Computers and Mathematics with Applications 71 (2016) 1990–2000

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Riemann–Hilbert problems for null-solutions to iterated generalized Cauchy–Riemann equations in axially symmetric domains

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ARTICLE INFO

Article history:
Received 10 October 2015
Received in revised form 17 March 2016
Accepted 20 March 2016
Available online 9 April 2016

Keywords:
Quaternion analysis
Iterated generalized Cauchy–Riemann operator
Axial symmetry
Riemann–Hilbert problems
Variable coefficients

ABSTRACT

We consider Riemann–Hilbert boundary value problems (for short RHBVPs) with variable coefficients for axially symmetric poly-monogenic functions, i.e., null-solutions to iterated generalized Cauchy–Riemann equations, defined in axially symmetric domains. This extends our recent results about RHBVPs with variable coefficients for axially symmetric monogenic functions defined in four-dimensional axially symmetric domains. First, we construct the Almansi-type decomposition theorems for poly-monogenic functions of axial type. Then, making full use of them, we give the integral representation solutions to the RHBVP considered. As a special case, we derive solutions to the corresponding Schwarz problem. Finally, we generalize the result obtained to functions of axial type which are null-solutions to perturbed iterated generalized Cauchy–Riemann equations $\mathcal{D}_\alpha^k\phi = 0$, $k \geqslant 2 (k \in \mathbb{N})$, $\alpha \in \mathbb{R}$.

1. Introduction

The theory of classical Riemann–Hilbert boundary value problems (for short RHBVPs) for poly-analytic functions closely links to theory of singular integral equations, and has been widely applied in other fields, such as theory of quantum mechanics, numerical computation of initial value problems for integral systems, statistical physics, theory of asymptotic analysis in random matrices, and time–frequency analysis (cf. [1–6]). Its investigation has been systematically done by many authors (cf. [7–9]). Thus, the natural question arises as what this type of boundary value problem in higher dimensions would look like. This will be not only a purely theoretical question, since such problems are closely linked to applications in mathematical physics like monogenic signal analysis or three dimensional problems in the magneto-hydodynamics and fluid mechanics (cf. [10]). Clifford analysis, in particular quaternionic analysis, is an elegant generalization of the classical complex analysis to higher dimensions. It focuses on monogenic functions, i.e., null-solutions to Dirac operator or generalized Cauchy–Riemann operator, whose factorization of the higher Laplace operator leads to refine real harmonic analysis (cf. [11,12,10,13–15]). Moreover, making full use of Clifford analysis RHBVPs, particularly Riemann boundary value problems, with constant coefficients for poly-monogenic functions were widely discussed, see [16–23]. Here, a direct

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http://dx.doi.org/10.1016/j.camwa.2016.03.017
0898-1221/

application of the properties of the Cauchy-type integral operator and existing power series expansions allows to represent the solution in terms of integral operators and Taylor series expansion. This kind of Riemann boundary value problem is also closely connected with other types of boundary value problems for different partial differential equations in higher dimensions, see [24–27]. However, as far as we know, almost no results about RHBVPs with variable coefficients even for monogenic functions of axial type could be seen before our recent research [28]. There, we observe the fact that the point-wise product of monogenic functions of axial type is still a monogenic function of axial type (cf. [11,14,15]) and, therefore, we solve RHBVPs with variable coefficients for monogenic functions of axial type by transferring it to RHBVPs for analytic functions on the complex plane. This makes an initial and very crucial step to solve RHBVPs for monogenic functions which is full of progress. In spite of this, nothing is said of poly-monogenic functions of axial type (see Section 2). So, in the underlying context, we will generalize our approach used in [28] to RHBVPs for poly-monogenic functions of axial type.

Our idea is to construct the Almansi-type decomposition theorems for poly-monogenic function of axial type, and apply them to transform RHBVPs with variable coefficients for poly-monogenic functions with axial symmetry in $\mathbb{R}^4$ to those for monogenic functions with axial symmetry in $\mathbb{R}^4$. In this way we will obtain the solutions to RHBVPs considered in an explicit form. Furthermore, we will also discuss RHBVPs for null-solutions to $\mathcal{D}_\alpha^k\phi = 0$, $k \geqslant 2\big(k \in \mathbb{N}\big)$, $\alpha \in \mathbb{R}$ (see Section 4). As a corollary, we present the solution to the Schwarz problem for poly-monogenic functions with axial symmetry. Here, what our contribution is to find the explicit integral representation solutions to RHBVPs for a special type of iterated Vekua system (see Remark 5). Moreover, by decomposing a monogenic functions into a sum of monogenic functions with axial symmetry, associated with the Almansi-type decomposition theorems for poly-monogenic function of axial type, we will extend this idea to RHBVPs with variable coefficients for poly-monogenic functions in quaternion analysis in the future.

The paper is organized as follows. In Section 2, we will recall the necessary facts about quaternion analysis. In Section 3, we derive the Almansi-type decomposition theorems for poly-monogenic functions of axial type. Applying them, in Section 4, we will focus on RHBVPs with variable coefficients for poly-monogenic functions with axial symmetry. We will give the solution to the RHBVP with boundary data belonging to a Hölder space in terms of an integral representation. Then, we will give the solutions to the Schwarz problem for poly-monogenic functions with axial symmetry. Furthermore, we will generalize our approach to functions of axial type which are null-solutions to $\mathcal{D}_\alpha^k\phi = 0$, $k \geqslant 2\big(k \in \mathbb{N}\big)$, $\alpha \in \mathbb{R}$.

2. Preliminaries

Detailed introductions to Clifford algebras, Quaternions, and Clifford analysis can be found in [12,10,13,14,27].

Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis of $\mathbb{H}$. These basic vectors satisfy:

$$e_ie_j + e_je_i = -2\delta_{i,j}, \quad i, j = 1, 2, 3, \qquad e_1e_2 = e_3, \qquad e_2e_3 = e_1, \qquad e_3e_1 = e_2,$$

where δ denotes the Kronecker delta, and $e_0 = 1$ denotes the identity element of the algebra of quaternions $\mathbb{H}$. Thus $\mathbb{H}$ is a real linear, associative, but non-commutative algebra.

