consequently, $f = 0$. Since $\mathcal{W}_-$ and $\mathcal{M}$ have dimension κ we obtain

$$Q_-T\mathcal{M} = \mathcal{W}_-.$$

Now, let $w_1, \ldots, w_\kappa$ be a basis for $\mathcal{W}_-$ and let $F_i : D(T) \mapsto \mathbb{R}_{0,n}, i = 1, \ldots, \kappa$, be linear functionals such that

$$Q_-Tf = \sum_{i=1}^{\kappa} w_i F_i(f), \qquad f \in D(T).$$

Obviously, no F_i can vanish identically. We are going to show that all $F_i's$ are bounded. To this end assume that F_1 is unbounded. Take $f_1 \in D(T)$ such that $F_1(f_1) = 1$ and consider the decomposition

$$D(T) = \operatorname{span}\{f_1\} + \ker F_1.$$

Then, there exists a sequence $(g_n)_{n=1}^\infty$ in $D(T)$ such that $g_n \to 0$ and $F_1(g_n) = 1$ for all n. Hence, $(f_1 - g_n)_{n=1}^\infty$ is sequence in $\ker F_1$ which converges to f_1. Therefore, $f_1 \in \overline{\ker F_1} = \overline{D(T)} = \mathcal{V}$. But then there exists a κ-dimensional submodule of $\mathcal{V}$ which is also a subset of the kernel of F_1 and, consequently, F_1 vanishes identically which is a contradiction. Therefore, F_1 is bounded. Proceeding iteratively we obtain that F_i is bounded for all $i = 1, \cdots, \kappa$.

Let $J_{\mathcal{V}}$ and $J_{\mathcal{W}}$ be the fundamental symmetries for $\mathcal{V}$ and $\mathcal{W}$, respectively. For all $f \in D(T)$ we have

$$[\langle\langle Q_+Tf, Q_+Tf\rangle\rangle_{\mathcal{W}}]_0 + [\langle\langle Q_-Tf, Q_-Tf\rangle\rangle_{\mathcal{W}}]_0 = [\langle\langle Tf, Tf\rangle\rangle_{\mathcal{W}}]_0 \leq [\langle\langle f, f\rangle\rangle_{\mathcal{V}}]_0 \leq [\langle\langle J_{\mathcal{V}}f, f\rangle\rangle_{\mathcal{V}}]_0$$

and, hence, there exists a constant C such that

$$[\langle\langle Q_+Tf, Q_+Tf\rangle\rangle_{\mathcal{W}}]_0 \leq [\langle\langle J_{\mathcal{V}}f, f\rangle\rangle_{\mathcal{V}}]_0 + [\langle\langle J_{\mathcal{W}}Q_-Tf, Q_-Tf\rangle\rangle_{\mathcal{W}}]_0 \leq C[\langle\langle J_{\mathcal{V}}f, f\rangle\rangle_{\mathcal{V}}]_0.$$

Consequently, the operator Q_+T is bounded on $D(T)$ and $T + Q_+T + Q_-T$ is bounded on $D(T)$. Therefore, T has a continuous extension to an operator $\tilde{T} \in L(\mathcal{V}, \mathcal{W})$ which is a contraction. □

Theorem 7. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Pontryagin modules with the same index κ and let $R \subset \mathcal{H}_1 \times \mathcal{H}_2$ be a densely defined contractive relation. Then R extends to the graph of a continuous contraction from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. The proof follows similar lines as in [6] and [4]. We start with the remark that the domain of the relation contains a maximum negative submodule since every dense linear submodule contains a negative submodule of maximum dimension κ. Let $\mathcal{H}_-$ be the submodule of the domain of R,

$$\mathrm{dom(R)} = \{h_1 \in \mathcal{H}_1 : (h_1, h_2) \in R \text{ for some } h_2 \in \mathcal{H}_2\}.$$

Now, consider $(h_1, w_2) \in R$ and $h_1 \in \mathcal{H}_-$. Since R is a contractive relation, that is,

$$[\langle\langle w_2, w_2\rangle\rangle_{\mathcal{H}_2}]_0 \leq [\langle\langle h_1, h_1\rangle\rangle_{\mathcal{H}_1}]_0,$$

whenever $(h_1, w_2) \in R$, we have

$$[\langle\!\langle w_2, w_2\rangle\!\rangle_{\mathcal{H}_2}]_0 \leq [\langle\!\langle h_1, h_1\rangle\!\rangle_{\mathcal{H}_1}]_0 \leq 0.$$

In case of $h_1 \neq 0$ we have even $[\langle\!\langle h_1, h_1\rangle\!\rangle_{\mathcal{H}_1}]_0 < 0$ which implies that $[\langle w_2, w_2\rangle_{\mathcal{H}_2}]_0 < 0$. Furthermore, let (h, w) and $(\tilde{h}, w)$ belong to R with $h, \tilde{h} \in \mathcal{H}_-$ and $w \in \mathcal{H}_2$. This means that $(h - \tilde{h}, 0) \in R$ and $[\langle\!\langle 0, 0\rangle\!\rangle_{\mathcal{H}_2}]_0 \leq [\langle\!\langle h - \tilde{h}, h - \tilde{h}\rangle\!\rangle_{\mathcal{H}_1}]_0$. Because $\mathcal{H}_-$ is a strictly negative submodule we get $h = \tilde{h}$ and, consequently, that R has a zero kernel and the image of $\mathcal{H}_-$ is a strictly negative submodule of $\mathcal{H}_2$ with dimension κ.

Now, let us take a basis $h_1, \ldots, h_\kappa$ in $\mathcal{H}_-$. Then, there are uniquely defined elements $w_1, \ldots, w_\kappa$ in $\mathcal{H}_2$ such that $(h_i, w_i) \in R$ for all i. Denote by $\mathcal{W}_-$ the linear span of $w_1, \ldots, w_\kappa$. We know that

$$\mathrm{dim}\mathcal{H}_- = \mathrm{dim}\mathcal{W}_- = \mathrm{ind}\mathcal{H}_1 = \mathrm{ind}\mathcal{H}_2$$

which mean that we have the fundamental decompositions $\mathcal{H}_1 = \mathcal{H}_+ + \mathcal{H}_-$ and $\mathcal{H}_2 = \mathcal{W}_+ + \mathcal{W}_-$. Now let us take $w = w_+ + w_-$ with $w_+ \in \mathcal{W}_+$ and $w_- \in \mathcal{W}_-$. Then $(v_-, w_-) \in R$ and $(0, w) = (v_-, w_-) + (-v_-, w_-) \in R$. Consequently, we have that $(-v_-, w_-) \in R$ and due to the contractivity of R we have also $[\langle\!\langle w_+, w_+\rangle\!\rangle_{\mathcal{H}_2}]_0 \leq [\langle\!\langle v_-, v_-\rangle\!\rangle_{\mathcal{H}_2}]_0 \leq 0$ from which follows that $w_+ = 0$. Therefore, $(0, w_-) \in R$ and since R is one-to-one on $\mathcal{H}_-$ we get $w_- = 0$.

