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Results in Mathematics

On the Riemann Boundary Value Problem for Null Solutions to Iterated Generalized Cauchy–Riemann Operator in Clifford Analysis

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Abstract. In this paper we consider a kind of Riemann boundary value problem (for short RBVP) for null solutions to the iterated generalized Cauchy–Riemann operator and the polynomially generalized Cauchy–Riemann operator, on the sphere of $\mathbb{R}^{n+1}$ with Hölder-continuous boundary data. Making full use of the poly-Cauchy type integral operator in Clifford analysis, we give explicit integral expressions of solutions to this kind of boundary value problems over the sphere of $\mathbb{R}^{n+1}$. As special cases solutions of the corresponding boundary value problems for the classical poly-analytic and meta-analytic functions are also derived, respectively.

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1. Introduction

Riemann boundary value problems for poly-analytic and meta-analytic functions in the complex plane are important for applications in mathematical physics and engineering such as the theory of elasticity, special problems for Maxwell equations, and general relativity theory. They are widely discussed (see, for instance, references [1–5] or elsewhere). Making full use of complex analytic methods explicit integral representations of solutions to these boundary value problems were given (see, e.g., [3–5] or elsewhere).

As an elegant generalization of analytic functions from the complex plane to higher-dimensions, Clifford analysis (see e. g. [6–8]) concentrates on the study of the so-called monogenic functions, i.e. null solutions to the Dirac or the generalized Cauchy–Riemann operator (see Definition 2.1 in Sect. 2), which represent higher-dimensional generalizations of the classic Cauchy–Riemann operator. It can be seen as a refinement of classic harmonic analysis due to fact that these differential operators factorize the Laplacian. Using methods of Clifford analysis different kinds of partial differential equations over the various domains and their corresponding boundary value problems were investigated, e.g. in [9–23] and [24–31]. In references [9–16] and [17–23] solutions to such partial differential equations on bounded and unbounded domains of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ were given, respectively, in virtue of integral representations and Taylor series. In references [24–29], a kind of Riemann boundary value problem for monogenic functions and poly-monogenic functions, i.e. null solutions to iterated Dirac operator or generalized Cauchy–Riemann operator (see Definition 2.2 in Sect. 2), on bounded subdomains and half space of $\mathbb{R}^n$, were studied. In references [30,31], by applying Almansi/Fischer-type decomposition theorems (decompositions of poly-monogenic functions into monogenic functions) and integral representation formulae, we established explicit expressions of solutions to a kind of Riemann boundary value problems for the polynomially monogenic functions over the sphere and half space of $\mathbb{R}^{n+1}$, i.e. functions which are annihilated by a polynomial generalized Cauchy–Riemann operator. In this paper, based on ideas of the higher order Cauchy-type integral in complex analysis contained in [5,3], we first introduce the poly-Cauchy type integral operator, and then consider some kind of Riemann boundary value problems for the poly-monogenic functions on the sphere of $\mathbb{R}^{n+1}$. Using potential-theoretical arguments which is new and different from the approach in reference [30], we also present explicit expressions of solutions to Riemann boundary value problems on the sphere of $\mathbb{R}^{n+1}$. As special cases we derive solutions to Riemann boundary value problems for poly-analytic functions and meta-analytic functions in the complex plane (see references e.g. [1–5]), correspondingly.

The paper is organized as follows. In Sect. 2 we recall some basic facts about Clifford analysis which will be required in the sequel. In Sect. 3 we introduce the poly-Cauchy type integral operator and study its boundary behaviour including Plemelj–Sokhotski formulae. In the last section we consider a kind of Riemann boundary value problems for null solutions to the iterated generalized Cauchy–Riemann operator $\mathcal{D}^k$, the operator $(\mathcal{D}-\lambda)^k$ with $\lambda \in \mathbb{C}$ and a polynomially generalized Cauchy–Riemann operator $p(\mathcal{D})$ on the sphere of $\mathbb{R}^{n+1}$. We give explicit integral representations of solutions to the boundary value problems on the sphere of $\mathbb{R}^{n+1}$, respectively. As special cases we also derive the solutions to the corresponding Riemann boundary value problems for polyanalytic, meta-analytic (see e.g. [3–5]) and polynomially analytic functions in the complex plane.

2. Preliminaries and Notations

In this section we recall some basic facts about Clifford analysis which will be needed in the sequel. More details can be found in the literature, for instance [6–8,15–17,20–22,32,33].

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of the Euclidean space $\mathbb{R}^n$. Let $\mathbb{R}^n$ be endowed with a non-degenerate quadratic form of signature $(0, n)$. We denote by $\mathbb{R}_{0,n}$ the 2^n-dimensional real Clifford algebra constructed over $\mathbb{R}^n$ with basis $\{e_{\mathcal{A}} : \mathcal{A} = \{h_1, \ldots, h_r\} \in \mathcal{PN}, 1 \leqslant h_1 < h_r \leqslant n\}$, where $\mathcal{N}$ stands for the set $\{1, 2, \ldots, n\}$ and $\mathcal{PN}$ denotes for the family of all order-preserving subsets of $\mathcal{N}$. We denote $e_{\emptyset}$ as the identity element 1 and $e_{\mathcal{A}}$ as $e_{h_1 \cdots h_r}$ for $\mathcal{A} = \{h_1, \ldots, h_r\} \in \mathcal{PN}$. The product in $\mathbb{R}_{0,n}$ is defined by

$$\begin{cases} e_{\mathcal{A}} e_{\mathcal{B}} = (-1)^{N(\mathcal{A}\cap\mathcal{B})}(-1)^{P(\mathcal{A},\mathcal{B})} e_{\mathcal{A}\Delta\mathcal{B}}, & \text{if } \mathcal{A}, \mathcal{B} \in \mathcal{PN}, \\ \lambda\mu = \sum\limits_{\mathcal{A},\mathcal{B}\in\mathcal{PN}} \lambda_{\mathcal{A}} \mu_{\mathcal{B}} e_{\mathcal{A}} e_{\mathcal{B}}, & \text{if } \lambda = \sum\limits_{\mathcal{A}\in\mathcal{PN}} \lambda_{\mathcal{A}} e_{\mathcal{A}}, \mu = \sum\limits_{\mathcal{B}\in\mathcal{PN}} \mu_{\mathcal{B}} e_{\mathcal{B}}, \end{cases}$$

where $N(\mathcal{A})$ is the cardinal number of the set $\mathcal{A}$ and $P(\mathcal{A}, \mathcal{B}) = \sum_{j\in\mathcal{B}} P(\mathcal{A}, j)$, $P(\mathcal{A}, j) = N(\mathcal{Z})$ and $\mathcal{Z} = \{i : i \in \mathcal{A}, i > j\}$. It follows that in particular $e_i^2 = -1$ if $i = 1, 2, \ldots, n$ and $e_i e_j + e_j e_i = 0$ if $1 \leqslant i < j \leqslant n$. Thus the real Clifford algebra $\mathbb{R}_{0,n}$ is a real linear, associative, but non-commutative algebra.

For arbitrary $a \in \mathbb{R}_{0,n}$ we have $a = \sum_{k=0}^{n} \sum_{N(\mathcal{A})=k} a_{\mathcal{A}} e_{\mathcal{A}} = \sum_{k=0}^{n} [a]_k$, $a_{\mathcal{A}} \in \mathbb{R}$, where $[a]_k = \sum_{N(\mathcal{A})=k} a_{\mathcal{A}} e_{\mathcal{A}}$ is the so-called k-vector part of a ($k = 1, 2, \ldots, n$). The Euclidean space $\mathbb{R}^{n+1}$ is embedded in $\mathbb{R}_{0,n}$ by identifying $(x_0, x_1, x_2, \ldots, x_n)$ with the Clifford vector x given by $x = \sum_{j=0}^{n} e_j x_j$. The conjugation in $\mathbb{R}_{0,n}$ is defined by $\bar{a} = \sum_{\mathcal{A}} a_{\mathcal{A}} \bar{e}_{\mathcal{A}}, \bar{e}_{\mathcal{A}} = (-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}, N(\mathcal{A}) = k, a_{\mathcal{A}} \in \mathbb{R}$, and hence $\overline{ab} = \bar{b}\bar{a}$ for arbitrary $a, b \in \mathbb{R}_{0,n}$.

