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Ricardo Abreu Blaya, Juan Bory Reyes, Alí Guzmán Adán & Uwe Kähler

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On the φ-Hyperderivative of the ψ-Cauchy-Type Integral in Clifford Analysis

Ricardo Abreu Blaya[1] · **Juan Bory Reyes**[2] · **Alí Guzmán Adán**[3] · **Uwe Kähler**[4]

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Abstract The aim of this paper is to introduce, in the framework of Clifford analysis, the notions of φ-hyperdifferentiability and φ-hyperderivability for ψ-hyperholomorphic functions where (φ, ψ) are two arbitrary orthogonal bases (called structural sets) of a Euclidean space. In this study we will also show how to exchange the integral sign and the φ-hyperderivative of the ψ-Cliffordian Cauchy-type integral. Thereby, we generalize, in a natural way, the corresponding quaternionic antecedent as well as the standard Clifford predecessor.

Keywords Clifford analysis · Hyperderivative · Hyperholomorphy · Cauchy-type integral

Mathematics Subject Classification 30G35

1 Introduction

At the heart of one-dimensional complex analysis lies the notion of a holomorphic function, which can be introduced by different equivalent approaches, for example,

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✉ Uwe Kähler
ukaehler@ua.pt

1 Facultad de Informática y Matemática, Universidad de Holguín, Holguín 80100, Cuba

2 SEPI-ESIME-Zacatenco, Instituto Politécnico Nacional, Mexico, D.F. 07738, Mexico

3 Clifford Research Group, Department of Mathematical Analysis, Faculty of Engineering and Architecture, Ghent University, Galglaan 2, 9000 Ghent, Belgium

4 Departamento de Matemática, Universidade de Aveiro, Aveiro, Portugal

the derivative of a complex function as a limit of a quotient, the Cauchy–Riemann conditions, complex differentiability, and others.

Standard Clifford analysis is a higher dimensional generalization of holomorphic function theory, and a refinement of harmonic analysis, but its main object of interest, the class of hyperholomorphic functions, is defined, almost always, in terms of the Cauchy–Riemann conditions.

Recently, a series of works [3,9,13–16,18,19] appeared which dealt with the notion of a hyperderivative as the limit of a quotient where the numerator and the denominator represent "increments" of a Clifford algebra-valued function and of the independent variable, respectively, as well as the directional derivative, where a direction means a hyperplane in a Euclidean space for a Clifford algebra. In the framework of quaternionic analysis, papers [20,26] preceded these developments. The consideration of $\overline{D}$ as the hyperderivative of a monogenic function led to an increased interest in the description of geometric properties of a hyperholomorphic function via the $\overline{D}$-operator. But there are many circumstances where such a description by means of other operators would be easier to apply. The reason is that the $\overline{D}$-operator has a preferential direction in the real axis, i.e., $Du + \overline{D}u = \partial_0 u$, which is not necessarily the most interesting one. One could overcome this problem by rotating $\overline{D}u$, as in [4], but this is usually cumbersome. One of the major discussions in this paper is what kind of operator can replace $\overline{D}$ as a hyperderivative, where we give the necessary conditions in terms of structural sets (orthonormal bases) φ and ψ. Although we do not discus the representation of geometric properties by means of the φ-hyperderivative, with the basic setting constructed in this paper, the transfer of the proofs from the classic case of the $\overline{D}$-operator to that case is straightforward. Furthermore, in many applications, one needs to know how the hyperderivative acts on the Cauchy-type integral operator. Therefore, we show how to interchange the integral sign and the φ-Cauchy–Riemann operator acting on the ψ-Cliffordian Cauchy-type integral.

It is worth mentioning that the directional derivative becomes crucial when one wants to realize in which sense the density of the Cauchy-type integral should be derived if one tries to exchange the integral sign and the hyperderivative of the Cauchy-type integral as a hyperholomorphic function.

2 Preliminaries

Given $m \in \mathbb{N}$, let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$. Consider the 2^m-dimensional real Clifford algebra $\mathbb{R}_{0,m}$ generated by $e_1, e_2, \ldots, e_m$ according to the multiplication rules $e_i e_j + e_j e_i = -2\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker's symbol. The elements $e_A : A \subseteq \mathbb{N}_m := \{1, 2, \ldots, m\}$ define a basis of $\mathbb{R}_{0,m}$, where $e_A = e_{h_1} \cdots e_{h_k}$ if $A = \{h_1, \ldots, h_k\}$ $(1 \le h_1 < \cdots < h_k \le m)$ and $e_\emptyset = e_0 = 1$.

Any $a \in \mathbb{R}_{0,m}$ may, thus, be written as $a = \sum_{A \subseteq \mathbb{N}_m} a_A e_A$ where $a_A \in \mathbb{R}$ or also as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector ($k \in \mathbb{N}_m^0 := \mathbb{N}_m \cup \{0\}$). If we denote the space of k-vectors by $\mathbb{R}_{0,m}^{(k)}$, it is obvious that $\mathbb{R}_{0,m} = \sum_{k=0}^{m} \oplus \mathbb{R}_{0,m}^{(k)}$. The conjugate of a is defined by $\bar{a} = \sum_{A \subseteq \mathbb{N}_m} a_A \bar{e}_A$, where

$$\bar{e}_A := (-1)^k e_{h_k} \cdots e_{h_1} = (-1)^{\frac{k(k+1)}{2}} e_A, \text{ if } e_A = e_{h_1} \cdots e_{h_k}.$$

One of the most elementary properties of this conjugation is that for every pair $a, b \in \mathbb{R}_{0,m}$ we have

$$\overline{ab} = \overline{b}\,\overline{a}. \tag{1}$$

The spaces $\mathbb{R}$ and $\mathbb{R}^m$ will be identified with $\mathbb{R}^{(0)}_{0,m}$ and $\mathbb{R}^{(1)}_{0,m}$, respectively. Moreover, each element $x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$ can be written as

$$x = x_0 + \sum_{i=1}^{m} x_i e_i \in \mathbb{R}^{(0)}_{0,m} \oplus \mathbb{R}^{(1)}_{0,m}.$$

For each $x \in \mathbb{R}^{(0)}_{0,m} \oplus \mathbb{R}^{(1)}_{0,m}$ it is worth noting that

$$x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_m^2 = |x|^2. \tag{2}$$

The extension of (2) to a norm of $a \in \mathbb{R}_{0,m}$ is straightforward and leads to

$$|a|^2 = [a\bar{a}]_0 = [\bar{a}a]_0 = \sum_A a_A^2.$$

In this paper, we will consider bounded domains[1] $\Omega \subset \mathbb{R}^{m+1}$ with smooth boundaries $\Gamma := \partial\Omega$. We will be interested in functions defined on subsets of $\mathbb{R}^{m+1}$ taking values in $\mathbb{R}_{0,m}$ which might be written as $f(x) = \sum_A f_A(x) e_A$ with f_A being $\mathbb{R}$-valued. Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed also by all components f_A. In this way, we obtain the following functions sets, for a suitable subset E of $\mathbb{R}^{m+1}$.