Finally, let us consider the orthogonal projection from $\mathcal{H}_2$ onto $\mathcal{W}_-$ and let us denote by T the densely defined contraction which has R as its graph. Then there exist right linear functionals $c_1, \ldots, c_\kappa$ defined on the domain of R such that

$$Tv = \sum_{n=1}^{\kappa} w_n c_n(v) + w_+$$

with $w_+ \in \mathcal{W}_+$ satisfies $[\langle\!\langle w_n, w_+\rangle\!\rangle_{\mathcal{H}_2}]_0$ for $n = 1, 2, \ldots, \kappa$. Suppose that c_1 is not bounded on its domain and take v_+ such that $c_1(v_+) = 1$ and $v_n \in \mathcal{V}_+$ such that $c_1(v_n) = 1$, for all $n = 1, 2, \cdots$, and $\lim_{n\to\infty}[\langle\!\langle v_+ - v_n, v_+ - v_n\rangle\!\rangle_{\mathcal{H}_1}]_0 = 0$. This means that v_+ is in the closure of $\mathrm{ker}c_1$ and, consequently, $\mathrm{ker}c_1 = \mathcal{V}_+$ and $\mathrm{ker}c_1$ contains a strictly negative subspace of dimension κ denoted by $\mathcal{V}_{+-}$. But then, we have

$$Tv = \sum_{n=2}^{\kappa} w_n c_n(v)$$

for $v \in \mathcal{V}_{+-}$ since also $v \in \mathrm{ker}c_1$, i.e. the dimension is smaller than κ which is a contradiction. This leaves us only the question of continuity. Here, we can use

$$\begin{aligned}
[\langle\!\langle w_+, w_+\rangle\!\rangle_{\mathcal{H}_2}]_0 &= [\langle\!\langle Tv, Tv\rangle\!\rangle_{\mathcal{H}_2}]_0 - [\langle\!\langle w_-, w_-\rangle\!\rangle_{\mathcal{H}_2}]_0 \\
&\leq [\langle\!\langle v, v\rangle\!\rangle_{\mathcal{H}_1}]_0 - [\langle\!\langle w_-, w_-\rangle\!\rangle_{\mathcal{H}_2}]_0 \\
&= [\langle\!\langle v_+, v_+\rangle\!\rangle_{\mathcal{H}_1}]_0 + [\langle\!\langle v_-, v_-\rangle\!\rangle_{\mathcal{H}_1}]_0 - [\langle\!\langle w_-, w_-\rangle\!\rangle_{\mathcal{H}_2}]_0 \\
&\leq [\langle\!\langle v_+, v_+\rangle\!\rangle_{\mathcal{H}_1}]_0 - [\langle\!\langle v_-, v_-\rangle\!\rangle_{\mathcal{H}_1}]_0 - [\langle\!\langle w_-, w_-\rangle\!\rangle_{\mathcal{H}_2}]_0
\end{aligned}$$

where $w_- = \sum_{n=2}^{\kappa} w_n c_n(v)$. From this we obtain that the densely defined map $v \mapsto w_+$ is bounded between Hilbert modules due to the fact that $v \mapsto w_-$ is bounded and the inner product $[\langle\!\langle v_+, v_+\rangle\!\rangle_{\mathcal{H}_1}]_0 + [\langle\!\langle v_-, v_-\rangle\!\rangle_{\mathcal{H}_1}]_0$ induces the topology of $\mathcal{H}_1$. Therefore, the mapping $v \mapsto w_+$ has an everywhere defined continuous extension. □

5. Krein modules again

Let us now come back to the study of Krein modules. Hereby, we are adapting ideas from [17] and [24] to the case of Clifford-Krein modules. One of the first questions in this case is of course if there exists also a version of the theorem of Moore-Aronszajn. Since positivity of the inner product is not an issue one could believe that any Hermitean kernel will give rise to a reproducing kernel. That this is not true in the case of a Clifford-Klein module can be seen from the complex-valued case; it is easily seen that a necessary condition for a function of two variables to be the reproducing kernel of a reproducing kernel Krein space is that it can be expresses as a difference of two positive functions. The following particular example appears in Schwartz paper [26], but also in Aronszajn's paper [10], and in Bognar's book [13], and present an Hermitean function which cannot be expressed as difference of two positive functions. Let E be a Banach bi-module that does not allow a Hilbert structure, and let $\Omega = E' \times E$. Then the kernel

$$\begin{aligned} k : E' \times E &\mapsto E \times E' \\ ((e_1', e_1), (e_2', e_2)) &\mapsto e_1'(e_2) + e_2'(e_1) \end{aligned}$$

cannot be the difference of two positive kernels (or as we will state in the sequel it cannot admit a Kolmogorov decomposition) since if we could write $k = k_+ - k_-$ then $(E, E') = H_+ \oplus H_-$ in terms of sets and E could be endowed with a Hilbert structure.

This means that a Hermitean kernel function is in general not a reproducing kernel of a Clifford-Krein module. Furthermore, there may be more than one reproducing kernel Krein module with a given reproducing kernel; see [26, 1], One needs additional (different) sets of conditions and concepts, like the concept of Kolmogorov decomposition, to insure existence and unicity. To introduce Kolmogorov decomposition, we write a Hermitean kernel as

$$K = [K_{i,j}]_{i,j\in\mathbb{J}}, \quad K_{i,j} \in \mathcal{L}(\mathcal{V}_i, \mathcal{V}_j), \tag{24}$$

where $K_{i,j} = K_{j,i}^*$, for all $i, j \in \mathbb{J}$, and $\mathcal{V}_j$'s are Krein modules. This allows us to give the following definition.

Definition 5. A Hermitean kernel $K = [K_{i,j}]_{i,j\in\mathbb{J}}, K_{i,j} \in \mathcal{L}(\mathcal{V}_i, \mathcal{V}_j)$, admits a Kolmogorov decomposition if there exists a Krein module $\mathcal{K}$ and operators $V_j \in \mathcal{L}(\mathcal{V}_j, \mathcal{K}),\ j \in \mathbb{J}$, such that

i) $K_{i,j} = V_i^* V_j, \quad i, j \in \mathbb{J}$;
ii) $\mathcal{K} = \vee_j V_j \mathcal{V}_j$.

Two Kolmogorov decompositions $\mathcal{K}_i, i = 1, 2$, with operators $V_{i,j} \in \mathcal{L}(\mathcal{V}_j, \mathcal{K}_i),\ j \in \mathbb{J}$, are said equivalent if there is an isomorphism $W \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $V_{2,j} = W V_{1,j},\ j \in \mathbb{J}$. If any two Kolmogorov decompositions are equivalent then we say that the kernel K has an essentially unique Kolmogorov decomposition.

Let $\mathfrak{F}$ denote the linear space of all finitely nonzero indexed sets $f = (f_j)_{j\in\mathbb{J}}$ where f_j are vectors in $\mathcal{V}_j$. We define a K-inner product on $\mathfrak{F}$ as

$$\langle\!\langle f, g \rangle\!\rangle_K := \sum_{i,j\in\mathbb{J}} \langle\!\langle K_{i,j} f_j, g_i \rangle\!\rangle_{\mathcal{V}_i}, \quad f, g \in \mathfrak{F}. \tag{25}$$

We say that K is nonnegative and write $K \geq 0$ if the K-inner product (25) is nonnegative, i.e. the inner product $\langle f, g\rangle_K = [\langle\!\langle f, g \rangle\!\rangle_K]_0$ is nonnegative. We write that two Hermitean kernels verify $K_1 \leq K_2$ whenever $K_2 - K_1 \geq 0$.

