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## Advances in Applied Clifford Algebras

ISSN 0188-7009
Volume 24
Number 4
Adv. Appl. Clifford Algebras (2014) 24:995-1004
DOI 10.1007/s00006-014-0495-8

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# Generating Functions for Spherical Harmonics and Spherical Monogenics 

P. Cerejeiras, U. Kähler and R. Lávička*

To K. Gürlebeck


#### Abstract

In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in $\mathbb{R}^{m}$. Here spherical monogenics are polynomial solutions of the Dirac equation in $\mathbb{R}^{m}$. In particular, we obtain the recurrence formula which expresses the generating function in dimension $m$ in terms of that in dimension $m-1$. Hence we can find closed formulæ of generating functions in $\mathbb{R}^{m}$ by induction on the dimension $m$.


Keywords. Spherical harmonics, spherical monogenics, Gelfand-Tsetlin basis, orthogonal basis, generating function.

## 1. Introduction

It is well-known that classical orthogonal polynomials can be defined by generating functions. This close relationship allows for an indirect study of a given family of orthogonal polynomials by means of formal manipulations of its generating function. A classical example is the shifted Newtonian potential which is the generating function of spherical harmonics (see [30]). Not only one obtains several properties and recursion formulas of spherical harmonics by manipulation of the generating function but it also allows to establish new relationships with other families of orthogonal polynomials (see, for instances, [2], [25]). For example, the Gegenbauer polynomials $C_{k}^{\nu}$ are uniquely determined by the generating function

$$
\begin{equation*}
\frac{1}{\left(1-2 x h+h^{2}\right)^{\nu}}=\sum_{k=0}^{\infty} C_{k}^{\nu}(x) h^{k} \tag{1}
\end{equation*}
$$

where $\nu>0, x, h \in \mathbb{R},|x| \leq 1$ and $|h|<1$ (see e.g. [16, p. 18] or [27, p.173]). The classic approach to generating functions for spherical harmonics is to separate the angular part and construct the generating function of the

[^0]associated Legendre polynomials [32, 1]. As will be clear in the sequel this approach to the generating function is not suitable for our purposes and we will present a different construction.

In [27], a general framework is developed for a study of properties of polynomial sequences, including the Appell property and generating functions. One of the principal advantages of a generating function is that instead of studying the action of an operator on each basis function one only needs to study the action of said operator on the generating function. This was used to great effect in many areas, such as Umbral calculus, quantum mechanics, and others $[27,30,28]$. Furthermore, generating functions form a bridge between analysis and discrete mathematics, by providing a really efficient tool for solving difference equations [31].

In this paper, we deal with generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in $\mathbb{R}^{m}$. Possible applications of this theory lie in the study of monogenic operators, noncommutative combinatorics, and structural mechanics. Also, studying difference equations over the set of monogenic functions or their boundary values is closely linked with the study of problems in image processing by means of the so-called monogenic signal.

Orthogonal bases of spherical harmonics are well-known and have been studied for a long time. Spherical harmonics are useful in many theoretical areas and on applications such as structural mechanics, etc. In Clifford analysis, a similar role is played by spherical monogenics. Monogenic functions are defined as Clifford algebra valued solutions $f$ of the equation $\partial f=0$ where $\partial$ is the Dirac operator on $\mathbb{R}^{m}$. Spherical monogenics are polynomial solutions of the Dirac equation. Since the Dirac operator $\partial$ factorizes the Laplace operator $\Delta$ in the sense that $\Delta=-\partial^{2}$ Clifford analysis can be understood as a refinement of harmonic analysis. On the other hand, monogenic functions are at the same time a higher dimensional analogue of holomorphic functions of one complex variable. See $[5,15,19,18]$ for an account of Clifford analysis.