An arbitrary $x \in \mathbb{H}$ can be written as $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \triangleq x_0 + \underline{x}$, where $\mathrm{Sc}(x) \triangleq x_0$ and $\mathrm{Vec}(x) \triangleq \underline{x}$ are the scalar and vector part of $x \in \mathbb{H}$, respectively. Elements $x \in \mathbb{R}^4$ can be identified with quaternions $x \in \mathbb{H}$. The conjugation is defined by $\bar{x} = \sum_{j=0}^{3} x_j\bar{e}_j$ with $\bar{e}_0 = e_0$ and $\bar{e}_j = -e_j$, $j = 1, 2, 3$, and hence, $\overline{xy} = \bar{y}\bar{x}$. The algebra of quaternions possesses an inner product $\big(y, x\big) = \mathrm{Sc}(y\bar{x}) = \big(y\bar{x}\big)_0$ for all $x, y \in \mathbb{H}$.

The corresponding norm is $|x| = \Big(\sum_{j=0}^{3}\big|x_j\big|^2\Big)^{\frac{1}{2}} = \sqrt{\big(x, x\big)}$. Any element $x \in \mathbb{H} \setminus \{0\}$ is invertible with inverse element $x^{-1} \triangleq \bar{x}|x|^{-2}$, i.e., $xx^{-1} = x^{-1}x = 1$. Furthermore, we can introduce the set

$$[x] = \big\{y : y = \mathrm{Sc}(x) + \mathit{l}|\underline{x}|,\ \mathit{l} \in S^2\big\},$$

where $S^2 = \big\{\underline{x} \subset \mathbb{R}^3 : |\underline{x}| = 1\big\}$.

We also need to define axially symmetric open sets.

Definition 2.1 (*Axially Symmetric Open Set*)**.** Let Ω be a non-empty open subset of $\mathbb{R}^4$. We say that Ω is axially symmetric if for any $x \in \Omega$, the subset $[x]$ is contained in $\Omega \subset \mathbb{R}^4$.

Remark 1. The unit ball of $\mathbb{R}^4$ and the upper half space $\mathbb{R}^4_+ = \big\{x \in \mathbb{R}^4 \big| x_0 > 0\big\}$ are examples of axially symmetric domains.

Let Ω be an axially symmetric domain of $\mathbb{R}^4$ with smooth boundary $\partial\Omega$. An $\mathbb{H}$-valued function $\phi = \sum_{j=0}^{3}\phi_je_j$ is continuous, Hölder continuous, p-integrable, continuously differentiable and so on if all components ϕ_j have that property. The corresponding function spaces, considered as either right-Banach or right-Hilbert modules, are denoted by $C\big(\Omega, \mathbb{H}\big)$, $H^\mu\big(\Omega, \mathbb{H}\big)\big(0 < \mu \leqslant 1\big)$, $L_p\big(\Omega, \mathbb{H}\big)\big(1 < p < +\infty\big)$, $C^1\big(\Omega, \mathbb{H}\big)$, respectively.

Applying Fueter's theorem (see [13,17]) a function of axial type is given by

$$\phi(x) = A\big(x_0, r\big) + \underline{\omega}B\big(x_0, r\big), \tag{1}$$

where $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in \mathbb{R}^4, r = |\underline{x}|, \underline{\omega} \in \left\{\underline{x} = \sum_{j=1}^{3} x_j e_j : |\underline{x}| = 1, x_j \in \mathbb{R}(j = 1, 2, 3)\right\}$, $A(x_0, r)$ and $B(x_0, r)$ are scalar-valued functions. It should be mentioned that in [13] a function of axial type is also called a function with axial symmetry.

In the whole of the paper any functions defined on $\Omega \cup \partial\Omega \subset \mathbb{R}^4$ with values in $\mathbb{H}$ are supposed to be of axial type unless otherwise stated.

In this context we will consider the generalized Cauchy–Riemann operator $\mathcal{D} = \sum_{j=0}^{3} e_j \partial_{x_j}$ in $\mathbb{R}^4$. The generalized Cauchy–Riemann operator factorizes the Laplacian in the sense $\overline{\mathcal{D}}\mathcal{D} = \sum_{j=0}^{3} \partial_{x_j}^2 = \Delta$ where Δ denotes the Laplacian in $\mathbb{R}^4$.

Definition 2.2. A function $\phi \in C^k(\Omega, \mathbb{H}), k \geqslant 1(k \in \mathbb{N})$ is called (left-) poly-monogenic if and only if $\mathcal{D}^k\phi = 0$, where $\mathcal{D}^k\phi \triangleq \mathcal{D}(\mathcal{D}^{k-1}\phi)$. A poly-monogenic function of axial type is called axially poly-monogenic. The set of all axially poly-monogenic functions defined in Ω forms a right-module, denoted by $M_k(\Omega, \mathbb{H})$. In particular, when $k = 1$, it reduces to (left-) monogenic functions of axial type, seen in Ref. [28]. In what follows we assume $k \geqslant 2(k \in \mathbb{N})$ in order to avoid the triviality.

Definition 2.3. For a function of axial type $\phi : \Omega \to \mathbb{H}$, we define the real part as $\mathrm{Re}\,\phi = A$. In dimension two, i.e., the case of $\mathbb{R}_{0,1} = \left\{x = x_0 + x_1 e_1 : e_1^2 = -1, x_j \in \mathbb{R}(j = 0, 1)\right\} \cong \mathbb{C}$, we have $\mathrm{Re}\,\phi = \mathrm{Sc}(\phi)$, which exactly corresponds to the usual understanding in complex analysis.

Remark 2. In [14,15] it is shown that the equation $\mathcal{D}^k\phi = 0, k \geqslant 2(k \in \mathbb{N})$ for functions of axial type is equivalent to a special kind of the iterated Vekua system [10].

In what follows $D \subset \mathbb{C}_+$ is the projection of the axially symmetric sub-domain $\Omega \subset \mathbb{R}^4$ into the (x_0, r)-plane, where $\mathbb{C}_+$ is the upper half of the (x_0, r)-plane.

3. Lemmas

Let us start with the following lemma. We will give the Almansi-type decomposition theorem for null-solutions of the equation $\mathcal{D}_\alpha^k\phi = 0, k \geqslant 2, k \in \mathbb{N}$ ($\mathcal{D}_\alpha = \mathcal{D} - \alpha I$, I being the identity operator), which have axial symmetry and are defined over axial domains of $\mathbb{R}^4$.