- $C^k(E, \mathbb{R}_{0,m})$—the set of all $\mathbb{R}_{0,m}$-valued functions, k-times continuously differentiable in E and $C^\infty(E, \mathbb{R}_{0,m}) := \bigcap_{k=0}^{\infty} C^k(E, \mathbb{R}_{0,m})$. For $f \in C^k(E, \mathbb{R}_{0,m})$ we will write

$$D_\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_0^{\alpha_0} \cdots \partial x_n^{\alpha_n}}, \ |\alpha| \le k,$$

  where $\alpha = (\alpha_0, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^{n+1}$ is a multi-index and $|\alpha| = \alpha_0 + \cdots + \alpha_n$.
- $C^{0,\mu}(E, \mathbb{R}_{0,m})$, $\mu \in (0, 1]$– the set of all μ-Hölder continuous and $\mathbb{R}_{0,m}$-valued functions in E. By $C^{k,\mu}(E, \mathbb{R}_{0,m})$, $k \in \mathbb{N}$, we will denote the set of functions $f \in C^{0,\mu}(E, \mathbb{R}_{0,m})$ whose partial derivatives $D_\alpha f \in C^{0,\mu}(E, \mathbb{R}_{0,m})$ for $|\alpha| \le k$.

Let $\psi := \{\psi^0, \psi^1, \dots, \psi^m\} \subset \mathbb{R}^{(0)}_{0,m} \oplus \mathbb{R}^{(1)}_{0,m}$. For brevity, we let $\overline{\psi} := \{\overline{\psi^0}, \overline{\psi^1}, \dots, \overline{\psi^m}\}$ stand for the conjugate of ψ. On the set $C^1(\Omega, \mathbb{R}_{0,m})$ the left and the right ψ-Cauchy–Riemann operators are defined by

[1] Open and connected sets.

$$ {}^{\psi}D[f] := \sum_{i=0}^{m} \psi^i \frac{\partial f}{\partial x_i},\ D^{\psi}[f] := \sum_{i=0}^{m} \frac{\partial f}{\partial x_i} \psi^i. \tag{3}$$

To fulfill the Laplacian factorization

$$ {}^{\psi}D\, {}^{\overline{\psi}}D = {}^{\overline{\psi}}D\, {}^{\psi}D = D^{\psi}\, D^{\overline{\psi}} = D^{\overline{\psi}}\, D^{\psi} = \Delta_{m+1}, \tag{4}$$

the following condition is required

$$ \psi^i \cdot \overline{\psi^j} + \psi^j \cdot \overline{\psi^i} = 2\delta_{i,j}, i, j \in \mathbb{N}_m^0. \tag{5}$$

Note that this last equality yields

$$ 2\delta_{i,j} = \psi^i \cdot \overline{\psi^j} + \psi^j \cdot \overline{\psi^i} = \psi^i \cdot \overline{\psi^j} + \overline{\psi^i \cdot \overline{\psi^j}} = 2\left[\psi^i \cdot \overline{\psi^j}\right]_0 = 2\left\langle \psi^i, \psi^j \right\rangle_{\mathbb{R}^{m+1}}, \tag{6}$$

thus, factorization (4) is true if and only if ψ represents an orthonormal basis of $\mathbb{R}^{m+1} \cong \mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}$.

A set ψ satisfying (5) is called a *structural set*. It is clear that ψ and $\overline{\psi}$ are structural sets simultaneously. Basic properties of structural sets can be found in [23,24].

Definition 1 Let ψ be a structural set. A function $f \in C^1(\Omega, \mathbb{R}_{0,m})$ is called left-ψ-hyperholomorphic if ${}^{\psi}D[f](x) = 0$ in Ω. We set ${}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,m}) := \ker {}^{\psi}D$. Similarly, f is right-ψ-hyperholomorphic if $D^{\psi}[f](x) = 0$, and we set $\mathfrak{M}^{\psi}(\Omega, \mathbb{R}_{0,m}) := \ker D^{\psi}$.

One of the most important examples of a two-sided ψ-hyperholomorphic function is

$$ K_{\psi}(x) = \frac{x_{\overline{\psi}}}{|\mathbb{S}^n| \cdot |x|^{n+1}}, $$

where $x_{\psi} := \sum_{i=0}^{n} x_i \psi^i$ if $x = \sum_{i=0}^{n} x_i e_i$ and $|\mathbb{S}^n|$ is the area of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. This function is known as the ψ-Cauchy kernel.

Left and right ψ-Cauchy–Riemann operators are connected by the relations

$$ {}^{\psi}D[f] = \overline{D^{\overline{\psi}}[\bar{f}]} \text{ and } D^{\psi}[f] = \overline{{}^{\overline{\psi}}D[\bar{f}]}. \tag{7}$$

Hence, it is sufficient to confine the discussion to the theory of left- (or right-) ψ-hyperholomorphic functions. In this work we will restrict our attention to left-ψ-hyperholomorphic (ψ-hyperholomorphic for short) functions. In the same way as [2], it can be proved that

$$ {}^{\varphi}\mathfrak{M}(\Omega, \mathbb{R}_{0,m}) = {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,m}) \quad \text{if and only if} \quad \varphi^0\overline{\psi^0} = \varphi^1\overline{\psi^1} = \cdots = \varphi^i\overline{\psi^i} = \cdots = \varphi^m\overline{\psi^m}. $$

If we denote this common value for every $\varphi^i\overline{\psi^i}$ by h, we can express this relation as $\varphi = h\psi$. These pairs of structural sets are called *equivalent pairs of structural sets*.

2.1 Integral Operators

In the following, we need some statements about integral operators resulting from the Cauchy kernel. For the sake of self-sufficiency of the paper we present them here.

There has been considerable effort (e.g., [5,21]) to establish a very general measure theoretic version of a Stokes formula. One of the most crucial facts of standard Clifford analysis is the existence of a version of the Stokes formula in this context. In [17], see also [10,11], was presented an approach under some mild measure theoretic background assumptions on the boundaries of the domains. In particular, domains of locally finite perimeter as well as those with Ahlfors regular boundaries were considered.

We follow [23] in formulating such a Stokes formula for an arbitrary structural set, in a sufficiently smooth context, but remark, taking into account the above comments, the smoothness conditions can be relaxed. The presented version goes back to Ryan [22]. It is however, not the aim of the paper to go into a deeper discussion of these generalizations (the reader is again advised to consult [10,11,17] for more details); we will use this fact only when the special case of parallelepiped domains are treated.

Theorem 1 (Strokes formula) *Let Ω be a closed bounded domain with a piecewise C^1-boundary. For $f, g \in C^1(\overline{\Omega}, \mathbb{R}_{0,m})$*

$$\int_\Gamma g(\xi)\, n_\psi(\xi)\, f(\xi)\, \mathrm{d}S_\xi = \int_\Omega (D^\psi[g](\xi)\, f(\xi) + g(\xi)\, {}^\psi D[f](\xi))\mathrm{d}V_\xi, \tag{8}$$

where $\mathrm{d}V_\xi$ denotes the volume element, $\mathrm{d}S_\xi$ is the surface element in $\mathbb{R}^{m+1}$ and $n_\psi(\xi) = \sum_{i=0}^n n_i(\xi)\psi^i$ where $n_i(\xi)$ is the i-th component of the outward unit normal vector on Γ at the point $\xi \in \Gamma$.

The Stokes formula leads immediately to two important consequences, which are widely known and can be found in many sources.