The first construction of orthogonal bases of spherical monogenics valid for any dimension was given by F. Sommen, see [29, 15]. In dimension 3, explicit constructions using the standard bases of spherical harmonics were done also by K. Gürlebeck, H. Malonek, I. Cação and S. Bock (see e.g. $[3,8,9,10,11,12,13])$. From the point of view of representation theory, the standard bases of spherical harmonics are nothing else than examples of the so-called Gelfand-Tsetlin bases, see [26]. V. Souček proposed studying these bases in Clifford analysis. In particular, in [4], it is observed that the complete orthogonal system in $\mathbb{R}^{3}$ of [3] and F. Sommen's bases [29, 15] can be both considered as Gelfand-Tsetlin bases. Actually, it turns out that Gelfand-Tsetlin bases in all cases so far studied in Clifford analysis are, by construction, uniquely determined and orthogonal and, in addition, they possess the so-called Appell property, see [24] for a recent survey, [21, 22] for the classical Clifford analysis, $[14,23]$ for Hodge-de Rham systems and [6, 7] for Hermitian Clifford analysis. Therefore we call them the standard orthogonal
bases in the sequel. For a detailed historical account of this topic, we refer to [4].

In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in $\mathbb{R}^{m}$. We obtain the recurrence formula which expresses the generating function in dimension $m$ in terms of that in dimension $m-1$, see below Theorem 1 for spherical harmonics and Theorem 2 for spherical monogenics. Using the recurrence formula, we can obtain closed formulæ of generating functions in $\mathbb{R}^{m}$ by induction on the dimension $m$. This is based on the generating function (1) for the Gegenbauer polynomials. It seems that analogous results can be obtained also for Hodgede Rham systems [23] and even in Hermitian Clifford analysis [7]. But, in the hermitian case, the generating function for the Jacobi polynomials should be used instead of (1).

## 2. Spherical Harmonics

In this section, we study generating functions for spherical harmonics in $\mathbb{R}^{m}$. Denote by $\mathbb{B}_{m}$ the unit ball in $\mathbb{R}^{m}$. Let us recall the standard construction of an orthogonal basis in the complex Hilbert space $L^{2}\left(\mathbb{B}_{m}, \mathbb{C}\right) \cap \operatorname{Ker} \Delta$ of $L^{2}$-integrable harmonic functions $g: \mathbb{B}_{m} \rightarrow \mathbb{C}$.

One proceeds by induction on the dimension $m$. Of course, in $\mathbb{R}^{2}$ the polynomials $\operatorname{harm}_{k_{2}}^{ \pm}, k_{2} \in \mathbb{N}_{0}$, given by

$$
\begin{equation*}
\operatorname{harm}_{k_{2}}^{ \pm}(x)=\left(x_{1} \pm i x_{2}\right)^{k_{2}} /\left(k_{2}!\right), x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2} \tag{2}
\end{equation*}
$$

form an orthogonal basis of the space $L^{2}\left(\mathbb{B}_{2}, \mathbb{C}\right) \cap \operatorname{Ker} \Delta$.
Now let $m \geq 3$. To construct the bases in higher dimensions, we need to introduce the following embedding factors. For $k, j \in \mathbb{N}_{0}$, we define

$$
\begin{equation*}
F_{m, j}^{(k)}(x)=|x|_{m}^{k} C_{k}^{m / 2+j-1}\left(x_{m} /|x|_{m}\right), x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m} \tag{3}
\end{equation*}
$$

where $|x|_{m}=\sqrt{x_{1}^{2}+\cdots+x_{m}^{2}}$. Then, it is well-known that in $\mathbb{R}^{m}$ an orthogonal basis of the space $L^{2}\left(\mathbb{B}_{m}, \mathbb{C}\right) \cap \operatorname{Ker} \Delta$ is formed by the polynomials $\operatorname{harm}_{k}^{ \pm}, k=\left(k_{2}, \ldots, k_{m}\right) \in \mathbb{N}_{0}^{m-1}$, given by

$$
\begin{equation*}
\operatorname{harm}_{k}^{ \pm}(x)=\operatorname{harm}_{k_{2}}^{ \pm}\left(x_{1}, x_{2}\right) \prod_{r=3}^{m} F_{r, k_{r-1}^{*}}^{\left(k_{r}\right)}\left(x_{1}, \ldots, x_{r}\right) \tag{4}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ and $k_{r}^{*}=k_{2}+\cdots+k_{r}$. See e.g. [16, p. 35] or [22]. In difference to [22], we use another normalization of the embedding factors $F_{m, j}^{\left(k_{m}\right)}$ and we also change the notation for indices which in turns provides a more elegant expression for generating functions.