The complexified Clifford algebra $\mathbb{C}_n = \mathbb{R}_{0,n} \otimes \mathbb{C}$ is given by having complex-valued coefficients, i.e. $\mathbb{C}_n = \mathbb{R}_{0,n} \oplus i\mathbb{R}_{0,n}$. Arbitrary $\lambda \in \mathbb{C}_n$ may be written as $\lambda = a + ib, a, b \in \mathbb{R}_{0,n}$, leading to the conjugation $\bar{\lambda} = \bar{a} - i\bar{b}$, where the bar denotes the usual Clifford conjugation in $\mathbb{R}_{0,n}$. This leads to the inner product and its associated norm in $\mathbb{C}_n$ given by $(\lambda, \mu) = \left[\bar{\lambda}\mu\right]_0$

and $|\lambda| = \sqrt{[\bar{\lambda}\lambda]_0} = \left(\sum_{\mathcal{A}} |\lambda_{\mathcal{A}}|^2\right)^{\frac{1}{2}}$. This leads to $x\bar{x} = |x|^2$ for arbitrary $x \in \mathbb{R}^{n+1}$.

The first-order differential operator $\mathcal{D} = \sum_{j=0}^{n} e_j \partial_{x_j}$ is called the generalized Cauchy–Riemann operator and $\mathcal{D}\overline{\mathcal{D}} = \Delta_{n+1}$, where Δ_{n+1} is the Laplace operator in the space of $\mathbb{R}^{n+1}$.

Let Ω be a bounded subdomain of $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$. In what follows, we denote the interior of Ω by Ω^+, the exterior of Ω by Ω^-. Continuity, Hölder-continuity, continuous differentiability and so on, are defined for a $\mathbb{C}_n$-valued function $\phi = \sum_{\mathcal{A}} \phi_{\mathcal{A}} e_{\mathcal{A}} : \Omega(\subset \mathbb{R}^{n+1}) \to \mathbb{C}_n$ where $\phi_{\mathcal{A}} : \Omega(\subset \mathbb{R}^{n+1}) \to \mathbb{C}$, by being ascribed to each component $\phi_{\mathcal{A}}$. The corresponding spaces are denoted, respectively, by $\mathcal{C}(\Omega, \mathbb{C}_n)$, $\mathbb{H}^{\mu}(\Omega, \mathbb{C}_n)(0 < \mu \leqslant 1)$, $\mathcal{C}^1(\Omega, \mathbb{C}_n)$ and so on.

Definition 2.1. Null solutions to the generalized Cauchy–Riemann operator $\mathcal{D}$, that is, $\mathcal{D}\phi = 0$, are called (left-) monogenic functions. They are called right-monogenic functions in case where the generalized Cauchy–Riemann operator is applied from the right. The set of left-monogenic functions in Ω forms a right-module, denoted by $\mathbb{M}_{(r)}(\Omega, \mathbb{C}_n)$.

Definition 2.2. Null solutions to the iterated generalized Cauchy–Riemann operator $\mathcal{D}^k$, that is, $\mathcal{D}^k\phi = 0(k \geqslant 2, k \in \mathbb{N})$, are called poly-monogenic functions.

3. Poly-Cauchy Type Integral Operator

In this section we introduce the poly-Cauchy type integral operator and state several of its properties. In particular, we study its boundary behaviour including the Plemelj–Sokhotski formula.

In the following for arbitrary $k \geqslant 2, k \in \mathbb{N}$, we introduce the functions

$$E_{\lambda}^{j}(x) = e^{\lambda x_0} \frac{1}{w_{n+1}} \frac{x_0^j \overline{x}}{j!|x|^{n+1}} (j = 0, 1, 2, \ldots, k-1),\ x \in \mathbb{R}^{n+1} \backslash \{0\},\ \lambda \in \mathbb{C}, \quad (1)$$

$$\text{when } \lambda = 0, E^{j}(x) = \frac{1}{w_{n+1}} \frac{x_0^j \overline{x}}{j!|x|^{n+1}} (j = 0, 1, 2, \ldots, k-1),\ x \in \mathbb{R}^{n+1} \backslash \{0\}, \quad (2)$$

where w_{n+1} is the surface area of the unit sphere in $\mathbb{R}^{n+1}$ and $E_{\lambda} \triangleq E_{\lambda}^{0}, E \triangleq E^{0}$. We will show that these functions can be used as kernels for an integral operator, the so-called poly-Cauchy type integral operator. Let us first remark that for $j = 0$ and $\lambda = 0$ we have the usual Cauchy kernel, i.e. the fundamental solution of the generalized Cauchy–Riemann operator.

These functions have the following properties which can be easily checked by direct calculation.

Lemma 3.1. *Suppose the functions $E_\lambda^j(x)(j=0,1,2,\ldots,k-1)$ and $\lambda\in\mathbb{C}$ as above. Then*

$$\begin{cases}(\mathcal{D}_\lambda E_\lambda)(x)=E_\lambda(x)\mathcal{D}_\lambda=0, \quad x\in\mathbb{R}^{n+1}\backslash\{0\},\\ (\mathcal{D}_\lambda E_\lambda^{j+1})(x)=E_\lambda^{j+1}(x)\mathcal{D}_\lambda=E_\lambda^j(x), \quad x\in\mathbb{R}^{n+1}\backslash\{0\}, \quad 0\leqslant j<k-1,\\ (\mathcal{D}_\lambda^k E_\lambda^{k-1})(x)=E_\lambda^{k-1}(x)\mathcal{D}_\lambda^k=0, \quad x\in\mathbb{R}^{n+1}\backslash\{0\},\end{cases}$$

when $\lambda=0$,

$$\begin{cases}(\mathcal{D}E)(x)=E(x)\mathcal{D}=0, \quad x\in\mathbb{R}^{n+1}\backslash\{0\},\\ (\mathcal{D}E^{j+1})(x)=E^{j+1}(x)\mathcal{D}=E^j(x), \quad x\in\mathbb{R}^{n+1}\backslash\{0\}, \quad 0\leqslant j<k-1,\\ (\mathcal{D}^k E^{k-1})(x)=E^{k-1}(x)\mathcal{D}^k=0, \quad x\in\mathbb{R}^{n+1}\backslash\{0\},\end{cases}$$

where $\mathcal{D}_\lambda\triangleq\mathcal{D}-\lambda,(\mathcal{D}_\lambda^k E^{k-1})(x)\triangleq\mathcal{D}_\lambda^{k-1}(\mathcal{D}_\lambda E^{k-1}(x))$ and $E^{k-1}(x)\mathcal{D}_\lambda^k\triangleq\mathcal{D}_\lambda^{k-1}(E^{k-1}(x)\mathcal{D}_\lambda),(\mathcal{D}^k E^{k-1})(x)\triangleq\mathcal{D}^{k-1}(\mathcal{D}E^{k-1}(x))$ and $E^{k-1}(x)\mathcal{D}^k\triangleq\mathcal{D}^{k-1}(E^{k-1}(x)\mathcal{D})$. Again, we would like to point out that the second property means that these functions are fundamental solutions of the respective operators.