Theorem 2 (Borel–Pompeiu formula) *Let $f \in C^1(\overline{\Omega}, \mathbb{R}_{0,m})$. Then*

$$\int_\Gamma K_\psi(\xi - x)\, n_\psi(\xi)\, f(\xi)\, \mathrm{d}S_\xi - \int_\Omega K_\psi(\xi - x)\, {}^\psi D[f](\xi)\, \mathrm{d}V_\xi = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1}\backslash\overline{\Omega}. \end{cases} \tag{9}$$

Theorem 3 (Cauchy integral formula) *Let $f \in {}^\psi\mathfrak{M}(\Omega, \mathbb{R}_{0,m}) \cap C^0(\overline{\Omega}, \mathbb{R}_{0,m})$. Then*

$$\int_\Gamma K_\psi(\xi - x)\, n_\psi(\xi)\, f(\xi)\, \mathrm{d}S_\xi = \begin{cases} f(x) & \text{if } \ x \in \Omega, \\ 0 & \text{if } \ x \in \mathbb{R}^{n+1}\backslash\overline{\Omega}. \end{cases}$$

From the above, we can see that the ψ-Cauchy kernel generates the following two important integrals:

$$^{\psi}T_{\Omega}[f](x) := -\int_{\Omega} K_{\psi}(\xi - x)\, f(\xi)\, \mathrm{d}V_{\xi}, \quad x \in \mathbb{R}^{n+1},$$

and

$$^{\varphi,\psi}K_{\Gamma}[f](x) := \int_{\Gamma} K_{\varphi}(\xi - x)\, n_{\psi}(\xi)\, f(\xi)\, \mathrm{d}S_{\xi}, \quad x \notin \Gamma.$$

While the first is a generalization of the usual Teodorescu transform, the second represents a boundary "exotic" operator which connects two arbitrary structural sets φ and ψ. When $\psi = \varphi$, $^{\varphi,\psi}K_{\Gamma}$ reduces to the usual Cauchy type integral

$$^{\psi}K_{\Gamma}[f](x) := \int_{\Gamma} K_{\psi}(\xi - x)\, n_{\psi}(\xi)\, f(\xi)\, \mathrm{d}S_{\xi}.$$

The singular version of $^{\varphi,\psi}K_{\Gamma}[f]$ on Γ, denoted by $^{\varphi,\psi}S_{\Gamma}[f]$, is given, as usual, by

$$^{\varphi,\psi}S_{\Gamma}[f] := 2\mathrm{tr}\,^{\varphi,\psi}K_{\Gamma}[f],$$

where tr denotes the trace operator, i.e., taking the limits of $x \in \Omega$ to the boundary.

The integral defining the operator $^{\varphi,\psi}S_{\Gamma}[f]$ is taken in the sense of the Cauchy principal value. For $\varphi = \psi$, a relation between the boundary value of $^{\psi}K_{\Gamma}[f]$ and $^{\psi}S_{\Gamma}[f] := {}^{\psi,\psi}S_{\Gamma}[f]$ is given, see [23].

Theorem 4 (Sokhotski–Plemelj formulas) *Let* $f \in C^{0,\mu}(\Gamma, \mathbb{R}_{0,m})$, $\mu \in (0, 1]$. *Then we have:*

$$^{\psi}K^{\pm}_{\Gamma}[f](t) := \lim_{\Omega^{\pm} \ni x \to t \in \Gamma} {}^{\psi}K_{\Gamma}[f](x) = \frac{1}{2}\left[^{\psi}S_{\Gamma}[f](t) \pm f(t)\right], \tag{10}$$

where $\Omega^{+} := \Omega$ *and* $\Omega^{-} := \mathbb{R}^{n+1} \backslash \overline{\Omega}$.

The following is an immediate consequence of the previous result and it was obtained in [1].

Corollary 1 *If* Γ *is a Lyapunov surface, for* $f \in C^{0,\mu}(\Gamma, \mathbb{R}_{0,m})$, $\mu \in (0, 1]$ *we have*

$$^{\varphi,\psi}K^{\pm}_{\Gamma}[f](t) := \lim_{\Omega^{\pm} \ni x \to t \in \Gamma} {}^{\varphi,\psi}K_{\Gamma}[f](x) = \frac{1}{2}\left[^{\varphi,\psi}S_{\Gamma}[f](t) \pm n_{\overline{\varphi}}(t) n_{\psi}(t) f(t)\right].$$

The work [25], followed by [6–8,12], initiated the study of the hypercomplex $^{\varphi,\psi}\Pi_{\Omega}$-operator, which is defined by $^{\varphi,\psi}\Pi_{\Omega} := {}^{\varphi}D\,^{\psi}T_{\Omega}$. One of its essential properties, which can be seen as a far-reaching generalization of the Borel–Pompeiu formula, is obtained in [1,2].

Theorem 5 (Generalized Borel–Pompeiu formula) *Let* $f \in C^1(\overline{\Omega}, \mathbb{R}_{0,m})$. *Then:*

$$^{\varphi,\psi}\Pi_\Omega[f](x) = \int_\Gamma K_{\overline{\varphi}}(\xi - x)\, n_{\overline{\psi}}(\xi)\, f(\xi)\, \mathrm{d}S_\xi - \int_\Omega K_{\overline{\varphi}}(\xi - x)\, {}^{\overline{\psi}}D[f](\xi)\, \mathrm{d}V_\xi,\ x \notin \Gamma. \tag{11}$$

Remark 2.1 The above formula may be written in an abbreviated form way as:

$$^{\varphi,\psi}\Pi_\Omega[f](x) = {}^{\overline{\varphi},\overline{\psi}}K_\Gamma[f](x) + {}^{\overline{\varphi}}T_\Omega{}^{\overline{\psi}}D[f](x),\ x \notin \Gamma.$$

Note that for $\varphi = \psi$ the formula (11) becomes in the classic Borel–Pompeiu formula (9).

For our purposes, let us reformulate a very important theorem from real analysis in hypercomplex form: the Whitney extension theorem (see [27])

Theorem 6 *Let* $E \subset \mathbb{R}^{m+1}$ *be a compact subset and let* $f \in C^k(E, \mathbb{R}_{0,m})$, $(k \in \mathbb{N})$. *Then there is a function* $\tilde{f} \in C^k(\mathbb{R}^{m+1}, \mathbb{R}_{0,m})$ *such that*

(i) $\tilde{f}|_E = f$,
(ii) $D_\alpha \tilde{f}|_E = D_\alpha f$ *for* $|\alpha| \leq k$,
(iii) $\tilde{f} \in C^\infty(\mathbb{R}^{m+1} \backslash E, \mathbb{R}_{0,m})$.

3 ψ-Hyperdifferentiability, ψ-Hyperderivability and ψ-Hyperholomorphy in Clifford Analysis

For the sake of simplicity and without any loss of generality in the discussion of the theory of ψ-hyperderivation we restrict ourselves to structural sets with $\psi^0 = 1$. By virtue of (5), we have for this case that

$$\overline{\psi^i} = -\psi^i \quad \text{for every } \ i \in \mathbb{N}_m. \tag{12}$$

The ψ-hyperderivation when $\psi^0 \notin \mathbb{R}$ can be treated in the same way after first applying a rotation given by $\overline{\psi^0}$.

In what follows we will use the following differential forms: the volume element in $\mathbb{R}^{m+1}$ given by the real valued $(m+1)$-form

$$\mathrm{d}V_x := \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$$

and the surface element induced by a structural set ψ

$$\sigma_{\psi,x} := \sum_{i=0}^{m} (-1)^i \psi^i \mathrm{d}\hat{x}_i,$$

where $\mathrm{d}\hat{x}_i$ is the differential m-form obtained from $\mathrm{d}V_x$ by omitting the factor $\mathrm{d}x_i$, for $i \in \mathbb{N}_m^0$.