Definition 1. We define the generating function $H_{m}^{ \pm}$of the orthogonal basis $\operatorname{harm}_{k}^{ \pm}, k \in \mathbb{N}_{0}^{m-1}$ of spherical harmonics in $\mathbb{R}^{m}$ by

$$
H_{m}^{ \pm}(x, h)=\sum_{k \in \mathbb{N}_{0}^{m-1}} \operatorname{harm}_{k}^{ \pm}(x) h^{k}
$$

whenever the series on the right-hand side converges absolutely. Here $x \in \mathbb{R}^{m}$, $h=\left(h_{2}, \ldots, h_{m}\right) \in \mathbb{R}^{m-1}$ and $h^{k}=h_{2}^{k_{2}} \cdots h_{m}^{k_{m}}$.

Obviously, the following result follows easily from (1).
Lemma 1. For $x \in \mathbb{R}^{m}$ and $h_{m} \in \mathbb{R}$, we have that

$$
\sum_{k_{m}=0}^{\infty} F_{m, j}^{\left(k_{m}\right)}(x) h_{m}^{k_{m}}=\frac{1}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\frac{m}{2}-1+j}}
$$

whenever $|x|_{m} \leq 1,\left|h_{m}\right|<1$ and $j \in \mathbb{N}_{0}$.
Now we prove basic properties of the generating functions $H_{m}^{ \pm}$.
Theorem 1. For each $m \geq 2$ there is a neighborhood $U_{m}$ of 0 in $\mathbb{R}^{m-1}$ such that the following statements hold true.
(i) The generating functions $H_{m}^{ \pm}(x, h)$ are defined if $|x|_{m} \leq 1$ and $h \in U_{m}$. Here $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ and $h=\left(h_{2}, \ldots, h_{m}\right) \in \mathbb{R}^{m-1}$.
(ii) For each $k=\left(k_{2}, \ldots, k_{m}\right) \in \mathbb{N}_{0}^{m-1}$, we have that

$$
\operatorname{harm}_{k}^{ \pm}(x)=\left.\frac{1}{k!} \partial^{k} H_{m}^{ \pm}(x, h)\right|_{h=0}, \quad|x|_{m} \leq 1
$$

where $k!=\left(k_{2}!\right) \cdots\left(k_{m}!\right)$ and $\partial^{k}=\partial_{h_{2}}^{k_{2}} \cdots \partial_{h_{m}}^{k_{m}}$.
(iii) For $m \geq 3,|x|_{m} \leq 1$ and $h \in U_{m}$, we have that

$$
H_{m}^{ \pm}(x, h)=d_{m}^{1-\frac{m}{2}} H_{m-1}^{ \pm}\left(\underline{x}, \underline{h} / d_{m}\right)
$$

where $d_{m}=1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}, \underline{x}=\left(x_{1}, \ldots, x_{m-1}\right)$ and $\underline{h} / d_{m}=$ $\left(h_{2} / d_{m}, \ldots, h_{m-1} / d_{m}\right)$.

Proof. We prove this theorem by induction on the dimension $m$. It is easily seen that the theorem is true for $m=2$. Indeed, we have that

$$
H_{2}^{ \pm}\left(x_{1}, x_{2}, h_{2}\right)=\sum_{k_{2}=0}^{\infty} \frac{\left(x_{1} \pm i x_{2}\right)^{k_{2}}}{k_{2}!} h_{2}^{k_{2}}=\exp \left(\left(x_{1} \pm i x_{2}\right) h_{2}\right)
$$

Now assume that the theorem is true for $m-1$. Let $H_{m-1}^{ \pm}(\underline{x}, \underline{h})$ be defined for $\underline{h} \in U_{m-1}=\left(-\delta_{2}, \delta_{2}\right) \times \cdots \times\left(-\delta_{m-1}, \delta_{m-1}\right)$ and $|\underline{x}|_{m-1} \leq 1$ and let $|x|_{m} \leq 1$. It is easy to see that

$$
\begin{equation*}
H_{m}^{ \pm}(x, h)=\sum_{\underline{k}}\left(\sum_{k_{m}=0}^{\infty} F_{m, k_{m-1}^{*}}^{\left(k_{m}\right)}(x) h_{m}^{k_{m}}\right) \operatorname{harm}_{\underline{k}}^{ \pm}(\underline{x}) \underline{h}^{\underline{k}} \tag{5}
\end{equation*}
$$