In reference [21], the iterated generalized Cauchy–Riemann equation $\mathcal{D}^k\phi=0,k\geqslant 2,k\in\mathbb{N}$ on the unbounded subdomains of $\mathbb{R}^{n+1}$ with the condition $k<n+1$ was discussed. In fact, all of the related results in [21] still hold when the condition $k<n+1$ is cut.

Let us now introduce the poly-Cauchy type integral operators for $k\geqslant 2,k\in\mathbb{N}$,

$$\Phi_\lambda(x)=\sum_{j=0}^{k-1}\int_{\partial\Omega}E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0}f_j(y), \quad x\notin\partial\Omega, \quad \lambda\in\mathbb{C}, \tag{3}$$

with $\lambda\neq 0$, while, when $\lambda=0$, we consider the operator

$$\Phi(x)=\sum_{j=0}^{k-1}\int_{\partial\Omega}E^j(y-x)d\sigma_y f_j(y), \quad x\notin\partial\Omega, \tag{4}$$

where function $f_j\in\mathcal{C}(\partial\Omega,\mathbb{C}_n)$ for $l=1,2,\ldots,k-1$. For these integral operators we can get the following properties, immediately.

Lemma 3.2. *The above defined function Φ_λ is well-defined in $\mathbb{R}^{n+1}\backslash\partial\Omega$. Moreover, for $l=0,1,2,\ldots,k-1$, we have*

$$(\mathcal{D}_\lambda^l\Phi_\lambda)(x)=\sum_{j=0}^{k-l-1}\int_{\partial\Omega}E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0}f_{j+l}(y), \quad x\notin\partial\Omega, \tag{5}$$

$$(\mathcal{D}^l\Phi)(x)=\sum_{j=0}^{k-l-1}\int_{\partial\Omega}E^j(y-x)d\sigma_y f_{j+l}(y), \quad x\notin\partial\Omega. \tag{6}$$

For the case $l=k$ we have

$$\mathcal{D}_\lambda^k\Phi_\lambda(x)=0,x\notin\partial\Omega, \quad \textit{i.e. } \Phi_\lambda \textit{ is a solution to } \mathcal{D}_\lambda^k\phi(x)=0 \textit{ in } \mathbb{R}^{n+1}\backslash\partial\Omega,$$
$$\mathcal{D}^k\Phi(x)=0,x\notin\partial\Omega, \quad \textit{i.e. } \Phi \textit{ is a solution to } \mathcal{D}^k\phi(x)=0 \textit{ in } \mathbb{R}^{n+1}\backslash\partial\Omega.$$

Proof. Remember that Ω is a bounded subdomain with smooth boundary $\partial\Omega$. By applying the properties of the functions $E_\lambda^j(x)(j = 0, 1, 2, \ldots, k-1)$ in Lemma 3.1 we get the terms (5) and (6). □

Lemma 3.3. *Suppose $f_j \in \mathbb{H}^\mu(\partial\Omega, \mathbb{C}_n)(j = 0, 1, 2, \ldots, k-1)$. Then we get for the above defined function Φ_λ the following properties*

$$(i) \quad \sum_{j=0}^{k-l-1} \int_{\partial\Omega} E_\lambda^{j+l}(y-t) d\sigma_y e^{-\lambda y_0} f_{j+l}(y) \textit{ is well-defined on } \partial\Omega,$$

$$(ii) \textit{ Moreover, } \left(\mathcal{D}_\lambda^l \Phi_\lambda\right)^\pm(t) \triangleq \lim_{x \to t \in \partial\Omega} (\mathcal{D}_\lambda^l \Phi_\lambda)(x)$$
$$= \pm\frac{1}{2} f_l(t) + \sum_{j=0}^{k-l-1} \int_{\partial\Omega} E_\lambda^{j+l}(y-t) d\sigma_y e^{-\lambda y_0} f_{j+l}(y), x \notin \partial\Omega, \tag{7}$$

where for $j = 0$ the related singular integral in (i) exists in the sense of the Cauchy principle value. Especially, when $\lambda = 0$ we have

$$(iii) \quad \sum_{j=0}^{k-l-1} \int_{\partial\Omega} E^{j+l}(y-t) d\sigma_y f_{j+l}(y) \textit{ is well-defined on } \partial\Omega,$$

$$(iv) \textit{ Moreover, } (\mathcal{D}^l \Phi)^\pm(t) \triangleq \lim_{x \to t \in \partial\Omega} (\mathcal{D}^l \Phi)(x)$$
$$= \pm\frac{1}{2} f_l(t) + \sum_{j=0}^{k-l-1} \int_{\partial\Omega} E^{j+l}(y-t) d\sigma_y f_{j+l}(y), x \notin \partial\Omega, \tag{8}$$

where again for $j = 0$ the related singular integral in (iii) exists in the sense of the Cauchy principle value.

Proof. It is sufficient to consider the case of (i) and (ii). The case of (iii) and (iv) follows immediately as a special case.

For (i) we can first state that when $j = 0$, since $e^{-\lambda t_0} f_0 \in \mathbb{H}^\mu(\partial\Omega, \mathbb{C}_n)$, the singular integral operator

$$\int_{\partial\Omega} E_\lambda(y-t) d\sigma_y e^{-\lambda y_0} f_0(y),$$
$$t \in \partial\Omega \quad \text{exits in the sense of the Cauchy principle value.}$$

Furthermore, when $j = 1, 2, \ldots, k-1$, the integral operator

$$\int_{\partial\Omega} E_\lambda^j(y-t) d\sigma_y e^{-\lambda y_0} f_{j+l}(y) \text{ only has a weak singularity.}$$

Hence, for $l = 0, 1, 2, \ldots, k-1$, we obtain that

$$\sum_{j=0}^{k-l-1} \int_{\partial\Omega} E_\lambda^{j+l}(y-t) d\sigma_y e^{-\lambda y_0} f_{j+l}(y), \text{ is well-defined on } \partial\Omega.$$

For (ii), when $l = 0$, it is necessary to prove that for arbitrary $x \in \mathbb{R}^{n+1}\backslash\partial\Omega$, it holds

$$\Phi_\lambda^{\pm}(t) = \lim_{x\to t\in\partial\Omega} \Phi_\lambda(x) = \pm\frac{1}{2}f_0(t) + \sum_{j=0}^{k-1}\int_{\partial\Omega} E_\lambda^j(y-t)d\sigma_y e^{-\lambda y_0} f_j(y).$$

For arbitrary $x \notin \partial\Omega$, we have

$$\begin{aligned}\Phi_\lambda(x) &= \int_{\partial\Omega} E_\lambda(y-x)d\sigma_y e^{-\lambda y_0} f_0(y) + \sum_{j=1}^{k-1}\int_{\partial\Omega} E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0} f_j(y)\\ &= \Phi_{\lambda,0}(x) + \widehat{\Phi}_\lambda(x).\end{aligned}$$

On the one hand, associating $e^{-\lambda t_0} f_0 \in \mathbb{H}^\mu(\partial\Omega, \mathbb{C}_n)$, we get

$$\Phi_{\lambda,0}^{\pm}(t) = \lim_{x\to t\in\partial\Omega} \Phi_{\lambda,0}(x) = \pm\frac{1}{2}f_0(t) + \int_{\partial\Omega} E_\lambda(y-t)d\sigma_y e^{-\lambda y_0} f_0(y), x \notin \partial\Omega.$$

On the other hand, by the Lebesgue dominated convergence theorem, we have

$$\lim_{x\to t\in\partial\Omega} \widehat{\Phi}_\lambda(x) = \sum_{j=1}^{k-1}\int_{\partial\Omega} E_\lambda^j(y-t)d\sigma_y e^{-\lambda y_0} f_j(y), x \notin \partial\Omega.$$

