This article was downloaded by: [b-on: Biblioteca do conhecimento online UA] On: 06 November 2013, At: 09:34 Publisher: Taylor & Francis Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



# Complex Variables and Elliptic Equations: An International Journal

Publication details, including instructions for authors and subscription information:

http://www.tandfonline.com/loi/gcov20

# Lax pairs for 3D spatial problems based on the Dirac operator

S. Bernstein<sup>a</sup>, P. Cerejeiras<sup>b</sup> & U. Kähler<sup>b</sup>

<sup>a</sup> TU Bergakademie Freiberg, Institute of Applied Analysis, Prüferstr. 9, 09596 Freiberg, Germany

<sup>b</sup> Center for Research and Development in Mathematics and Applications, University of Aveiro, Department of Mathematics, Aveiro, Portugal Published online: 10 Jul 2013.

To cite this article: S. Bernstein , P. Cerejeiras & U. Kähler , Complex Variables and Elliptic Equations (2013): Lax pairs for 3D spatial problems based on the Dirac operator, Complex Variables and Elliptic Equations: An International Journal, DOI: 10.1080/17476933.2013.804516

To link to this article: <u>http://dx.doi.org/10.1080/17476933.2013.804516</u>

## PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms &

Conditions of access and use can be found at <u>http://www.tandfonline.com/page/terms-and-conditions</u>



### Lax pairs for 3D spatial problems based on the Dirac operator

S. Bernstein<sup>a\*</sup>, P. Cerejeiras<sup>b</sup> and U. Kähler<sup>b</sup>

<sup>a</sup>TU Bergakademie Freiberg, Institute of Applied Analysis, Prüferstr. 9, 09596 Freiberg, Germany; <sup>b</sup>Department of Mathematics, Center for Research and Development in Mathematics and Applications, University of Aveiro, Aveiro, Portugal

Communicated by I. Sabadini

(Received 4 Febraury 2013; final version received 6 May 2013)

In this paper, we show that Lax pairs can be constructed using the Dirac operator in the context of Clifford analysis. Since Lax pairs are closely linked to spectral decompositions they are not easily obtainable in the context of Dirac operators due to the non-commutativity of the underlying algebraic structure. The main idea is to substitute the classic Lax approach by the so-called AKNS method. We demonstrate that it is possible to obtain Lax pairs for both linear and non-linear PDE's in this way.

**Keywords:** quaternions; biquaternions; Dirac operator; Lax pairs; S-integrable; inverse scattering transform

AMS Subject Classifications: 30G35; 35Q41; 37K15

#### 1. Introduction

Lax pairs are a well-established tool for the study of instationary non-linear PDEs. Given a pair of linear operators acting on a certain Hilbert space  $\mathcal{H}$  of complex-valued functions, we say that they form a Lax pair for an instationary non-linear PDE if that PDE arises as a compatibility condition between the two given operators. A key point in the theory is the fact that one of the operators (say,  $\mathcal{L}$ ) is time-dependent with known spectra while the second operator (say,  $\mathcal{A}$ ) controls the time evolution of the eigenvalues of the first operator. In [1], Fokas showed that one can also find Lax pairs for linear PDEs making it an unifying theory for both linear and non-linear PDEs. However, and although there exist some techniques for the construction of such a pair if one has a previous knowledge of the operator  $\mathcal{L}$ (see [2]) up to date there is no systematic way for obtaining a Lax pair associated to a given PDE.

Another method to derive non-linear PDEs is the AKNS method from Ablowitz, Kaup, Newel and Segur (see [3]). Again, a compatibility equation between two operators (the so-called AKNS pair) is interpreted as a (in general, non-linear) integrable PDE. A crucial difference between these two methods is the fact that the eigenvalues no longer play a role in the later one.

<sup>\*</sup>Corresponding author. Email: swanhild.bernstein@math.tu-freiberg.de

The process of generating Lax pairs or, equivalently, AKNS pairs, is of crucial interest here. On one hand, there exist large solvable classes of non-linear PDE (also denoted S-integrable equations) to which these generating methods can be applied; on the other hand, several theories have been developed during the last decades for solving equations arising as compatibility between such pairs. Two standard examples are the inverse spectral transformation (IST) (c.f. [4]) and the dressing method (by Zakharov and Shabat, c.f. [5,6]). This last method consists of generating new solutions of a given equation departing from a known solution of a related equation. It has its roots in the theory of Riemann–Hilbert problems, involving the  $\overline{\partial}$  operator and complex function theory (hence, being also known as  $\overline{\partial}$ –method).

The higher dimensional equivalent to the  $\overline{\partial}$ -formalism in complex function theory is the Dirac operator in Clifford or quaternionic analysis. Here, we restrict ourselves to the case of quaternions and biquaternions, a restriction which allows us to consider 3D spatial problems. For the choice of the Dirac operator as a replacement of the  $\overline{\partial}$ -operator, we justify our choice with the following observations. First of all, the Dirac operator appears in the description of the massless electron. It has a much wider appearance as the first-order differential operator which factorizes the Laplacian and it is covariant under the action of elements in the Spin-group, i.e. rotations. Furthermore, it also appears in connection with the conductivity equation and the Schrödinger equation with potentials of conductivity type. Therefore, it seems natural that any higher dimensional generalization of the abovementioned methods would involve a higher dimensional function theory based on the Dirac operator. In fact, there exist already works in that direction, for example [7].