where the first sum is taken over all $\underline{k}=\left(k_{2}, \ldots, k_{m-1}\right) \in \mathbb{N}_{0}^{m-2}$. By Lemma 1, we have that

$$
\sum_{k_{m}=0}^{\infty} F_{m, k_{m-1}^{*}}^{\left(k_{m}\right)}(x) h_{m}^{k_{m}}=d_{m}^{1-\frac{m}{2}-\left(k_{2}+\cdots+k_{m-1}\right)}
$$

if $\left|h_{m}\right|<1$. Using this formula and (5), we have that

$$
H_{m}^{ \pm}(x, h)=d_{m}^{1-\frac{m}{2}} \sum_{\underline{k}} h \operatorname{arm}_{\underline{k}}^{ \pm}(\underline{x})\left(\underline{h} / d_{m}\right)^{\underline{k}}=d_{m}^{1-\frac{m}{2}} H_{m-1}^{ \pm}\left(\underline{x}, \underline{h} / d_{m}\right)
$$

whenever $h \in U_{m}=\left(-\delta_{2} / 4, \delta_{2} / 4\right) \times \cdots \times\left(-\delta_{m-1} / 4, \delta_{m-1} / 4\right) \times(-1 / 2,1 / 2)$. Indeed, $d_{m} \geq\left(1-h_{m}|x|_{m}\right)^{2}>1 / 4$ if $\left|h_{m}\right|<1 / 2$. Hence, if $h \in U_{m}$ we have that $\underline{h} / d_{m} \in U_{m-1}$ and, by (5), we can easily see that some rearrangement of the power series defining $H_{m}^{ \pm}(x, h)$ converges at $h$. Then Abel's Lemma [20, Proposition 1.5.5, p. 23] proves that this power series converges absolutely on the whole $U_{m}$, which finishes the proof of the theorem.

Using the recurrence formula (iii) of Theorem 1, we can find closed formulæ of generating functions for spherical harmonics in $\mathbb{R}^{m}$ by induction on the dimension $m$.

Corollary 1. In particular, we have the following formula

$$
H_{3}^{ \pm}(x, h)=\frac{1}{\left(1-2 x_{3} h_{3}+h_{3}^{2}|x|_{3}^{2}\right)^{1 / 2}} \exp \left(\frac{\left(x_{1} \pm i x_{2}\right) h_{2}}{1-2 x_{3} h_{3}+h_{3}^{2}|x|_{3}^{2}}\right) .
$$

Here $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$ and $h=\left(h_{2}, h_{3}\right) \in \mathbb{R}^{2}$.
Remark 1. It is well-known that an orthogonal basis of real valued spherical harmonics in $\mathbb{R}^{m}$ is formed by the polynomials $\Re$ harm $_{k}^{+}$, $\Im$ harm ${ }_{k}^{+}, k \in \mathbb{N}_{0}^{m-1}$. Here $\Re z$ and $\Im z$ are the real and imaginary part of the complex number $z$. Hence the corresponding generating functions are $\Re H_{m}^{+}, \Im H_{m}^{+}$.

Remark 2. If one replaces in the definition of the orthogonal basis (4) the polynomials $\operatorname{harm}_{k_{2}}^{ \pm}\left(x_{1}, x_{2}\right)=\left(x_{1} \pm i x_{2}\right)^{k_{2}} /\left(k_{2}!\right)$ with

$$
\begin{equation*}
\overline{\operatorname{harm}}_{k_{2}}^{ \pm}\left(x_{1}, x_{2}\right)=\left(x_{1} \pm i x_{2}\right)^{k_{2}} \tag{6}
\end{equation*}
$$

the corresponding generating functions $\bar{H}_{m}^{ \pm}$are definitely different from $H_{m}^{ \pm}$ but they obviously satisfy again Theorem 1. In particular, we have that

$$
\bar{H}_{2}^{ \pm}\left(x, h_{2}\right)=\sum_{k_{2}=0}^{\infty}\left(x_{1} \pm i x_{2}\right)^{k_{2}} h_{2}^{k_{2}}=\frac{1-\left(x_{1} \mp x_{2} i\right) h_{2}}{1-2 x_{1} h_{2}+h_{2}^{2}|x|_{2}^{2}}
$$

Here $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ and $h_{2} \in \mathbb{R}$.

