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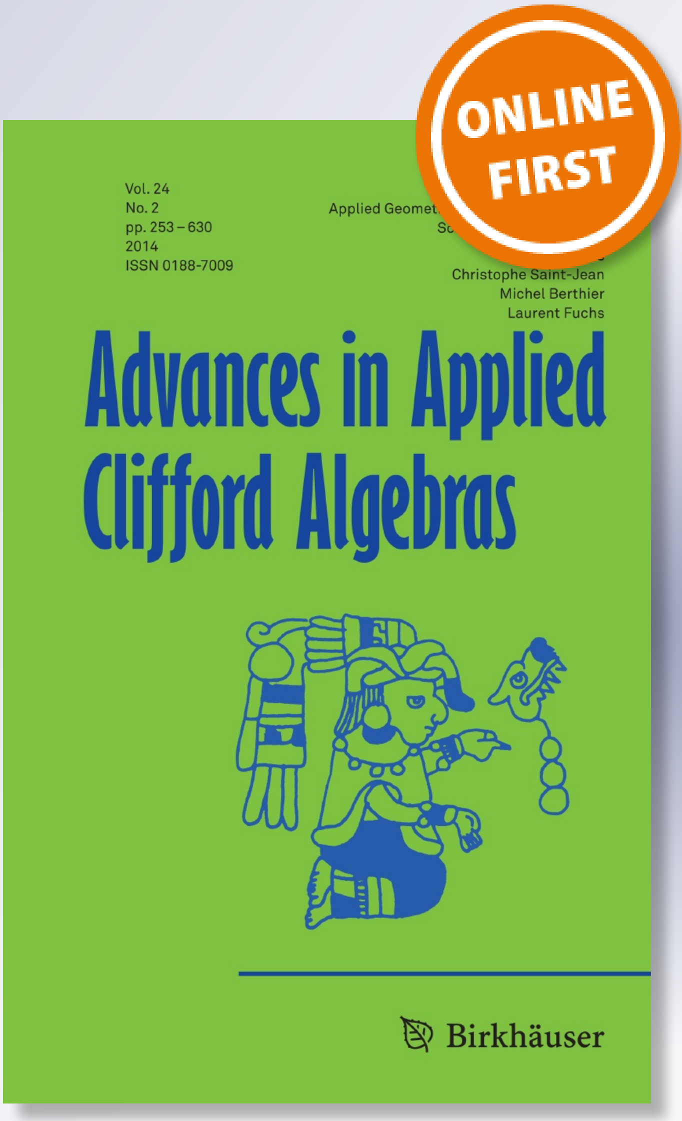

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A Short Note on the Local Solvability of the Quaternionic Beltrami Equation

Juan Bory Reyes, Paula Cerejeiras, Alí Guzmán Adán and Uwe Kähler*

Dedicated to Klaus Gürlebeck on the occasion of his 60th birthday

Abstract. In this paper we discuss the local solvability of the inhomogeneous Beltrami equations in Quaternionic Analysis. We give an example of a Beltrami equation with no distributional solution and deduce the compatibility condition. This study is closely linked to the study of Dirac operators with non-constant coefficients.

Keywords. Quaternionic Beltrami equation, local solvability.

1. Introduction

Beltrami equations in higher dimensions are a quite old and at the same time quite recent topic. While A. Newlander and L. Nirenberg already studied a Beltrami equation in several complex variables in 1957 [15] and V. Shevshenko [16] proposed a quaternionic version in 1962 the study of Beltrami systems in Clifford analysis started effectively only in the nineties with the paper by K. Gürlebeck and U. Kähler [7] where the authors gave a solution for $q \in L_2(\Omega)$ under the natural condition $\|q\|_{L_2} < q_0 \leq 1$. In fact the original condition was given as $\|q\| < q_0 \leq 1/\|\Pi\|$ where Π denotes the generalized Π-operator, but as they later showed the norm of this operator as an operator from $L_2(\Omega)$ into $L_2(\Omega)$ equals to one. The question of the L_p-norm for the Π-operator (or Beurling-Ahlfors operator) is still an open question (the famous conjecture of Ivaniec). Also, K. Gürlebeck, U. Kähler, and M. Shapiro studied quaternionic Beltrami equations based on structural sets [8].