This raises an important and interesting question: can Lax pairs be obtained from equations involving the Dirac operator? The answer is not obvious. From the non-commutativity of the setting, one expects the operator  $\mathcal{L}$  to have left- and right-spectra which cannot be interchanged freely. This problem affects the classic Lax pair method sufficiently as to become impracticable in the case of non-linear equations. Therefore, we circumvent this problem by applying the more general AKNS method.

The structure of the paper is as follows. We first show that one can obtain Lax pairs for linear PDEs involving the Dirac operator. Hereby, we also highlight the difficulties which arise from the non-commutativity of the underlying algebraic structure. Afterwards, we adapt the AKNS method in order to obtain non-linear PDEs and we establish the restrictions on the operators due to the involvement of the Dirac operator. Finally, we conclude with some applications of this method to the construction of Lax pairs for higher dimensional non-linear PDEs of KdV-type.

#### 2. Toolboxes

#### 2.1. Lax pairs and AKNS method

We start with giving a more detailed description of both methods mentioned above. As stated before, two operators form a Lax pair for an instationary non-linear PDE if that PDE arises as a compatibility condition between them. More precisely, consider a time-dependent linear operator  $\mathcal{L}$  acting on a complex-valued Hilbert space with known spectra, say  $\mathcal{L}\psi = \lambda\psi$ , and an operator  $\mathcal{A}$  (also time-dependent and linear) which controls the time-evolution of the eigenfunctions of the previous operator. This corresponds to have the following system of linear PDEs

$$\mathcal{L}\psi - \lambda\psi = 0 \& \psi_t - \mathcal{A}\psi = 0, \tag{1}$$

or, in other word, each eigenfunction  $\psi$  of  $\mathcal{L}$  has to satisfy the evolution equation  $\psi_t = \mathcal{A}\psi$ . Then, if

- (i) the spectral parameter  $\lambda$  is time-independent,
- (ii)  $(\mathcal{L}\psi)_t = \mathcal{L}_t \psi + \mathcal{L}\psi_t$ , for all  $\psi \in \mathcal{H}$ ,

we have that the pair  $(\mathcal{L}, \mathcal{A})$  of linear operators satisfy the identity

$$\begin{aligned} (\mathcal{L}\psi)_t &= \lambda\psi_t \ \Leftrightarrow \ \mathcal{L}_t\psi + \mathcal{L}\psi_t = \lambda\psi_t \\ &\Leftrightarrow \ \mathcal{L}_t\psi + \mathcal{L}\mathcal{A}\psi = \mathcal{A}\mathcal{L}\psi \end{aligned}$$

i.e. the Lax pair  $(\mathcal{L}, \mathcal{A})$  is linked to the (in general) non-linear PDE arising from the compatibility condition

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{A}] = 0 \tag{2}$$

between the operators, where  $[\cdot, \cdot]$  denotes the commutator. This method is called in the literature the *Lax method* (see e.g. [3]). To give an example, we consider the operators

$$\mathcal{L} = i \begin{pmatrix} 1+k & 0\\ 0 & 1-k \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \overline{u}\\ u & 0 \end{pmatrix},$$
(3)

$$\mathcal{A} = ik \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx}^2 + \begin{pmatrix} \frac{-i|u|^2}{1+k} & \overline{u}_x \\ -u_x & \frac{-i|u|^2}{1-k} \end{pmatrix}, \tag{4}$$

where the complex-valued function u = u(x, t) acts as a parameter. Then, we have that these operators satisfy identity (2) whenever u = u(x, t) is a solution of the non-linear Schrödinger equation

$$iu_t + u_{xx}^2 + \kappa u^2 \overline{u} = 0,$$

where  $\kappa = \frac{2}{1-k^2}$ . The importance of this technique resides in the fact that it allows to compute the parameter u = u(x, t) (solution of the non-linear PDE) in terms of the scattered data of the eigenfunctions  $\psi$  (see [8]). Obviously, this fact makes the inverse problem (to obtain a Lax pair from a given PDE) to be an interesting (and in general challenging) problem. Also, in [1] Fokas showed that one can also find Lax pairs for linear PDEs making it an unifying theory for both linear and non-linear PDEs. However, we repeat that up to date there is no systematic way for obtaining a Lax pair associated to a given PDE.

It may be worth pointing out that (at least in the linear case) the notion of Lax pair is somewhat included in the larger concept of syzygies for systems of differential operators (see [9]). In consequence, results in the Section 3.2 could also be obtained by an adaptation of these methods.

The AKNS method (see [3]) works in a slightly different way: consider a system

$$v_x = \mathcal{X}v \& v_t = \mathcal{T}v \tag{5}$$

where  $\mathcal{X}, \mathcal{T}$  are linear matricial operators  $(n \times n)$  and v is a *n*-dimensional vector. From  $v_{xt} = v_{tx}$ , we obtain the compatibility equation

$$\mathcal{X}_t - \mathcal{T}_x = [\mathcal{T}, \mathcal{X}],$$

which, as in the Lax method, is a (non-linear) integrable PDE. This method was used by its authors to solve initial value problems by means of the inverse scattering transform.

As long as the operators involved are linear the two methods are equivalent. However, the AKNS method can be regarded as more general than the Lax one since it is not restricted to the eigenvalue dependence equation  $\mathcal{L}\psi = \lambda\psi$ . As we will show later, this modification gives us enough freedom to overcome the non-commutative problem in the compatibility condition. Therefore, we restrict our attention to the AKNS method for deriving non-linear systems linked to non-linear multidimensional PDEs. In addition to this, a close observation of example (3)–(4), linked to the Schrödinger equation, reinforces the observation that multi-dimensional problems should be deal in terms of the Dirac operator, that is, of the first-order operator which factorizes the Laplacian. In the next subsection, we will give a short overview of the non-commutative setting for the Dirac operator.