Since Γ is a smooth surface in $\mathbb{R}^{m+1}$ then we can write

$$\sigma_{\psi,x} = n_{\psi}\, \mathrm{d}S_x, \tag{13}$$

where $\mathrm{d}S_x$ is the elementary surface element in $\mathbb{R}^{m+1}$. In what follows, we shall not distinguish between the two expressions of $\sigma_{\psi,\xi}$.

Moreover, we will use the $(m-1)$-differential form

$$\tau_{\psi,x} := \sum_{i=1}^{m} (-1)^i \psi^i\, \mathrm{d}\hat{x}_{0,i}.$$

Here $\mathrm{d}\hat{x}_{0,i}$ denotes the $(m-1)$-differential form obtained from $\mathrm{d}\hat{x}_0$ by omitting the factor $\mathrm{d}x_i$, $i \in \mathbb{N}_m$.

In the preliminaries we already gave a global definition (by means of the ψ-Cauchy–Riemann operator) of ψ-hyperholomorphy associated to an arbitrary structural set ψ. This is a global definition in the sense that it is related to the whole domain Ω. Now let us concentrate on a local version of ψ-hyperholomorphy.

Definition 2 A function $f \in C^1(\Omega, \mathbb{R}_{0,m})$ is called ψ-hyperholomorphic at $x^0 \in \Omega$ if f is ψ-hyperholomorphic in some open neighborhood $V(x^0) \subset \Omega$ of x^0.

Definition 3 Given $x \in \Omega$, a function $f \in C^1(\Omega, \mathbb{R}_{0,m})$ is called ψ-hyperdifferentiable at x if there is a Clifford number denoted by $f'_{\psi}(x)$, such that

$$d(\tau_{\psi,x}\, f(x)) = \sigma_{\psi,x}\, f'_{\psi}(x). \tag{14}$$

The Clifford number $f'_{\psi}(x)$ is named the ψ-hyperderivative of f at x. The function f is called ψ-hyperdifferentiable in Ω if it is ψ-hyperdifferentiable at every $x \in \Omega$.

In quaternionic and standard Clifford cases, the above notions are fully discussed in [13–16,18–20] and are well known. The following result can be found in the works cited and it is the basis for the justification of the consideration of ${}^{\overline{\psi}}D[f]$ as the ψ-hyperderivative of f.

Proposition 1 *Let* $f \in C^1(\Omega, \mathbb{R}_{0,m})$. *Then*

$$d(\tau_{\psi,x}\, f(x)) = \frac{1}{2}\sigma_{\psi,x}\, {}^{\overline{\psi}}D[f](x) - \frac{1}{2}\sigma_{\overline{\psi},x}\, {}^{\psi}D[f](x). \tag{15}$$

From the relation (15) it is possible to establish the connection between ψ-hyperholomorphy and ψ-hyperdifferentiability.

Theorem 7 *Let* $f \in C^1(\Omega, \mathbb{R}_{0,m})$. *Then* f *is* ψ*-hyperholomorphic at* $x^0 \in \Omega$ *if and only if* f *is* ψ*-hyperdifferentiable at* x^0 *and for such functions we have*

$$f'_{\psi}(x^0) = \frac{1}{2}\, {}^{\overline{\psi}}D[f](x^0). \tag{16}$$

Remark 3.1 When $\psi^0 \notin \mathbb{R}$, formula (16) takes the form

$$f'_\psi(x^0) = \frac{1}{2}\overline{\psi}\, D[\psi^0 f](x^0).$$

3.1 The ψ-Hyperderivative as a Limit of a Quotient of Increments

In [13, Sec. 2.4] the hypercomplex derivative (in Standard Clifford analysis) is also studied in terms of the limit of the quotient of increments both of the function and of the variable. The task is now to obtain an analogue in our case.

Let us define a non-degenerate m-dimensional parallelepiped with vertex $x^0 \in \mathbb{R}^{m+1}$ and edge vectors $\{v_1, \dots, v_m\} \subset \mathbb{R}^{m+1}$ (these vectors are linearly independent over $\mathbb{R}$ as vectors of $\mathbb{R}^{m+1}$) by

$$\Pi = \left\{ x^0 + \sum_{i=1}^{m} t_i v_i \, \in \mathbb{R}^{m+1} : \, (t_1, \dots, t_m) \in [0, 1]^m \right\},$$

and its boundary by

$$\partial\Pi = \left\{ x^0 + \sum_{i=1}^{m} t_i v_i \, \in \mathbb{R}^{m+1} : \, (t_1, \dots, t_m) \in \partial[0, 1]^m \right\}.$$

The following result relates the number $f'_\psi(x) \in \mathbb{R}_{0,m}$ with a limit of a "quotient of increments". It can be proved following standard arguments, see for example [13].

Theorem 8 *Let $f : \Omega \to \mathbb{R}_{0,m}$ be ψ-hyperholomorphic at x^0 and let $f'_\psi(x^0)$ be its ψ-hyperderivative. Then for every sequence $\{\Pi_k\}_{k\in\mathbb{N}}$ of non-degenerate oriented m-parallelepiped with vertex x^0 the equality*

$$\lim_{k\to\infty} \left[\left(\int_{\Pi_k} \sigma_{\psi,x} \right)^{-1} \left(\int_{\partial\Pi_k} \tau_{\psi,x} \cdot f(x) \right) \right] = f'_\psi(x^0) \tag{17}$$

holds if $\lim_{k\to\infty}$ *diam* $\Pi_k = 0$.

The above theorem can be used to introduce a directional ψ-hyperderivative. The "directions" to be considered are given by hyperplanes $L \subset \mathbb{R}^{m+1}$ with equation

$$\gamma(x) := \sum_{i=0}^{m} n_i x_i + d = 0,$$

where $(n_0, \dots, n_m)$ is the unit normal vector to L and $d \in \mathbb{R}$.

Definition 4 Let $x^0 \in L \cap \Omega$. A function $f : \Omega \to \mathbb{R}_{0,m}$ is called ψ-hyperderivable at x^0 along L, if for any sequence $\{\Pi_k\}_{k\in\mathbb{N}}$, $\Pi_k \subset L$, and such that $\lim_{k\to\infty} \operatorname{diam}\Pi_k = 0$, of non-degenerate m-parallelepipeds with vertex x^0, the limit

$$\lim_{k\to\infty}\left[\left(\int_{\Pi_k} \sigma_{\psi,x}\right)^{-1}\left(\int_{\partial\Pi_k} \tau_{\psi,x}\cdot f(x)\right)\right], \tag{18}$$

exists and does not depend on the choice of the sequence $\{\Pi_k\}_{k\in\mathbb{N}}$. If it exists, this limit is called the m-dimensional directional ψ-hyperderivative of f along the hyperplane L and it will be denoted by $f'_{\psi,L}(x^0)$.

Notice that, this definition works only for families of parallelepipeds Π_k fully contained in L, in contrast with the conditions in Theorem 8 where the parallelepipeds are free to move in $\mathbb{R}^{m+1}$.