*Corresponding Author
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Later on the same authors together with P. Cerejeiras and H. Malonek studied the question of local homeomorphic solutions in the quaternionic and Clifford setting [4], [3]. That the topic is still under discussion can be seen in the paper by A. Koski [12] where Beltrami equations with q being a VMO function are discussed.

Lately, there also appeared several papers which discuss Dirac operators with non-constant coefficients. One of the principal problems in discussing such kind of operators is the question of local solvability, i.e. the question when $Du = f$ with D being a Dirac operator with non-constant coefficients and Ω being a neighborhood of a point x_0, has for any $f \in C_0^\infty(\Omega)$ a (distributional) solution $u \in \mathcal{D}'(\Omega)$.

From the famous theorem of Malgrange-Ehrenpreis we know that in the case of differential equations involving the Dirac operator and constant coefficients we always have a solution. This is not anymore true in the case of non-constant coefficients, of which the Beltrami equation is a simple example. In fact the Beltrami equation can always we rewritten as a Dirac equation with non-constant coefficients. In this paper we will adapt the famous Lewy's example to construct an inhomogeneous Beltrami equation which has no solution in any open subset of $\mathbb{R}^3$. Afterwards we will discuss the (Ψ) condition and derive the condition for local solvability.

2. Preliminaries

The algebra of quaternions $\mathbb{H}$ is a four-dimensional real associative division algebra with unit 1 spanned by the elements $\{e_1, e_2, e_3\}$ endowed with the relations

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

$$e_1e_2 = -e_2e_1 = e_3, \; e_2e_3 = -e_3e_2 = e_1, \; e_1e_3 = -e_3e_1 = e_2.$$

This algebra is a non-commutative field. The real and vectorial parts of a given quaternion

$$q = x_0 1 + x_1e_1 + x_2e_2 + x_3e_3$$

are defined as $\mathrm{Re}(q) = q_0 := x_0 \in \mathrm{Sc}\mathbb{H}$, and $\mathrm{Vec}(q) = \underline{q} := x_1e_1 + x_2e_2 + x_3e_3 \in \mathrm{Vec}\mathbb{H}$. Therefore, in contrast to complex numbers, $\underline{q}$ is not a real number. We have natural embeddings of the real numbers and of $\mathbb{R}^3$ into quaternions given by

$$x_0 \in \mathbb{R} \to x_0 1 \in \mathrm{Sc}\mathbb{H} \quad \text{and} \quad (x_1, x_2, x_3) \in \mathbb{R}^3 \to x_1e_1 + x_2e_2 + x_3e_3 \in \mathrm{Vec}\mathbb{H}.$$

Therefore, we have the identifications $\mathbb{H} \equiv \mathbb{R}^4, \; \mathrm{Vec}\mathbb{H} \equiv \mathbb{R}^3, \; \mathrm{Sc}\mathbb{H} \equiv \mathbb{R}$, where $\mathrm{Vec}\mathbb{H}$ denotes the three dimensional space of imaginary quaternions, and $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$.

Also of interest is the algebra of complexified quaternions, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$, with elements of type

$$q = z_0 1 + z_1e_1 + z_2e_2 + z_3e_3 = z_0 + \underline{q}, \quad z_j \in \mathbb{C}.$$

There is a suitable conjugation on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$, given by the operating rules

$$q = z_0 + \underline{q} \quad \mapsto \quad \overline{q} = \overline{z}_0 - \overline{z}_1 e_1 - \overline{z}_2 e_2 - \overline{z}_3 e_3$$

and satisfying to the involution property $\overline{qp} = \overline{p}\ \overline{q}$. The Euclidean scalar product is defined by $\langle q, p\rangle = \mathrm{Re}(q\overline{p}) = \frac{1}{2}(q\overline{p} + p\overline{q})$ and the corresponding norm $\|q\|^2 = \langle q, q\rangle$ verifies $\|qp\| = \|q\|\ \|p\|$.

