

The Balian-Low theorem for a new kind of Gabor system

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The Balian–Low theorem expresses the fact that time–frequency concentration is incompatible with non-redundancy for Gabor systems. In this article, the Balian–Low theorem is established for a new kind of Gabor systems $\{e^{im\theta(2\pi t)}g(t-n)\}_{m,n\in\mathbb{Z}}$ associated with a phase function $\theta(t)$ satisfying certain assumptions. Meanwhile, some properties of the corresponding generalized Zak transform are shown and explicit examples of the phase function $\theta(t)$ are provided.

Keywords: Balian–Low theorem; generalized Zak transform; Gabor system; frame; phase function

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1. Introduction

The Balian–Low theorem, which is a key result in time–frequency analysis, has been of interest to many researchers for some years [1–10]. It expresses the fact that time–frequency concentration and non-redundancy are incompatible properties for Gabor systems $\{e^{2\pi i m t}g(t-n)\}_{m,n \in \mathbb{Z}}$. Specifically, if $g \in L^2(\mathbb{R})$ has the property that the functions $g_{m,n}(t) = e^{2\pi i m t}g(t-n)$ constitute a frame for $L^2(\mathbb{R})$, i.e.,

$$A\|f\|^{2} \leq \sum_{m,n \in \mathbb{Z}} |\langle f, g_{m,n} \rangle|^{2} \leq B\|f\|^{2},$$

then either

$$\int_{-\infty}^{+\infty} t^2 |g(t)|^2 \mathrm{d}t = \infty \quad \text{or} \quad \int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 \mathrm{d}\xi = \infty,$$

where the Fourier transform \hat{g} of g is formally defined by

$$\hat{g}(\omega) = \mathcal{F}g(\omega) = \int_{\mathbb{R}} g(t)e^{-2\pi it\omega} \mathrm{d}t.$$

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It is well-known that the Zak transform is an important tool for studying the frame given by Gabor systems [8,11–13]. The Zak transform was independently introduced by J. Zak in 1967 and defined by

$$(Zf)(t,\omega) = \sum_{k \in \mathbb{Z}} e^{2\pi i k\omega} f(t-k), \quad (t,\omega) \in [0,1)^2.$$

This defines a unitary operator from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$. In abstract harmonic analysis the Zak transform is called the Weil–Brezin map.

Note that the harmonic waves $e^{2\pi i n\omega}$, $n \in \mathbb{Z}$ in the Zak transform have constant frequencies, which can be seen as the derivative of the linear phase $\phi(\omega) = 2\pi n\omega$, such a purely monochromatic signal cannot expose the time-varying property of nonstationary signals [14–16]. Recently, a kind of specific nonlinear phase function $\theta_a(2\pi\omega)$ are proposed [17–21]. For different *a*, the shapes of $\cos \theta_a(2\pi\omega)$ (also those of $\sin \theta_a(2\pi\omega)$) are different. It is observed that the closer |a| gets to 1, the sharper the graph of $\cos \theta_a(2\pi\omega)$ is. The nontrivial harmonic waves $e^{i\theta_a(2\pi\omega)}$, which represent a conformal re-scaling of classic Fourier atoms, have positive time-varying frequencies and are expected to be better suitable and adaptable, along with different choices of *a*, to nonlinear and non-stationary time–frequency analysis. Moreover, in [22], associated with a kind of phase function $\theta(t)$ and $\varphi(t)$ satisfying certain assumptions, the authors study the Chirp transform with the kernel $e^{i\theta(t)\varphi(\omega)}$, get some new phenomena on the Shannon sampling theorem by dealing with sampling points which may non-equally distributed and solve certain differential equations with variable coefficients.

Motivated by these points, this article studies a new kind of Gabor systems generated by g

$$\{e^{im\theta(2\pi t)}g(t-n)\}_{m,n\in\mathbb{Z}}$$

by replacing the harmonic waves $e^{2\pi i m t}$ in the Zak transform by $e^{i m \theta(2\pi t)}$ where $\theta(t)$ satisfies certain assumptions. The proposed Gabor system $\{e^{im\theta(2\pi t)}g(t-n)\}_{m,n\in\mathbb{Z}}$ can be related to already existing cases. Trivially, if we assume $\theta(2\pi t) = 2\pi\lambda t$, for a fixed parameter $\lambda > 0$, then the proposed Gabor system reduces to the classical cases [8,23]. Moreover, in the case of $\theta(2\pi t) = \theta_a(2\pi t)$, i.e., using the nonlinear Fourier atoms in [17–21], we have that the frequency modulation $e^{im\theta_a(2\pi t)}$ represents a conformal dilation of the classical modulation $e^{i2\pi mt}$ on the unit circle. If we take the proposed Gabor systems with different parameters a, we can obtain a dictionary of Gabor frames with different dilation parameters in the modulation part. A simple change of variables can establish a clear relation between this system and the system generated by the affine Weyl-Heisenberg group with dilation on the window function [24,25]. The nonlinear Fourier atoms $e^{i m \theta_a(2\pi t)}$ discussed in [17–21] correspond to conformal re-scalings of the classical Fourier atoms $e^{2\pi i m t}$ and therefore are better adapted to capture non-stationary features of band-limited signals. Those atoms are a particular case of the ones in the proposed Gabor system in so far as we are not restricted to conformal phase functions $\theta_a(t)$. This freedom allows us to choose phase functions adequate to the necessary non-uniform sampling of the signal [22]. Applications of non-uniform sampling range from communication theory (missing data problem) and astronomical measurements to medical imaging such as computerized tomography and magnetic resonance imaging, as they require the use of Gabor systems with nonlinear phase functions [26].