We call a Hermitean kernel L associated to the same Krein spaces $\mathcal{V}_j$ with $L \geq 0$ and $-L \leq K \leq L$ a nonnegative majorant for a Hermitean kernel K. Hereby, we associate a Hilbert space $\mathcal{H}_L$ with L by standard construction. Then the quotient space $\mathfrak{F}/\mathfrak{H}_L$, where $\mathfrak{H}_L$ is the subspace of all elements orthogonal to $\mathfrak{F}$ in the L-inner product, is dense in $\mathcal{H}_L$. For $f \in \mathfrak{F}$ denote by $[f]$ the corresponding coset in $\mathfrak{F}/\mathfrak{H}_L$. An inner product in $\mathfrak{H}_L$ is given on the dense set by

$$\langle\!\langle [f], [g] \rangle\!\rangle_{\mathfrak{H}_L} := \langle\!\langle f, g \rangle\!\rangle_L, \quad f, g \in \mathfrak{F}.$$

Furthermore, there exists a unique operator $J \in \mathcal{L}(\mathfrak{H}_L)$ such that

$$\langle\!\langle J[f], [g] \rangle\!\rangle_{\mathfrak{H}_L} := \langle\!\langle f, g \rangle\!\rangle_K, \quad f, g \in \mathfrak{F}.$$

The operator J is selfadjoint and satisfies $\|J\| \leq 1$. It is called Gram operator of the kernel K for the majorant L.

Before we continue let us give an important lemma for Clifford-Krein modules which is based on the fact that every selfadjoint operator on a Krein module is congruent to a selfadjoint operator on a Hilbert module, i.e. if $C \in \mathcal{L}(\mathcal{K})$ there is a Hilbert module $\mathfrak{H}$, a selfadjoint operator $B \in \mathcal{L}(\mathfrak{H})$ and an invertible operator $X \in \mathcal{L}(\mathfrak{H}, \mathcal{K})$ such that $C = X^* B X$. This allows us to state the following theorem.

Theorem 8. *Let K be a Hermitean kernel then the following statements are equivalent*

i) K has a Kolmogorov decomposition;
ii) K has a nonnegative majorant;
iii) $K = K_+ - K_-$ for some Hermitean kernels $K_+ \geq 0, K_- \geq 0$.

In this case, iii) can be chosen such that the only Hermitean kernel M such that $0 \leq M \leq K_\pm$ is $M = 0$.

Proof. *i)* $\Rightarrow$ *ii)* Suppose K has a Kolmogorov decomposition and we construct a Hilbert module $\mathcal{M}$ and an invertible operator $X \in \mathcal{L}(\mathcal{K}, \mathcal{M})$ such that

$$|\langle f, f\rangle_{\mathcal{K}}| = |[\langle\langle f, f\rangle\rangle_{\mathcal{K}}]_0| \leq \langle Xf, Xf\rangle_{\mathcal{M}}, \qquad f \in \mathcal{K}.$$

Consider now

$$L = \{L_{ij}\}_{i,j\in\mathbb{J}}, \quad \text{s.t.} \quad L_{ij} = V_i^* X^* X V_j \in \mathcal{L}(\mathcal{V}_i, \mathcal{V}_j), \quad i, j \in \mathbb{J}, \tag{26}$$

with $\mathcal{V}_i$'s Krein modules - recall Definition 5. Obviously, L is a nonnegative majorant for K.

ii) $\Rightarrow$ *i)* Let K have a nonnegative majorant L. We denote by $J \in \mathcal{L}(\mathfrak{H}_L)$ the associated Gram operator. Using the fact that every self-adjoint operator on a Clifford-Hilbert module has a factorization $J = AA^*$ with $\ker A = \{0\}$ we have that there is a natural continuous embedding operator E_j from $\mathcal{V}_j$ to $\mathfrak{H}_L$ for all $j \in \mathbb{J}$. In particular, $E_j u = [f_u]$, where f_u is the element of $\mathfrak{F}$ which has u as its j-th component and has zero in all other components. Now, setting $V_j = A^* E_j, j \in \mathbb{J}$ leads to a Kolmogorov decomposition.

ii) $\Rightarrow$ *iii)* Let us assume that K has a nonnegative majorant L with corresponding Gram operator J. Since J is Hermitean it has real eigenvalues. Then, using the above defined embedding operators $E_j, j \in \mathbb{J}$, we have $K_{ij} = E_i^* J E_j, i, j \in \mathbb{J}$. We denote by $P_0, P_\pm$ the projection operators given by

$$P_0 J[f] = 0, \quad [\langle\langle P_+ J[f], [f]\rangle\rangle_{\mathfrak{H}_L}]_0 > [\langle\langle f, f\rangle\rangle_K]_0, \quad [\langle\langle P_- J[f], [f]\rangle\rangle_{\mathfrak{H}_L}]_0 < [\langle\langle f, f\rangle\rangle_K]_0,$$

for all $f \in \mathfrak{F}$. Then we can define kernels $K_\pm$ via

$$K_{\pm ij} = E_i^* (\pm P_\pm) J E_j, \qquad i, j \in \mathbb{J}, \tag{27}$$

such that $K_\pm \geq 0$ and $K = K_+ - K_-$.

We can now point out that the kernels $K_\pm$ are minimal in the sense of the theorem. For this we suppose there exist a Hermitean kernel M with $0 \leq M \leq K_\pm$. Due to $\|J\| \leq 1$ and $K_\pm \leq L$ we have $0 \leq M \leq L$. Now, the Gram operator $H \in \mathcal{L}(\mathfrak{H}_L)$ associated to M relative to L satisfies

$$0 \leq \langle H[f], [f]\rangle_{\mathfrak{H}_L} = [\langle\langle H[f], [f]\rangle\rangle_{\mathfrak{H}_L}]_0 \leq [\langle\langle \pm P_\pm J[f], [f]\rangle\rangle_{\mathfrak{H}_L}]_0 = \langle \pm P_\pm J[f], [f]\rangle_{\mathfrak{H}_L}, \qquad f \in \mathfrak{F}.$$

But since $P_+ J$ and $P_- J$ are supported on orthogonal submodules of $\mathfrak{H}_L$ we get $H = 0$ and $\langle\langle f, g\rangle\rangle_M = 0$ for all $f, g \in \mathfrak{F}$, and, therefore, $M = 0$.

iii) $\Rightarrow$ *ii)* Suppose that $K = K_+ - K_-$. Then $L = K_+ + K_-$ is a nonnegative majorant for K. □

The first part of the above proof provides us with a constructive way to obtain a majorant L (26) which we can explore further. To this end let us now consider K to be a Hermitean kernel where a nonnegative majorant L exists. We say that a Kolmogorov decomposition for K is L-continuous if the mapping which maps $[f]$ on $\mathfrak{F}/\mathfrak{H}_L$ into $\sum_{j\in\mathbb{J}} V_j f_j$ on $\mathcal{K}$ extends to a continuous operator from on $\mathfrak{H}_L$ into $\mathcal{K}$. Here, we can state the following theorem.

Theorem 9. *Suppose K is a Hermitean kernel.*

i) if K has a Kolmogorov decomposition, the decomposition is L-continuous with respect to the nonnegative majorant L given by (26);
ii) if K has a nonnegative majorant L , the Kolmogorov decomposition of K constructed via the kernels (27) is L-continuous.