Theorem 9 *Let $V(x^0)$ be an $(m+1)$-dimensional neighborhood of $x^0 \in \mathbb{R}^{m+1}$. Let $f \in C^1\left(V(x^0), \mathbb{R}_{0,m}\right)$. Then, f is ψ-hyperderivable at x^0 along any hyperplane $L \ni x^0$.*

Proof Using standard techniques we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\left[\left(\int_{\Pi_k} \sigma_{\psi,x}\right)^{-1}\left(\int_{\partial\Pi_k} \tau_{\psi,x}\cdot f(x)\right)\right]\\
&\quad= \frac{1}{2}\lim_{k\to\infty}\left[\left(\int_{\Pi_k} \sigma_{\psi,x}\right)^{-1}\cdot\left(\int_{\Pi_k} \sigma_{\psi,x}\left({}^{\overline{\psi}}D[f](x) - n^2_{\overline{\psi}}\,{}^{\psi}D[f](x)\right)\right)\right]. \qquad (19)
\end{aligned}$$

Since $f \in C^1\left(V(x^0), \mathbb{R}_{0,m}\right)$ the limit on the left side exists and it does not depend on the choice of $\{\Pi_k\}_{k\in\mathbb{N}}$, which completes the proof. $\square$

The equality (19) immediately gives the following two results, which contain generalizations of [14, Corollaries 3.3.1, 3.3.2].

Corollary 2 *Under the conditions of Theorem 9 we have*

$$f'_{\psi,L}(x^0) = \frac{1}{2}\left({}^{\overline{\psi}}D[f](x^0) - n^2_{\overline{\psi}}\,{}^{\psi}D[f](x^0)\right). \tag{20}$$

Corollary 3 *Let $f \in C^1\left(V(x^0), \mathbb{R}_{0,m}\right)$. Then f is ψ-hyperholomorphic at x^0 if and only if $f'_{\psi,L}(x^0)$ does not depend on the hyperplane L.*

4 The Notion of φ-Hyperderivative for ψ-Hyperholomorphic Functions

In the previous section, we have stated the building blocks of a ψ-hyperderivation theory which can be obtained directly by means of standard arguments, see [13]. The notion of hyperderivative was given, as in the classical case, by the conjugate Cauchy–Riemann operator. But by using a second structural set φ we have more possibilities to define the notion of hyperderivability for certain kinds of hyperholomorphic functions. The main goal now is to extend our approach to the notion of φ-hyperderivative for ψ-hyperholomorphic functions for a pair of structural sets (φ, ψ). To do this, we need the following generalization of the differential form $\tau_{\psi,x}$:

$$\tau_{\varphi,\psi,x} = \sum_{0\le i<j\le m} (-1)^{i+j+1} \left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right) \mathrm{d}\hat{x}_{i,j}.$$

Remark 4.1 Observe that

- We need the condition $\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j \neq 0$ for at least one pair of $0 \le i < j \le m$. But this will always hold if the choice of φ and ψ is not trivial, i.e., non left-equivalent structural sets. In fact, if we write $\varphi^i\overline{\psi^i} = h_i$ for every $i = 0, \ldots m$, we obtain from the condition $\overline{\varphi^i}\varphi^j = \overline{\psi^i}\psi^j$ that $\overline{\psi^i}\,\overline{h_i}h_j\psi^j = \overline{\psi^i}\psi^j$ and equivalently that $h_i = h_j$. From now on, we will consider only non-equivalent pairs of structural sets.
- When $\varphi = \overline{\psi}$ and $\psi^0 = 1$ we have for $0 < i < j$ that:

$$\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j = \psi^i\overline{\psi^j} - \overline{\psi^i}\psi^j = -\psi^i\psi^j + \psi^i\psi^j = 0.$$

  And for $i = 0$ we obtain: $\overline{\varphi^0}\varphi^j - \overline{\psi^0}\psi^j = \overline{\psi^j} - \psi^j = -2\psi^j$. Then,

$$\tau_{\overline{\psi},\psi,x} = 2\sum_{j=1}^{m}(-1)^j\psi^j\,\mathrm{d}\hat{x}_{0,j} = 2\tau_{\psi,x}.$$

The introduction of this new differential form allows us to obtain the following key result which involves two arbitrary structural sets.

Theorem 10 *Let φ, ψ be an arbitrary pair of structural sets. Then, for every $f \in C^1(\Omega, \mathbb{R}_{0,m})$*

$$d(\tau_{\varphi,\psi,x} f(x)) = \sigma_{\overline{\varphi},x}\,{}^{\varphi}D[f](x) - \sigma_{\overline{\psi},x}\,{}^{\psi}D[f](x) \tag{21}$$

holds.

Proof

$$\begin{aligned}
\mathrm{d}(\tau_{\varphi,\psi,x} f(x)) &= (-1)^{m-1}\tau_{\varphi,\psi,x} \wedge d(f(x)) \\
&= (-1)^{m-1}\left(\sum_{0\le i<j\le m} (-1)^{i+j+1}\left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right)\mathrm{d}\hat{x}_{i,j}\right) \wedge \left(\sum_{k=0}^{m}\frac{\partial f}{\partial x_k}(x)\,\mathrm{d}x_k\right) \\
&= (-1)^{m-1}\sum_{\substack{0\le i<j\le m\\ 0\le k\le m}} (-1)^{i+j+1}\left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right)\frac{\partial f}{\partial x_k}(x)\,\mathrm{d}\hat{x}_{i,j}\wedge \mathrm{d}x_k.
\end{aligned}$$

But $d\hat{x}_{i,j} \wedge dx_k = 0$ for $k \neq i, k \neq j$; and $\begin{cases} d\hat{x}_{i,j} \wedge dx_i = (-1)^{m-i-1} d\hat{x}_j, \\ d\hat{x}_{i,j} \wedge dx_j = (-1)^{m-j} d\hat{x}_i. \end{cases}$

Hence, we get

$$
\begin{aligned}
&d(\tau_{\varphi,\psi,x} f(x)) \\
&= (-1)^{m-1} \sum_{0\le i<j\le m} (-1)^{i+j+1} \left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right) \\
&\quad \times \left[(-1)^{m-i-1} \frac{\partial f}{\partial x_i}(x) d\hat{x}_j + (-1)^{m-j} \frac{\partial f}{\partial x_j}(x) d\hat{x}_i\right] \\
&= \sum_{i=0}^{m} \left(\sum_{j=0}^{i-1} (-1)^{i-1} \left(\overline{\varphi^j}\varphi^i - \overline{\psi^j}\psi^i\right) \frac{\partial f}{\partial x_j}(x) \right. \\
&\quad \left. + \sum_{j=i+1}^{m} (-1)^i \left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right) \frac{\partial f}{\partial x_j}(x) \right) d\hat{x}_i \\
&= \sum_{i=0}^{m} (-1)^i \left(\sum_{j=0}^{i-1} \left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right) \frac{\partial f}{\partial x_j}(x) + \sum_{j=i+1}^{m} \left(\overline{\varphi^i}\varphi^j - \overline{\psi^i}\psi^j\right) \frac{\partial f}{\partial x_j}(x) \right) d\hat{x}_i \\
&= \sum_{i=0}^{m} (-1)^i \left(\overline{\varphi^i} \sum_{j\neq i} \varphi^j \frac{\partial f}{\partial x_j}(x) - \overline{\psi^i} \sum_{j\neq i} \psi^j \frac{\partial f}{\partial x_j}(x) \right) d\hat{x}_i \\
&= \sum_{i=0}^{m} (-1)^i \left(\overline{\varphi^i}\, {}^{\varphi}D[f](x) - \overline{\psi^i}\, {}^{\psi}D[f](x) \right) d\hat{x}_i \\
&= \left(\sum_{i=0}^{m} (-1)^i \overline{\varphi^i}\, d\hat{x}_i \right) {}^{\varphi}D[f](x) - \left(\sum_{i=0}^{m} (-1)^i \overline{\psi^i}\, d\hat{x}_i \right) {}^{\psi}D[f](x) \\
&= \sigma_{\overline{\varphi},x}\, {}^{\varphi}D[f](x) - \sigma_{\overline{\psi},x}\, {}^{\psi}D[f](x).
\end{aligned}
$$