The rest of this article is organized as follows: Section 2 is devoted to giving some assumptions on the phase function $\theta(t)$ and providing some explicit phase functions satisfying the given assumptions. In Section 3, we depict some properties of the generalized Zak transform. In Section 4, we prove the Balian–Low theorem for the Gabor systems $\{e^{im\theta(2\pi t)}g(t-n)\}_{m,n \in \mathbb{Z}}$. Some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we introduce some assumptions on the phase function θ necessary for our study and give some explicit examples satisfying the assumptions. Firstly, we fix the notations to be used later on. For any arbitrary measure μ in \mathbb{R} , consider the function spaces $L^p(\mathbb{R}, d\mu)$, with 0 , of*p* $-integrable functions in <math>\mathbb{R}$ with respect to the measure μ and with finite norm

$$\|f\|_{p,\mu} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(t)|^p \mathrm{d}\mu(2\pi t)\right)^{\frac{1}{p}}.$$

In addition, for p=2, denote its norm as $||f||_{2,\mu} = ||f||_{\mu}$, equipped with the inner product

$$\langle f,g\rangle_{2,\mu} := \langle f,g\rangle_{\mu} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\overline{g(x)} \mathrm{d}\mu(2\pi x).$$

We also denote by $L^2([0, 1)^2, d\mu)$ the Hilbert space with inner product

$$\langle f,g\rangle_{2,\mu} := \langle f,g\rangle_{\mu} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 f(t,\omega) \overline{g(t,\omega)} \mathrm{d}\mu(2\pi t) \,\mathrm{d}\mu(2\pi \omega).$$

Secondly, we introduce our main assumptions as follows.

Assumption 2.1 Consider the set of measures $\mu : \mathbb{R} \to \mathbb{R}$ of class C^2 satisfying $\mu' > 0$.

Due to the reason that we need to consider the Hilbert space $L^2([0, 1)^2, d\theta)$ for the phase function $\theta(2\pi t)$ defined in the interval [0, 1) and being strictly increasing, we extend $\theta(2\pi t)$ from [0, 1) to the whole line by keeping the property of a strictly increasing function. For this reason, the following restriction on the phase function θ is very natural.

Assumption 2.2 Assume that $\theta(t)$ is a function satisfying Assumption 2.1 and, furthermore,

$$\theta(t+2k\pi) = \theta(t) + 2k\pi, \qquad (2.1)$$

for any $t \in \mathbb{R}, k \in \mathbb{Z}$.

In what follows, under Assumption 2.2, we can get some properties of the phase function θ . On the one hand, θ satisfies Assumption 2.2 if and only if it is uniquely determined by its restriction $\theta|_{[0,2\pi]}$. The restriction map

$$\theta|_{[0,2\pi]}: [0,2\pi] \to [c_0,c_0+2\pi]$$

is a bijection, where $c_0 = \theta(0)$. Thus $\theta([0, 2\pi])$ is a closed interval of length 2π . On the other hand, θ' is a 2π -period function and $\theta' \simeq 1$. Indeed by the periodicity

$$0 < \min_{x \in [0,2\pi]} \theta'(x) \le \theta'(t) \le \max_{x \in [0,2\pi]} \theta'(x) \quad \forall t \in \mathbb{R},$$
(2.2)

as desired.

Assume that θ satisfies Assumption 2.2, by Qian's theorem [21], $d\theta(t)$ is a sum of a number of *n* harmonic measures on the unit disc if and only if

$$H(e^{i\varphi(t)}) = -ie^{i\varphi(t)} + \frac{i}{2\pi} \int_0^{2\pi} \varphi(t) \mathrm{d}t,$$

where H is the circular Hilbert transform on $L^2([0, 2\pi])$, satisfying

$$H(e^{ikt}) = -i\operatorname{sgn}(k)e^{ikt} \quad \forall k \in \mathbb{Z}.$$

At last, we shall provide explicit phase functions satisfying Assumption 2.2. Nonlinear Fourier atoms can be understood as boundary values of Blaschke products [17–19]. In the simplest case they are defined by

$$e^{i\theta_a(t)} := \tau_a(e^{it})$$

with τ_a being the Möbius transformation

$$\tau_a(z) = \frac{z-a}{1-\overline{a}z}, \quad |a| < 1.$$

For any complex number $a = |a|e^{it_a}$ with |a| < 1, the nonlinear phase function $\theta_a(t)$ is given by

$$\theta_a(t) := t + 2 \arctan \frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)} \quad \forall t \in \mathbb{R},$$
(2.3)