This theorem means that L-continuous Kolmogorov decompositions always exist whereby uniqueness depends on the Gram operator.

Proof. To prove the first item we note that

$$\langle\langle [f],[g]\rangle\rangle_{\mathfrak{H}_L} = \langle\langle X\sum_{i\in\mathbb{J}} V_i f_i, X\sum_{j\in\mathbb{J}} V_j g_j\rangle\rangle_{\mathcal{M}}, \qquad f,g\in\mathfrak{F}.$$

Therefore, the mapping which maps $[f]$ into $X\sum_{i\in\mathbb{J}} V_i f_i$ is an isometry from $\mathfrak{H}_L$ into $\mathcal{M}$ and the mapping which maps $[f]$ into $\sum_{i\in\mathbb{J}} V_i f_i$ is the composition of the previous mapping and X^{-1}, thus continuous.

For the second item we have to show that the mapping which maps $[f]$ into $\sum_{i\in\mathbb{J}} V_i f_i$ extends to a continuous operator from $\mathfrak{H}_L$ into the Krein module $\mathcal{K}$. By the previous theorem, the associated Gram operator J admits a factorization $J = AA^*$ with $\ker A = \{0\}$. Because $\mathcal{K}$ is the closed span of the ranges of the operators V_i if we show that for any $f,g\in\mathfrak{F}$

$$\langle\langle [f], A\sum_{j\in\mathbb{J}} V_j g_j\rangle\rangle_{\mathfrak{H}_L} = \langle\langle \sum_{i\in\mathbb{J}} V_i f_i, \sum_{j\in\mathbb{J}} V_j g_j\rangle\rangle_{\mathcal{K}}$$

then the continuous extension is simply A^*. For the operators $V_i, i\in\mathbb{J}$, we have $V_i = A^*E_i$ and $\sum_{i\in\mathbb{J}} E_i g_i = [g]$. Therefore, the above equality can be written as

$$\langle\langle [f], AA^*[g]\rangle\rangle_{\mathfrak{H}_L} = \langle\langle \sum_{i\in\mathbb{J}} V_i f_i, \sum_{i\in\mathbb{J}} V_i g_i\rangle\rangle_{\mathcal{K}},$$

which holds true due to $AA^* = J$ and, consequently, both sides are equal to $\langle\langle f,g\rangle\rangle_K$. □

One of the first questions now is if in the present case of Clifford-Krein modules Kolmogorov decompositions are unique. To this end we have to introduce the concept of a unique factorization property.

Definition 6. A self-adjoint operator $C\in\mathcal{L}(\mathcal{K})$ on a Clifford-Krein module has the unique factorization property if for any two factorizations

$$C = A_1A_1^* = A_2A_2^*$$

with $A_j\in\mathcal{L}(\mathcal{K}_j,\mathcal{K}), \ker A_j = \{0\}, j = 1,2$ for some Clifford-Krein modules $\mathcal{K}_j$, there exists an isomorphism $U\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ such that $A_1 = A_2U$.

This definition allows us to state the following theorem.

Theorem 10. *Let K be a Hermitean kernel with nonnegative majorant L and Gram operator J. Any two L-continuous Kolmogorov decompositions are equivalent if and only if J has the unique factorization property.*

Proof. Assume that the Gram operator J has the unique factorization property. Consider two L-continuous Kolmogorov decompositions:

$$K_{ij} = V_{1,i}^*V_{1,j}, \quad V_{1,j}\in\mathcal{L}(\mathcal{V}_j,\mathcal{K}_1), \quad i,j\in\mathbb{J},$$
$$K_{ij} = V_{2,i}^*V_{2,j}, \quad V_{2,j}\in\mathcal{L}(\mathcal{V}_j,\mathcal{K}_2), \quad i,j\in\mathbb{J}.$$

Since we know that the mapping which maps $[f]$ into $\sum_{j\in\mathbb{J}} V_{1,j}f_j$ extends to a continuous operator A from $\mathfrak{H}_L$ into $\mathcal{K}_1$ with adjoint $A_1 = A^*\in\mathcal{L}(\mathcal{K}_1,\mathfrak{H}_L)$. For all $f,g\in\mathfrak{F}$ we have

$$\langle\langle [f], A_1\sum_{i\in\mathbb{J}} V_{1,i}g_i\rangle\rangle_{\mathfrak{H}_L} = \langle\langle \sum_{j\in\mathbb{J}} V_{1,j}f_j, \sum_{i\in\mathbb{J}} V_i g_i\rangle\rangle_{\mathcal{K}_1}$$

and, therefore,

$$\langle\langle J[f],[g]\rangle\rangle_{\mathfrak{H}_L} = \langle\langle f,g\rangle\rangle_{\mathcal{K}_1} = \langle\langle \sum_{j\in\mathbb{J}} V_{1,j}f_j, \sum_{i\in\mathbb{J}} V_i g_i\rangle\rangle_{\mathcal{K}_1} = \langle\langle [f], A_1A_1^*[g]\rangle\rangle_{\mathfrak{H}_L}.$$

The latter means that $J = A_1A_1^*$. In the same way we can get the factorization $J = A_2A_2^*$, and due to the unique factorization property we have $A_1 = A_2W$ for some unitary operator $W\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. This leads to

$$V_{2,j} = A_2^*E_j = WA_1^*E_j = WV_{1,j},$$

for all $j \in \mathbb{J}$ and the two Kolmogorov decompositions are equivalent. Let us assume now that any two L-continuous Kolmogorov decompositions of K are equivalent. Consider

$$J = A_1 A_1^* = A_2 A_2^*$$

with $A_1 \in \mathcal{L}(\mathcal{K}_1, \mathfrak{H}_L)$, $A_2 \in \mathcal{L}(\mathcal{K}_2, \mathfrak{H}_L)$, $\ker A_1 = \{0\}$, and $\ker A_2 = \{0\}$. Using Theorem 9 we can construct L-continuous Kolmogorov decompositions by putting $V_{1,j} = A_1^* E_j$ and $V_{2,j} = A_2^* E_j$ for all $j \in \mathbb{J}$. Now, using the condition that the decompositions are equivalent we have that there exist a unitary operator $W \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $V_{2,j} = WV_{1,j}$ for each $j \in \mathbb{J}$. Since $\mathcal{K}_1 = \vee_j V_{1,j}\mathcal{V}_j$ and $\mathcal{K}_2 = \vee_j V_{2,j}\mathcal{V}_j$ we get $WA_1^* = A_2^*$ and, consequently, $A_1 = A_2 W$. With other words, J possesses the unique factorization property. □

We can give a better result on uniqueness if we assume stronger conditions.,

Theorem 11. *An essentially unique Kolmogorov decomposition for a Hermitean kernel K exists if and only if for all nonnegative majorants the Gram operators have the unique factorization property.*