## 3. Spherical Monogenics

In this section, we introduce and investigate generating functions for spherical monogenics. For an account of Clifford analysis, we refer to [5, 15, 19, 18]. Denote by $\mathcal{C} \ell_{m}$ either the real Clifford algebra $\mathbb{R}_{0, m}$ or the complex one $\mathbb{C}_{m}$, generated by the elements $\mathbf{e}_{1}, \ldots, \mathbf{e}_{m}$ such that $\mathbf{e}_{j}^{2}=-1$ for $j=1, \ldots, m$. As usual, a vector $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ corresponds to the element $\mathbf{x}=$ $x_{1} \mathbf{e}_{1}+\cdots+x_{m} \mathbf{e}_{m}$ of the Clifford algebra $\mathcal{C} \ell_{m}$. Let $G \subset \mathbb{R}^{m}$ be open. Then a continuously differentiable function $f: G \rightarrow \mathcal{C} \ell_{m}$ is called monogenic if it satisfies the equation $\partial f=0$ on $G$ where the Dirac operator $\partial$ is defined as

$$
\begin{equation*}
\partial=\mathbf{e}_{1} \partial_{x_{1}}+\cdots+\mathbf{e}_{m} \partial_{x_{m}} . \tag{7}
\end{equation*}
$$

Denote by $L^{2}\left(\mathbb{B}_{m}, \mathcal{C} \ell_{m}\right) \cap \operatorname{Ker} \partial$ the space of $L^{2}$-integrable monogenic functions $g: \mathbb{B}_{m} \rightarrow \mathcal{C} \ell_{m}$. It is well-known that $L^{2}\left(\mathbb{B}_{m}, \mathcal{C} \ell_{m}\right) \cap \operatorname{Ker} \partial$ forms the
right $\mathcal{C} \ell_{m}$-linear Hilbert space. Let us recall a construction of an orthogonal basis in this space which is quite analogous to the harmonic case described in the previous section, see [22] for more details.

It is easy to see that in $\mathbb{R}^{2}$ the polynomials mon $_{k_{2}}, k_{2} \in \mathbb{N}_{0}$, given by

$$
\begin{equation*}
\operatorname{mon}_{k_{2}}(x)=\left(x_{1}-\mathbf{e}_{12} x_{2}\right)^{k_{2}} /\left(k_{2}!\right), x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}, \tag{8}
\end{equation*}
$$

form an orthogonal basis of the space $L^{2}\left(\mathbb{B}_{2}, \mathcal{C} \ell_{2}\right) \cap \operatorname{Ker} \partial$. Here we write $\mathbf{e}_{12}=$ $\mathbf{e}_{1} \mathbf{e}_{2}$ as usual. Now let $m \geq 3$. To construct the bases in higher dimensions, we need to introduce the embedding factors $X_{m, j}^{(k)}$ for $k, j \in \mathbb{N}_{0}$, defined as

$$
\begin{equation*}
X_{m, j}^{(k)}(x)=\frac{m-2+k+2 j}{m-2+2 j} F_{m, j}^{(k)}(x)+F_{m, j+1}^{(k-1)}(x) \underline{\mathbf{x}} \mathbf{e}_{m} \tag{9}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ and $\underline{\mathbf{x}}=x_{1} \mathbf{e}_{1}+\cdots+x_{m-1} \mathbf{e}_{m-1}$. Here $F_{m, j}^{(k)}$ are given in (3) and we put $F_{m, j+1}^{(-1)}=0$. Then it is well-known that an orthogonal basis of the space $L^{2}\left(\mathbb{B}_{m}, \mathcal{C} \ell_{m}\right) \cap \operatorname{Ker} \partial$ is formed by the polynomials

$$
\begin{equation*}
\operatorname{mon}_{k}(x)=X_{m, k_{m-1}^{*}}^{\left(k_{m}\right)} X_{m-1, k_{m-2}^{*}}^{\left(k_{m-1}\right)} \cdots X_{3, k_{2}^{*}}^{\left(k_{3}\right)} \operatorname{mon}_{k_{2}}\left(x_{1}, x_{2}\right), x \in \mathbb{R}^{m} \tag{10}
\end{equation*}
$$

where $k=\left(k_{2}, \ldots, k_{m}\right) \in \mathbb{N}_{0}^{m-1}$ and $k_{r}^{*}=k_{2}+\cdots+k_{r}$. Here

$$
X_{r, k_{r-1}^{*}}^{\left(k_{r}\right)}=X_{r, k_{r-1}^{*}}^{\left(k_{r}\right)}\left(x_{1}, \ldots, x_{r}\right)
$$

for $r=3, \ldots, m$. Let us remark that due to non-commutativity of the Clifford multiplication the order of factors in the product (10) is important. See [22] for more details. In comparison with [22], we use another normalization of the embedding factors $X_{m, j}^{\left(k_{m}\right)}$ and we also change the notation for indices to get a nice expression for generating functions.