Hence

$$\Phi_\lambda^{\pm}(t) = \pm\frac{1}{2}f_0(t) + \sum_{j=0}^{k-1}\int_{\partial\Omega} E_\lambda^j(y-t)d\sigma_y e^{-\lambda y_0} f_j(y).$$

Similarly, for $l = 1, 2, \ldots, k-1$, associating Lemma 3.2, we get for arbitrary $x \in \mathbb{R}^{n+1}\backslash\partial\Omega$,

$$\begin{aligned}(\mathcal{D}_\lambda^l \Phi_\lambda)^{\pm}(t) &= \lim_{x\to t\in\partial\Omega} (\mathcal{D}_\lambda^l \Phi_\lambda)(x)\\ &= \pm\frac{1}{2}f_l(t) + \sum_{j=0}^{k-l-1}\int_{\partial\Omega} E_\lambda^{j+l}(y-t)d\sigma_y e^{-\lambda y_0} f_{j+l}(y).\end{aligned}$$

□

Lemma 3.4. [30,31] *Suppose* $\phi \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ *is a solution to the equation* $\mathcal{D}_\lambda^k \phi(x) = 0, \lambda \in \mathbb{C}$, *then there exist the unique monogenic functions* $\phi_j \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$ *such that*

$$\phi(x) = \phi_0(x) + x_0 e^{\lambda x_0}\phi_1(x) + \cdots + x_0^{k-1} e^{\lambda x_0}\phi_{k-1}(x). \tag{9}$$

Lemma 3.5. [30,31,22] $\mathcal{D}_\lambda^k \phi(x) = 0, x \in \mathbb{R}^{n+1}$ *with* $\lambda \in \mathbb{C}$ *and for* $j = 0, 1, 2, \ldots, k-1$ *with* $k \in \mathbb{N}, k \geqslant 2$,

$$\liminf_{R\to+\infty} \frac{M(R, \mathcal{D}^j \phi_\lambda)}{R^{r-j}} = L_j < +\infty, r \geqslant k-1,$$

where $M(R, \mathcal{D}^j\phi_\lambda) = \max\limits_{|x|=R} \left|\mathcal{D}^j\phi_\lambda(x)\right|, \mathcal{D}^0\phi_\lambda(x) = \phi_\lambda(x)$ *with* $\phi_\lambda(x) = e^{-\lambda x_0}\phi(x)$ *and* r *is a non-negative integer, then* $\phi(x) = e^{\lambda x_0}P_r(x)$ *where function* $P_r(x)$ *is a polynomial function of total degree no greater than* r *on the variables* $x_i(i = 0, 1, \ldots, n)$*. Moreover, when* $r = 0$*, we have* $\phi(x) = de^{\lambda x_0}$*, where* d *is a* $\mathbb{C}_n$*-valued constant.*

4. Boundary Value Problems

In this section we consider a kind of Riemann boundary value problem for null solutions to a polynomially generalized Cauchy–Riemann operator, which includes the cases of powers of the generalized Cauchy–Riemann operator and $(\mathcal{D} - \lambda)^k(k \in \mathbb{N}, k \geqslant 2, \lambda \in \mathbb{C})$, on the ball centred at the origin with boundary values given by Hölder-continuous functions in Clifford analysis. Applying the poly-Cauchy type integral operator from the previous section, we get the explicit integral representations of their solutions. As special cases we also derive the solutions to Riemann boundary value problems for polyanalytic, metaanalytic (see e.g. [3–5]) and polynomially analytic functions in the complex plane.

In the sequel we denote the open unit ball centered at the origin by $B(1)$, for short B_+, whose closure is $\overline{B}(1)$, its boundary by S^n and $B_- = \mathbb{R}^{n+1}\backslash\overline{B}(1)$. We remark that $\omega \in S^n$ is the outward pointing unit normal vector of S^n. Furthermore, while we will only consider the Riemann boundary value problem on the ball centred at the origin the approach works for all bounded Lipschitz domains. For the case of half space of $\mathbb{R}^{n+1}$, it could be seen more details in Reference e.g. [31].

We are first interested in the following boundary value problem.

RBVP I. *Given the boundary data* $f_j \in \mathbb{H}^\mu(S^n, \mathbb{C}_n)(j = 0, 1, 2, \ldots, k-1)$ *with* $k \in \mathbb{N}, k \geqslant 2$*, find a function* $\phi \in \mathcal{C}^k(B_\pm, \mathbb{C}_n)$ *such that* $\mathcal{D}^l\phi(l = 1, 2, \ldots, k-1)$ *and* ϕ *are continuously extendable from* $B_\pm$ *to* S^n *and it holds*

$$(i)\begin{cases}\mathcal{D}^k\phi(x) = 0, x \in B_\pm,\\ \phi^+(t) = \phi^-(t), t \in S^n,\\ (\mathcal{D}\phi)^+(t) = (\mathcal{D}\phi)^-(t), t \in S^n\\ \qquad\vdots \qquad\qquad \vdots\\ (\mathcal{D}^l\phi)^+(t) = (\mathcal{D}^l\phi)^-(t), t \in S^n\\ \qquad\vdots \qquad\qquad \vdots\\ (\mathcal{D}^{k-1}\phi)^+(t) = (\mathcal{D}^{k-1}\phi)^-(t), t \in S^n.\end{cases}$$

The problem (i) *is also called jump problem.*

For this boundary value problem we can state the following theorem.

Theorem 4.1. *Boundary value problem* (i) *is solvable and the solution is given by*

$$\phi(x)=\sum_{j=0}^{k-1}\sum_{l_j=0}^{+\infty}x_0^j P_{l_j}(x),\quad x\in B_\pm, \tag{10}$$

where function $P_{l_j}(x)$ *is an inner spherical monogenic polynomial of order* $l_j(l_j=0,1,2,\dots)$ *on the variable* $x\in\mathbb{R}^{n+1}$.

Proof. Our approach is to transfer boundary value problem (i) into k mutually independent boundary value problems. Since $\mathcal{D}^k\phi(x)=0, x\in B_\pm$ by applying Lemma 3.4, we get the following decomposition into unique monogenic functions $\phi_j\in\mathcal{C}^1(\Omega,\mathbb{C}_n)(j=0,1,2,\dots,k-1)$, satisfying

$$\phi(x)=\phi_0(x)+x_0\phi_1(x)+\cdots+x_0^{k-1}\phi_{k-1}(x).$$

Since $\mathcal{D}(x_0^j\phi_j)=jx_0^{j-1}\phi_j, j=1,2,\dots,k-1$, one can easily show that boundary value problem (i) is equivalent to

$$\begin{cases}\mathcal{D}\phi_j(x)=0, x\in B_\pm, j=0,1,2,\dots,k-1,\\ \phi_0^+(t)=\phi_0^-(t), t\in S^n,\\ \phi_1^+(t)=\phi_1^-(t), t\in S^n\\ \qquad\vdots \qquad\qquad \vdots\\ \phi_l^+(t)=\phi_l^-(t), t\in S^n, 2\leqslant l\leqslant k-2,\\ \qquad\vdots \qquad\qquad \vdots\\ \phi_{k-1}^+(t)=\phi_{k-1}^-(t), t\in S^n.\end{cases}$$

Using Lemma 3.4 we have that the function ϕ satisfies the equation $\mathcal{D}^k\phi(x)=0$ in $\mathbb{R}^{n+1}$.