Lemma 3.1. *Suppose that $\phi \in C^k(\Omega, \mathbb{H})$ of axial type is a solution to the equation $\mathcal{D}^k\phi(x) = 0(k \geqslant 2, k \in \mathbb{N})$, then there uniquely exist monogenic functions $\phi_j \in C^1(\Omega, \mathbb{H})$ of axial type, such that*

$$\phi(x) = \phi_0(x) + x_0\phi_1(x) + \cdots + x_0^{k-1}\phi_{k-1}(x),$$

where

$$\begin{cases} \phi_{k-1}(x) = \dfrac{1}{(k-1)!}\mathcal{D}^{k-1}\phi(x), \\ \phi_{k-2}(x) = \dfrac{1}{(k-2)!}\mathcal{D}^{k-2}\left(I - \dfrac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1}\right)\phi(x), \\ \quad\vdots \qquad\qquad\qquad \vdots \\ \phi_1(x) = \mathcal{D}\left(I - \dfrac{1}{2}x_0^2\mathcal{D}^2\right)\dots\left(I - \dfrac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1}\right)\phi(x), \\ \phi_0(x) = \left(I - x_0\mathcal{D}\right)\left(I - \dfrac{1}{2}x_0^2\mathcal{D}^2\right)\dots\left(I - \dfrac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1}\right)\phi(x). \end{cases} \tag{2}$$

I in (2) denotes the identity operator, and $(\mathcal{D}^k\phi)(x) \triangleq \mathcal{D}(\mathcal{D}^{k-1}\phi)(x)(k \geqslant 2, k \in \mathbb{N})$.

Proof. Since $\phi \in C^k(\Omega, \mathbb{H})$ is a solution to $\mathcal{D}^k\phi(x) = 0(k \geqslant 2, k \in \mathbb{N})$, we can apply Lemma 3.2 in [20] (see also [21,25,26]). This means that there exist monogenic functions $\phi_j \in C^1(\Omega, \mathbb{H})$ which satisfy

$$\phi(x) = \phi_0(x) + x_0\phi_1(x) + \cdots + x_0^{k-1}\phi_{k-1}(x), \tag{3}$$

and $\phi_j,\ j = 0, 1, 2, \dots, k-1$, are explicitly given by (2).

As $\phi \in C^k(\Omega, \mathbb{H})$ is of axial type, i.e.,

$$\phi(x) = A_0(x_0, r) + \underline{\omega}B_0(x_0, r),$$

where $A_0, B_0 \in C^k(\Omega, \mathbb{R})$ and $x = x_0 + r\underline{\omega} \in \mathbb{R}^4$ we can apply $\mathcal{D}$ to ϕ

$$\begin{aligned}\mathcal{D}\phi(x) &= \left(\partial_{x_0} + \underline{\omega}\partial_r + \frac{1}{r}\partial_\omega\right)\left(A_0(x_0, r) + \underline{\omega}B_0(x_0, r)\right)\\ &= \left(\partial_{x_0}A_0(x_0, r) - \partial_r B_0(x_0, r)\right) + \underline{\omega}\left(\partial_r A_0(x_0, r) + \partial_{x_0}B_0(x_0, r) + \frac{2}{r}B_0(x_0, r)\right)\\ &\triangleq A_1(x_0, r) + \underline{\omega}B_1(x_0, r),\end{aligned} \tag{4}$$

where $A_1(x_0, r), B_1(x_0, r) \in C^{k-1}(\Omega, \mathbb{R})$ and see that $\mathcal{D}\phi$ is of the same type.

Repeating the procedure l-times we obtain

$$\mathcal{D}^l\phi(x) = A_l(x_0, r) + \underline{\omega}B_l(x_0, r),$$

where $A_l(x_0, r), B_l(x_0, r) \in C^{k-l}(\Omega, \mathbb{R}), l = 2, 3, \dots, k-1(k \geqslant 2, k \in \mathbb{N})$.

Finally, by applying $\mathcal{D}^l, l = 1, 2, 3, \dots, k-1(k \geqslant 2, k \in \mathbb{N})$ on both sides of (3), we get

$$\mathcal{D}^l\phi(x) = \phi_l + x_0\phi_{l+1} + \dots + x_0^{k-1-l}\phi_{k-1}, \quad l = 1, 2, 3, \dots, k-1(k \geqslant 2, k \in \mathbb{N})$$

and, therefore, $\phi_j(j = 0, 1, \dots, k-1)$ is still of axial type. This completes the proof. □

Thanks to Lemma 3.1 we have the following result.

Lemma 3.2. *Suppose that $\phi \in C^k(\Omega, \mathbb{H})(k \geqslant 2, k \in \mathbb{N})$ of axial type is a solution to the equation $\mathcal{D}_\alpha^k\phi(x) = 0, \alpha \in \mathbb{R}$, then there uniquely exist monogenic functions $\phi_j \in C^1(\Omega, \mathbb{H})$ of axial type, such that*

$$\phi(x) = e^{\alpha x_0}\phi_0(x) + x_0e^{\alpha x_0}\phi_1(x) + \dots + x_0^{k-1}e^{\alpha x_0}\phi_{k-1}(x),$$

where $\mathcal{D}_\alpha = \mathcal{D} - \alpha I$ with I being identity operator, and $\phi_j(j = 0, 1, 2, \dots, k-1)$ are given as in Lemma 3.1.

Proof. Since $\mathcal{D}_\alpha\phi = \mathcal{D}(e^{-\alpha x_0}\phi), \alpha \in \mathbb{R}$, then

$$\mathcal{D}_\alpha^k\phi = 0 \text{ is equivalent to } \mathcal{D}^k(e^{-\alpha x_0}\phi) = 0, \quad x \in \Omega.$$

So it follows the result. □

Remark 3. The key point in the study of poly-monogenic functions of axial type is the decomposition established in Lemma 3.1. It extends the so-called Almansi-type decomposition theorems [23–26] to the case of poly-monogenic functions of axial type.

4. RHBVPs for poly-monogenic functions of axial type

In this section we discuss RHBVPs with variable coefficients for null-solutions of the equation $\mathcal{D}_\alpha^k\phi = 0(k \geqslant 2, k \in \mathbb{N}, \alpha \in \mathbb{R})$. Hereby, we will restrict ourselves to functions which have axial symmetry and are defined over axial domains of $\mathbb{R}^4$. We first solve the RHBVP in $\mathbb{R}^4$ by transforming it into a RHBVP for complex analytic functions. Afterwards, in the case of boundary values belonging to a Hölder space we represent its solution in terms of explicit integral representation formula. As a corollary we obtain the solution to the corresponding Schwarz problem in quaternion analysis.

Let us begin with restating RHBVP with variable coefficients for monogenic functions of axial type, discussed in Ref. [28].