Definition 2. We define the generating function $M_{m}$ of the orthogonal basis mon $_{k}, k \in \mathbb{N}_{0}^{m-1}$ of spherical monogenics in $\mathbb{R}^{m}$ by

$$
M_{m}(x, h)=\sum_{k \in \mathbb{N}_{0}^{m-1}} \operatorname{mon}_{k}(x) h^{k}
$$

whenever the series on the right-hand side converges absolutely. Here $x \in \mathbb{R}^{m}$ and $h=\left(h_{2}, \ldots, h_{m}\right) \in \mathbb{R}^{m-1}$.

In particular, it is easily seen that

$$
M_{2}\left(x_{1}, x_{2}, h_{2}\right)=\sum_{k_{2}=0}^{\infty} \frac{\left(x_{1}-\mathbf{e}_{12} x_{2}\right)^{k_{2}}}{k_{2}!} h_{2}^{k_{2}}=\exp \left(\left(x_{1}-\mathbf{e}_{12} x_{2}\right) h_{2}\right) .
$$

Here $\exp \left(\left(x_{1}-\mathbf{e}_{12} x_{2}\right) h_{2}\right)=\exp \left(x_{1} h_{2}\right)\left(\cos \left(x_{2} h_{2}\right)-\mathbf{e}_{12} \sin \left(x_{2} h_{2}\right)\right)$. To study the generating functions in higher dimensions we need to know the generating function of the embedding factors $X_{m, j}^{\left(k_{m}\right)}$.
Lemma 2. For $x \in \mathbb{R}^{m}$ and $h_{m} \in \mathbb{R}$, we have that

$$
\sum_{k_{m}=0}^{\infty} X_{m, j}^{\left(k_{m}\right)}(x) h_{m}^{k_{m}}=\frac{1+h_{m} \mathbf{x e}_{m}}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{m / 2+j}}
$$

whenever $|x|_{m} \leq 1,\left|h_{m}\right|<1$ and $j \in \mathbb{N}_{0}$. Here $\mathbf{x}=x_{1} \mathbf{e}_{1}+\cdots+x_{m} \mathbf{e}_{m}$.
Proof. Put $\nu=m / 2-1+j$. By (9), the series we want to sum up is equal to

$$
\sum_{k_{m}=0}^{\infty} \frac{k_{m}+2 \nu}{2 \nu} F_{m, j}^{\left(k_{m}\right)}(x) h_{m}^{k_{m}}+\sum_{k_{m}=1}^{\infty} F_{m, j+1}^{\left(k_{m}-1\right)}(x) h_{m}^{k_{m}} \underline{\mathbf{x}} \mathbf{e}_{m}=\Sigma_{1}+\Sigma_{2}
$$

Obviously, by Lemma 1, we get that

$$
\Sigma_{2}=\frac{h_{m} \underline{\underline{\mathbf{x}}}{ }_{m}}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\nu+1}}
$$

Moreover, using Lemma 1 again, we have that

$$
\Sigma_{1}=\frac{h_{m}}{2 \nu} \sum_{k_{m}=1}^{\infty} F_{m, j}^{\left(k_{m}\right)}(x) k_{m} h_{m}^{k_{m}-1}+\frac{1}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\nu}}
$$

and hence

$$
\Sigma_{1}=\frac{h_{m}}{2 \nu} \frac{d}{d h_{m}} \frac{1}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\nu}}+\frac{1}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\nu}}
$$

which gives

$$
\Sigma_{1}=\frac{1-x_{m} h_{m}}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{\nu+1}}
$$

Finally, using $\mathbf{x}=\underline{\mathbf{x}}+x_{m} \mathbf{e}_{m}$ we conclude that

$$
\Sigma_{1}+\Sigma_{2}=\frac{1+h_{m} \mathbf{x e}_{m}}{\left(1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}\right)^{m / 2+j}}
$$

which finishes the proof.
Now we can prove basic properties of the generating functions $M_{m}$ quite similarly as in the harmonic case if, in this case, we use Lemma 2 instead of Lemma 1. Then we obtain the following result.