The quaternionic multiplication can be expressed in terms of the usual scalar and vector product on $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Vec}\mathbb{H} \equiv \mathbb{C}^3$ by

$$qp = (z_0 + \underline{q})(w_0 + \underline{p}) = z_0 w_0 - \underline{q} \cdot \underline{p} + z_0 \underline{p} + w_0 \underline{q} + \underline{q} \times \underline{p}.$$

For more details we refer to the books [9, 6].

Consider now a basis $\varphi = \{\varphi_0, \ldots, \varphi_3\} \in \mathbb{H}^4$, also called a structural set (originally introduced by M. Shapiro and his co-authors, e.g. [14]) and the operators:

$$^{\varphi}Du = \sum_{k=0}^{3} \varphi_k \partial_k u \tag{2.1}$$

$$^{\varphi}D_l u = \sum_{k=0}^{l-1} \varphi_k \partial_k u + \overline{\varphi_l} \partial_l u + \sum_{k=l+1}^{3} \varphi_k \partial_k u. \tag{2.2}$$

In 1962 Shevchenko considered the following Beltrami equation [16]:

$$^{\varphi}Du = q_1 {}^{\varphi}D_1 u + q_2 {}^{\varphi}D_2 u + q_3 {}^{\varphi}D_3 u.$$

which contains as a special case the usual Beltrami equation $^{\varphi}Du - q {}^{\psi}Du = 0$, where ψ denotes another (in general) different basis. Let us now consider in the rest of the paper its inhomogeneous version $^{\varphi}Du - q {}^{\psi}Du = F$. The principal question is of course does this equation has a (at least local) solution if F is not analytic, e.g. $F \in C^\infty$?

To keep the example simple we restrict ourselves to the three-dimensional case, without any loss of generality. Consider now the following bases $\varphi = \{1, e_1, e_2\}$ and $\psi = \{1, e_1, -e_2\}$ in $\mathbb{H}^3$.

Theorem 2.1. *Consider the equation*

$$^{\varphi}Du + \left(\frac{\frac{1}{2} + e_3 x_0 - e_2 x_1}{\frac{1}{4} + x_0^2 + x_1^2}\right)^2 {}^{\psi}Du = \left(-\frac{1}{2} + e_3 x_0 - e_2 x_1\right) F$$

for $F \in C^1$ in a neighbourhood of $(0, 0, x_3^0)$. If u is solution in a neighbourhood of $(0, 0, x_3^0)$ which belongs to C^1 then $F(x_3)$ is analytic at x_3^0.

Note that by the Cauchy-Kovalevskaya theorem we will have a solution if F is analytic.

The above equation can be rewritten as

$$\left(-\frac{1}{2} + e_3 x_0 - e_2 x_1\right) {}^{\varphi}Du - \left(\frac{1}{2} + e_3 x_0 - e_2 x_1\right) {}^{\varphi}D_3 u = F(x_3). \tag{2.3}$$

Let us consider the following change of variable $x_0 + e_1x_1 = re^{e_1\theta}$. Then we can rewrite

$$\partial_{x_0} + e_1\partial_{x_1} = \frac{x_0 + e_1x_1}{r^2}\left(\partial_{\ln r} + e_1\partial_\theta\right)$$

and integrate $(\partial_{x_0} + e_1\partial_{x_1})\, u$ over circles $x_0^2 + x_1^2 = r^2 = \text{const}$. This leads to

$$\int_0^{2\pi} (\partial_{x_0} + e_1\partial_{x_1})\, ud\theta \quad = \quad \int_0^{2\pi} \frac{e^{e_1\theta}}{r}\left(\partial_{\ln r} + e_1\partial_\theta\right) ud\theta.$$

Via integration by parts we get

$$\int_0^{2\pi} \frac{e^{e_1\theta}}{r} e_1\partial_\theta ud\theta = \int_0^{2\pi} \frac{e^{e_1\theta}}{r} ud\theta.$$

Furthermore, since we have $2\partial_{r^2}\left(\frac{1}{r}u\right) = \frac{1}{r}\left(\partial_{\ln r}u + u\right)$ we get

$$\int_0^{2\pi} \frac{e^{e_1\theta}}{r}\left(\partial_{\ln r} + e_1\partial_\theta\right) ud\theta = 2\partial_{r^2}\int_0^{2\pi} e^{e_1\theta} rud\theta.$$