Proof. The necessity of the condition is immediate since otherwise the previous theorem ensures the existence of nonequivalent Kolmogorov decompositions. Let us assume now that each Gram operators possesses the unique factorization property. By Theorem 9 we know that any two Kolmogorov decompositions are continuous relative to the majorants L_1 and L_2, thus $L = L_1 + L_2$ is a majorant for the kernel K. Because of $L_1, L_2 \leq L$ we have that the corresponding mappings are densely defined contractions from $\mathfrak{H}_L$ into $\mathfrak{H}_{L_1}$ and $\mathfrak{H}_{L_2}$, respectively. But these spaces are Hilbert spaces and, therefore, the mappings which map $[f]$ into $\sum_{j\in\mathbb{J}} V_{1,j} f_j$ and $\sum_{j\in\mathbb{J}} V_{2,j} f_j$, respectively, are compositions of continuous operators, and, therefore, L-continuous. By the previous theorem the two Kolmogorov decompositions are equivalent. □

Theorem 12. *Let K be a Hermitean kernel, and assume that there exists a Kolmogorov decomposition such that the linear span of the submodules $\mathcal{V}_j, j \in \mathbb{J}$, contains one of the submodules $\mathcal{K}_\pm$ in some fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. Then K has an essentially unique Kolmogorov decomposition.*

Proof. Consider two Kolmogorov decompositions and define a linear relation R from $\mathcal{K}_1$ into $\mathcal{K}_2$ via

$$R = \{(\sum_{j\in\mathbb{J}} V_{1,j} f_j, \sum_{j\in\mathbb{J}} V_{2,j} f_j) : f \in \mathfrak{F}\}.$$

From the definition of a Kolmogorov decomposition we know that R has to have dense domain and dense range. For all $f \in \mathfrak{F}$ we have

$$\langle\!\langle \sum_{j\in\mathbb{J}} V_{1,j} f_j, \sum_{j\in J} V_{2,j} f_j \rangle\!\rangle_{\mathcal{K}_1} = \langle\!\langle f, f \rangle\!\rangle_K = \langle\!\langle \sum_{j\in\mathbb{J}} V_{2,j} f_j, \sum_{j\in\mathbb{J}} V_{2,j} f_j \rangle\!\rangle_{\mathcal{K}_2}.$$

Now, the domain of R contains one of the submodules $\mathcal{K}_{1,+}$ in some fundamental decomposition $\mathcal{K}_1 = \mathcal{K}_{1,+} \oplus \mathcal{K}_{1,-}$. This means that the closure of R is the graph a unitary operator $W \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. Therefore, we get $V_{2,j} = WV_{1,j}$ for all $j \in \mathbb{J}$ and with it the equivalence of the two Kolmogorov decomposition. □

We also get the following corollary.

Corollary 1. *If a Hermitean kernel K has a Kolmogorov decomposition such that $\mathcal{K}$ is either a Clifford-Pontryagin module or the antimodule of a Pontryagin module, then K has an essentially unique Kolmogorov decomposition.*

From the above we can see that as a sufficient condition we can reduce the question of uniqueness to the question of self-adjoint operator $C \in \mathcal{L}(\mathcal{K})$ having the unique factorization property. For this property we can state the following theorem.

Theorem 13. *Let $\mathcal{K}$ be a Krein module and let $C \in \mathcal{L}(\mathcal{K})$ be a self-adjoint operator then the following statements are equivalent:*

(1) C has the unique factorization property
(2) for some factorization $C = AA^*$, $\operatorname{ran}(A^*)$ contains one of the submodules $\mathcal{K}_\pm$ in some fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$.

The proof will be omitted since it is an easy adaptation of the proof of Theorem 2.8 in [15].

There are some simple sufficient conditions for the unique factorization property given by the next theorem.

Theorem 14. *Let $\mathcal{K}$ be a Krein module and let $C \in \mathcal{L}(\mathcal{K})$ be a self-adjoint operator then the following conditions are sufficient for C having the unique factorization property:*
(1) C is non-negative, i.e. $C \geq 0$,
(2) One of $\operatorname{ind}_\pm C$ is finite,
(3) $C^2 \leq C$.

Proof. Obviously, the first condition implies the second. For the second we can assume $\operatorname{ind}_- C < \infty$. Now, let $C = AA^*$ be any factorization. Then we have $\operatorname{ind}_-\mathcal{K}_1 = \operatorname{ind}_- C < \infty$ and due to $\ker A = \{0\}$, $\operatorname{ran}(A^*)$ is dense in A and condition 2 follows from Theorem 3. □

6. Examples of Krein modules with reproducing kernels in Clifford analysis

Let us start with the usual characterization of a reproducing kernel.

Theorem 15. *Let $\mathcal{K}$ be a Clifford-Krein module of functionals defined on a set Ω and taking values in a Clifford-Krein module $\mathcal{V}$. Then $\mathcal{K}$ has a reproducing kernel if and only if all evaluation mappings $E(x), x \in \Omega$ belong to $\mathcal{L}(\mathcal{K}, \mathcal{V})$. The reproducing kernel is uniquely determined by the module $\mathcal{K}$ and given by*

$$K(x, y) = E(x)E(y)^*, \qquad x, y \in \Omega.$$

Now, for a Clifford-Krein module to have a reproducing kernel is equivalent to the existence of a Kolmogorov decomposition with $V_j = E_j^*$. Yet this reproducing Clifford-Krein module is not unique, i.e. we can have two Clifford-Krein modules with the same reproducing kernel. To restore the uniqueness we need that the Krein module has an essentially unique Kolmogorov decomposition. This observation results in the following theorem.

Theorem 16. *If $K(x, y), x, y \in \Omega$ is a Hermitean kernel with values in $\mathcal{L}(\mathcal{V})$ for some Krein module $\mathcal{V}$, then the following statements are equivalent:*
(1) $K(x, y)$ is the reproducing kernel for some Krein module $\mathcal{K}$ of functions on Ω.
(2) $K(x, y)$ has a nonnegative majorant $L(x, y)$ on $\Omega \times \Omega$.
(3) $K(x, y) = K_+(x, y) - K_-(x, y)$ for some nonnegative kernels $K_\pm(x, y)$ on $\Omega \times \Omega$.

Furthermore, under the above conditions we have
(1) For a given nonnegative majorant $L(x, y)$ for $K(x, y)$ there is a Krein module $\mathcal{K}$ with reproducing kernel $K(x, y)$ which is contained continuously in the Hilbert module $\mathfrak{H}_L$ with reproducing kernel $L(x, y)$.
(2) There is a continuous self-adjoint operator J on $\mathfrak{H}_L$ such that $J : L(x, \cdot)f \mapsto K(x, \cdot)f, x \in \Omega, f \in \mathcal{V}$. The module $\mathcal{K}$ is unique if and only if J has the unique factorization property.

Proof. The equivalence of statements 1 to 3 is already proven in Theorem 8. For the proof of statement 1 we can construct a reproducing kernel Clifford-Krein module $\mathcal{K}$ with kernel K as in the proof of Theorem 8 based on the nonnegative majorant $L(x, y)$. The corresponding reproducing kernel Hilbert module $\mathfrak{H}_L$ appears naturally and the associated Gram operator J satisfies the mapping condition in (5). Now, Theorem 9 implies that there exist a continuous operator A^* mapping $\mathfrak{H}_L$ into $\mathcal{K}$ such that

$$A^* : L(x, \cdot)f \mapsto K(x, \cdot)f, \quad x \in \Omega, f \in \mathcal{V}.$$

Since the adjoint operator is just the inclusion mapping from $\mathcal{K}$ into $\mathfrak{H}_L$ we have that $J = AA^*$ and $\mathcal{K}$ is continuously contained in $\mathfrak{H}_L$.