#### 2.2. Quaternions and biquaternions

The algebra of quaternions  $\mathbb{H}$  generalizes complex numbers into higher dimensions. An arbitrary quaternion is given by

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_j \in \mathbb{R}, \quad i = 0, 1, 2, 3, j \in \mathbb{R}$$

where  $e_0$  is the unit element of the algebra and can be identified with 1. The generalized imaginary units  $e_1$ ,  $e_2$ ,  $e_3$  (sometimes also denoted by **i**, **j**, **k**) satisfy the following multiplication rules:

$$e_j e_0 = e_0 e_j, \quad j = 1, 2, 3, \quad e_0^2 = 1,$$
  
 $e_i e_j + e_j e_i = -2\delta_{ij}$  (Kronecker delta),

A quaternion  $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$  is a sum of a scalar part Sc  $q = [q]_0 = q_0e_0 = q_0$  and a vector part Vec  $q = \underline{q} = q_1e_1 + q_2e_2 + q_3e_3$ . A quaternion is a pure quaternion if its real part vanishes,  $q_0 = 0$ , so that  $q = \underline{q}$ . The conjugate quaternion  $\overline{q}$  of a quaternion q is given by

$$\overline{q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 = \operatorname{Sc} q - \operatorname{Vec} q.$$

Biquaternions  $\mathbb{B}$  are complexified quaternions, i.e.  $q_j \in \mathbb{C}$ , j = 0, 1, 2, 3, whereas  $e_j$  satisfies the same multiplication rules as mentioned before. The algebra  $\mathbb{B}$  also supports quaternion conjugation, acting as a complex linear anti-automorphism together with the usual complex conjugation, i.e. the conjugated quaternion  $\overline{a}$  of  $a \in \mathbb{B}$  is

$$\overline{a} = \sum_{i=0}^{3} \overline{a}_i \overline{e}_i,$$

whereas  $\overline{a}_i$  is the standard complex conjugation and  $\overline{e}_i$  is the quaternion conjugation. As in the quaternion case, we define its *scalar* and *vectorial parts* as  $Sc(a) := a_0e_0$  and  $Vec(a) := a_1e_1 + a_2e_2 + a_3e_3$ , and one has a *pure biquaternion* if its real part is zero. For a pure biquaternion  $a \in \mathbb{B} \sim \mathbb{C}^3$ , we have

$$a\overline{a} = \sum_{i,j=1}^{3} a_i \overline{a}_j e_i \overline{e}_j = \sum_{j=1}^{3} |a_j|^2 - \sum_{i < j} (a_i \overline{a}_j - a_j \overline{a}_i) e_i e_j,$$

so that

$$[a\overline{a}]_0 = \sum_{j=1}^3 |a_j|^2 = ||a||^2,$$

the Euclidean norm of  $a \in \mathbb{C}^3$ .

The quaternion product ax can always be decomposed into its symmetric and antisymmetric parts, via the anti-commutator and commutator operators,

$$2ax = (ax + xa) + (ax - xa) = \{a, x\} + [a, x].$$
(6)

Moreover, if both *a*, *b* are pure biquaternions then the anti-commutator yields  $\{a, b\} = (ab + ba) = -2a \cdot b$ , where  $a \cdot b$  is the usual Euclidean inner product between the vectors  $a, b \in \mathbb{C}^3$ , while in  $\mathbb{R}^3$  the commutator yields  $[a, b] = (ab - ba) = 2a \times b$ , the classical Gibbs wedge product or cross product. For more details on quaternions and biquaternions, we refer to [10].

For constructing an analytic theory for functions of 3 real variables  $x = (x_1, x_2, x_3)$  as a higher dimensional analogue to the function theory of one complex variable we introduce the Dirac operator

$$D = \sum_{j=1}^{3} e_j \partial_{x_j}.$$

We will write  $D_x$  whenever explicit mention of the variable on which the partial derivatives act is needed.

A function *u* from an open set  $\Omega \subset \mathbb{R}^3$  into  $\mathbb{B}$  is said to be *left-monogenic* (resp., *right-monogenic*) if it holds

$$Du = 0 \quad (\text{resp. } uD = 0) \text{ in } \Omega. \tag{7}$$

Equation (7) generalizes the Cauchy–Riemann equations of complex analysis and can be written as the system:

$$\operatorname{Sc} (Du) = [Du]_0 = \sum_{j=1}^3 \partial_{x_j} u_j e_j^2 = -\nabla \cdot \underline{u} = -\operatorname{div} \underline{u} = 0,$$
  

$$\operatorname{Vec} (Du) = \underline{Du} = \sum_{j=1}^3 \partial_{x_j} u_0 e_j + \sum_{i < j} (\partial_{x_i} u_j - \partial_{x_j} u_i) e_i e_j$$
  

$$= \nabla u_0 + \nabla \times \underline{u} = \operatorname{grad} u_0 + \operatorname{curl} \underline{u} = 0.$$

Moreover, we have  $\Delta f = -D^2 u$ , that is, the Dirac operator factorizes the Euclidean Laplacian.

For more details on Clifford analysis we refer to [11], [12] or [13].