which has the unique decomposition: the sum of a linear part and a periodic part. One can easily check that the nonlinear phase function $\theta_a(t)$ satisfy Assumption 2.2 based on the following facts. By direct calculation, one can find

$$\theta_a'(t) = \frac{1 - |a|^2}{1 - 2|a|\cos(t - t_a) + |a|^2} = \frac{1 - |a|^2}{|1 - \overline{a}e^{it}|^2} = 2\pi p_a(t) > 0,$$

where $p_a(t)$ is the Poisson kernel for the point *a*, and

$$\theta_a(t+2\pi) = \theta_a(t) + 2\pi, \quad p_a(t+2\pi) = p_a(t).$$

Since $1 - |a| \le |1 - \overline{a}e^{it}| \le |1 + |a|$, one can obtain the bounds

$$\frac{1-|a|}{1+|a|} \le p_a(t) \le \frac{1+|a|}{1-|a|},$$

or rather, $p_a(t) \simeq 1$.

The above concerns the case of a nonlinear Fourier atom based on a single Möbius transform. This can be extended to the case of finite Blaschke product. For some fixed $N \in \mathbb{N}$, the nonlinear Fourier atom generated by the finite Blaschke product

$$B_{\vec{a}}(z) = \prod_{k=1}^{N} \tau_{a_k}(z), \quad |a_k| < 1$$

is given by

$$e^{i\sum_{k=1}^{N}\theta_{a_k}(t)}=B_{\vec{a}}(e^{it}).$$

Consider the nonlinear phase function

$$\theta_{\vec{a}}(t) = \frac{1}{N} \sum_{k=1}^{N} \theta_{a_k}(t).$$
(2.4)

For $\theta'_{\vec{a}}(t)$ we have

$$\theta'_{\vec{a}}(t) = \frac{1}{N} \sum_{k=1}^{N} \theta'_{a_k}(t) = 2\pi \frac{1}{N} \sum_{k=1}^{N} p_{a_k}(t) = 2\pi p_{\vec{a}}(t),$$

which is always positive. We can check that the phase function $\theta_{\bar{a}}(t)$ defined by (2.4) also satisfies Assumption 2.2.

3. Properties of the generalized Zak transform

We want to establish the Balian-Low theorem for a new kind of Gabor systems

$$g_{m,n}(t) := e^{im\theta(2\pi t)}g(t-n), \quad t \in \mathbb{R}, \quad m, n \in \mathbb{Z},$$
(3.1)

for the spaces $L^2(\mathbb{R}, d\theta)$, where the phase function $\theta(t)$ satisfies Assumption 2.2.

To this end, we consider the generalized Zak transform Z_{θ} defined by

$$(Z_{\theta}f)(t,\omega) := \sum_{k \in \mathbb{Z}} e^{ik\theta(2\pi\omega)} f(t-k), \quad (t,\omega) \in [0,1)^2$$

It can be shown that the series above converges in the norm of $L^2([0, 1)^2, d\theta)$.

Some properties of the generalized Zak transform Z_{θ} are discussed here. In what follows, we will focus on considering that the generalized Zak transform Z_{θ} is a unitary map from $L^2(\mathbb{R}, d\theta)$ to $L^2([0, 1)^2, d\theta)$. One way of seeing this is the following. Let us consider the function $e_{m,n} = e^{im\theta(2\pi x)}e(x-n)$, with e(x) = 1 for 0 < x < 1, e(x) = 0 otherwise. Simple calculation offers that the system $\{e^{in\theta(2\pi x)}\}_{n=-\infty}^{+\infty}$ constitutes an orthonormal basis for $L^2([0, 1), d\theta)$. Thus $e_{m,n}$ constitutes an orthonormal basis for $L^2(\mathbb{R}, d\theta)$. Direct calculation tells us that

$$(Z_{\theta}e_{m,n})(t,\omega) = e^{i\,m\theta(2\pi t)}e^{-i\,n\theta(2\pi\omega)}(Z_{\theta}e)(t,\omega),$$

and $(Z_{\theta}e)(t, \omega) = 1$ almost everywhere on $[0, 1)^2$. It follows that Z_{θ} maps an orthonormal basis of $L^2(\mathbb{R}, d\theta)$ to an orthonormal basis of $L^2([0, 1)^2, d\theta)$, so that Z_{θ} is unitary.

By using the identity (2.1), it is easy to see the properties of time and frequency shifts of the generalized Zak transform.

THEOREM 3.1 The generalized Zak transform Z_{θ} satisfies the following two equations:

$$(Z_{\theta}f)(t,\omega+n) = (Z_{\theta}f)(t,\omega)$$

and

$$(Z_{\theta}f)(t+n,\omega) = e^{i n\theta(2\pi\omega)}(Z_{\theta}f)(t,\omega).$$

From Theorem 3.1, we can extend $Z_{\theta}f$ outside $[0, 1)^2$. Another way of seeing this fact is shown in the following theorem.