□

Remark 4.2 Observe that

- (21) is a generalization of (15). In fact, (15) can be obtained from (21) by taking $\varphi = \overline{\psi}$ and $\psi^0 = 1$. It also constitutes a generalization of the corresponding theorems of [26], Mitelman/Shapiro and Gürlebeck/Malonek since (21) is not only restricted to D and $\bar{D}$.
- For the conjugation of $\tau_{\varphi,\psi,x}$ we have:

$$
\begin{aligned}
\overline{\tau_{\varphi,\psi,x}} &= \sum_{0\le i<j\le m} (-1)^{i+j+1} \left(\overline{\varphi^j}\varphi^i - \overline{\psi^j}\psi^i\right) d\hat{x}_{i,j} \\
&= \sum_{0\le i<j\le m} (-1)^{i+j+1} \left(\overline{\psi^i}\psi^j - \overline{\varphi^i}\varphi^j\right) d\hat{x}_{i,j} = \tau_{\psi,\varphi,x}.
\end{aligned}
$$

Applying (21) for the pair of structural sets ψ, φ and $f=\overline{g}$ we obtain,

$$\mathrm{d}(\tau_{\psi,\varphi,x}\overline{g}(x))=\sigma_{\overline{\psi},x}{}^{\psi}D[\overline{g}](x)-\sigma_{\overline{\varphi},x}{}^{\varphi}D[\overline{g}](x).$$

By conjugation we have:

$$\mathrm{d}(g(x)\tau_{\varphi,\psi,x})=D^{\overline{\psi}}[g](x)\sigma_{\psi,x}-D^{\overline{\varphi}}[g](x)\sigma_{\varphi,x}. \tag{22}$$

Then, using (21) and (22) we obtain:

$$\begin{aligned}\mathrm{d}\left(g(x)\tau_{\varphi,\psi,x}f(x)\right)&=\mathrm{d}\left(g(x)\tau_{\varphi,\psi,x}\right)\wedge f(x)+(-1)^{m-1}g(x)\tau_{\varphi,\psi,x}\wedge\mathrm{d}(f(x))\\&=\mathrm{d}\left(g(x)\tau_{\varphi,\psi,x}\right)f(x)+g(x)\mathrm{d}\left(\tau_{\varphi,\psi,x}f(x)\right)\\&=D^{\overline{\psi}}[g](x)\sigma_{\psi,x}f(x)-D^{\overline{\varphi}}[g](x)\sigma_{\varphi,x}f(x)\\&\quad+g(x)\sigma_{\overline{\varphi},x}{}^{\varphi}D[f](x)-g(x)\sigma_{\overline{\psi},x}{}^{\psi}D[f](x).\end{aligned} \tag{23}$$

Now, we are in a position to introduce the notion of φ-hyperderivative for ψ-hyperholomorphic functions.

Definition 5 A function $f\in C^1(\Omega,\mathbb{R}_{0,m})$ is called φ-hyperdifferentiable in the ψ-sense in Ω if for any $x\in\Omega$ there is a Clifford number denoted by $f'_{\varphi,\psi}(x)$, such that

$$d(\tau_{\varphi,\psi,x}\,f(x))=\sigma_{\overline{\varphi},x}\;f'_{\varphi,\psi}(x).$$

The Clifford number $f'_{\varphi,\psi}(x)$ is named the φ-hyperderivative in the ψ-sense of f at x.

Theorem 11 *Let $f\in C^1(\Omega,\mathbb{R}_{0,m})$. Then f is ψ-hyperholomorphic at $x^0\in\Omega$ if and only if f is φ-hyperdifferentiable in the ψ-sense at x^0 and*

$$f'_{\varphi,\psi}(x^0)={}^{\varphi}D[f](x^0).$$

Proof The function f is φ-hyperdifferentiable in the ψ-sense if and only if $\sigma_{\overline{\varphi},x}{}^{\varphi}D[f](x)-\sigma_{\overline{\psi},x}{}^{\psi}D[f](x)=\sigma_{\overline{\varphi},x}\;f'_{\varphi,\psi}(x)$ which is equivalent to,

$$\begin{aligned}&\sigma_{\overline{\varphi},x}\left[{}^{\varphi}D[f](x)-f'_{\varphi,\psi}(x)\right]-\sigma_{\overline{\psi},x}{}^{\psi}D[f](x)\\&\quad=\sum_{i=0}^{m}(-1)^i\left[\overline{\varphi}^i\left({}^{\varphi}D[f](x)-f'_{\varphi,\psi}(x)\right)-\overline{\psi^i}{}^{\psi}D[f](x)\right]\mathrm{d}\hat{x}_i=0.\end{aligned}$$

Hence,

$$\left({}^{\varphi}D[f](x)-f'_{\varphi,\psi}(x)\right)-\varphi^i\overline{\psi^i}{}^{\psi}D[f](x)=0,\ \text{ for every } i\in\mathbb{N}_m^0.$$

But, since φ and ψ are not equivalent structural sets we have that $\varphi^i\overline{\psi^i}\neq\varphi^j\overline{\psi^j}$ for at least one pair $0\le i,j\le m$. Therefore, ${}^{\psi}D[f](x)=0$ and $f'_{\varphi,\psi}(x)={}^{\varphi}D[f](x)$. □

From (21) it is possible to express the φ-hyperderivative in the ψ-sense of a ψ-hyperholomorphic function in terms of an specific quotient of increments. This follows by the same methods as in the previous section. Moreover, we can define a notion of a m-directional φ-hyperderivative in the ψ-sense with analogous properties to the single m-directional ψ-hyperderivartive.

5 Hyperderivation of the Cauchy Type Integral

Using our concept of φ-hyperdifferentiation in the ψ-sense we can study the operator ${}^{\varphi}D^{\psi}K_{\Gamma}$.

As before let $\Omega \subset \mathbb{R}^{m+1}$ be a bounded domain, which is now assumed to be simply connected and let $\Gamma := \{\xi \in \mathbb{R}^{m+1} : \varrho(\xi) = 0\}$, where $\varrho \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$ and grad $\varrho(\xi) \neq 0$ for each $\xi \in \Gamma$. This assumption means that there exists an outward normal vector at every point $\xi \in \Gamma$ and as a consequence the tangent plane is well defined everywhere in Γ. If we replace $g(\xi)$ by $K_{\overline{\varphi}}(\xi - x)$ in (23) and assume $f \in C^1(\Gamma, \mathbb{R}_{0,m})$, we get for each $x \notin \Gamma$ that

$$\begin{aligned} d_{\xi}\left(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)\right) = D_{\xi}^{\overline{\psi}}[K_{\overline{\varphi}}(\xi - x)]\sigma_{\psi,\xi} f(\xi) + K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\varphi},\xi}\, {}^{\varphi}D[f](\xi) \\ -K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\psi},\xi}\, {}^{\psi}D[f](\xi). \end{aligned} \tag{24}$$