If we denote $y = r^2$ and consider the function

$$U(y, x_3) = \int_0^{2\pi} e^{e_1\theta} rud\theta$$

as well as the function $G(x_3)$ such that $G'(x_3) = F(x_3)$ then we get

$$(\partial_{x_3} + e_1\partial_y)\, U(x_3, y) = \pi G'(x_3)$$

with $U(x_3, 0) = 0$. But this implies that the function

$$V(y, x_3) = U(y, x_3) - \pi G$$

is analytic in a domain which includes all points (y, x_3) with $y > 0$ being sufficiently small. Since $V = -\pi G$ for $y = 0$ with G being real-valued we can extend V analytically across $y = 0$. Therefore, V is analytic in zero and, consequently, so is G. But this also implies that F is analytic.

3. Example of Beltrami Equation with No Solution

The previous example can be modified such that our Beltrami equation has no solution in the Hölder space $C^{1,\alpha}$. To this end we need to construct a new right-hand side F. We consider a set of points $\{P_j\}_{j\in\mathbb{N}}$ which is dense in $\mathbb{R}^3$ and $P_j = (p_j, q_j, s_j)$. Furthermore we consider a sequence of spheres U_j each centered at P_j and with radius ρ_j, such that $\lim_{j\to\infty}\rho_j = 0$. Any arbitrary set in $\mathbb{R}^3$ contains at least some of the spheres U_j. Let us denote by $c_j = \max(j, |p_j|, |q_j|)$.

Consider now a sequence $\epsilon = (\epsilon_j)_{j=1}^\infty$ with

$$\|\epsilon\| = \limsup |\epsilon_j| < \infty.$$

Keep in mind that the space of these sequences with the above norm forms a Banach space and, therefore, cannot be exhausted by a countable sum of non-dense sets.

Given a periodic real-valued C^∞-function Φ, depending only on x_2 and being nowhere analytic we can consider the following function

$$F_\epsilon(x_0,x_1,x_2)=\sum_{j=1}^{\infty}\epsilon_j c_j^{-\epsilon_j}\Phi'(x_2+2q_jx_0-2p_jx_1).$$

Is is easy to see that $F_\epsilon\in C^\infty$ since we have that for any multi-index $\nu=(\nu_1,\nu_2)$

$$\partial^{\nu}_{x_0,x_1}F_\epsilon(x_0,x_1,x_2)=\sum_{j=1}^{\infty}\epsilon_j c_j^{-\epsilon_j}\Phi^{|\nu|+1}(x_22q_jx_0-2p_jx_1)q_j^{\nu_1}(-p_j)^{\nu_2}2^{|\nu|}.$$

Note that the series converges absolutely and uniformly. This happens because $\Phi^{|\nu|+1}$ is periodic and, therefore, bounded while $\sum_{j=1}^{\infty}c_j^{-c_j+\nu}<\infty$.

All of this leads to the point that it is enough to show the existence of a sequence $\epsilon^\star$ such that for $F=F^\star$ there is no open set in $\mathbb{R}^3$ on which our equation has a solution in $C^{1,\alpha}$.

Let us denote by $C_m^{1,1/n}$ the set of all functions whose first derivatives satisfy a Hölder condition of exponent $1/n$ and constant m. Clearly, one has $C^{1,\alpha}=\sum_{n,m}C_m^{1,1/n}$. Furthermore, all functions $f\in C_m^{1,1/n}$ which vanish at P_j form a compact subset, which implies that all functions $f\in C^{1,\alpha}(U_j)$ which vanish at P_j belong to a countable sum of compact sets. Now, let $E_{j,n,m}$ denote the set of all sequences ϵ such that for $F=F_\epsilon$ our Beltrami equation has a solution in U_j which belongs to $C_m^{1,1/n}$. It is easy to see that $E_{j,n,m}$ is closed.