Furthermore, suppose that J has the unique factorization property and denote by $\mathcal{K}_1$ and $\mathcal{K}_2$ two Clifford-Krein modules with the same reproducing kernel K which are contained continuously in $\mathfrak{H}_L$. Since the two Kolmogorov decomposition are equivalent by Theorem 10 we have that the identity mapping on the linear span of all functions $K(x,\cdot)f,\ x \in \Omega, f \in \mathcal{V}$ extends to a unitary operator from $\mathcal{K}_1$ onto $\mathcal{K}_2$. Due to the fact that the evaluation mappings are bounded for any reproducing kernel Clifford-Krein module we have that the modules $\mathcal{K}_1$ and $\mathcal{K}_2$ are identical. On the other hand, if we assume that we have two distinct Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ with reproducing kernel K contained continuously in $\mathfrak{H}_L$ then we have two nonequivalent L-continuous Kolmogorov decompositions. From Theorem 10 we have that the Gram operator J cannot possess the unique factorization property. $\square$

Let us now take a closer look at some examples. As stated already before an obvious example for a Pontryagin space is the space $\mathbb{R}^{p,q}$ with the sesquilinear form

$$< x, y >= \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i,$$

which gives rise to the Clifford module $\mathbb{R}_{p,q}$ with sesquilinear form

$$\langle\!\langle x, y\rangle\!\rangle = \sum_{A,B} \langle x_A, y_B\rangle \overline{e_A} e_B.$$

For this Clifford-Pontryagin module we have a fundamental decomposition in the form

$$x_+ = \sum_{\#\{i\in A:1\leq i\leq p\}\ even} e_A x_A, \quad \text{and } x_- = \sum_{\#\{i\in A:1\leq i\leq p\}\ odd} e_A x_A$$

with $x_A \in \mathbb{R}_{p,q}$. To this decomposition we associate the norms

$$\|x_+\|_+^2 := [\langle\!\langle x_+, x_+\rangle\!\rangle]_0, \quad \|x_-\|_-^2 := -[\langle\!\langle x_-, x_-\rangle\!\rangle]_0.$$

Now, the resulting function space $L_2(\mathbb{R}^n, \mathbb{R}_{p,q}) = L_2(\mathbb{R}^n) \otimes \mathbb{R}_{p,q}$ is a classic example of a Clifford-Krein module with an associated fundamental decomposition $L_2(\mathbb{R}^n, \mathbb{R}_{p,q}) = V_+ \oplus V_-$ with $f = f_+ + f_-$, $f_+ \in V_+$, $f_- \in V_-$ where $f = \sum_A e_A f_A$, and

$$f_+ = \sum_{\#\{i\in A:1\leq i\leq p\}\ even} e_A f_A \in (L_2(\mathbb{R}^n) \otimes \mathbb{R}_{p,q})_+,$$

and

$$f_- = \sum_{\#\{i\in A:1\leq i\leq p\}\ odd} e_A f_A \in (L_2(\mathbb{R}^n) \otimes \mathbb{R}_{p,q})_-.$$

For the above decomposition of the Clifford-Krein module we introduce the norms

$$\|f_+\|_{+,2}^2 := [\langle\!\langle f_+, f_+\rangle\!\rangle]_0, \quad f_+ \in (L_2(\mathbb{R}^n) \otimes \mathbb{R}_{p,q})_+, \tag{28}$$

and

$$\|f_-\|_{-,2}^2 := -[\langle\!\langle f_-, f_-\rangle\!\rangle]_0, \quad f_- \in (L_2(\mathbb{R}^n) \otimes \mathbb{R}_{p,q})_-. \tag{29}$$

Furthermore, the above canonical decomposition gives rise to the signature operator $J_V(f_+ + f_-) = f_+ - f_-$. Obviously, for this operator we have $J_V^2 = I$ and $J_V^* = J$. Hereby, J^* denotes the adjoint operator with respect to the indefinite inner product

$$\langle\!\langle f, g\rangle\!\rangle = \sum_{i=1}^{n} < f_A, g_B > \bar{e}_A e_B.$$

To provide a more specific example let us take a look at the Clifford algebra $\mathbb{R}_{1,1}$. Representing a general element in this algebra $x = x_0 + e_1x_1 + e_2x_2 + e_1e_2x_{12}$ as a vector in $\mathbb{R}^4$ we observe that it splits into

$$x = x_+ + x_- := (x_0 + e_2x_2) + (e_1x_1 + e_1e_2x_{12})$$

so that the operator J can be represented by the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $\mathrm{ind}_\pm J$ are both finite J has the unique factorization property by Theorem 14.

Let us consider the application of our study to two examples of reproducing kernel Krein modules in this context.

In the first example we consider the ultrahyperbolic Dirac operator. For simplicity we abbreviate $x = \sum_{i=1}^{p} e_i x_i + \sum_{i=p+1}^{p+q} e_i x_i := T + X$, where $T = \sum_{i=1}^{p} e_i x_i$ and $X = \sum_{i=p+1}^{p+q} e_i x_i$. The ultrahyperbolic Dirac operator is given by $\partial_{p,q} = \partial_T - \partial_X$. It factorizes the ultrahyperbolic operator $\partial_{p,q}^2 = \Delta_T - \Delta_X$. Elements of its kernel are called ultrahyperbolic monogenic functions. If we additionally assume ultrahyperbolic monogenic functions to be $\alpha-$homogeneous we arrive at the system

$$\begin{cases} \partial_{p,q} u = \partial_T u - \partial_X u = 0, \\ \mathbb{E}u = \alpha u, \end{cases}$$

where $\mathbb{E} = \sum_{i=1}^{p+q} x_i \partial_i$ is the Euler operator which measures the degree of homogeneity. The solutions of this system were studied in [19] and presented in the form of the following theorem. Here, we denote $T = |T|\epsilon, X = |X|\omega$.

Theorem 17. [19] *Given two inner spherical monogenics $V_\lambda(\epsilon)$ and $V_\kappa(\omega)$, i.e. restrictions of homogeneous null solutions for the Dirac operator on $\mathbb{R}^{p,0}$, respective $\mathbb{R}^{0,q}$, to S^{p-1}, respective S^{q-1} a null-solution for the ultrahyperbolic Dirac operator is given by*

$$u(T, X) = \partial_{p,q} |T|^{\alpha+1} f(|X|^2) \epsilon V_\lambda(\epsilon) V_\kappa(\omega)$$

where $f(t) = f(|X|^2)$ satisfies the hypergeometric differential equation

$$t(1-t)\frac{d^2 f}{dt^2} + \left[\kappa + \frac{q}{2} - (\kappa - \alpha - \frac{p}{2} + 1)t\right] \frac{df}{dt} - \frac{(\kappa + \lambda - \alpha)(\kappa - \lambda - \alpha - p)}{4} f = 0.$$

If we restrict now to $r_T S^{p-1} \times r_X S^{q-1} := |T| S^{p-1} \times |X| S^{q-1}$ these functions have the form

$$u_{\lambda,\kappa}(\epsilon, \omega) = -2r_T^2 \left[A(r_X) + \omega\epsilon B(r_X)\right] V_\lambda(\epsilon) V_\kappa(\omega),$$

where $A(r_X), B(r_X)$ are constants depending on r_X.