THEOREM 3.2 The operator Z_{θ} is unitary from $L^2(\mathbb{R}, d\theta)$ to $L^2([0, 1)^2, d\theta)$, i.e.,

$$\langle Z_{\theta}f, Z_{\theta}g \rangle_{\theta} = \langle f, g \rangle_{\theta}, \quad f, g \in L^{2}(\mathbb{R}).$$
 (3.2)

Proof We only need to show the result for $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ since the general case can be treated by limit procedure. By the definition of the nonlinear Zak transform and interchanging the order of the integral and summation, we have that

$$\begin{split} \langle Z_{\theta}f, Z_{\theta}g \rangle_{\theta} \\ &= \frac{1}{4\pi^2} \int_0^1 \int_0^1 Z_{\theta}f(t,\omega) \overline{Z_{\theta}g(t,\omega)} d\theta(2\pi t) d\theta(2\pi \omega) \\ &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \int_0^1 f(t-n) \sum_{m \in \mathbb{Z}} \overline{g(t-m)} \left(\int_0^1 e^{i(n-m)\theta(2\pi\omega)} d\theta(2\pi\omega) \right) d\theta(2\pi t). \end{split}$$

By a simple calculation, we know that

$$\int_0^1 e^{i(n-m)\theta(2\pi\omega)} \mathrm{d}\theta(2\pi\omega) = 2\pi\delta_{n,m},$$

where $\delta_{n,m}$ denotes Kronecker delta and here we have used the fact $\theta(2\pi) = \theta(0) + 2\pi$. By noting this and the 2π -periodicity of $\theta'(t)$, we have that

$$\begin{split} \langle Z_{\theta}f, Z_{\theta}g \rangle_{\theta} &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{0}^{1} f(t-n) \overline{g(t-n)} \mathrm{d}\theta(2\pi t) \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{-n}^{-n+1} f(t) \overline{g(t)} \mathrm{d}\theta(2\pi t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \overline{g(t)} \mathrm{d}\theta(2\pi t). \end{split}$$

The following theorem states that we can reconstruct the original signal from its generalized Zak transform. We remark that the reconstruction formula of signal f can also be given in discrete form by the generalized Zak transform similar to the classical case. Although that is not in the scope of this article, we refer to [27,28] for more details.

THEOREM 3.3 If $f \in L^2(\mathbb{R})$, then the following relations hold true:

$$f(t) = \frac{1}{2\pi} \int_0^1 Z_{\theta} f(t, \omega) d\theta(2\pi\omega), \quad t \in \mathbb{R},$$
(3.3)

$$\hat{f}(\omega) = \int_0^1 Z_{\theta} f\left(t, \frac{1}{2\pi} \theta^{-1}(2\pi\omega)\right) e^{-2\pi i t \omega} \mathrm{d}t, \quad \omega \in \mathbb{R},$$
(3.4)

where \hat{f} is the Fourier transform of f and $\theta^{-1}(\omega)$ is the inverse of the phase function θ . *Proof* We first show (3.3). The definition of the generalized Zak transform implies that

$$\frac{1}{2\pi} \int_0^1 Z_{\theta} f(t,\omega) \mathrm{d}\theta(2\pi\omega)$$

= $\frac{1}{2\pi} \int_0^1 f(t) \mathrm{d}\theta(2\pi\omega) + \frac{1}{2\pi} \int_0^1 \sum_{k \neq 0} f(t-k) e^{ik\theta(2\pi\omega)} \mathrm{d}\theta(2\pi\omega).$

Note that the first integral of the right side of above equation is just f(t). To calculate the second integral of the right side of above equation, we interchange the order of the integral and summation and get that

$$\int_0^1 \sum_{k \neq 0} f(t-k) e^{ik\theta(2\pi\omega)} d\theta(2\pi\omega)$$

= $\sum_{k \neq 0} f(t-k) \int_0^1 e^{ik\theta(2\pi\omega)} d\theta(2\pi\omega)$
= $\sum_{k \neq 0} f(t-k) \int_{\theta(0)}^{\theta(2\pi)} e^{ik\xi} d\xi = 0.$

Here we have used the fact $\theta(2\pi) = \theta(0) + 2\pi$ again.

Now we turn to show (3.4). The definition of the generalized Zak transform leads to

$$\int_0^1 Z_{\theta} f\left(t, \frac{1}{2\pi} \theta^{-1}(\omega)\right) e^{-2\pi i t \omega} \mathrm{d}t$$

=
$$\int_0^1 \sum_{k \in \mathbb{Z}} f(t-k) e^{-2\pi i \omega (t-k)} \mathrm{d}t$$

=
$$\sum_{k \in \mathbb{Z}} \int_{-k}^{-k+1} f(x) e^{-2\pi i \omega x} \mathrm{d}x = \hat{f}(\omega).$$