Theorem 4.1 ([28]). *The Riemann–Hilbert boundary value problem is as follows: to find a function $\phi \in C^1(\Omega, \mathbb{H})$ of axial type which satisfies the condition*

$$(\star)\begin{cases}\mathcal{D}\phi(x) = 0, & x \in \Omega,\\ Re\left\{\lambda(t)\phi(t)\right\} = g(t), & t \in \partial\Omega,\end{cases}$$

where $g \in H^\mu(\partial\Omega, \mathbb{R})$, and $D = \left\{z : |z - a| < 1\right\} \subset \mathbb{C}_+$ with $z = x_0 + ir$, $a = a_0 + ia_1 \in \mathbb{C}_+$ being the corresponding domain of $\mathbb{C}_+$ with boundary ∂D.

(i) *If $\lambda \in H^\mu(\partial\Omega, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $z \in \partial D$, then the Riemann–Hilbert boundary value problem $(\star)$ is solvable and its solution is given by*

$$\phi(x) = \Delta\left(Re(f)(x_0, |\underline{x}|) + \underline{\omega}Im(f)(x_0, |\underline{x}|)\right), \quad x \in \Omega,$$

where Re(f) and Im(f) denote the real and imaginary part of the complex-valued function f, respectively. f itself is given by

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{g}(\zeta)}{\zeta-\xi} d\zeta d\xi + \sum_{n=0}^{+\infty} t_n \big(z-a\big)^n, \quad z \in D,$$

with $\widetilde{g} = \lambda^{-1}\frac{r}{2}g$, $t_n \in \mathbb{C}$.

(ii) *Suppose that* $\lambda = \Pi\widehat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^m \big(x - \widehat{\alpha}_i\big)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\Omega$ *and* $\nu_i \in \mathbb{N}$. *Furthermore, if* $\widehat{\lambda} \in H^\mu\big(\partial\Omega, \mathbb{R}\big)$ *and* $\widehat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\Omega$, *then the Riemann–Hilbert boundary value problem* $(\star)$ *is solvable, and its solution is given again by*

$$\phi(x) = \Delta\Big(Re(f)\big(x_0, |\underline{x}|\big) + \underline{\omega} Im(f)\big(x_0, |\underline{x}|\big)\Big), \quad x \in \Omega,$$

where f is given by

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)}\frac{\widehat{g}(\zeta)}{\zeta-\xi} d\zeta d\xi + \sum_{n=0}^{+\infty} \int_a^z \frac{l_n\big(\xi-a\big)^n}{\Pi(\xi)} d\xi, \quad z \in D,$$

with $\widehat{g} = \widehat{\lambda}^{-1}\frac{r}{2}g$, $l_n \in \mathbb{C}$.

D and ∂D are the same with those defined by Problem $(\star)$ of Theorem 4.1 unless stated. We will continue to discuss RHBVPs with variable coefficients for poly-monogenic functions of axial type. Then we will generalize our approach to functions of axial type defined over an axial domain of $\mathbb{R}^4$, which are null-solutions to $\mathcal{D}_\alpha^k\phi = 0$, $k \geqslant 2\big(k \in \mathbb{N}\big)$, $\alpha \in \mathbb{R}$. We begin with the following extension of Problem $(\star)$.

Problem 1. Find a function $\phi \in C^k\big(\Omega, \mathbb{H}\big)\big(k \geqslant 2, k \in \mathbb{N}\big)$ of axial type, which satisfies the following condition

$$(\star\star)\begin{cases} \mathcal{D}^k\phi(x) = 0, & x \in \Omega, \\ \mathrm{Re}\Big\{\lambda(t)\phi(t)\Big\} = g_0(t), & t \in \partial\Omega, \\ \mathrm{Re}\Big\{\lambda(t)\big(\mathcal{D}\phi\big)(t)\Big\} = g_1(t), & t \in \partial\Omega, \\ \quad\vdots & \quad\vdots \\ \mathrm{Re}\Big\{\lambda(t)\big(\mathcal{D}^l\phi\big)(t)\Big\} = g_l(t), & t \in \partial\Omega,\ 0 < l < k-1, \\ \quad\vdots & \quad\vdots \\ \mathrm{Re}\Big\{\lambda(t)\big(\mathcal{D}^{k-1}\phi\big)(t)\Big\} = g_{k-1}(t), & t \in \partial\Omega, \end{cases}$$

where $g_j : \Omega \to \mathbb{R}\big(j = 0, 1, \ldots, k-1\big)$ are scalar-valued functions, and Ω is a bounded axial domain of $\mathbb{R}^4$.

Theorem 4.2. *Suppose* $g_j \in H^\mu\big(\partial\Omega, \mathbb{R}\big)\big(j = 0, 1, 2, \ldots, k-1\big)$, *and D is the projection of* $\Omega \subset \mathbb{R}^4$ *into* $\mathbb{C}_+$. *Then the Riemann–Hilbert boundary value problem* $(\star\star)$ *is solvable, and its solution is given by*

$$\phi(x) = \phi_0(x) + x_0\phi_1(x) + \cdots + x_0^{k-1}\phi_{k-1}(x), \quad x \in \Omega,$$

where $\phi_j(x) = \Delta\Big(Re(f_j)\big(x_0, |\underline{x}|\big) + \underline{\omega} Im(f_j)\big(x_0, |\underline{x}|\big)\Big)$, $j = 0, 1, 2, \ldots, k-1$. *For the functions* f_j *we have to consider two cases:*

(i) *If* $\lambda \in H^\mu\big(\partial\Omega, \mathbb{R}\big)$, *and* $\lambda \neq 0$ *for arbitrary* $x \in \partial\Omega$, *we have*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{G}_j(\zeta)}{\zeta-\xi} d\zeta d\xi + \sum_{n=0}^{+\infty} t_{j,n}\big(z-a\big)^n, \quad z \in D,\ t_n \in \mathbb{C},$$

where $\widetilde{G}_j \triangleq \lambda^{-1}\frac{r}{2}\widetilde{g}_j$, *with* $\widetilde{g}_j = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} x_0^{i-j}\frac{r}{2}g_i$.