Let us now consider the reproducing kernels

$$K(x,y) = \sum_{|\lambda|+|\kappa|=0}^{\infty} \overline{a(\omega_y, \epsilon_y) V_\lambda(\epsilon_y) V_\kappa(\omega_y)} a(\omega_x, \epsilon_x) V_\lambda(\epsilon_x) V_\kappa(\omega_x) c_{\lambda,\kappa},$$

where $a(\omega_x, \epsilon_x) = A(r_X) + \omega\epsilon B(r_X)$ and $c_{\lambda,\kappa} \in \mathbb{R}$ for all λ, κ. When $c_{\lambda,\kappa} = 1$, for all κ, λ, we recover the corresponding "Hardy" space, i.e.

$$f(x) = \int_{S^{p-1}} \int_{S^{q-1}} K(x,y) f(y) dS(\omega_y) dS(\epsilon_y).$$

For a given sequence $c = (c_{\lambda,\kappa})$, where $c_{\lambda,\kappa} \geq 0$ for $\lambda \in \mathbb{N}_0^p, \kappa \in \mathbb{N}_0^q$, we define its support as

$$\mathrm{supp}(c) := \{(\lambda, \kappa) \in \mathbb{N}_0^{p+q} : c_{\lambda,\kappa} \neq 0\}.$$

We now consider the space with the reproducing kernel given by

$$\begin{aligned} K_c(x,y) &= \sum_{(\lambda,\kappa)\in \mathrm{supp}(c): |\lambda|+|\kappa|=0}^{\infty} \overline{a(\omega_y, \epsilon_y) V_\lambda(\epsilon_y) V_\kappa(\omega_y)} a(\omega_x, \epsilon_x) V_\lambda(\epsilon_x) V_\kappa(\omega_x) c_{\lambda,\kappa} \\ &:= \sum_{(\lambda,\kappa)\in \mathrm{supp}(c): |\lambda|+|\kappa|=0}^{\infty} \Psi_{\lambda,\kappa}((\omega_x, \epsilon_x), (\omega_y, \epsilon_y)) c_{\lambda,\kappa}, \end{aligned} \tag{30}$$

where $\Psi_{\lambda,\kappa}((\omega_x,\epsilon_x),(\omega_y,\epsilon_y)) := \overline{a(\omega_y,\epsilon_y)V_\lambda(\epsilon_y)V_\kappa(\omega_y)}a(\omega_x,\epsilon_x)V_\lambda(\epsilon_x)V_\kappa(\omega_x)$ with $a(\omega_x,\epsilon_x) = A(r_X)+\omega\epsilon B(r_X)$. The components $\Psi_{\lambda,\kappa}$ decompose into

$$\Psi_{\lambda,\kappa} = \Psi^+_{\lambda,\kappa} - \Psi^-_{\lambda,\kappa}, \tag{31}$$

with $\Psi^\pm_{\lambda,\kappa}$ taking values in $\mathcal{V}_\pm$. The kernel (30) defines the reproducing kernel right Krein module (RKrKM) $\mathcal{H}_c$ containing all functions

$$f(y) = \sum_{(\lambda,\kappa)\in\mathrm{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \Psi_{\lambda,\kappa}((\omega_x,\epsilon_x),(\omega_y,\epsilon_y))f_{\lambda,\kappa}, \quad f_{\lambda,\kappa}\in\mathbb{R}_{p,q},$$

for which it holds

$$\|f\|_c^2 := \sum_{(\lambda,\kappa)\in\mathrm{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \frac{\|f_{\lambda,\kappa}\|^2_{|\mathbb{R}_{p,q}|}}{c_{\lambda,\kappa}} < \infty. \tag{32}$$

The RKrKM $\mathcal{H}_c$ is associated to the domain

$$\Omega_c := \left\{ y\in\mathbb{R}^{p,q} : \sum_{(\lambda,\kappa)\in\mathrm{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \left(\|\Psi^+_{\lambda,\kappa}\|^2_{+,2} + \|\Psi^-_{\lambda,\kappa}\|^2_{-,2}\right) c_{\lambda,\kappa} < \infty \right\}, \tag{33}$$

where these norms are given by (28) and (29). Moreover, the following reproducing formula holds.

$$f(x) = \int_{(\omega_y,\epsilon_y)\in S^{p-1}\times S^{q-1}} K_c(x,y)f(y)dS(\omega_y)dS(\epsilon_y). \tag{34}$$

A majorant for the reproducing kernel K_c is given by the kernel

$$L(x,y) = \sum_{(\lambda,\kappa)\in\mathrm{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \Psi^+_{\lambda,\kappa}((\omega_x,\epsilon_x),(\omega_y,\epsilon_y))c_{\lambda,\kappa} + \Psi^-_{\lambda,\kappa}((\omega_x,\epsilon_x),(\omega_y,\epsilon_y))c_{\lambda,\kappa}$$

which allows us to apply Theorem 16. The Gram operator J in Theorem 16 acts on f as $Jf_+ = f_+$ and $Jf_- = -f_-$ and has a trivial factorization $J = AA^*$ with $A = J$ and $A^* = I$ whereby A^* denotes the J-adjoint operator. Since $\mathcal{V}_+ \subset \mathrm{ran}J^* = \mathrm{ran}I$ by Theorem 13 J has the unique factorization property.

Furthermore, we want to point out that also in this case the same reproducing kernel can lead to different RKrKM's. To this end we recall that the matrices

$$T = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

are J-unitary, i.e. $TJT^* = J$. Now, for functions $f(\epsilon,\omega) = \sum_{|\lambda|+|\kappa|=0}^{\infty} u_{\lambda,\kappa}(\epsilon,\omega)f_{\lambda,\kappa}$ we can introduce the norms

$$\begin{aligned} \|f\|_{\mathcal{K}_1} &= \sum_{n=0}^{\infty}\sum_{|\lambda|+|\kappa|=n} [\ \overline{f_{\lambda,\kappa}}f_{\lambda,\kappa}\ ]_0 \\ \|f\|_{\mathcal{K}_2} &= \sum_{n=0}^{\infty}\sum_{|\lambda|+|\kappa|=n} [\ \overline{f_{\lambda,\kappa}}T^{2n}f_{\lambda,\kappa}\ ]_0 \end{aligned}$$

together with the inner product $\langle\!\langle f,g\rangle\!\rangle_J = \sum_{n=0}^{\infty}\sum_{|\lambda|+|\kappa|=n}\overline{f_{\lambda,\kappa}}Jg_{\lambda,\kappa}$. Now, it is easy to show that the modules $\mathcal{K}_1$ and $\mathcal{K}_2$ are different, c.f. [1].

An interesting case is when the coefficients $c_{\lambda,\kappa} = 1$ which correspond to the (counterpart of the) Hardy space. The coefficients $c_{\lambda,\kappa} = \frac{\lambda!}{|\lambda|!}\frac{\kappa!}{|\kappa|!}$ correspond to the Arveson space while $c_{\lambda,\kappa} = \lambda!\kappa!$ to the Fock space.

The study of the corresponding tensor products and associated Krein spaces $\mathcal{H}_c$ will be presented in a future publication.