#### 3. Lax pairs for the linearized NLS equation

The key point for a successful construction of Lax pairs is an adequate factorization of second-order operators in terms of first-order operators. Based on this observation, and

the fact that the spatial one-dimensional Schrödinger equation is linked to the following factorization of the Helmholtz operator,

$$\partial_x^2 + \kappa = (\partial_x - ik)(\partial_x + ik), \quad k \in \mathbb{C} \text{ such that } k^2 = \kappa \in \mathbb{R},$$

it is only natural to consider a similar factorization of its higher dimensional counterpart

$$\Delta_x + \kappa, \quad \kappa \in \mathbb{R},$$

in terms of the Dirac operator. For that, an obvious candidate is the operator

$$D_x + ik$$

where  $k \in \mathbb{B}$  is such that  $k^2 = -\kappa$ . Unfortunately, this naïve replacement does not work due to the non-commutative nature of Clifford algebras. In fact,

$$(D_x - ik)(D_x + ik)u = (D_x - ik)(D_xu + iku) = D_x^2u + iD_x(ku) - ikD_xu + k^2u$$
  
=  $(-\Delta + k^2)u + i[D_x(ku) - k(D_xu)].$ 

Due to (6), we have

$$D_{x}(ku) = \sum_{j=i}^{n} e_{j}\partial_{x_{j}}(ku) = \sum_{j=i}^{n} e_{j}k(\partial_{x_{j}}u)$$
  
=  $\sum_{j=i}^{n} (-ke_{j} + \{e_{j}, k\})(\partial_{x_{j}}u) = -k(D_{x}u) + \sum_{j=i}^{n} \{e_{j}, k\}(\partial_{x_{j}}u),$ 

so that we get

$$(D_x - ik)(D_x + ik)u = -(\Delta - k^2)u - 2ik(D_x u) + i\sum_{j=i}^n \{e_j, k\}(\partial_{x_j} u)$$

This means that such a decomposition only provides a Helmholtz operator if k and  $D_x$  commute, that is to say, when k is a scalar.

#### 3.1. Lax pair for the instationary Schrödinger equation

One way to overcome the problem of non-commutativity is to introduce the multiplication operator  $M^k$ , for  $k \in \mathbb{B}$  (see [14]), acting on functions as

$$M^k: u \to M^k u := uk.$$

Then a factorization of the Helmholtz operator is possible, i.e.

$$-i(D_x + M^{-ik})(D_x + M^{ik})u = -i(D_x + M^{-ik})(D_x u + iuk)$$
  
=  $-iD_x^2 u + D_x(uk) - (D_x u)k - ik^2 u = i(\Delta + \kappa)u,$ 

for the case where  $\kappa = -k^2 \in \mathbb{R}$ .

Based on this decomposition, and following the concept of the algorithm described in [1], we are able to construct a Lax pair for the instationary Schrödinger equation.

THEOREM 3.1 The instationary Schrödinger equation

$$(i\partial_t + \Delta)\mu = 0$$

possesses the Lax pair

$$\begin{cases} (D_x + M^{ik})\mu = u\\ (i\partial_t - \lambda)\mu = (D_x + M^{-ik})u, \end{cases}$$
(8)

where  $\mu = \mu(x, t)$  is a quaternion-valued function and  $\lambda = (ik)^2 \in \mathbb{R}$ .

*Proof* By inserting the first equation into the second equation of (8), we obtain

$$i\partial_t \mu - \lambda \mu = (D_x + M^{-ik})u = (D_x + M^{-ik})(D_x + M^{ik})\mu = -\Delta \mu + \mu k^2,$$

which implies that  $\mu$  satisfies the desired PDE

$$i\partial_t \mu + \Delta \mu = 0,$$

for any  $k \in \mathbb{H}$  such that  $(ik)^2 = \lambda \in \mathbb{R}$ , the spectral parameter.

#### 3.2. Modified algorithm

Based on the previous observations, we can now present a first algorithm for the construction of Lax Pairs in the case of linear PDEs.

THEOREM 3.2 Let  $P_1(D)$ ,  $P_2(D)$  be two polynomial Dirac operators with constant coefficients. Then

$$\begin{cases} P_1(D)\mu = u \\ P(\partial_t, D)\mu + P_2(D)u = 0 \end{cases}$$
(9)

is a Lax pair for the instationary PDE

$$[P(\partial_t, D) + P_2(D)P_1(D)]\mu = 0,$$
(10)

where P(x, y) denotes a polynomial of two variables.

*Proof* In fact, rather elementary calculations give

$$0 = P(\partial_t, D)\mu + P_2(D)u = P(\partial_t, D)\mu + P_2(D)P_1(D)\mu.$$

We remark that the above construction is independent of the nature of the coefficients or of its solutions  $\mu = \mu(x, t)$ , u = u(x, t). Hence, PDEs with quaternion-valued coefficients can also be considered. Theorem 3.1 is a direct application of this theorem. Let us give an example involving a third-order PDE.

Example 3.3 The system

$$\begin{cases} (D - M^k)\mu = u \\ (\partial_t + M^{k^3})\mu + [D^2u + (Du)k + uk^2] = 0 \end{cases}$$

is a Lax pair for the instationary PDE with constant coefficients

$$(\partial_t - D\Delta)\mu = 0.$$

We have  $P_2(D) := D^2 + M^k D + M^{k^2}$  which, together with  $P_1(D) := D - M^k$ , satisfies the relation

$$P_2(D)P_1(D)\mu = (D^2 + M^k D + M^{k^2})(D - M^k)\mu = (D^3 - M^{k^3})\mu.$$

Hence, the above example represents a Lax pair for

$$0 = (\partial_t - D\Delta)\mu = [(\partial_t + M^{k^3}) + (D^3 - M^{k^3})]\mu.$$

Next, we present a generalization of the algorithm developed in [1] to the general case of PDEs with quaternion-valued coefficients.