(ii) *If* $\lambda = \Pi\widehat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^m\big(x - \widehat{\alpha}_i\big)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\Omega$, *and* $\widehat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\Omega$, *with* $\nu_i \in \mathbb{N}$, *and* $\widehat{\lambda} \in H^\mu\big(\partial\Omega, \mathbb{R}\big)$, *we have*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)}\frac{\widehat{G}_j(\zeta)}{\zeta-\xi} d\zeta d\xi + \sum_{n=0}^{+\infty} \int_a^z \frac{l_n\big(\xi-a\big)^n}{\Pi(\xi)} d\xi, \quad z \in D,\ l_n \in \mathbb{C},$$

where $\widehat{G}_j \triangleq \widehat{\lambda}^{-1}\frac{r}{2}\widetilde{g}_j$, *with* $\widetilde{g}_j = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} x_0^{i-j}\frac{r}{2}g_i$.

Furthermore, suppose that $D_1 = \left\{z : |z-a| < R\right\} \subset D$, $0 < R < 1$ *is the projection of* $\Omega_1 \subset \Omega$ *into the* $\left(x_0, r\right)$*-plane. Then the solution to the Riemann–Hilbert boundary value problem* $(\star\star)$ *is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j \phi_j(x), \quad x \in \Omega_1,$$

where, for $j = 0, 1, 2, \ldots, k-1$,

$$\phi_j(x) = \frac{2i}{\pi}\int_{\partial D_1} \left(z-\overline{x}\right)\left(z^2 - 2Sc(x)z + |x|^2\right)^{-2} f_j(z)dz, \quad x \in \Omega_1, \tag{5}$$

with f_j *being given in accordance to the above cases* (i) *or* (ii).

Proof. Since $\mathcal{D}^k\phi(x) = 0$, $x \in \Omega$ by Lemma 3.1 there exist uniquely defined functions ϕ_j of axial type, which are monogenic in Ω and satisfy

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j \phi_j(x).$$

Hence, for arbitrary $1 \leqslant l \leqslant k-1$, we get

$$\mathcal{D}^l\phi(x) = c_l\phi_l(x) + c_{l+1}x_0\phi_{l+1}(x) + \cdots + c_{k-1}x_0^{k-1-l}\phi_{k-1}(x), \quad x \in \Omega,$$

where $c_j = j\left(j-1\right)\cdots\left(j-l+1\right)$, $j = l, \ldots, k-1$.

Let us introduce the following vector and matrix functions

$$\mathbf{\Phi}(x) = \left(\phi_0(x), \phi_1(x), \ldots, \phi_{k-1}(x)\right)^T, \quad x \in \Omega,$$

$$\operatorname{Re}\left\{\lambda(t)\mathbf{\Phi}(t)\right\} = \left(\operatorname{Re}\left\{\lambda(t)\phi_0(t)\right\}, \operatorname{Re}\left\{\lambda(t)\phi_1(t)\right\}, \ldots, \operatorname{Re}\left\{\lambda(t)\phi_{k-1}(t)\right\}\right)^T, \quad t \in \partial\Omega,$$

$$\mathbf{g}(t) = \left(g_0(t), g_1(t), \ldots, g_{k-1}(t)\right)^T, \quad t \in \partial\Omega,$$

and

$$\mathbf{A}(t_0) = \begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^{k-1} \\ 0 & 1! & 2t_0 & \cdots & (k-1)t_0^{k-2} \\ 0 & 0 & 2! & \cdots & (k-1)(k-2)t_0^{k-3} \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)! \end{pmatrix}_{k\times k}, \quad t_0 \in \partial\Omega.$$

With the above introduced functions we can rewrite our boundary value problem $(\star\star)$ as the following matrix boundary value problem

$$(*)\begin{cases} \mathcal{D}\phi_j(x) = 0, \quad j = 0, 1, \ldots, k-1, \ x \in \Omega, \\ \mathbf{A}(t_0)\operatorname{Re}\left\{\lambda(t)\mathbf{\Phi}(t)\right\} = \mathbf{g}(t), \quad t \in \partial\Omega. \end{cases}$$

Again, by the procedure of proof of Theorem 3.3 in [28], the boundary value problem $(*)$ is equivalent to the following problem

$$(**)\begin{cases} \partial_{\bar{z}}f_j(z) = 0, \quad j = 0, 1, \ldots, k-1, z \in D, \\ \mathbf{A}(x_0)\operatorname{Re}\left\{\lambda(z)\left(\mathbf{\Delta u}\right)(z)\right\} = \mathbf{g}(z), \quad z \in \partial D, \end{cases}$$

where

$$\left(\mathbf{\Delta u}\right)(z) = \left(\left(\Delta u_0\right)(z), \left(\Delta u_1\right)(z), \ldots, \left(\Delta u_{k-1}\right)(z)\right)^T, \quad z \in D.$$

For $x_0 \in \partial D$ we have $\left|\mathbf{A}(x_0)\right| \neq 0$. Hence, $\mathbf{A}(x_0)$ is invertible. The inverse of $\mathbf{A}(x_0)$ is denoted by $\mathbf{A}^{-1}(x_0)$. Next, we define

$$\widetilde{\mathbf{g}}(z) = \mathbf{A}^{-1}(x_0)\mathbf{g}(z) \triangleq \left(\widetilde{g}_0(z), \widetilde{g}_1(z), \ldots, \widetilde{g}_{k-1}(z)\right)^T,$$

where $\widetilde{g}_j = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} x_0^{i-j}\frac{r}{2}g_i$, $j = 0, 1, 2, \ldots, k-1$.

Furthermore, let $h_j = \partial_r f_j$. Then we can rewrite our boundary value problem $(*)$ as

$$(***)\begin{cases}\partial_{\bar z} h_j(z) = 0, & j = 0, 1, \ldots, k-1,\ z \in D,\\ \mathrm{Re}\Big\{\lambda(z)\mathbf{h}(z)\Big\} = \widetilde{\mathbf{g}}(z), & z \in \partial D,\end{cases}$$

where $\mathrm{Re}\Big\{\lambda(t)\mathbf{h}(z)\Big\} = \Big(\mathrm{Re}\big\{\lambda(z)h_0(z)\big\}, \mathrm{Re}\big\{\lambda(z)h_1(z)\big\}, \ldots, \mathrm{Re}\big\{\lambda(z)h_{k-1}(z)\big\}\Big)^T,\ z \in \partial D$.

In the same way as above we can now apply Theorem 4.1 to this boundary value problem. As before we have to consider two cases depending on λ having no zeros or a finite number of zeros. Let us start with the case of no zeros.