Theorem 2. For each $m \geq 2$ there is a neighborhood $U_{m}$ of 0 in $\mathbb{R}^{m-1}$ such that the following statements hold true.
(i) The generating functions $M_{m}(x, h)$ are defined if $|x|_{m} \leq 1$ and $h \in U_{m}$. Here $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ and $h=\left(h_{2}, \ldots, h_{m}\right) \in \mathbb{R}^{m-1}$.
(ii) For each $k \in \mathbb{N}_{0}^{m-1}$, we have that

$$
\operatorname{mon}_{k}(x)=\left.\frac{1}{k!} \partial^{k} M_{m}(x, h)\right|_{h=0}, \quad|x|_{m} \leq 1
$$

where $k!=\left(k_{2}!\right) \cdots\left(k_{m}!\right)$ and $\partial^{k}=\partial_{h_{2}}^{k_{2}} \cdots \partial_{h_{m}}^{k_{m}}$.
(iii) For $m \geq 3,|x|_{m} \leq 1$ and $h \in U_{m}$, we have that

$$
M_{m}(x, h)=\left(1+h_{m} \mathbf{x e}_{m}\right) d_{m}^{-\frac{m}{2}} M_{m-1}\left(\underline{x}, \underline{h} / d_{m}\right)
$$

where $d_{m}=1-2 x_{m} h_{m}+h_{m}^{2}|x|_{m}^{2}, \underline{x}=\left(x_{1}, \ldots, x_{m-1}\right)$ and $\underline{h} / d_{m}=$ $\left(h_{2} / d_{m}, \ldots, h_{m-1} / d_{m}\right)$.

Using the recurrence formula (iii) of Theorem 2, we can find closed formulæ of generating functions for spherical monogenics in $\mathbb{R}^{m}$ by induction on the dimension $m$.

Corollary 2. In particular, we have the following formula

$$
M_{3}(x, h)=\frac{1+h_{3} \mathbf{x e}_{3}}{\left(1-2 x_{3} h_{3}+h_{3}^{2}|x|_{3}^{2}\right)^{3 / 2}} \exp \left(\frac{\left(x_{1}-\mathbf{e}_{12} x_{2}\right) h_{2}}{1-2 x_{3} h_{3}+h_{3}^{2}|x|_{3}^{2}}\right)
$$

Here $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$ and $h=\left(h_{2}, h_{3}\right) \in \mathbb{R}^{2}$.
Remark 3. If one replaces in the definition of the orthogonal basis (10) the polynomials mon $_{k_{2}}\left(x_{1}, x_{2}\right)=\left(x_{1}-\mathbf{e}_{12} x_{2}\right)^{k_{2}} /\left(k_{2}!\right)$ with

$$
\begin{equation*}
\overline{m o n}_{k_{2}}\left(x_{1}, x_{2}\right)=\left(x_{1}-\mathbf{e}_{12} x_{2}\right)^{k_{2}} \tag{11}
\end{equation*}
$$

the corresponding generating functions $\bar{M}_{m}$ are different from $M_{m}$ but they obviously satisfy again Theorem 2. In particular, we have that

$$
\bar{M}_{2}\left(x, h_{2}\right)=\sum_{k_{2}=0}^{\infty}\left(x_{1}-\mathbf{e}_{12} x_{2}\right)^{k_{2}} h_{2}^{k_{2}}=\frac{1-\left(x_{1}+\mathbf{e}_{12} x_{2}\right) h_{2}}{1-2 x_{1} h_{2}+h_{2}^{2}|x|_{2}^{2}} .
$$

Here $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ and $h_{2} \in \mathbb{R}$.

## Acknowledgements

We would like to thank V. Souček for useful discussions on this topic. The work of the first and second authors was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology("FCT - Fundação para a Ciência e a Tecnologia"), within project PEstOE/MAT/UI4106/2014. We would like to thank the anonymous referee for his/her helpful comments.

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Received: April 15, 2014.
Accepted: July 23, 2014.


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