THEOREM 3.4 Let  $P(\partial_t, D)$ ,  $P^*(\partial_t, D)$  be two operators with constant coefficients satisfying the commutation relation

$$P(\partial_t, D)(\partial_t + M^{ik})\mu = (\partial_t + M^{ik})P^*(\partial_t, D)\mu.$$
(11)

Then the system

$$\begin{cases} P^*(\partial_t, D)\mu = 0\\ (\partial_t + M^{ik})\mu = u \end{cases}$$
(12)

yields a Lax pair for the instationary PDE

$$P(\partial_t, D)u = 0. \tag{13}$$

Proof Indeed, we get

$$P(\partial_t, D)u = P(\partial_t, D)(\partial_t + M^{ik})\mu = (\partial_t + M^{ik})P^*(\partial_t, D)\mu = 0.$$

COROLLARY 3.5 If  $P(\partial_t, D)$  has scalar-valued coefficients then we have

$$P^*(\partial_t, D) = P(\partial_t, D).$$

As already mentioned in the introduction these results can also be obtained using the method of Syzygies in [9].

#### 3.3. Scalar plane waves associated to the Helmholtz operator

The process of linearization of the Schrödinger equation relays on the factorization of the Helmholtz operators

$$(\Delta - k^2)\mu = -(D_x + M^{-ik})(D_x + M^{ik})\mu.$$

In the 1D case, the first-order operator  $\partial_x - ik$  is linked to the well-known scalar plane waves  $u_k(x) = e^{ikx}$ . Unfortunately, those solutions do not transit easily to the nD case. In fact, we have  $D_x = (ik)u_k(x) \neq u_k(x)(ik)$ , due to non-commutativity. We recall that the spectral parameter in the Helmholtz equation is given by  $\kappa = (ik)^2$ , with  $k \in \mathbb{H}$ .

#### 4. Extension to non-linear systems

While the Lax pair method can be applied to linear PDEs it will show its real power when applied to the case of non-linear PDEs. Here, this means that we have to consider non-linear PDEs involving the Dirac operator. But, as we have seen in the previous section when we assume the standard Lax pair method, the non-commutativity creates quite some difficulties for the spectral equation. Therefore, in what follows we propose to exploit the AKNS method. Since it is more general it frees us from the restrictions imposed by the spectral equation.

In order to obtain non-linear systems by means of a convenient choice of parameters for a time-dependent Clifford valued function u = u(x, t), with  $x \in \mathbb{R}^n$ ,  $t \in \mathbb{R}^+$ , we consider the matrices

$$\mathcal{X} = \begin{bmatrix} 0 & u - \lambda \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix}.$$

where  $\lambda$  is scalar-valued. Nothing is said, at the moment, on the nature of parameters  $\alpha$ ,  $\beta$ ,  $\xi$ ,  $\eta$ . To study the nature of these parameters we need to investigate the compatibility condition.

#### 4.1. The compatibility condition

Recall the compatibility condition from the AKNS method:

$$\mathcal{X}_t - D_x \mathcal{T} = [\mathcal{T}, \mathcal{X}]. \tag{14}$$

For the left-hand side we obtain

$$\mathcal{X}_t - D_x \mathcal{T} = \begin{bmatrix} -D_x \alpha & u_t - D_x \beta \\ -D_x \xi & -D_x \eta \end{bmatrix},$$

while for the right-hand side, we get

$$\begin{bmatrix} \mathcal{T}, \mathcal{X} \end{bmatrix} = \mathcal{T}\mathcal{X} - \mathcal{X}\mathcal{T} \\ = \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix} \begin{bmatrix} 0 & u - \lambda \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & u - \lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix}.$$

Therefore, compatibility condition (14) leads to

$$\begin{bmatrix} -D_x \alpha & \partial_t u - D_x \beta \\ -D_x \xi & -D_x \eta \end{bmatrix} = \begin{bmatrix} \beta - (u-\lambda)\xi & \alpha(u-\lambda) - (u-\lambda)\eta \\ \eta - \alpha & \xi(u-\lambda) - \beta \end{bmatrix},$$

and we obtain the following system of four non-linear PDEs

$$\beta = (u - \lambda)\xi - D_x \alpha \tag{15}$$

$$\eta = \alpha - D_x \xi \tag{16}$$

$$D_{x}\beta = \partial_{t}u + (u - \lambda)\eta - \alpha(u - \lambda)$$
(17)

$$D_x \eta = \beta - \xi (u - \lambda). \tag{18}$$

Substituting (15) into (17) we obtain

$$0 = \partial_t u - D_x \beta + (u - \lambda)\eta - \alpha(u - \lambda)$$
  
=  $\partial_t u - D_x [(u - \lambda)\xi - D_x \alpha] + (u - \lambda)\eta - \alpha(u - \lambda).$ 