(i) If $\lambda, g \in H^\mu\big(\partial\Omega, \mathbb{R}\big)$ with $\lambda \neq 0$ for arbitrary $x \in \partial\Omega$ we have $\widetilde{G}_j \triangleq \lambda^{-1}\frac{r}{2}\widetilde{g}_j \in H^\mu\big(\partial D, \mathbb{R}\big)$ and can write f_j in the form

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{G}_j(\zeta)}{\zeta - \xi} d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_{j,n}\big(z-a\big)^n, \quad z \in D,\ t_{j,n} \in \mathbb{C}, n \in \mathbb{N}. \tag{6}$$

In the second case we again remove the zeros from λ.

(ii) If $\lambda = \Pi\widehat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^m\big(x - \widehat{\alpha}_i\big)^{\nu_i}$, where $\widehat{\lambda} \neq 0$ for arbitrary $x \in \partial\Omega$, $\widehat{\alpha}_i \in \partial\Omega$, with $\nu_i \in \mathbb{N}$ then we have

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)}\frac{\widehat{G}_j(\zeta)}{\zeta - \xi} d\zeta\, d\xi + \sum_{n=0}^{+\infty}\int_a^z \frac{l_n\big(\xi - a\big)^n}{\Pi(\xi)} d\xi, \quad z \in D, l_n \in \mathbb{C}. \tag{7}$$

Therefore, for $j = 0, 1, \ldots, k-1$, we obtain

$$\phi_j(x) = \Delta\Big(\mathrm{Re}(f_j)\big(x_0, |\underline{x}|\big) + \underline{\omega}\mathrm{Im}(f_j)\big(x_0, |\underline{x}|\big)\Big), \quad x \in \Omega,$$

and the solution to the Riemann–Hilbert boundary value problem $(*)$ is given by

$$\phi(x) = \phi_0(x) + x_0\phi_1(x) + \cdots + x_0^{k-1}\phi_{k-1}(x), \quad x \in \Omega.$$

Now, by Theorems 3.1 and 3.9 in [15] we get the desired result. □

As a direct corollary of Theorem 4.2 we have the following statement.

Corollary 4.3. *Let $g_j \in H^\mu\big(\partial\Omega, \mathbb{R}\big)\big(j = 0, 1, 2, \ldots, k-1\big)$, and $D \subset \mathbb{C}_+$ is the projection of $\Omega \in \mathbb{R}^4$. Then the Schwarz problem*

$$(\sharp\sharp)\begin{cases}\mathcal{D}^k\phi(x) = 0, & x \in \Omega,\\ Re\Big\{\phi(t)\Big\} = g_0(t), & t \in \partial\Omega,\\ Re\Big\{\big(\mathcal{D}\phi\big)(t)\Big\} = g_1(t), & t \in \partial\Omega,\\ \quad\vdots & \quad\vdots\\ Re\Big\{\big(\mathcal{D}^l\phi\big)(t)\Big\} = g_l(t), & t \in \partial\Omega,\ 0 < l < k-1,\\ \quad\vdots & \quad\vdots\\ Re\Big\{\big(\mathcal{D}^{k-1}\phi\big)(t)\Big\} = g_{k-1}(t), & t \in \partial\Omega,\end{cases}$$

is uniquely solvable, and the solution is given by

$$\phi(x) = \phi_0(x) + x_0\phi_1(x) + \cdots + x_0^{k-1}\phi_{k-1}(x), \quad x \in \Omega,$$

where

$$\phi_j(x) = \Delta\Big(Re(f_j)\big(x_0, |\underline{x}|\big) + \underline{\omega}Im(f_j)\big(x_0, |\underline{x}|\big)\Big), \quad j = 0, 1, 2, \ldots, k-1, x \in \Omega,$$

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{G}_{1,j}(\zeta)}{\zeta - \xi} d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_{j,n}\big(z-a\big)^n, \quad z \in D,\ t_n \in \mathbb{C},$$

and $\widetilde{G}_{1,j} \triangleq \frac{r}{2}\widetilde{g}_{1,j}$ with $\widetilde{g}_{1,j} = \sum_{i=j}^{k-1}\frac{(-1)^{i+j}}{(i-j)!j!}x_0^{i-j}\frac{r}{2}g_i$.

Remark 4. In the 2-dimensional case, the Riemann–Hilbert boundary value problem $(\star\star)$ reduces to the Riemann–Hilbert boundary value problem for poly-analytic functions defined on the complex plane [3,7–9]. Moreover, in case of $\lambda \equiv 1, x \in \partial\Omega$, the Riemann–Hilbert boundary value problem $(\star\star)$ reduces to the Schwarz problem for poly-analytic functions defined over the complex plane [6,7].

Furthermore, we next have the study of Problem $1(\alpha)$ described below, which is an extension of Problem 1.

Problem $1(\alpha)$**.** Find a function $\phi \in C^k\big(\Omega, \mathbb{H}\big)\big(k \geqslant 2, k \in \mathbb{N}\big)$ of axial type defined on a bounded domain Ω of $\mathbb{R}^4$, satisfying the conditions

$$(\hbar)\begin{cases} \mathcal{D}_\alpha^k \phi(x) = 0, \quad x \in \Omega, \\ \mathrm{Re}\Big\{\lambda(t)\phi(t)\Big\} = g_0(t), \quad t \in \partial\Omega, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ \mathrm{Re}\Big\{\lambda(t)\big(\mathcal{D}_\alpha^l \phi\big)(t)\Big\} = g_l(t), \quad t \in \partial\Omega, 1 \leqslant l < k-1, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ \mathrm{Re}\Big\{\lambda(t)\big(\mathcal{D}_\alpha^{k-1} \phi\big)(t)\Big\} = g_{k-1}(t), \quad t \in \partial\Omega, \end{cases}$$

where $\mathcal{D}_\alpha = \mathcal{D} - \alpha I$ with I being the identity operator, $\alpha \in \mathbb{R}$, $g_j : \Omega \to \mathbb{R}\big(j = 0, 1, \ldots .k-1\big)$.

For this type of problem we can state the following theorem.