Moreover, by Theorem 11.3.4 in [6] we can use the expansion of monogenic functions into inner spherical monogenics to obtain our expression (10). □

By combining Theorem 4.1 with Lemma 3.5, we directly get the following corollary

Corollary 4.1. *Consider boundary value problem* (i). *If for* $j=0,1,2,\dots,k-1$ *with* $k\in\mathbb{N}, k\geqslant 2$,

$$\liminf_{R\to+\infty}\frac{M(R,\mathcal{D}^j\phi)}{R^{r-j}}=L_j<+\infty, r\geqslant k-1,$$

where $M(R,\mathcal{D}^j\phi)=\max_{|x|=R}\left|\mathcal{D}^j\phi(x)\right|, \mathcal{D}^0\phi(x)=\phi(x)$ *and* r *is a non-negative integer, then the solution to boundary value problem* (i) *is* $\phi(x)=P_r(x)$ *where* $P_r(x)$ *is a polynomial function of total degree no greater than* r *on the variable* $x\in\mathbb{R}^{n+1}$. *Moreover, when* $r=0$, *the solution to boundary value problem* (i) *is* $\phi(x)=d$, *where* $d\in\mathbb{C}_n$ *is a constant.*

Next, we will consider the following boundary value problem.

RBVP II. *Given the boundary data* $f_j \in \mathbb{H}^\mu(S^n, \mathbb{C}_n)(j = 0, 1, 2, \ldots, k-1)$ *with* $k \in \mathbb{N}, k \geqslant 2$, *find a function* $\phi \in \mathcal{C}^k(B_\pm, \mathbb{C}_n)$ *such that* $\mathcal{D}^l\phi(l = 1, 2, \ldots, k-1)$ *and* ϕ *are continuously extendable from* $B_\pm$ *to* S^n *and satisfy*

$$(ii) \begin{cases} \mathcal{D}^k\phi(x) = 0, x \in B_\pm, \\ \phi^+(t) = \phi^-(t)G + f_0(t), t \in S^n, \\ (\mathcal{D}\phi)^+(t) = (\mathcal{D}\phi)^-(t)G + f_1(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}^l\phi)^+(t) = (\mathcal{D}^l\phi)^-(t)G + f_l(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}^{k-1}\phi)^+(t) = (\mathcal{D}^{k-1}\phi)^-(t)G + f_{k-1}(t), t \in S^n, \end{cases}$$

where $G \in \mathbb{C}_n$ *is a constant and has an inverse denoted by* G^{-1}.

Theorem 4.2. *Boundary value problem* (ii) *is solvable and the solution can be expressed via*

$$\phi(x) = \begin{cases} \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x)d\sigma_y f_j(y) + \sum\limits_{j=0}^{k-1} \sum\limits_{l_j=0}^{+\infty} x_0^j P_{l_j}(x), x \in B_+, \\ \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x)d\sigma_y f_j(y)G^{-1} + \sum\limits_{j=0}^{k-1} \sum\limits_{l_j=0}^{+\infty} x_0^j P_{l_j}(x)G^{-1}, x \in B_-. \end{cases}$$

where function $P_{l_j}(x)$ *is an inner spherical monogenic polynomial of order* $l_j(l_j = 0, 1, 2, \ldots)$ *on the variable* $x \in \mathbb{R}^{n+1}$.

Proof. Using the poly-Cauchy type integral operator we can reduce boundary value problem (ii) into (i). By applying Lemmas 3.2 and 3.3 we have that

$$\Phi(x) = \sum_{j=0}^{k-1} \int_{S^n} E^j(y-x)d\sigma_y f_j(y), x \in B_\pm$$

is a solution to boundary value problem (ii).

Now, consider the function

$$\psi(x) = \begin{cases} \phi(x) - \Phi(x), x \in B_+, \\ \phi(x)G - \Phi(x), x \in B_-. \end{cases}$$

Then boundary value problem (ii) reduces to the previous case

$$\begin{cases} \mathcal{D}^k\psi(x) = 0, x \in B_\pm, \\ \psi^+(t) = \psi^-(t), t \in S^n, \\ (\mathcal{D}\psi)^+(t) = (\mathcal{D}\psi)^-(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}_\lambda^l\psi)^+(t) = (\mathcal{D}_\lambda^l\psi)^-(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}^{k-1}\psi)^+(t) = (\mathcal{D}^{k-1}\psi)^-(t), t \in S^n, \end{cases}$$

By virtue of Theorem 4.1 we obtain Theorem 4.2. □

Corollary 4.2. *Consider boundary value problem* (i). *If for* $j = 0, 1, 2, \ldots, k-1$ *with* $k \in \mathbb{N}, k \geqslant 2$,

$$\liminf_{R \to +\infty} \frac{M(R, \mathcal{D}^j \phi)}{R^{r-j}} = L_j < +\infty, r \geqslant k-1,$$

where $M(R, \mathcal{D}^j \phi) = \max_{|x|=R} \left|\mathcal{D}^j \phi(x)\right|, \mathcal{D}^0 \phi(x) = \phi(x)$, *and* r *is a non-negative integer, then the solution to boundary value problem* (ii) *is given by*

$$\phi(x) = \begin{cases} \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) + P_r(x), x \in B_+, \\ \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) G^{-1} + P_r(x) G^{-1}, x \in B_-, \end{cases}$$

where function $P_r(x)$ *is a polynomial function of total degree no greater than* r *on the variable* $x \in \mathbb{R}^{n+1}$. *Moreover, when* $r = 0$, *the solution to boundary value problem* (ii) *is as follows*

$$\phi(x) = \begin{cases} \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) + d, x \in B_+, \\ \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) G^{-1} + d G^{-1}, x \in B_-, \end{cases}$$

where $d \in \mathbb{C}_n$ *is a constant.*

RBVP III. *Given the boundary data* $f_j \in \mathbb{H}^\mu(S^n, \mathbb{C}_n)(j = 0, 1, 2, \ldots, k-1)$ *with* $k \in \mathbb{N}, k \geqslant 2$, *find a function* $\phi \in \mathcal{C}^k(B_\pm, \mathbb{C}_n)$ *such that* $\mathcal{D}_\lambda^l \phi(l = 1, 2, \ldots, k-1), \lambda \in \mathbb{C} \backslash \{0\}$ *and* ϕ *are continuously extendable from* $B_\pm$ *to* S^n *and satisfy*

$$(iii) \begin{cases} \mathcal{D}_\lambda^k \phi(x) = 0, x \in B_\pm, \\ \phi^+(t) = \phi^-(t) G + f_0(t), t \in S^n, \\ (\mathcal{D}_\lambda \phi)^+(t) = (\mathcal{D}_\lambda \phi)^-(t) G + f_1(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}_\lambda^l \phi)^+(t) = (\mathcal{D}_\lambda^l \phi)^-(t) G + f_l(t), t \in S^n, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathcal{D}_\lambda^{k-1} \phi)^+(t) = (\mathcal{D}_\lambda^{k-1} \phi)^-(t) G + f_{k-1}(t), t \in S^n, \end{cases}$$

where $G \in \mathbb{C}_n$ *is an invertible constant.*

Theorem 4.3. *Boundary value problem* (iii) *is solvable and its solution is expressed by*

$$\phi(x)=\begin{cases}\sum_{j=0}^{k-1}\int_{S^n}E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0}f_j(y)+\sum_{j=0}^{k-1}\sum_{l_j=0}^{+\infty}e^{\lambda x_0}x_0^jP_{l_j}(x), x\in B_+,\\ \sum_{j=0}^{k-1}\int_{S^n}E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0}f_j(y)G^{-1}\\ \quad+\sum_{j=0}^{k-1}\sum_{l_j=0}^{+\infty}e^{\lambda x_0}x_0^jP_{l_j}(x)G^{-1}, x\in B_-.\end{cases}$$

where function $P_{l_j}(x)$ is an inner spherical monogenic polynomial of order $l_j(l_j=0,1,2,\ldots)$ on the variable $x\in\mathbb{R}^{n+1}$.