From (16), (15), and (18) we get

$$\begin{cases} D_x \eta = D_x \alpha - D_x^2 \xi \\ D_x \eta = [(u-\lambda)\xi - D_x \alpha] - \xi(u-\lambda) \end{cases} \Rightarrow 2D_x \alpha = D_x^2 \xi + u\xi - \xi u, \quad (19)$$

which leads to

$$0 = \partial_{t}u - D_{x}[(u-\lambda)\xi - D_{x}\alpha] + (u-\lambda)\eta - \alpha(u-\lambda)$$
  

$$= \partial_{t}u - D_{x}(u\xi) + \lambda D_{x}\xi + D_{x}^{2}\alpha + (u-\lambda)(\alpha - D_{x}\xi) - \alpha(u-\lambda)$$
  

$$= \partial_{t}u - D_{x}(u\xi) + \lambda D_{x}\xi + \frac{1}{2}[D_{x}^{3}\xi + D_{x}(u\xi - \xi u)] + u\alpha - \alpha u - (u-\lambda)D_{x}\xi$$
  

$$= \partial_{t}u + \frac{1}{2}D_{x}^{3}\xi - \frac{1}{2}D_{x}(u\xi + \xi u) + u\alpha - \alpha u - (u-2\lambda)D_{x}\xi$$
  

$$= \partial_{t}u + \frac{1}{2}D_{x}^{3}\xi - \frac{1}{2}D_{x}\{u,\xi\} + [u,\alpha] - (u-2\lambda)D_{x}\xi,$$
(20)

where [a, b] := ab - ba and  $\{a, b\} = ab + ba$  denote the commutator and the anticommutator, respectively. The last equation allows us to study possible choices for the parameters  $\alpha$ ,  $\beta$ ,  $\xi$ ,  $\eta$ .

#### 4.2. A variant of the transport equation

Let us first assume  $\alpha$  to be a scalar-valued constant. Then (20) reduces to

$$0 = \partial_t u + \frac{1}{2} D_x^3 \xi - \frac{1}{2} D_x \{u, \xi\} - (u - 2\lambda) D_x \xi.$$

Moreover, from (19) we have  $\xi u = u\xi + D_x^2 \xi$ , and hence, we obtain

$$0 = \partial_t u + \frac{1}{2} D_x^3 \xi - \frac{1}{2} D_x (2u\xi + D_x^2 \xi) - (u - 2\lambda) D_x \xi$$
  
=  $\partial_t u - D_x (u\xi) - u D_x \xi + 2\lambda D_x \xi.$  (21)

#### 4.2.1. A multi-dimensional transport equation

If, additionally, we assume  $\xi \in \mathbb{C}$  to be a constant, that is to say,  $D_x \xi = 0$ , then (20) reduces to the differential form of a *n*-dimensional transport equation (with  $f(u) = -\xi u$ )

$$0 = \partial_t u - \xi D_x u,$$

an equation which admits the vectorial solution

$$u(x,t) = \sum_{j=1}^{3} e^{-\xi t} \exp(x_j e_j) e_j = \sum_{j=1}^{3} e^{-\xi t} [-\sin(x_j) + e_j \cos(x_j)].$$

Moreover, (19) implies that  $D_x \alpha = 0$ , that is,  $\alpha$  must be a monogenic function. Below, we show some examples of the scalar and vectorial parts of *u* (Figures 1 and 2).

The first example refers to the 1D solution  $u(x, t) = e^{-\xi t} [-\sin(x) + e_1 \cos(x)]$ . For the second example, we consider  $\xi = -1$  at time t = 4.

#### 4.2.2. A non-linear evolution equation

Since, we are interested in non-linear equations, it is reasonable to require that  $\xi$  depends on *u*.

At this stage, we would like to point our that if  $\xi$  and u commute (e.g. if u is scalarvalued) then (19) means that  $\xi$  has to be harmonic or pure vector-valued. This also implies that the case where  $\xi = u$  is trivial. Therefore, let us consider instead the case where  $\xi = D_x u$ . Here, from (21) we obtain the equation

$$0 = \partial_t u - D_x (u D_x u) + u \Delta u - 2\lambda \Delta u.$$
<sup>(22)</sup>

We observe that

$$D_{x}(uD_{x}u) = \sum_{j=1}^{3} e_{j}\partial_{x_{j}}(uD_{x}u)$$
  
=  $\left(\sum_{j=1}^{3} e_{j}\partial_{x_{j}}u\right)D_{x}u + \sum_{j=1}^{3} e_{j}u(\partial_{x_{j}}D_{x}u)$   
=  $(D_{x}u)^{2} + \sum_{j=1}^{3}(ue_{j} + [e_{j}, u])(\partial_{x_{j}}D_{x}u)$   
=  $(D_{x}u)^{2} + uD_{x}^{2}u + \sum_{j=1}^{3}[e_{j}, u](\partial_{x_{j}}D_{x}u).$ 

Therefore, a scalar solution u of (22) will satisfy the non-linear PDE

$$0 = \partial_t u - (D_x u)^2 + 2(u - \lambda)\Delta u.$$

#### 4.3. A variant of the KdV equation

Let us now turn our attention to a case where we obtain a variant of the well-known KdV equation. To this end we assume  $\xi = 4\lambda + 2u$  and  $\alpha = D_x u$ . Now, from (16) we obtain  $\eta = -D_x$ . This means that Equation (20) reduces to

$$0 = \partial_{t}u + \frac{1}{2}D_{x}^{3}\xi - \frac{1}{2}D_{x}\{u,\xi\} + [u,\alpha] - (u-2\lambda)D_{x}\xi$$
  

$$= \partial_{t}u + \frac{1}{2}D_{x}^{3}(4\lambda + 2u) - \frac{1}{2}D_{x}\{u,4\lambda + 2u\} + [u,D_{x}u] - (u-2\lambda)D_{x}(4\lambda + 2u)$$
  

$$= \partial_{t}u + D_{x}^{3}u - 4\lambda D_{x}u - 2D_{x}(u^{2}) + u(D_{x}u) - (D_{x}u)u - 2(u-2\lambda)D_{x}u$$
  

$$= \partial_{t}u + D_{x}^{3}u - 2D_{x}(u^{2}) - u(D_{x}u) - (D_{x}u)u.$$
(23)