Let us prove now that

$$\int_{\Gamma} d_{\xi}(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)) = 0. \tag{25}$$

In fact, if $x \in \Omega$, we can take $\epsilon > 0$ such that $B[x,\epsilon] \subset \Omega$, where $B[x,\epsilon]$ is the ball of the radius ϵ centered in x. Then, combining Stokes formula with Theorem 6 and taking into account that the coefficients of the $(m-1)$-differential form $K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi}\widetilde{f}(\xi)$ belong to $C^2(\mathbb{R}^{m+1}\backslash\{\Gamma \cup \{x\}\}, \mathbb{R}_{0,m})$, we have

$$\begin{aligned} &\int_{\Gamma} d_{\xi}(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)) \\ &= \int_{\Omega\backslash B[x,\epsilon]} d_{\xi}^2(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi}\widetilde{f}(\xi)) + \int_{\partial B[x,\epsilon]} d_{\xi}(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi}\widetilde{f}(\xi)) \\ &= \int_{\partial B[x,\epsilon]} d_{\xi}(K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi}\widetilde{f}(\xi)). \end{aligned}$$

But for $\xi \in \partial B[x,\epsilon]$ it is clear that $K_{\overline{\varphi}}(\xi - x) = \dfrac{(\xi - x)_{\varphi}}{|\mathbb{S}^n|\epsilon^{m+1}}$. Keeping in mind that the coefficients of the $(m-1)$-differential form $(\xi - x)_{\varphi}\ \tau_{\varphi,\psi,\xi}\widetilde{f}(\xi)$ belong to $C^2(\mathbb{R}^{m+1}\backslash\Gamma, \mathbb{R}_{0,m})$, we obtain

$$\int_\Gamma d_\xi (K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)) = \frac{1}{|\mathbb{S}^n|\epsilon^{m+1}} \int_{\partial B[x,\epsilon]} d_\xi((\xi - x)_\varphi \ \tau_{\varphi,\psi,\xi} \widetilde{f}(\xi))$$
$$= \frac{1}{|\mathbb{S}^n|\epsilon^{m+1}} \int_{B(x,\epsilon)} d_\xi^2((\xi - x)_\varphi \ \tau_{\varphi,\psi,\xi} \widetilde{f}(\xi)) = 0.$$

On the contrary, if $x \notin \overline{\Omega}$, we have

$$\int_\Gamma d_\xi (K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)) = \int_\Omega d_\xi^2 (K_{\overline{\varphi}}(\xi - x)\tau_{\varphi,\psi,\xi} f(\xi)) = 0.$$

Then, integrating over Γ in both sides of (24) and using

$$D_\xi^{\overline{\psi}}[K_{\overline{\varphi}}(\xi - x)] = D_\xi^{\overline{\psi}}\, {}^\varphi D_\xi[\theta_{m+1}(\xi - x)]$$
$$= {}^\varphi D_\xi D_\xi^{\overline{\psi}}[\theta_{m+1}(\xi - x)] = {}^\varphi D_\xi[K_\psi(\xi - x)] = -{}^\varphi D_x[K_\psi(\xi - x)],$$

we obtain

$$\begin{aligned} &{}^\varphi D_x \int_\Gamma K_\psi(\xi - x)\sigma_{\psi,\xi} f(\xi) \\ &= \int_\Gamma K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\varphi},\xi}\, {}^\varphi D[f](\xi) - \int_\Gamma K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\psi},\xi}\, {}^\psi D[f](\xi). \end{aligned} \tag{26}$$

But from (13) we have that

$$\int_\Gamma K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\psi},\xi}\, {}^\psi D[f](\xi) = \int_\Gamma K_{\overline{\varphi}}(\xi - x) n_{\overline{\psi}}(\xi)\, {}^\psi D[f](\xi)\, \mathrm{d}S_\xi$$
$$= \int_\Gamma K_{\overline{\varphi}}(\xi - x) n_{\overline{\varphi}}(\xi) n_\varphi(\xi) n_{\overline{\psi}}(\xi)\, {}^\psi D[f](\xi)\, \mathrm{d}S_\xi.$$

Then, (26) can be written in the following terms.

Theorem 12 *Let $\Omega \subset \mathbb{R}^{m+1}$ be a simply connected domain with boundary $\Gamma := \{\xi \in \mathbb{R}^{n+1} : \varrho(\xi) = 0\}$, where $\varrho \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$, grad $\varrho(\xi) \neq 0$ for all $\xi \in \Gamma$; and $f \in C^1(\Gamma, \mathbb{R}_{0,m})$. Then for all $x \notin \Gamma$*

$$\begin{aligned} &{}^\varphi D_x \int_\Gamma K_\psi(\xi - x)\sigma_{\psi,\xi} f(\xi) \\ &= \int_\Gamma K_{\overline{\varphi}}(\xi - x)\sigma_{\overline{\varphi},\xi} \left({}^\varphi D[f](\xi) - n_\varphi(\xi) n_{\overline{\psi}}(\xi)\, {}^\psi D[f](\xi)\right). \end{aligned} \tag{27}$$

Or, short,

$${}^\varphi D^\psi K_\Gamma[f](x) = {}^{\overline{\varphi}}K_\Gamma \left[{}^\varphi D[f] - n_\varphi n_{\overline{\psi}}\, {}^\psi D[f]\right](x)$$
$$= {}^{\overline{\varphi}}K_\Gamma\, {}^\varphi D[f](x) - {}^{\overline{\varphi},\overline{\psi}}K_\Gamma\, {}^\psi D[f](x),\ x \notin \Gamma.$$

Remark 5.1 When $\varphi = \overline{\psi}$ and $\psi^0 = 1$, the above result can be interpreted in terms of the m-directional ψ-hyperderivative introduced in Sect. 3.1: let $T(\xi)$ be the tangent hyperplane to Γ at a point $\xi \in \Gamma$. From (20), we have that (27) may be rewritten in a more evident and palpable form as

$$^{\overline{\psi}}D_x \int_\Gamma K_\psi(\xi - x)\, \sigma_{\psi,\xi}\, f(\xi) = 2 \int_\Gamma K_\psi(\xi - x)\, \sigma_{\psi,\xi}\ f'_{\psi,T(\xi)}(\xi). \tag{28}$$

In addition, taking into account Theorem 7, we proved that

$$({}^\psi K_\Gamma[f])'_\psi(x) = {}^\psi K_\Gamma[f'_{\psi,T}](x). \tag{29}$$

Formula (29) says that the ψ-hyperderivative of the Cliffordian ψ-Cauchy-type integral (the latter is a ψ-hyperholomorphic function, hence its ψ-hyperderivative is well defined) is again a ψ-Cauchy-type integral but now its density is the ψ-directional hyperderivative along the tangent hyperplanes.

It is obvious that for a ψ-hyperholomorphic function f, the ψ-hyperderivatives of any order $k \geq 1$: $f_\psi^{(k)} := ({}^{(k-1)}f_\psi)'_\psi$, $f_\psi^{(0)} = f$ are well defined; similarly for ψ-directional hyperderivatives. Thus, by (28)–(29), the following results are obtained by induction.