Let us therefore define the space \mathcal{Z} by

$$\mathcal{Z} := \left\{ \phi : \mathbb{R}^2 \to \mathbb{C}; \, \phi(t+n,\omega) = e^{in\theta(2\pi\omega)}\phi(t,\omega), \, \phi(t,\omega+n) \\ = \phi(t,\omega), \, \|\phi\|_{\theta}^2 = \frac{1}{4\pi^2} \int_0^1 \int_0^1 |\phi(t,\omega)|^2 \mathrm{d}\theta(2\pi t) \mathrm{d}\theta(2\pi\omega) < \infty \right\}, \quad (3.5)$$

then the generalized Zak transform Z_{θ} is unitary between $L^2(\mathbb{R}, d\theta)$ and \mathcal{Z} . By (3.3), we know the inverse map is easy as well: for any $\phi \in \mathcal{Z}$,

$$(Z_{\theta}^{-1}\phi)(t) = \frac{1}{2\pi} \int_0^1 \phi(t,\omega) \mathrm{d}\theta(2\pi\omega).$$

4. Generalized Balian-Low theorem

In this section, based on the properties of the generalized Zak transform, we mainly discuss the time-frequency localization properties of Gabor systems $g_{m,n}(t)$ which are the content of the generalized Balian-Low theorem. At first, we prove some useful lemmas as follows.

LEMMA 4.1 If $g_{m,n}(t)$ is defined as (3.1), then

$$(Z_{\theta}g_{m,n})(t,\omega) = e^{im\theta(2\pi t)}e^{-in\theta(2\pi\omega)}(Z_{\theta}g)(t,\omega).$$
(4.1)

Proof From the definition of the generalized Zak transform and $g_{m,n}$, we get that

$$(Z_{\theta}g_{m,n})(t,\omega) = \sum_{k\in\mathbb{Z}} e^{ik\theta(2\pi\omega)}g_{m,n}(t-k)$$

= $\sum_{k\in\mathbb{Z}} e^{ik\theta(2\pi\omega)}g(t-k-n)e^{im\theta(2\pi(t-k))}$
= $\sum_{k\in\mathbb{Z}} e^{i(k-n)\theta(2\pi\omega)}g(t-k)e^{im\theta(2\pi t)}$
= $e^{im\theta(2\pi t)}e^{-in\theta(2\pi\omega)}\sum_{k\in\mathbb{Z}} e^{ik\theta(2\pi\omega)}g(t-k).$

From the classical Parseval identity, we can conclude the generalized Parseval identity

$$\int_0^1 \int_0^1 |f(t,\omega)|^2 d\theta(2\pi t) d\theta(2\pi \omega)$$

= $\frac{1}{4\pi^2} \sum_{m,n \in \mathbb{Z}} \left| \int_0^1 \int_0^1 f(t,\omega) e^{im\theta(2\pi t)} e^{in\theta(2\pi \omega)} d\theta(2\pi t) d\theta(2\pi \omega) \right|^2$.

Based on Lemma 4.1 and the generalized Parseval identity, we get the following result.

LEMMA 4.2 For any function $f \in L^2(\mathbb{R})$, the following relation holds true:

$$\sum_{m,n\in\mathbb{Z}} |\langle f,g_{m,n}\rangle_{\theta}|^{2}$$

= $\frac{1}{4\pi^{2}} \int_{0}^{1} \int_{0}^{1} |Z_{\theta}f(t,\omega)|^{2} |Z_{\theta}g(t,\omega)|^{2} \mathrm{d}\theta(2\pi t) \mathrm{d}\theta(2\pi\omega).$ (4.2)

Proof As already shown in Theorem 3.2, the operator Z_{θ} is unitary. This implies

$$\begin{split} &\sum_{m,n\in\mathbb{Z}} |\langle f,g_{m,n}\rangle_{\theta}|^{2} = \sum_{m,n\in\mathbb{Z}} |\langle Z_{\theta}f,Z_{\theta}g_{m,n}\rangle_{\theta}|^{2} \\ &= \sum_{m,n\in\mathbb{Z}} \left| \frac{1}{4\pi^{2}} \int_{0}^{1} \int_{0}^{1} Z_{\theta}f(t,\omega) \overline{Z_{\theta}g(t,\omega)} e^{-im\theta(2\pi t)} e^{in\theta(2\pi\omega)} \mathrm{d}\theta(2\pi t) \mathrm{d}\theta(2\pi\omega) \right|^{2} \\ &= \frac{1}{4\pi^{2}} \int_{0}^{1} \int_{0}^{1} |Z_{\theta}f(t,\omega)|^{2} |Z_{\theta}g(t,\omega)|^{2} \mathrm{d}\theta(2\pi t) \mathrm{d}\theta(2\pi\omega). \end{split}$$

From this Lemma, we get that $Z_{\theta}(F^*F)Z_{\theta}^{-1}$ corresponds to a multiplication by $|Z_{\theta}g(t,\omega)|^2$ on the space \mathcal{Z} defined in (3.5), where *F* is the coefficient operator from $L^2(\mathbb{R}, d\theta)$ to

$$l^{2}(\mathbb{Z}) := \{ c = (c_{m,n})_{m,n \in \mathbb{Z}}; \ \|c\|^{2} = \sum_{m,n \in \mathbb{Z}} |c_{m,n}|^{2} < \infty \}$$

defined by

$$(Ff)_{m,n} = \langle f, g_{m,n} \rangle_{\theta},$$

and the frame operator F^*F is defined by

$$F^*Ff = \sum_{m,n \in \mathbb{Z}} \langle f, g_{m,n} \rangle_{\theta} g_{m,n}, \qquad (4.3)$$

where F^* is the adjoint operator of F,

$$F^*c=\sum_{m,n\in\mathbb{Z}}c_{m,n}g_{m,n}.$$

Based on general frame theory, if the functions $g_{m,n}(t) = g(t-n)e^{im\theta(2\pi t)}$ constitute a frame for $L^2(\mathbb{R}, d\theta)$, then the frame operator F^*F defined by (4.3) is a bounded, positive and invertible mapping from $L^2(\mathbb{R}, d\theta)$ onto itself. The associated dual frame $\tilde{g}_{m,n}$ given by $(F^*F)^{-1}g_{m,n}$ yields an exact frame expansion of f of the form