Immediately, one observes that if the solution u is assumed to be scalar then it will satisfy

$$0 = [\partial_t u + D_x^3 u - 2D_x(u^2) - u(D_x u) - (D_x u)u]_0 = \partial_t u$$

thus generating a time-independent solution of our Equation (23). In a similar way, a pure vectorial solution u has the drawback that its time-dependence is expressed by the vectorial



Figure 1. Graphics of  $[u(x, t)]_0 = -e^{-t} \sin(x)$  and  $[u(x, t)]_1 = e^{-t} \cos(x)]$ , with  $\xi = 1$ ,  $0 \le x \le 4\pi$  and  $0 \le t \le 2$ .



Figure 2. Graphics of scalar and vectorial parts of  $u(x, y, 4) = -e^4[\sin(x) + \sin(y)] + e^4\cos(x)e_1 + e^4\cos(y)e_2$ .

part of (23), thus reducing our equation to

$$0 = [\partial_t u + D_x^3 u - 2D_x(u^2) - u(D_x u) - (D_x u)u]_1$$
  
=  $\partial_t u - 2D_x(u^2).$ 

Therefore, we will now direct our attention to *u* being a combination of scalar and vectorial solutions.

#### 4.3.1. The one-dimensional case

Let us start with the one-dimensional case, i.e. we consider  $u(x, t) = u_0(x, t) + e_1u_1(x, t)$ , for  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ . In this setting the Dirac operator is reduced to  $D = e_1 \partial_x$ . Calculating the different terms we obtain

$$0 = \partial_t u + (e_1 \partial_x)^3 u - 2e_1 \partial_x (u^2) - u(e_1 \partial_x u) - (e_1 \partial_x u) u$$
  
=  $\partial_t (u_0 + e_1 u_1) - e_1 \partial_x^3 (u_0 + e_1 u_1) - 2e_1 \partial_x (u_0^2 - u_1^2 + 2e_1 u_0 u_1)$   
 $- \{(u_0 + e_1 u_1), (-\partial_x u_1 + e_1 \partial_x u_0)\}$ 

so that, separating the scalar and vectorial parts, we get a system of coupled PDEs of KdV-type

$$\begin{cases} \partial_t u_0 + \partial_x^3 u_1 + 6u_0 \partial_x u_1 + 6u_1 \partial_x u_0 = 0, \\ \partial_t u_1 - \partial_x^3 u_0 - 4u_0 \partial_x u_0 + 4u_1 \partial_x u_1 = 0. \end{cases}$$

#### 4.3.2. The quaternionic case

Now, let us turn our attention to the higher dimensional case. For the sake of simplicity, we restrict ourselves to the quaternionic case where  $x = x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$ . Here, the corresponding first-order operator is a modification of the Dirac operator, namely

$$D = e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3},$$

and, thus, satisfies  $D^2 = -\Delta$ ,  $\Delta$  being the Laplace operator in  $\mathbb{R}^3$ . Moreover, a quaternionic solution

$$u(x,t) = u_0(x,t) + u_1(x,t)e_1 + u_2(x,t)e_2 + u_3(x,t)e_3$$
  
=  $u_0(x,t) + \underline{u}(x,t) \in \mathbb{H}$  (24)

satisfies

$$Du = D(u_0 + \underline{u}) = \operatorname{grad} u_0 - \operatorname{div} \underline{u} + \operatorname{curl} \underline{u}$$

We now compute the different terms in (23). First of all, we have

$$D^{3}u = -D(\Delta u)$$
  
= -grad (\Delta u\_{0}) + div (\Delta u) - curl (\Delta u), (25)

and

$$-2D(u^{2}) = -2D(uu)$$
  
= 
$$-2\left[(Du)u - u(Du) - 2\sum_{j}u_{j}\partial_{x_{j}}u\right]$$
  
= 
$$-2[(Du)u - u(Du)] + 4\sum_{j}u_{j}\partial_{x_{j}}u.$$

Since  $-\{u, Du\} = -[(Du)u + u(Du)]$  we get

$$-2D(u^2) - \{u, Du\} = [(Du)u + u(Du)] - 4(Du)u + 4\sum_j u_j \partial_{x_j} u$$
  
= 2[(Du)u]\_0 - 4(Du)u + 4\sum\_j u\_j \partial\_{x\_j} (u\_0 + u).