Corollary 4 *Let $k \in \mathbb{N}$, $f \in C^k(\Gamma, \mathbb{R}_{0,m})$ and $\varrho \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$. Then for each $x \notin \Gamma$*

$$^{\overline{\psi}}D_x^k \left(\int_\Gamma K_\psi(\xi - x)\, \sigma_{\psi,\xi}\ f(\xi) \right) = 2^k\ {}^\psi K_\Gamma[f_{\psi,T}^{(k)}](x),$$

or short

$$({}^\psi K_\Gamma[f])_\psi^{(k)}(x) = {}^\psi K_\Gamma\left[f_{\psi,T}^{(k)}\right](x).$$

Corollary 5 *Let $f \in C^k(\Gamma, \mathbb{R}_{0,m})$, $\varrho \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$. Then for $t \in \Gamma$ the following two limits exist:*

$$({}^\psi K_\Gamma[f])_\psi^{(k)\pm}(t) := \lim_{x \in \Omega^\pm \to t \in \Gamma} ({}^\psi K_\Gamma[f])_\psi^{(k)}(x)$$

and they are given by

$$({}^\psi K_\Gamma[f])_\psi^{(k)\pm}(t) = \frac{1}{2}\left[f_{\psi,T(t)}^{(k)}(t) + {}^\psi S_\Gamma\left[f_{\psi,T}^{(k)}\right](t) \right].$$

Remark 5.2 The above results generalize the corresponding ones obtained in [13,20].

6 An Alternative Proof

In this section, we will provide an alternative proof for Theorem 12 without the necessity of using the hyperderivative approach. In this new approach, the generalized Borel–Pompeiu formula in Theorem 5 will be the building block. Let us point out that this proof while less direct depends only on the smoothness assumptions for the generalized Borel–Pompeiu formula.

Alternative Proof of Theorem 12
Applying our operator ${}^{\varphi}D$ to both sides of the Borel–Pompeiu formula (9), inside of Ω, we get that

$$ {}^{\varphi}D^{\psi}K_{\Gamma}[f] = {}^{\varphi}D[\tilde{f}] - {}^{\varphi}D^{\psi}T_{\Omega}{}^{\psi}D[\tilde{f}], $$

where $\tilde{f} \in C^1(\overline{\Omega}, \mathbb{R}_{0,m})$ is the continuous extension of f obtained by Theorem 6.
Next, by virtue of the generalized Borel–Pompeiu formula (11) one has

$$ \begin{aligned} {}^{\varphi}D^{\psi}K_{\Gamma}[f] &= {}^{\varphi}D[\tilde{f}] - {}^{\varphi,\psi}\Pi_{\Omega}{}^{\psi}D[\tilde{f}] = {}^{\varphi}D[\tilde{f}] - {}^{\overline{\varphi},\overline{\psi}}K_{\Gamma}{}^{\psi}D[f] - {}^{\overline{\varphi}}T_{\Omega}{}^{\overline{\psi}}D^{\psi}D[\tilde{f}] \\ &= {}^{\varphi}D[\tilde{f}] - {}^{\overline{\varphi},\overline{\psi}}K_{\Gamma}{}^{\psi}D[f] - {}^{\overline{\varphi}}T_{\Omega}{}^{\overline{\varphi}}D^{\varphi}D[\tilde{f}] \\ &= {}^{\varphi}D[\tilde{f}] - {}^{\overline{\varphi},\overline{\psi}}K_{\Gamma}{}^{\psi}D[f] - ({}^{\varphi}D[\tilde{f}] - {}^{\overline{\varphi}}K_{\Gamma}{}^{\varphi}D[f]) \\ &= {}^{\overline{\varphi}}K_{\Gamma}{}^{\varphi}D[f] - {}^{\overline{\varphi},\overline{\psi}}K_{\Gamma}{}^{\psi}D[f]. \end{aligned} $$

If $x \in \mathbb{R}^{m+1}\backslash\overline{\Omega}$, the proof is similar. □

Remark 6.1 Theorem 12 says that the left φ-Cauchy–Riemann operator acting over the ψ-Cauchy-type integral of the function f is again a Cauchy-type integral, but now with structural set $\overline{\varphi}$ and density ${}^{\varphi}D[f] - n_{\varphi}n_{\overline{\psi}}{}^{\psi}D[f]$. In the same way as in the case of a single structural set, this will be the m-dimensional φ-directional hyperderivative in the ψ-sense.

The following statements are simply obtained by induction as a consequence of Theorem 12.

Corollary 6 *Let $k \in \mathbb{N}$, $f \in C^k(\Gamma, \mathbb{R}_{0,m})$, $g \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$ and let $\psi, \varphi_1, \ldots, \varphi_k$ be $(k+1)$ arbitrary structural sets. Then for each $x \notin \Gamma$*

$$ {}^{\varphi_k}D^{\varphi_{k-1}}D \cdots {}^{\varphi_1}D\ {}^{\psi}K_{\Gamma}[f](x) = {}^{\overline{\varphi}_k}K_{\Gamma}\prod_{i=1}^{k}\left({}^{\varphi_i}D - n_{\varphi_i}n_{\varphi_{i-1}}{}^{\overline{\varphi}_{i-1}}D\right)[f](x) $$

or in an equivalent form

$$\prod_{i=1}^{k} {}^{\varphi_k}D_x\left(\int_\Gamma K_\psi(\xi-x)n_\psi(\xi)f(\xi)\,d\Gamma_\xi\right)$$
$$=\int_\Gamma K_{\overline{\varphi}_k}(\xi-x)n_{\overline{\varphi}_k}(\xi)\prod_{i=1}^{k}\left({}^{\varphi_i}D-n_{\varphi_i}n_{\varphi_{i-1}}{}^{\overline{\varphi}_{i-1}}D\right)[f](\xi)\,\mathrm{d}S_\xi,$$

here φ_0 denotes the structural set $\overline{\psi}$.

Theorems 6 and 12, Plemelj–Sokhotski formulae (10), and Corollary 1 combined give the following result.

Corollary 7 (Sokhotski-Plemelj formulae for the boundary values of ${}^{\varphi}D^{\psi}K_\Gamma$). *Let $\Omega\subset\mathbb{R}^{m+1}$ be a domain and let $\Gamma:=\partial\Omega$ be a Lyapunov surface. Then for $f\in C^{1,\mu}(\Gamma,\mathbb{R}_{0,m})$, $\mu\in(0,1]$, and $t\in\Gamma$ the following two limits exist:*

$$({}^{\varphi}D^{\psi}K_\Gamma[f])^{\pm}(t):=\lim_{x\in\Omega^{\pm}\to t\in\Gamma}{}^{\varphi}D^{\psi}K_\Gamma[f](x)$$

and they are given by

$$({}^{\varphi}D^{\psi}K_\Gamma[f])^{\pm}(t)=\frac{1}{2}[{}^{\overline{\varphi}}S_\Gamma{}^{\varphi}D[f](t)-{}^{\overline{\varphi},\overline{\psi}}S_\Gamma{}^{\psi}D[f](t)$$
$$\pm({}^{\varphi}D[f](t)-n_\varphi(t)n_{\overline{\psi}}(t){}^{\psi}D[f](t))]. \tag{30}$$

Finally, using the equality (30) we obtain an expression for the jump of the function ${}^{\varphi}D^{\psi}K_\Gamma[f]$ on Γ.

$$({}^{\varphi}D^{\psi}K_\Gamma[f])^{+}(t)-({}^{\varphi}D^{\psi}K_\Gamma[f])^{-}(t)={}^{\varphi}D[f](t)-n_\varphi(t)n_{\overline{\psi}}(t){}^{\psi}D[f](t),\ t\in\Gamma.$$

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