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, \widetilde{g}_{m,n} \rangle_{\theta} g_{m,n} = \sum_{m,n \in \mathbb{Z}} \langle f, g_{m,n} \rangle_{\theta} \widetilde{g}_{m,n},$$

which provides an explicit reconstruction of the signal from the Gabor frame $g_{m,n}$ [13].

To get the following lemma, let us recall the rule of thumb, the smooth-and decay principle: if the function f is smooth, then the Fourier transform $\mathcal{F}f$ decays quickly and vice versa. Here is the link between the function f and its Fourier transform $\mathcal{F}f$. Denote the multiplication and differentiation operators as

$$Qf(x) := xf(x), \quad Pf(x) := -if'(x).$$
 (4.4)

The Fourier transform \mathcal{F} turns differentiation operators into multiplication operators, i.e.,

$$\mathcal{F} \circ P = 2\pi Q \circ \mathcal{F}.$$

LEMMA 4.3 Suppose f_k , Pf_k , $Qf_k \in L^2(\mathbb{R}, d\theta)$, k = 1, 2, where the operators Q, P are defined as in (4.4). Then

$$\langle Pf_1, Qf_2 \rangle_{\theta} = \langle Qf_1, Pf_2 \rangle_{\theta} + i \langle f_1, f_2 \rangle_{\theta} + 2\pi i \int_{-\infty}^{+\infty} x f_1(x) \overline{f_2(x)} \theta''(2\pi x) \mathrm{d}x.$$

Proof As we all know, if $\varphi(x)$, $\psi(x)$ satisfy

$$|\varphi(x)| \le C(1+x^2)^{-1}, \quad |\psi(x)| \le C(1+x^2)^{-1},$$

then we have

$$\begin{split} \langle Q\varphi, P\psi \rangle_{\theta} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} x\varphi(x) i\overline{\psi'(x)} d\theta(2\pi x) \\ &= -\int_{-\infty}^{+\infty} i[x\varphi(x)\theta'(2\pi x)]' \overline{\psi(x)} dx \\ &= -i\langle \varphi, \psi \rangle_{\theta} + \langle P\varphi, Q\psi \rangle_{\theta} - 2\pi i \int_{-\infty}^{+\infty} x\varphi(x) \overline{\psi(x)} \theta''(2\pi x) dx. \end{split}$$

On the other hand, since f_k , Pf_k , $Qf_k \in L^2(\mathbb{R}, d\theta)$, by using (2.4), we get that f_k , Pf_k , $Qf_k \in L^2(\mathbb{R})$. Consequently, there exist $f_{k,n}$ satisfying

$$|f_{k,n}(x)| \le C_n(1+x^2)^{-1},$$

such that

$$||f_{k,n} - f_k||_2 \to 0, \quad ||Pf_{k,n} - Pf_k||_2 \to 0, \quad ||Qf_{k,n} - Qf_k||_2 \to 0, \quad n \to \infty.$$

In fact, for instance, we can take $f_{k,n}$ as

$$f_{k,n} = \sum_{l=0}^{n} \langle f_k, H_l \rangle H_l,$$

where H_l are the Hermite functions [7]. Due to (2.2), we get that

$$\|f_{k,n} - f_k\|_{\theta} \to 0, \quad \|Pf_{k,n} - Pf_k\|_{\theta} \to 0, \quad \|Qf_{k,n} - Qf_k\|_{\theta} \to 0, \quad n \to \infty.$$

Then

$$\begin{aligned} \langle Pf_1, Qf_2 \rangle_\theta \\ &= \lim_{n \to \infty} \langle Pf_{1,n}, Qf_{2,n} \rangle_\theta \\ &= \lim_{n \to \infty} [\langle Qf_{1,n}, Pf_{2,n} \rangle_\theta + i \langle f_{1,n}, f_{2,n} \rangle_\theta + 2\pi i \int_{-\infty}^{+\infty} x f_{1,n}(x) \overline{f_{2,n}(x)} \theta''(2\pi x) dx] \\ &= \langle Qf_1, Pf_2 \rangle_\theta + i \langle f_1, f_2 \rangle_\theta + 2\pi i \int_{-\infty}^{+\infty} x f_1(x) \overline{f_2(x)} \theta''(2\pi x) dx. \end{aligned}$$