Proof. Our method is to reduce boundary value problem (ii) to the previous case. Since $\lambda\in\mathbb{C}\backslash\{0\}$ and $e^{-\lambda t_0}f_j\in\mathbb{H}^\mu(S^n,\mathbb{C}_n)(j=0,1,2,\ldots,k-1)$, by applying Lemmas 3.2 and 3.3 we obtain

$$\Phi_\lambda(x)=\sum_{j=0}^{k-1}\int_{S^n}E_\lambda^j(y-x)d\sigma_y e^{-\lambda y_0}f_j(y), x\in B_\pm$$

is a solution to boundary value problem (iii). Consider the function

$$\Psi(x)=\begin{cases}\phi(x)-\Phi_\lambda(x), x\in B_+,\\ \phi(x)G-\Phi_\lambda(x), x\in B_-.\end{cases}$$

Then boundary value problem (iii) reduces to the case

$$(iv)\begin{cases}\mathcal{D}_\lambda^k\Psi(x)=0, x\in B_\pm,\\ \Psi^+(t)=\Psi^-(t), t\in S^n,\\ (\mathcal{D}_\lambda\Psi)^+(t)=(\mathcal{D}_\lambda\Psi)^-(t), t\in S^n,\\ \qquad\vdots\qquad\qquad\vdots\\ (\mathcal{D}_\lambda^l\Psi)^+(t)=(\mathcal{D}_\lambda^l\Psi)^-(t), t\in S^n,\\ \qquad\vdots\qquad\qquad\vdots\\ (\mathcal{D}_\lambda^{k-1}\Psi)^+(t)=(\mathcal{D}_\lambda^{k-1}\Psi)^-(t), t\in S^n.\end{cases}$$

Moreover, boundary value problem (iv) is equivalent to the case

$$\begin{cases}(\mathcal{D}^k e^{-\lambda x_0}\Psi)(x)=0, x\in B_\pm,\\ (e^{-\lambda t_0}\Psi)^+(t)=(e^{-\lambda t_0}\Psi)^-(t), t\in S^n,\\ (\mathcal{D}e^{-\lambda x_0}\Psi)^+(t)=(\mathcal{D}e^{-\lambda x_0}\Psi)^-(t), t\in S^n,\\ \qquad\vdots\qquad\qquad\vdots\\ (\mathcal{D}^l e^{-\lambda x_0}\Psi)^+(t)=(\mathcal{D}^l e^{-\lambda x_0}\Psi)^-(t), t\in S^n,\\ \qquad\vdots\qquad\qquad\vdots\\ (\mathcal{D}^{k-1}e^{-\lambda x_0}\Psi)^+(t)=(\mathcal{D}^{k-1}e^{-\lambda x_0}\Psi)^-(t), t\in S^n,\end{cases}$$

Now, by using Theorem 4.1, we obtain

$$\Psi(x)=e^{\lambda x_0}\sum_{j=0}^{k-1}\sum_{l_j=0}^{+\infty}x_0^jP_{l_j}(x), x\in B_\pm,$$

where function $P_{l_j}(x)$ is an inner spherical monogenic polynomial of order $l_j(l_j = 0, 1, 2, \ldots)$ on the variable $x \in \mathbb{R}^{n+1}$. Thus we get Theorem 4.3. $\square$

Corollary 4.3. *Consider boundary value problem (iii). If for $j = 0, 1, 2, \ldots, k-1$ with $k \in \mathbb{N}, k \geqslant 2$,*

$$\liminf_{R\to+\infty} \frac{M(R, \mathcal{D}^j \phi_\lambda)}{R^{r-j}} = L_j < +\infty, r \geqslant k-1,$$

where $M(R, \mathcal{D}^j \phi_\lambda) = \max_{|x|=R} \left|\mathcal{D}^j \phi_\lambda(x)\right|, \mathcal{D}^0 \phi_\lambda(x) = \phi_\lambda(x)$ with $\phi_\lambda(x) = e^{-\lambda x_0}\phi(x)$, and r is a non-negative integer, then the solution to boundary value problem (iii) is given by

$$\phi(x) = \begin{cases} \sum\limits_{j=0}^{k-1} \int_{S^n} E_\lambda^j(y-x) d\sigma_y e^{-\lambda y_0} f_j(y) + e^{\lambda x_0} P_r(x), x \in B_+, \\ \sum\limits_{j=0}^{k-1} \int_{S^n} E_\lambda^j(y-x) d\sigma_y e^{-\lambda y_0} f_j(y) G^{-1} + e^{\lambda x_0} P_r(x) G^{-1}, x \in B_-, \end{cases}$$

where function $P_r(x)$ is a polynomial function of total degree no greater than r on the variable $x \in \mathbb{R}^{n+1}$. Moreover, when $r = 0$, the solution to boundary value problem (iii) is given by

$$\phi(x) = \begin{cases} \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) + e^{\lambda x_0} d, x \in B_+, \\ \sum\limits_{j=0}^{k-1} \int_{S^n} E^j(y-x) d\sigma_y f_j(y) G^{-1} + e^{\lambda x_0} d G^{-1}, x \in B_-, \end{cases}$$

where $d \in \mathbb{C}_n$ is a constant.

Remark 1. When $n = 2$, boundary value problems (ii) and (iii) correspond to transmission problems for poly-monogenic functions and null solutions to iterated perturbed generalized Cauchy–Riemann operator $\mathcal{D}_\lambda, \lambda \in \mathbb{C}\backslash\{0\}$ on unit ball of $\mathbb{R}^3$, respectively. When $n = 1$ we have $\mathbb{R}^2 \approxeq \mathbb{C}$, i.e., $z = x_0 + x_1 e_1 \in \mathbb{C}$ and $\partial_{\bar{z}} = \partial_{x_0} + \partial_{x_1} e_1$ with $e_1^2 = -1$. $D = \{|z| = 1 : z \in \mathbb{C}\}$ denotes the unit disk in $\mathbb{C}$ with its boundary $\partial D = \{|z| = 1 : z \in \mathbb{C}\}$. In this case boundary value problems (ii) and (iii) reduce to the following cases, respectively,

$$(\ast) \begin{cases} \partial_{\bar{z}}^k \phi(z) = 0, z \in D, \\ \phi^+(t) = \phi^-(t) G + f_0(t), t \in \partial D, \\ \left[\partial_{\bar{z}} \phi\right]^+(t) = \left[\partial_{\bar{z}} \phi\right]^-(t) G + f_1(t), t \in \partial D, \\ \quad \vdots \qquad\qquad \vdots \\ \left[\partial_{\bar{z}}^l \phi\right]^+(t) = \left[\partial_{\bar{z}}^l \phi\right]^-(t) G + f_l(t), t \in \partial D, \\ \quad \vdots \qquad\qquad \vdots \\ \left[\partial_{\bar{z}}^{k-1} \phi\right]^+(t) = \left[\partial_{\bar{z}}^{k-1} \phi\right]^-(t) G + f_{k-1}(t), t \in \partial D, \\ \liminf\limits_{R\to+\infty} \frac{M(R, \partial_{\bar{z}}^j \phi)}{R^{r-j}} = L_j < +\infty, j = 0, 1, 2, \ldots, k-1, \end{cases}$$

with $f_l \in \mathbb{H}^\mu(\partial D, \mathbb{C})(l = 0, 1, 2, \ldots, k-1)$, $M(R, \partial_{\bar z}^l \phi) = \max_{|z|=R} \left|\partial_{\bar z}^l \phi(z)\right|$, and