Consider now the sequence $\epsilon=\delta^j=(\delta_{1,j},\delta_{2,j},\ldots)$. Then our equation (2.3) becomes

$$\left(-\frac{1}{2}+e_3x_0-e_2x_1\right){}^{\varphi}Du-\left(\frac{1}{2}+e_3x_0-e_2x_1\right){}^{\varphi}D_3u=\Phi'(x_2+2q_jx_0-2p_jx_1)$$

taking $F=F_\delta^j$. With the substitution

$$z_0=x_0-p_j,z_1=x_1-q_j,y_1=x_2-2q_jx_0+2p_jx_1$$

we get

$$\left(-\frac{1}{2}+e_3z_0-e_2z_1\right){}^{\varphi}Du-\left(\frac{1}{2}+e_3z_0-e_2z_1\right){}^{\varphi}D_3u=\Phi'(y_1)$$

and, since a C^1-solution (2.3) would imply that the above equation has a C^1-solution in a neighborhood of $(0,0,y_1)$ which we showed that it is not possible in the previous section.

In fact, using the arguments from the classic paper of Hörmander [10] we know that the considered Beltrami equation cannot even have a distributional solution.

4. Remark on the Beltrami Equation and the Hörmander Condition – Compatibility Relations

Now, let us take a closer look at the problem of local solvability. There is the well-known theorem about local solvability for scalar differential operators based on the (Ψ)-condition by Nirenberg and Trèves. Unfortunately, there is no similar condition for general systems of PDE's. But such a condition exists if the systems are of type $N \times N$, c.f. [5].

Our case here is simpler due to the quaternionic structure, since it provides linear independence, which in turns implies that this case reduces to a more classic approach. For the sake of simplicity we will only consider the case of ${}^{\psi}Du - q\overline{{}^{\psi}Du} = F$ and $\psi_0 = 1$, whose symbol is given by

$$p(\xi, x) = i\xi - q(x)\overline{\xi}.$$

The other cases are given by adding an orthogonal transformation to q which makes the calculations more complicated without adding any principal changes.

The interesting points are the (non-trivial) zeros of the symbol $p(\xi, x) = 0$. Since,

$$p(\xi, x) = 0 \Leftrightarrow \frac{\xi^2}{|\xi|^2} = q(x)$$

it is easy to see that there are no other zeros if $|q(x)| < 1$, i.e. the Beltrami equation is elliptic. But without that condition, there are other zeros and, therefore, no ellipticity.

Furthermore, we remark that the gradient in ξ, denoted by ∇_ξ, applied to p is given by

$$\nabla_\xi p = (\nabla_\xi \xi) + q\overline{(\nabla_\xi \xi)} = \begin{pmatrix} 1 + q_0 + \underline{q} \\ q_1 + (1 - q_0)\varphi_1 - q_3\varphi_2 + q_2\varphi_3 \\ q_2 + q_3\varphi_1 + (1 - q_0)\varphi_2 - q_1\varphi_3 \\ q_3 - q_2\varphi_1 + q_1\varphi_2 + (1 + q_0)\varphi_3 \end{pmatrix}. \tag{4.4}$$

It is is easy to see that the gradient never vanishes since this would require $1 + q_0 = 1 - q_0 = 0$. Therefore, our operator is of principal type. Even more, the components in the gradient in ξ of p are linearly independent which means that our operator is principally normal at any point x_0.

Let us take now a look at the Poisson bracket

$$\{\overline{p}, p\} := \langle \nabla_\xi \overline{p}, \nabla_x p\rangle - \langle \nabla_x \overline{p}, \nabla_\xi p\rangle,$$

where ∇_ξ and ∇_x denote the gradient with respect to ξ and x and $\langle\cdot,\cdot\rangle$ denotes the real scalar bilinear form applied to quaternionic vectors. Because we have in our case

$$\begin{aligned} \langle \nabla_\xi \overline{p}, \nabla_x p\rangle &= (-i)i \sum_j \left[\overline{(\nabla_\xi \xi)} + (\nabla_\xi \xi)\,\overline{q}\right]_j \left[(\nabla_x q)\,\overline{\xi}\right]_j \\ \langle \nabla_\xi \overline{p}, \nabla_x p\rangle &= (-i)i \sum_j \left[\xi\,(\nabla_x \overline{q})\right]_j \left[(\nabla_\xi \xi) + q\overline{(\nabla_\xi \xi)}\right]_j \end{aligned}$$

we get

$$\{\overline{p}, p\} = 2\mathrm{Vec}\sum_j \left[\overline{(\nabla_\xi \xi)} + (\nabla_\xi \xi)\,\overline{q}\right]_j \left[(\nabla_x q)\,\overline{\xi}\right]_j .$$