THEOREM 4.4 Let $g \in L^2(\mathbb{R}, d\theta)$ have the property that the functions $g_{m,n}(t) = g(t-n)e^{im\theta(2\pi t)}$ constitute a frame for $L^2(\mathbb{R}, d\theta)$, i.e.,

$$A\|f\|_{\theta}^{2} \leq \sum_{m,n \in \mathbb{Z}} |\langle f, g_{m,n} \rangle_{\theta}|^{2} \leq B\|f\|_{\theta}^{2}.$$

Then either

$$\int_{-\infty}^{+\infty} t^2 |g(t)|^2 \mathrm{d}\theta(2\pi t) = \infty \quad \text{or} \quad \int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 \mathrm{d}\theta(2\pi\xi) = \infty.$$

Proof The proof presented here is analogous with the proof of the classical assertion for the classical Balian–Low theorem. At last we regard the classical result as a special case on the construction of the contradiction in our proof. Since we have proved Lemma 4.2 and since the generalized Zak transform Z_{θ} is unitary, this implies

$$0 < A \le |Z_{\theta}g(s,t)|^2 \le B < \infty.$$

$$(4.5)$$

Now let us consider the dual frame vector $\tilde{g}_{m,n}$ given by

$$\tilde{g}_{m,n} = (F^*F)^{-1}g_{m,n}.$$
 (4.6)

Since $Z_{\theta}(F^*F)Z_{\theta}^{-1}$ corresponds to a multiplication by $|Z_{\theta}g|^2$ on \mathcal{Z} , it follows that

$$Z_{\theta}\tilde{g}_{m,n} = |Z_{\theta}g|^{-2} Z_{\theta}g_{m,n}$$

or

$$Z_{\theta}\tilde{g}_{m,n}(t,\omega) = |Z_{\theta}g(t,\omega)|^{-2}e^{im\theta(2\pi t)}e^{-in\theta(2\pi\omega)}(Z_{\theta}g)(t,\omega)$$
$$= e^{im\theta(2\pi t)}e^{-in\theta(2\pi\omega)}[\overline{(Z_{\theta}g)(t,\omega)}]^{-1},$$
(4.7)

which is in the space Z by (4.5). In particular, (4.7) implies that

$$\tilde{g}_{m,n}(x) = e^{i m \theta (2\pi x)} \tilde{g}(x-n)$$

with $Z_{\theta}\tilde{g} = [\overline{Z_{\theta}g}]^{-1}$.

Suppose now that

$$\int_{-\infty}^{+\infty} t^2 |g(t)|^2 \mathrm{d}\theta(2\pi t) < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 \mathrm{d}\theta(2\pi\xi) < \infty,$$

we can get that $Qg, Pg \in L^2(\mathbb{R}, d\theta)$, where the multiplication and differentiation operators Q, P are defined as in (4.4). This will lead to contradiction, which will prove the theorem. One checks that

$$[Z_{\theta}(Qg)](t,\omega) = t(Z_{\theta}g)(t,\omega) - \frac{1}{2\pi i \theta'(2\pi\omega)} \partial_{\omega}(Z_{\theta}g)(t,\omega),$$

which means that $Qg \in L^2(\mathbb{R}, d\theta)$ if and only if $\partial_{\omega}(Z_{\theta}g) \in L^2([0, 1)^2, d\theta)$. Similarly, $Pg \in L^2(\mathbb{R}, d\theta)$ if and only if $\partial_t(Z_{\theta}g) \in L^2([0, 1)^2, d\theta)$. Consequently,

$$\partial_t Z_\theta \tilde{g} = [\overline{Z_\theta g}]^{-2} \overline{\partial_t Z_\theta g}$$
 and $\partial_\omega Z_\theta \tilde{g} = [\overline{Z_\theta g}]^{-2} \overline{\partial_\omega Z_\theta g}$

are in $L^2([0, 1)^2, d\theta)$; hence $Q\tilde{g}, P\tilde{g} \in L^2(\mathbb{R}, d\theta)$. For the functions g and \tilde{g} , we shall next prove the fact

$$\langle Qg, P\tilde{g} \rangle_{\theta} = \langle Pg, Q\tilde{g} \rangle_{\theta}$$

where we derive the contradiction. In fact, we firstly have

$$\begin{split} \langle \tilde{g}, g_{m,n} \rangle_{\theta} \\ &= \langle Z_{\theta} \tilde{g}, Z_{\theta} g_{m,n} \rangle_{\theta} \\ &= \frac{1}{4\pi^2} \int_0^1 \int_0^1 Z_{\theta} \tilde{g}(t, \omega) \overline{Z_{\theta} g_{m,n}(t, \omega)} d\theta(2\pi t) d\theta(2\pi \omega) \\ &= \frac{1}{4\pi^2} \int_0^1 \int_0^1 Z_{\theta} \tilde{g}(t, \omega) \overline{Z_{\theta} g(t, \omega)} e^{-im\theta(2\pi t)} e^{in\theta(2\pi \omega)} d\theta(2\pi t) d\theta(2\pi \omega) \\ &= \delta_{m,0} \delta_{n,0}, \end{split}$$

where $\delta_{n,m}$ denotes Kronecker delta. Similarly, we can get that

$$\langle g, \tilde{g}_{m,n} \rangle_{\theta} = \delta_{m,0} \delta_{n,0}. \tag{4.8}$$