$$(\circledast\circledast)\quad \begin{cases} (\partial_{\bar z} - \lambda)^k \phi(z) = 0, z \in D,\\ \phi^+(t) = \phi^-(t)G + f_0(t), t \in \partial D,\\ \left[(\partial_{\bar z} - \lambda)\phi\right]^+(t) = \left[(\partial_{\bar z} - \lambda)\phi\right]^-(t)G + f_1(t), t \in \partial D,\\ \quad\vdots \qquad\qquad\qquad \vdots\\ \left[(\partial_{\bar z} - \lambda)^l\phi\right]^+(t) = \left[(\partial_{\bar z} - \lambda)^l\phi\right]^-(t)G + f_l(t), t \in \partial D,\\ \quad\vdots \qquad\qquad\qquad \vdots\\ \left[(\partial_{\bar z} - \lambda)^{k-1}\phi\right]^+(t) = \left[(\partial_{\bar z} - \lambda)^{k-1}\phi\right]^-(t)G + f_{k-1}(t), t \in \partial D,\\ \liminf_{R\to+\infty} \frac{M(R, \partial_{\bar z}^j \phi_\lambda)}{R^{r-j}} = L_j < +\infty, j = 0, 1, 2, \ldots, k-1, \end{cases}$$

where $f_l \in \mathbb{H}^\mu(\partial D, \mathbb{C})$ and $M(R, \partial_{\bar z}^l \phi_\lambda) = \max_{|z|=R} \left|\partial_{\bar z}^l e^{-\lambda x_0}\phi(z)\right|$. $\left[\partial_{\bar z}^l \phi\right]^\pm(t)$ and $\left[(\partial_{\bar z} - \lambda)^l \phi\right]^\pm(t)$ with $\lambda \in \mathbb{C}\backslash\{0\}$ are defined analogously to $\phi^\pm(t), t \in \partial D$ as above for $l = 0, 1, 2, \ldots, k-1$. Moreover, when $k = 2$, problems $(\circledast)$ and $(\circledast\circledast)$ further reduce to the cases for bi-analytic functions in [4].

This implies the corresponding classical Riemann boundary value problems for classic poly-analytic functions and meta-analytic functions on the unit disk in the complex plane(see references e.g. [2–5] or elsewhere) can be solved by the way of the poly-Cauchy type integral operator, which is different from the method in the reference [30].

In what follows we take a polynomial $p(\lambda) = \lambda^k + a_1\lambda^{k-1} + \cdots + a_k (a_i \in \mathbb{C}, i = 1, 2, \ldots, k)$ with $k \in \mathbb{N}, k \geqslant 2$, and consider a polynomially generalized Cauchy–Riemann operator

$$p(\mathcal{D}) = \mathcal{D}^k + a_1\mathcal{D}^{k-1} + \cdots + a_k I$$

where I denotes the identity operator. We call the polynomial $p(\lambda)$ the characteristic polynomial of $p(\mathcal{D})$. The solutions to the polynomially generalized Cauchy–Riemann equation $p(\mathcal{D})\phi = 0$ are the so-called polynomially monogenic functions. In the following we denote

$$\ker p(\mathcal{D}) = \{\phi : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{C}_n \big| p(\mathcal{D})\phi = 0\}.$$

Since p as a complex polynomial can be decomposed into

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_m)^{n_m},$$

where $\sum_{i=1}^m n_i = k, n_i \in \mathbb{N}\,(i = 1, 2, \cdots, m)$, $\lambda_i \in \mathbb{C}\,(i = 1, 2, \ldots, m)$ are the zeros of the characteristic polynomial $p(\lambda) = 0$, the associated polynomial operator $p(\mathcal{D})$ has the following decomposition

$$p(\mathcal{D}) = \mathcal{D}_{\lambda_1}^{n_1}\mathcal{D}_{\lambda_2}^{n_2} \cdots \mathcal{D}_{\lambda_m}^{n_m}. \tag{11}$$

Hereby, the operators $\mathcal{D}_{\lambda_i}^{n_i} \triangleq (\mathcal{D} - \lambda_i)^{n_i} (i = 1, 2, \ldots, m)$ commute with each other.

RBVP IV. *Find a function* $\phi \in \mathcal{C}^k(B_\pm, \mathbb{C}_n)$ *such that all functions* $\phi, \mathcal{D}^s_{\lambda_i}\left(\sum_{j=1}^{n_i} c_{i,j} l_{i,j}(\mathcal{D})\phi\right)$ *can be continuously extended to* S^n *from* $B_\pm$*, respectively, and*

$$(v)\quad \begin{cases} p(\mathcal{D})\phi(x) = 0, x \in B_\pm, \\ \left[\mathcal{D}^s_{\lambda_i}\left(\sum_{j=1}^{n_i} c_{i,j} l_{i,j}(\mathcal{D})\phi\right)\right]^+(t) \\ = \left[\mathcal{D}^s_{\lambda_i}\left(\sum_{j=1}^{n_i} c_{i,j} l_{i,j}(\mathcal{D})\phi\right)\right]^-(t)G + f_{i,s}(t), t \in S^n, \end{cases}$$

where $\left[\mathcal{D}^s_{\lambda_i}\left(\sum_{j=1}^{n_i} c_{i,j} l_{i,j}(\mathcal{D})\phi\right)\right]^\pm(t)$ *is defined similarly to* $\phi^\pm(t), t \in S^n$*, as above and*

$$\begin{aligned} & p(\mathcal{D}) = (\mathcal{D}-\lambda_1)^{n_1}(\mathcal{D}-\lambda_2)^{n_2}\cdots(\mathcal{D}-\lambda_m)^{n_m}, \\ & i = 1,2,\ldots,m, j = 1,2,\ldots,n_i, c_{i,j} = \frac{1}{(n_i-j)!}\left[\frac{d^{n_i-j}}{d\lambda^{n_i-j}}\frac{(\lambda-\lambda_i)^{n_i}}{p(\lambda)}\right]\Bigg|_{\lambda=\lambda_i}, \\ & l_{i,j}(\mathcal{D})\phi \triangleq (\mathcal{D}-\lambda_1)^{n_1}((\mathcal{D}-\lambda_2)^{n_2} \\ & \quad \cdots(\mathcal{D}-\lambda_i)^{n_i-j}(\mathcal{D}-\lambda_{i+1})^{n_{i+1}}\cdots(\mathcal{D}-\lambda_m)^{n_m}\phi), \end{aligned}$$

$\mathcal{D}^s_{\lambda_i}\phi(s = 0,1,2,\ldots,n_i-1)$ *is defined similarly to* $\mathcal{D}^k_\lambda\phi$ *for pairwise different* $\lambda_i \in \mathbb{C}$ *with* $\mathcal{D}_{\lambda_i} \triangleq \mathcal{D}-\lambda_i$*,* $f_{i,s} \in \mathbb{H}^\mu(S^n, \mathbb{C}_n)$*,* G *is an invertible constant of* $\mathbb{C}_n$*.*

Theorem 4.4. *Boundary value problem* (v) *is solvable and its solution can be expressed in the form*

$$\phi(x) = \sum_{i=1}^m \phi_i(x), \quad x \in B_\pm,$$

where the function ϕ_i *are given as*

$$\phi_i(x) = \begin{cases} \sum_{s=0}^{n_i-1}\int_{S^n} E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0} f_{i,s}(y) \\ \quad + \sum_{s=0}^{n_i-1}\sum_{l_{i,s}=0}^{+\infty} e^{\lambda_i x_0} x_0^s P_{l_{i,s}}(x), x \in B_+, \\ \sum_{s=0}^{n_i-1}\int_{S^n} E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0} f_{i,s}(y)G^{-1} \\ \quad + \sum_{s=0}^{n_i-1}\sum_{l_{i,s}=0}^{+\infty} e^{\lambda_i x_0} x_0^s P_{l_{i,s}}(x)G^{-1}, x \in B_-, \end{cases}$$

with function $P_{l_{i,s}}(x)$ *being an inner spherical monogenic polynomial of order* $l_{i,s}$ *on the variable* $x \in \mathbb{R}^{n+1}$*.*