A direct calculation gives

$$\{\overline{p}, p\} = 2\mathrm{Vec}\left(((1+\overline{q}) - \sum_{j=1}^{3} \varphi_j (1-\overline{q})\partial_{x_j} q)\overline{\xi}\right).$$

There is a better way to study the term $\{\overline{p}, p\}$ by using

$$(\xi\nabla_x \overline{q}) := (A_0^j + \underline{A}^j) = \xi_0 \partial_{x_j} q_0 + \underline{\xi}\partial_{x_j}\underline{q} - \xi_0\partial_{x_j}\underline{q} + (\partial_{x_j} q_0)\underline{\xi} - \xi \wedge (\partial x_j \underline{q})$$

and (4.4), i.e.

$$(\nabla_\xi \xi) + q\overline{(\nabla_\xi \xi)} = \begin{pmatrix} 1 + q_0 + \underline{q} \\ q_1 + (1-q_0)\varphi_1 - q_3\varphi_2 + q_2\varphi_3 \\ q_2 + q_3\varphi_1 + (1-q_0)\varphi_2 - q_1\varphi_3 \\ q_3 - q_2\varphi_1 + q_1\varphi_2 + (1+q_0)\varphi_3 \end{pmatrix}.$$

Denoting each component of the last vector by $B_0^j + \underline{B}^j$ we obtain

$$\{\overline{p}, p\} = 2\sum_j (A_0^j \underline{B}^j + B_0^j \underline{A}^j + \underline{A}^j \wedge \underline{B}^j). \tag{4.5}$$

In this form the term $\{\overline{p}, p\}$ is easier to study. For instance, example (2.3) gives us $\{\overline{p}, p\} = -8e_1\xi_3$.

In his paper [10] L. Hörmander gives a necessary condition for local solvability, which is given by

$$p(\xi, x_0) = 0 \Rightarrow \{\overline{p}, p\} = 0. \tag{4.6}$$

For principally normal operators that condition is also known to be sufficient. Applied to our case we arrive at the following theorem

Theorem 4.1. *The Beltrami equation* ${}^{\psi}Du = q\,{}^{\overline{\psi}}Du$ *is locally solvable at* x_0 *if and only if for* $\left(\frac{\xi}{|\xi|}\right)^2 = q(x_0)$ *we get*

$$\mathrm{Vec}\left(((1+\overline{q}) - \sum_{j=1}^{3} \varphi_j (1-\overline{q})\partial_{x_j} q)\overline{\xi}\right) = 0. \tag{4.7}$$

The condition of the theorem is easier to verify in the form (4.5), i.e.

$$\sum_j (A_0^j \underline{B}^j + B_0^j \underline{A}^j + \underline{A}^j \wedge \underline{B}^j) = 0.$$

In our example (2.3) we have $\{\overline{p}, p\} = -8e_1\xi_3 \neq 0$. Condition (4.7) can also be seen as the compatibility condition for our Beltrami equation, c.f. [15].

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Juan Bory Reyes
Departamento de Matemática
Universidad de Oriente
Santiago de Cuba 90500
Cuba
e-mail: `jbory@rect.uo.edu.cu`

Paula Cerejeiras
Departamento de Matemática
Universidade de Aveiro
P-3810-159 Aveiro
Portugal
e-mail: pceres@ua.pt

Alí Guzmán Adán
Departamento de Matemática
Universidad de Oriente
Santiago de Cuba 90500
Cuba
e-mail: ali.guzman@csd.uo.edu.cu

Uwe Kähler
Departamento de Matemática
Universidade de Aveiro
P-3810-159 Aveiro
Portugal
e-mail: ukaehler@ua.pt

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