Secondly, since $Qg, P\tilde{g} \in L^2(\mathbb{R}, d\theta)$ and since the $g_{m,n}, \tilde{g}_{m,n}$ constitute dual frames, we have

$$\langle Qg, P\tilde{g} \rangle_{\theta} = \sum_{m,n} \langle Qg, \tilde{g}_{m,n} \rangle_{\theta} \langle g_{m,n}, P\tilde{g} \rangle_{\theta}.$$

Due to (2.4) and (4.8), we have

$$\begin{split} \langle Qg, \tilde{g}_{m,n} \rangle_{\theta} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} xg(x) e^{-im\theta(2\pi x)} \overline{\tilde{g}(x-n)} \mathrm{d}\theta(2\pi x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x) e^{-im\theta(2\pi x)} (x-n) \overline{\tilde{g}(x-n)} \mathrm{d}\theta(2\pi x) \\ &= \langle g_{-m,-n}, Q\tilde{g} \rangle_{\theta}. \end{split}$$

Similarly, $\langle g_{m,n}, P\tilde{g} \rangle_{\theta} = \langle Pg, \tilde{g}_{-m,-n} \rangle_{\theta}$. Consequently,

$$\langle Qg, P\tilde{g} \rangle_{\theta} = \sum_{m,n} \langle Pg, \tilde{g}_{-m,-n} \rangle_{\theta} \langle g_{-m,-n}, Q\tilde{g} \rangle_{\theta} = \langle Pg, Q\tilde{g} \rangle_{\theta}.$$

Together with the result in Lemma 4.3 this implies

$$\langle g, \tilde{g} \rangle_{\theta} = -2\pi \int_{-\infty}^{+\infty} xg(x)\overline{\tilde{g}(x)}\theta''(2\pi x)\mathrm{d}x.$$
 (4.9)

However, from (4.8) we have

$$\langle g, \tilde{g} \rangle_{\theta} = 1. \tag{4.10}$$

Now, we can get that there exists a contradiction between (4.9) and (4.10), that is, we need to interpret that

$$-2\pi \int_{-\infty}^{+\infty} xg(x)\overline{\tilde{g}(x)}\theta''(2\pi x)\mathrm{d}x \equiv 1$$

is not true for arbitrary phase function $\theta(x)$ satisfying Assumption 2.2. We remark that for the left-hand equivalence above we have that it is zero or negative while on the right-hand side 1 is positive. In particular, let $\theta(x) = 1 + x$, which evidently satisfies Assumption 2.2, thus we get that

$$-2\pi \int_{-\infty}^{+\infty} xg(x)\overline{\tilde{g}(x)}\theta''(2\pi x)\mathrm{d}x = 0,$$

which results in a contradiction between (4.9) and (4.10). The proof is complete.

5. Conclusions

Associated with some properties of the well-defined generalized Zak transform, this article deals with the Balian–Low theorem for a new kind of Gabor systems $\{e^{im\theta(2\pi t)}g(t-n)\}_{m,n\in\mathbb{Z}}$, where the phase function $\theta(t)$ satisfies Assumption 2.2. Applying Theorem 4.4 to the explicit example for the phase function $\theta(t)$, the specific nonlinear phase function $\theta_a(t)$,

$$\theta_a(t) = t + 2 \arctan \frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)},$$

we can get that if $g \in L^2(\mathbb{R}, d\theta_a)$ has the property that the functions $g_{m,n}(t) = e^{im\theta_a(2\pi t)}g(t-n)$ constitute a frame for $L^2(\mathbb{R}, d\theta_a)$, then either

$$\int_{-\infty}^{+\infty} t^2 |g(t)|^2 \frac{1 - |a|^2}{|1 - \overline{a} e^{2\pi i t}|^2} dt = \infty \quad \text{or} \quad \int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 \frac{1 - |a|^2}{|1 - \overline{a} e^{2\pi i \xi}|^2} d\xi = \infty.$$

Especially, if we take a = 0, then the nonlinear phase function $\theta_0(t) = t$, and the result above reduces to the classical case.

More generally, applying Theorem 4.4 to the specific nonlinear phase function

$$\theta_{\vec{a}}(t) = \frac{1}{N} \sum_{k=1}^{N} \theta_{a_k}(t),$$

we can get that if $g \in L^2(\mathbb{R}, d\theta_{\bar{a}})$ has the property that the functions $g_{m,n}(t) = e^{im\theta_{\bar{a}}(2\pi t)}g(t-n)$ constitute a frame for $L^2(\mathbb{R}, d\theta_{\bar{a}})$, then either

$$\int_{-\infty}^{+\infty} t^2 |g(t)|^2 \sum_{k=1}^{N} \frac{1 - |a_k|^2}{|1 - \overline{a}_k e^{2\pi i t}|^2} dt = \infty$$

or

$$\int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 \sum_{k=1}^{N} \frac{1 - |a_k|^2}{|1 - \overline{a}_k e^{2\pi i \xi}|^2} d\xi = \infty.$$

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