Let us now consider the classic problem of interpolation for null solutions of the ultrahyperbolic Dirac operator:

Given the nodes and values $(x_j, y_j), j \in \mathbb{J}$ we want construct a function f, such that

$$f(x_j) = y_j, \qquad \text{for all } j \in \mathbb{J}. \tag{35}$$

Suppose that $f \in \mathcal{H}_c$ then we can write the interpolation condition as

$$y_j = f(x_j) = \sum_{(\lambda,\kappa)\in\mathrm{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \Psi_{\lambda,\kappa}(\omega_{x_j}, \epsilon_{x_j}) f_{\lambda,\kappa}, \quad f_{\lambda,\kappa} \in \mathbb{R}_{p,q}.$$

Now, restricting (λ, κ) to a finite set with cardinality equal to $\mathbb{J}$ we arrive at the following matrix problem

$$\underline{y} = (\Psi_{\lambda_l,\kappa_l}(\omega_{x_j}, \epsilon_{x_j}))_{j,l\in\mathbb{J}} \underline{f_{\lambda,\kappa}}.$$

Unfortunately, the matrix $(\Psi_{\lambda_l,\kappa_l}((\omega_{x_j}, \epsilon_{x_j}))_{j,l}$ is not positive definite, so that we cannot directly study the solution of this problem, but by using Theorem 16 via the fundamental decomposition we can split the above problem into two problems:

$$\begin{aligned} \underline{y^+} &= (\Psi^+_{\lambda_l,\kappa_l}(\omega_{x_j}, \epsilon_{x_j}))_{j,l} \underline{f^+_{\lambda,\kappa}}, \\ \underline{y^-} &= (\Psi^-_{\lambda_l,\kappa_l}(\omega_{x_j}, \epsilon_{x_j}))_{j,l} \underline{f^-_{\lambda,\kappa}}. \end{aligned} \tag{36}$$

While the first matrix $\Psi^+_{\lambda_l,\kappa_l}$ corresponds to the kernel matrix of a reproducing kernel over a Hilbert module and, therefore, the matrix is positive definite the negative of the second matrix corresponds to the kernel matrix of a reproducing kernel over an anti-Hilbert module and, consequently, $\Psi^-_{\lambda_l,\kappa_l}$ is also positive definite. We can get the solvability of these matrix equations in the same way as the solvability of equation (14) and arrive at the following theorem:

Theorem 18. *Let f be in $\mathcal{H}_c$ then the interpolation problem (35) has a unique solution whereby the coefficients $f_{\lambda,\kappa}$ satisfy $f_{\lambda,\kappa} = f^+_{\lambda,\kappa} - f^-_{\lambda,\kappa}$ and $f^+_{\lambda,\kappa}, f^-_{\lambda,\kappa}$ are solutions of the matrix systems (36).*

A second example is motivated by the spherical Radon transform which arises in the determination of the orientation density function f (ODF) of a polycrystalline specimen from given pole density data [14].

Definition 7. [Spherical Radon transform] [14]) Let f be a $L^1(S^3)$ function. We define the spherical Radon transform of f as the mean over all rotations q mapping the direction $\epsilon \in S^2$ into $\omega \in S^2$ and we write

$$\begin{aligned} (Rf)(\epsilon,\omega) &:= \frac{1}{2\pi} \int_{\{q\in S^3:\omega=\overline{q}\epsilon q\}} f(q)dq \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(q(\epsilon,\omega,t))dt, \end{aligned} \tag{37}$$

where $q(\epsilon,\omega,t) = \left(\cos\frac{\eta}{2} + \frac{\epsilon\times\omega}{\|\epsilon\times\omega\|}\sin\frac{\eta}{2}\right)\cos t + \frac{\epsilon+\omega}{\|\epsilon+\omega\|}\sin t$, with $\eta = \arccos(\langle\epsilon,\omega\rangle)$, denotes the great circle in S^3 of all unit quaternions q which rotates $\epsilon \in S^2$ into $\omega \in S^2$.

For the spherical Radon transform it is well-known that $R : L^2(S^3) \mapsto L^2(S^2 \times S^2)$ and $(\Delta_\epsilon - \Delta_\omega)(Rf)(\epsilon,\omega) = 0$. Denote $u = Rf$ we can write it in the form

$$u(\epsilon,\omega) = \sum_{l=0}^{\infty} \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} Y_l^{m_1}(\epsilon) Y_l^{m_2}(\omega) \lambda_l^{m_1 m_2}$$

where $\lambda_l^{m_1 m_2} = O(l^{-k})$ for any $k \in \mathbb{N}$.

Instead of using the standard inner product of L^2 for the study of these functions it is more appropriate to use the "energetic" inner product

$$\langle u, v\rangle_R = \langle \partial_{p,q} u, \partial_{p,q} v\rangle$$

using our ultrahyperbolic Dirac operator $\partial_{p,q}$. This inner product is indefinite and gives rise to a Krein space $L^2(S^2 \times S^2)$ with $\langle \cdot, \cdot \rangle_R$.

From the above considerations we can consider reproducing kernel modules $\mathcal{H}_c$ with kernels

$$K_c(x, y) = \sum_{l=0}^{\infty} \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} \overline{Y_l^{m_1}(\epsilon_y) Y_l^{m_2}(\omega_y)} Y_l^{m_1}(\epsilon_x) Y_l^{m_2}(\omega_x) c_l^{m_1,m_2}$$

with $c_l^{m_1,m_2} \in \mathbb{R}$.

An interesting case is when the coefficients $c_l^{m_1,m_2}$ factorize as

$$c_l^{m_1,m_2} = d_l^{m_1} d_l^{m_2}.$$

The corresponding kernel K_c factorizes as

$$K_c = (K_d)^2$$

where

$$K_d(x, y) = \sum_{l=0}^{\infty} \sum_{m_1=-l}^{l} \overline{Y_l^{m_1}(\epsilon_y)} Y_l^{m_1}(\epsilon_x) d_l^{m_1}.$$

The corresponding reproducing kernel Hilbert space is the restriction to the diagonal of the elements of the tensor product $H(K_d) \otimes H(K_d)$. See [9](S. 8, p. 357) for the classical case. The case where $d_\lambda = 1$ corresponds to the (counterpart of the) Hardy space, $d_\lambda = \frac{\lambda!}{|\lambda|!}$ to the Arveson space, $d_\lambda = \lambda!$ to the Fock space.

We can again consider the interpolation problem for functions u, i.e. the following question:

Find $u \in L^2(S^2 \times S^2)$, such that $u(\epsilon_j, \omega_j) = a_j$ where a_j is the given data.

As a first step we are looking to express u in terms of

$$u(\omega) = \sum_{\lambda \in \Lambda} K(\epsilon_\lambda, \omega) u_\lambda, \qquad u_\lambda \in \mathbb{R}.$$

From our fundamental decomposition we can write

$$K(\epsilon_\lambda, \omega) = K_+(\epsilon_\lambda, \omega) - K_-(\epsilon_\lambda, \omega),$$

where K_+ and $-K_-$ are positive kernels, and associate to each of these kernels a problem of the type (36). This leads to the following theorem.

Theorem 19. *Let f be in $\mathcal{H}_c$ then the above interpolation problem has a unique solution whereby the coefficients u_λ satisfy $u_\lambda = u_\lambda^+ - u_\lambda^-$ and $u_\lambda^+, u_{\lambda,}^-$ are solutions of the matrix systems arising from the positive kernels K_+ and K_-.*

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