Theorem 4.4. *Let $g_j \in H^\mu\big(\partial\Omega, \mathbb{R}\big)\big(j = 0, 1, 2, \ldots, k-1\big)$ and $D \subset \mathbb{C}_+$ be the projection of $\Omega \subset \mathbb{R}^4$. Then the Riemann–Hilbert boundary value problem $(\hbar)$ is uniquely solvable and the solution is given by*

$$\phi(x) = e^{\alpha x_0}\phi_0(x) + x_0 e^{\alpha x_0}\phi_1(x) + \cdots + x_0^{k-1} e^{\alpha x_0}\phi_{k-1}(x), \quad x \in \Omega, \tag{8}$$

where $\phi_j(x) = \Delta\Big(Re(f_j)\big(x_0, |\underline{x}|\big) + \underline{\omega} Im(f_j)\big(x_0, |\underline{x}|\big)\Big)$, $j = 0, 1, 2, \ldots, k-1$,

(i) *when $\lambda \in H^\mu\big(\partial\Omega, \mathbb{H}\big)$, and $\lambda \neq 0$ for all $x \in \partial\Omega$,*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{e^{\alpha x_0} G_j(\zeta)}}{\zeta - \xi} d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_{j,n}\big(z-a\big)^n, \quad z \in D, t_n \in \mathbb{C},$$

where $\widetilde{G}_j \triangleq \lambda^{-1}\frac{r}{2}\widetilde{g}_j$, with $\widetilde{g}_j = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} x_0^{i-j}\frac{r}{2} g_i$,

(ii) *when $\lambda = \Pi\widehat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^m\big(x - \widehat{\alpha}_i\big)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\Omega$, where $\widehat{\lambda} \neq 0$ for all $x \in \partial\Omega$, with $\nu_i \in \mathbb{N}$ and $\widehat{\lambda} \in H^\mu\big(\partial\Omega, \mathbb{H}\big)$,*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)}\frac{\widehat{e^{\alpha x_0} G_j(\zeta)}}{\zeta - \xi} d\zeta\, d\xi + \sum_{n=0}^{+\infty}\int_a^z \frac{l_n\big(\xi - a\big)^n}{\Pi(\xi)} d\xi, \quad z \in D, l_n \in \mathbb{C},$$

where $\widehat{G}_j \triangleq \widehat{\lambda}^{-1}\frac{r}{2}\widetilde{g}_j$, with $\widetilde{g}_j = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} x_0^{i-j}\frac{r}{2} g_i$.

Moreover, if D_1 is the projection of $\Omega_1 \subset \Omega$ into the $\big(x_0, r\big)$-plane with boundary ∂D_1 then the solution to the Riemann–Hilbert boundary value problem $(\hbar)$ has the form

$$\phi(x) = e^{\alpha x_0}\phi_0(x) + x_0 e^{\alpha x_0}\phi_1(x) + \cdots + x_0^{k-1} e^{\alpha x_0}\phi_{k-1}(x), \quad x \in \Omega_1,$$

with

$$\phi_j(x) = \frac{2i}{\pi}\int_{\partial D_1}\big(z - \bar{x}\big)\Big(z^2 - 2Sc(x)z + |x|^2\Big)^{-2} f_j(z)dz, \quad x \in \Omega_1,$$

and f_j as in the above cases (i) *or* (ii).

Proof. Applying Lemma 3.2, we get

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j e^{\alpha x_0}\phi_j(x),$$

where $\phi_j \in C^1$ is axially monogenic.

Let $\mathbf{E}(t) = \mathrm{diag}\big\{e^{\alpha t_0}, e^{\alpha t_0}, \ldots, e^{\alpha t_0}\big\}$. For $1 \leqslant l \leqslant k-1$, we have

$$\mathcal{D}_\alpha^l \phi(x) = \sum_{j=l}^{k-1} c_l x_0^{j-l} e^{\alpha x_0}\phi_j(x), \qquad c_j = j\big(j-1\big)\ldots\big(j-l+1\big), \quad j = l, \ldots, k-1.$$

Hence, we can write the problem in the form

$$\begin{cases} \mathcal{D}\left(e^{\alpha x_0}\phi_j\right)(x) = 0, \quad j = 0, 1, \ldots, k-1, \ x \in \Omega, \\ \mathbf{E}(t)\mathbf{A}(t_0)\mathrm{Re}\left\{\lambda(t)\mathbf{\Phi}(t)\right\} = \mathbf{E}(t)\mathbf{g}(t), \quad t \in \partial\Omega, \end{cases}$$

where $\mathbf{\Phi}$, $\mathbf{g}$ and $\mathbf{A}$, $t \in \Omega$ are as in Theorem 4.2.

Equivalently, we get

$$\begin{cases} \mathcal{D}\left(e^{\alpha x_0}\phi_j\right)(x) = 0, \quad j = 0, 1, \ldots, k-1, \ x \in \Omega, \\ \mathbf{E}(t)\mathbf{A}(t_0)\mathrm{Re}\left\{\lambda(t)e^{\alpha x_0}\mathbf{\Phi}(t)\right\} = \mathbf{E}(t)e^{\alpha x_0}\mathbf{g}(t), \quad t \in \partial\Omega, \end{cases}$$

where $\mathbf{\Phi}$, $\mathbf{g}$ and $\mathbf{A}$, $t \in \Omega$ are as in Theorem 4.2.

Since $D \subset \mathbb{C}_+$ is the projection of $\Omega \subset \mathbb{R}^4$ into the $\left(x_0, r\right)$-plane, then $e^{\alpha x_0}g_j \in H^{\mu}\left(\partial\Omega, \mathbb{R}\right)\left(j = 0, 1, 2, \ldots, k-1\right)$. Similar to the proof of Theorem 4.2, by applying Theorem 3.9 from [15] we obtain the result. □

As a direct consequence of Theorem 4.4 we get the following theorem.

Theorem 4.5. *Let $g_j \in H^{\mu}\left(\partial\Omega, \mathbb{R}\right)$, $j = 0, 1, 2, \ldots, k-1$, and $D \subset \mathbb{C}_+$ the projection of $\Omega \in \mathbb{R}^4$. Then the Schwarz problem*

$$(\sharp\sharp\sharp)\begin{cases} \mathcal{D}_{\alpha}^{k}\phi(x) = 0, \quad x \in \Omega, \\ Re\left\{\phi(t)\right\} = g_0(t), \quad t \in \partial\Omega, \\ \quad\vdots \qquad\qquad \vdots \\ Re\left\{\left(\mathcal{D}_{\alpha}^{l}\phi\right)(t)\right\} = g_l(t), \quad t \in \partial\Omega, \ 1 \leqslant l < k-1, \\ \quad\vdots \qquad\qquad \vdots \\ Re\left\{\left(\mathcal{D}_{\alpha}^{k-1}\phi\right)(t)\right\} = g_{k-1}(t), \quad t \in \partial\Omega, \end{cases}$$

is uniquely solvable, and the solution is given by

$$\phi(x) = e^{\alpha x_0}\phi_0(x) + x_0e^{\alpha x_0}\phi_1(x) + \cdots + x_0^{k-1}e^{\alpha x_0}\phi_{k-1}(x), \quad x \in \Omega,$$