For the first two terms on the right-hand side, we have

$$-4(Du)u = -4(\operatorname{grad} u_0 - \operatorname{div} \underline{u} + \operatorname{curl} \underline{u})(u_0 + \underline{u})$$
  
=  $4u_0\operatorname{div} \underline{u} - 4[(\operatorname{grad} u_0 + \operatorname{curl} \underline{u})\underline{u}]_0 + 4(\operatorname{div} \underline{u})\underline{u} - 4u_0(\operatorname{grad} u_0 + \operatorname{rot} \underline{u})$ (26)

and

$$2[(Du)u]_0 = 2[(u_0 + \underline{u})(\operatorname{grad} u_0 - \operatorname{div} \underline{u} + \operatorname{curl} \underline{u})]_0$$
  
= 2u\_0 \operatorname{div} \underline{u} + 2[\underline{u} (\operatorname{grad} u\_0 + \operatorname{curl} \underline{u})]\_0 (27)

Inserting these terms in (23), and separating the scalar and vectorial parts gives now the non-linear system of coupled PDEs

$$\begin{cases} \partial_t u_0 + \operatorname{div}\left(\Delta \underline{u}\right) + 6u_0 \operatorname{div} \underline{u} + 4\sum_{j=1}^3 u_j \partial_{x_j} u_0 - 2[\underline{u}(\operatorname{grad} u_0 + \operatorname{curl} \underline{u})]_0 = 0\\ \partial_t \underline{u} - \operatorname{grad} \Delta u_0 - \operatorname{curl}\left(\Delta \underline{u}\right) + 4\sum_{j=1}^3 u_j \partial_{x_j} \underline{u} + 4(\operatorname{div} \underline{u})\underline{u} - 4u_0(\operatorname{grad} u_0 + \operatorname{curl} \underline{u}) = 0 \end{cases}$$

or, in terms of the standard gradient operator and  $\underline{u} = (u_1, u_2, u_3)$ ,

$$\begin{bmatrix} \partial_t u_0 + \nabla(\Delta \underline{u}) + 6u_0 \nabla \cdot \underline{u} + 4(\underline{u} \cdot \nabla)u_0 + 2\underline{u} \cdot (\nabla u_0 + \nabla \times \underline{u}) = 0\\ \partial_t \underline{u} - \nabla \Delta u_0 - \nabla \times (\Delta \underline{u}) + 4(\underline{u} \cdot \nabla)\underline{u} + 4(\nabla \cdot \underline{u})\underline{u} - 4u_0(\nabla u_0 + \nabla \times \underline{u}) = 0 \end{bmatrix}$$

Also, we remark that, due to the nature of the quaternionic algebras, one can also consider a pure vectorial solution  $u = \underline{u} = u_1e_1 + u_2e_2 + u_3e_3$  in which case we obtain the non-linear PDE of KdV-type

$$\partial_t \underline{u} - \nabla \times (\Delta \underline{u}) + 4[(\nabla \cdot \underline{u}) + \underline{u} \cdot \nabla]\underline{u} = 0.$$

#### Acknowledgements

This work was supported by *FEDER* funds through *COMPETE* – Operational Programme Factors of Competitiveness Programa Operacional Factores de Competitividade and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology FCT–Fundação para a Ciência e a Tecnologia, within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690.

#### References

- Fokas AS. A unified transform method for solving linear and certain non-linear PDEs. Proc. Royal Soc. London A. 1997;453:1411–1443.
- [2] Lax PD. Integrals of non-linear equations of evolution and solitary waves. Commun. Pure Appl. Math. 1968;XXI:467–490.
- [3] Aktosun T. Inverse scattering transform, KdV, and solitons. In: Ball JA, Helton JW, Klaus M, Rodman L, editors. Current trends in operator theory and its applications. Vol. 149, Operator theory: advances and applications. Basel: Birkhuser; 2004. p. 1–22.
- [4] Novikov S, Manakov SV, Pitaevskii LP, Zakharov VE. Theory of solitons: the inverse scattering method. Monographs in contemporary mathematics. Springer: New York; 1984.
- [5] Zakharov VE, Shabat AB. A scheme for integrating the non-linear equations of mathematical physics by the method of inverse scattering problem. I. Funct. Anal. Appl. 1974;8:226–235.
- [6] Zakharov VE, Shabat AB. Integration of non-linear equations of mathematical physics by the method of inverse scattering. II. Funct. Anal. Appl. 1979;13:166–174.
- [7] Isozaki H. Inverse scattering theory for Dirac operators. Ann. Inst. Henri Poincare Sect. A. 1997;66:237–270.
- [8] Ablowitz MJ, Clarkson PA. Solitons, non-linear evolution equations and inverse scattering. Cambridge: Cambridge University Press; 1991.
- [9] Colombo F, Sabadini I, Sommen F, Struppa DC. Analysis of Dirac systems and computational algebra. Vol. 39, Progress in mathematical physics. Boston: Birkhäuser; 2004.
- [10] Kravchenko VV, Shapiro M. Integral representations for spatial models of mathematical physics. Vol. 351, Pitman research notes in mathematics series. Addison Wesley Longman Ltd: Edinburgh Gate Harlow; 1996.

- [11] Axelsson A, Grognard R, Hogan J, McIntosh A. Harmonic analysis of Dirac operators in Lipschitz domains. In: Brackx F, Chisholm JSR, Soucek, V, Prague, editors. Clifford analysis and its applications. Vol. 25, NATO Science Series II: Mathematics, Physics and Chemistry. Kluwer Academic; 2001. p. 231–246.
- [12] Gilbert JE, Murray MAM. Clifford algebras and Dirac operators in harmonic analysis. Cambridge University Press: Cambridge; 1991.
- [13] McIntosh A. Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains. In: Ryan J, editor. Clifford algebras in analysis and related topics. Boca Raton (FL): CRC Press; 1996. p. 33–88.
- [14] Bernstein S, Gürlebeck K. On a higher dimensional Miura-transform. Complex Variables Theory Appl. 1999;38:307–319.