Proof. Applying Lemma 4 in [13] or [22,21,30], we obtain

$$\phi(x)=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{1}{(n_i-j)!}\left[\frac{d^{n_i-j}}{d\lambda^{n_i-j}}\frac{(\lambda-\lambda_i)^{n_i}}{p(\lambda)}\right]\Bigg|_{\lambda=\lambda_i} l_{i,j}(\mathcal{D})\phi(x)\triangleq\sum_{i=1}^{m}\phi_i(x),$$

where $l_{i,j}(\mathcal{D})\phi(x)$ is defined as above and $p(\lambda)=\prod_{i=1}^{m}(\lambda-\lambda_i)^{n_i}$ is the characteristic polynomial of $p(\mathcal{D})$ with $\lambda_i, n_i(i=1,2,\ldots,m)$ as above. Let us remark that by construction we have $\mathcal{D}_{\lambda_i}^{n_i}\phi_i(x)=0$ for $i=1,2,\ldots,m$. Therefore, in view of Theorem 6 in [13] or [21,22,30], boundary value problem (v) is equivalent to the following m boundary value problems

$$(vi)\quad\begin{cases}\mathcal{D}_{\lambda_i}^{n_i}\phi_i(x)=0, x\in B_{\pm}, i=1,2,\ldots,m\\ \phi_i^{+}(t)=\phi_i^{-}(t)G+f_{i,0}(t), t\in S^n,\\ \left[\mathcal{D}_{\lambda_i}\phi_i\right]^{+}(t)=\left[\mathcal{D}_{\lambda_i}\phi_i\right]^{-}(t)G+f_{i,1}(t), t\in S^n,\\ \quad\vdots \qquad\qquad\qquad \vdots\\ \left[\mathcal{D}_{\lambda_i}^{s}\phi_i\right]^{+}(t)=\left[\mathcal{D}_{\lambda_i}^{s}\phi_i\right]^{-}(t)G+f_{i,s}(t), t\in S^n,\\ \quad\vdots \qquad\qquad\qquad \vdots\\ \left[\mathcal{D}_{\lambda_i}^{n_i-1}\phi_i\right]^{+}(t)\\ \quad=\left[\mathcal{D}_{\lambda_i}^{n_i-1}\phi_i\right]^{-}(t)G+f_{i,n_i}(t), t\in S^n,\end{cases}$$

where $\left[\mathcal{D}_{\lambda_i}^{s}\phi_i\right]^{\pm}(t)$ is defined similarly to $\phi^{\pm}(t), t\in S^n$, as above for $s=0,1,2,\ldots,n_i-1$.

As $\lambda_i\in\mathbb{C}(i=1,2,\ldots,m)$ and $f_{i,j}\in\mathbb{H}^{\mu}(S^n,\mathbb{C}_n)(i=1,2,\ldots,m, j=1,2,\ldots,n_i)$, using Theorems 4.2,4.3 we get the solution to boundary value problem (vi) as

$$\phi(x)=\sum_{i=1}^{m}\phi_i(x), x\in B_{\pm},$$

where the function ϕ_i given by

$$\phi_i(x)=\begin{cases}\sum\limits_{s=0}^{n_i-1}\int_{S^n}E_{\lambda_i}^{s}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)\\ \quad+\sum\limits_{s=0}^{n_i-1}\sum\limits_{l_{i,s}=0}^{+\infty}e^{\lambda_i x_0}x_0^{s}P_{l_{i,s}}(x), x\in B_+,\\ \sum\limits_{s=0}^{n_i-1}\int_{S^n}E_{\lambda_i}^{s}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)G^{-1}\\ \quad+\sum\limits_{s=0}^{n_i-1}\sum\limits_{l_{i,s}=0}^{+\infty}e^{\lambda_i x_0}x_0^{s}P_{l_{i,s}}(x)G^{-1}, x\in B_-.\end{cases}$$

and function $P_{l_{i,s}}$ being an inner spherical monogenic polynomial of order $l_{i,s}$ on the variable $x\in\mathbb{R}^{n+1}$. □

Corollary 4.4. *Consider boundary value problem* (v). *If it holds*

$$\liminf_{R\to+\infty}\frac{M(R,\mathcal{D}^s\phi_i)}{R^{r-s}}=L_{i,s}<+\infty, r\geqslant s-1, s=0,1,2,\ldots,n_i-1,$$
$$i=1,2,\ldots,m,$$

where $M(R,\mathcal{D}^s\phi_i)\triangleq\max_{|x|=R}\left|\mathcal{D}^s(e^{-\lambda_i x_0}\phi_i(x))\right|(s=0,1,2,\ldots,n_i-1)$ *and* r *is a non-negative integer, then the solution to problem* (v) *can be written as*

$$\phi(x)=\sum_{i=1}^{m}\phi_i(x), x\in B_\pm,$$

where

$$\phi_i(x)=\begin{cases}\sum\limits_{s=0}^{n_i-1}\int_{S^n}E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)+e^{\lambda_i x_0}P_r(x), x\in B_+,\\ \sum\limits_{s=0}^{n_i-1}\int_{S^n}E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)G^{-1}+e^{\lambda_i x_0}P_r(x)G^{-1}, x\in B_-,\end{cases}$$

and function P_r *being a polynomial function of total degree no greater than* r *on the variable* $x\in\mathbb{R}^{n+1}$. *Moreover, when* $r=0$, *the solution to boundary value problem* (v) *takes the form*

$$\phi(x)=\sum_{i=1}^{m}\phi_i(x), x\in B_\pm,$$

where

$$\phi_i(x)=\begin{cases}\sum\limits_{s=0}^{n_i-1}\int_{S^n}E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)+e^{\lambda_i x_0}d, x\in B_+,\\ \sum\limits_{s=0}^{n_i-1}\int_{S^n}E^s_{\lambda_i}(y-x)d\sigma_y e^{-\lambda_i y_0}f_{i,s}(y)G^{-1}+e^{\lambda_i x_0}dG^{-1}, x\in B_-,\end{cases}$$

and $d\in\mathbb{C}_n$ *is a constant.*

Remark 2. When $n=1$, $\mathbb{R}^2\approxeq\mathbb{C}$, i.e., $z=x_0+x_1e_1\in\mathbb{C}$ and $\partial_{\bar{z}}=\partial_{x_0}+\partial_{x_1}e_1$ with $e_1^2=-1$ and $D=\{|z|<1:z\in\mathbb{C}\}$ denotes the unit disk in $\mathbb{C}$ with boundary $\partial D=\{|z|=1:z\in\mathbb{C}\}$, boundary value problem (v) reduces to the corresponding Riemann boundary value problem for classic poly-analytic and meta-analytic functions (see references e.g. [2–5] or elsewhere) on the unit disk in the complex plane, respectively. Therefore, problem (v) provides a generalization of Riemann boundary value problems for classic poly-analytic and meta-analytic functions.

When $n=2$, the considered problem (v) corresponds exactly to the transmission problem for null solutions to polynomially generalized Cauchy–Riemann operator on the sphere of $\mathbb{R}^3$. This means that all of the boundary value problems are solved by the way of the poly-Cauchy type integral operator, which is new and different from the ideas presented in [30]. Moreover, the considered RBVPs have important applications in elasticity, fluid mechanics, electromagnetic field theory and so on.

Remark 3. In this paper all of the results about RBVPs are given for boundary data in spaces of Hölder-continuous functions. In fact the corresponding results about the same RBVPs still hold for boundary data in the Sobolev spaces.

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