where

$$\phi_j(x) = \Delta\left(Re(f_j)\left(x_0, |\underline{x}|\right) + \underline{\omega}Im(f_j)\left(x_0, |\underline{x}|\right)\right), \quad x \in \Omega, j = 0, 1, 2, \ldots, k-1,$$

with $f_j(z) = \frac{1}{2\pi}\int_a^z\int_{\partial D}\frac{\widetilde{e^{\alpha x_0}G_{1,j}(\zeta)}}{\zeta-\xi}d\zeta\, d\xi + \sum_{n=0}^{+\infty}t_{j,n}\left(z-a\right)^n$, $z \in D$, $t_n \in \mathbb{C}$, and $\widetilde{G}_{1,j} \triangleq \frac{r}{2}\widetilde{g}_{1,j}$, with $\widetilde{g}_{1,j} = \sum_{i=j}^{k-1}\frac{(-1)^{i+j}}{(i-j)!j!}x_0^{i-j}\frac{r}{2}g_i$.

More generally, in case of the Riemann–Hilbert boundary value problem $(\hbar\hbar)$: find a function $\phi \in \mathcal{C}^k\left(\Omega, \mathbb{H}\right)\left(k \geqslant 2, k \in \mathbb{N}\right)$ of axial type, satisfying

$$(\hbar\hbar)\begin{cases} \mathcal{D}_{\alpha}^{k}\phi(x) = 0, \quad x \in \Omega, \\ \mathrm{Re}\left\{\lambda_0(t)\phi(t)\right\} = g_0(t), \quad t \in \partial\Omega, \\ \quad\vdots \qquad\qquad \vdots \\ \mathrm{Re}\left\{\lambda_l(t)\left(\mathcal{D}_{\alpha}^{l}\phi\right)(t)\right\} = g_l(t), \quad t \in \partial\Omega, 1 \leqslant l < k-1, \\ \quad\vdots \qquad\qquad \vdots \\ \mathrm{Re}\left\{\lambda_{k-1}(t)\left(\mathcal{D}_{\alpha}^{k-1}\phi\right)(t)\right\} = g_{k-1}(t), \quad t \in \partial\Omega, \end{cases}$$

where $\alpha \in \mathbb{R}$, $g_j : \Omega \to \mathbb{R}$ we can state the following theorem.

Theorem 4.6. *If $D \subset \mathbb{C}_+$ is the projection of $\Omega \subset \mathbb{R}^4$, and $g_j \in H^{\mu}\left(\partial\Omega, \mathbb{R}\right)\left(j = 0, 1, 2, \ldots, k-1\right)$, then the Riemann–Hilbert boundary value problem $(\hbar\hbar)$ is uniquely solvable, and the solution is given by*

$$\phi(x) = e^{\alpha x_0}\phi_0(x) + x_0e^{\alpha x_0}\phi_1(x) + \cdots + x_0^{k-1}e^{\alpha x_0}\phi_{k-1}(x), \quad x \in \Omega,$$

where, for $j = 0, 1, 2, \ldots, k-1$, $\phi_j(x) = \Delta\left(Re(f_j)\left(x_0, |\underline{x}|\right) + \underline{\omega}Im(f_j)\left(x_0, |\underline{x}|\right)\right)$, $x \in \Omega$,

(i) *when* $\lambda_j \in H^\mu(\partial\Omega, \mathbb{R})$ *and* $\lambda \neq 0$ *for all* $x \in \partial\Omega$*, we have*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{e^{\alpha x_0} G_j}(\zeta)}{\zeta - \xi} d\zeta d\xi + \sum_{n=0}^{+\infty} t_{j,n}(z-a)^n, \quad z \in D, j = 0, 1, 2, \ldots, k-1, t_n \in \mathbb{C}, n \in \mathbb{N},$$

where $\widetilde{\mathrm{G}}_j \triangleq \lambda_j^{-1}\frac{r}{2}\widetilde{g}_j$*, with* $\widetilde{g}_j = \sum_{i=j}^{k-1}\frac{(-1)^{i+j}}{(i-j)!j!}x_0^{i-j}\frac{r}{2}g_i$,

(ii) *when* $\lambda_j = \Pi_j\widehat{\lambda}_j$, $\Pi_j(x) = \Pi_{i=1}^m\left(x - \widehat{\alpha}_{i,j}\right)^{\nu_{i,j}}$, $\widehat{\alpha}_{i,j} \in \partial\Omega$*, where* $\widehat{\lambda}_j \neq 0$ *for all* $x \in \partial\Omega$ *with* $\nu_{i,j} \in \mathbb{N}$*, and* $\widehat{\lambda}_j \in H^\mu(\partial\Omega, \mathbb{R})$*, we have*

$$f_j(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi_j(\xi)}\frac{\widehat{e^{\alpha x_0} G_j}(\zeta)}{\zeta - \xi} d\zeta d\xi + \sum_{n=0}^{+\infty}\int_a^z \frac{l_n(\xi - a)^n}{\Pi(\xi)} d\xi, \quad z \in D,\ l_n \in \mathbb{C}, n \in \mathbb{N},$$

where $\widehat{\mathrm{G}}_j \triangleq \widehat{\lambda_j}^{-1}\frac{r}{2}\widetilde{g}_j$, $\widetilde{g}_j = \sum_{i=j}^{k-1}\frac{(-1)^{i+j}}{(i-j)!j!}x_0^{i-j}\frac{r}{2}g_i$.

Moreover, if $D_1 \subset \mathbb{C}_+$ *is the projection of* $\Omega_1 \subset \Omega$ *into the* (x_0, r)*-plane, the uniquely determined solution to the Riemann–Hilbert boundary value problem* $(\hbar\hbar)$ *is given by*

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j e^{\alpha x_0}\phi_j(x),$$

where $\phi_j(x) = \frac{2i}{\pi}\int_{\partial D_1}(z - \bar{x})\left(z^2 - 2Sc(x)z + |x|^2\right)^{-2} f_j(z)dz$, $x \in \Omega_1$*, with* f_j *as in* Theorem 4.5 *according to the cases* (i) *or* (ii).

Remark 5. In fact, Corollary 4.3 gives us explicit solutions to the Schwarz problem for a special type of iterated Vekua system defined over the complex plane [10,11,14,15].

When the space dimension considered is 2, Problem $(\hbar)$ reduces to the Riemann–Hilbert boundary value problem for poly-analytic functions defined over the complex plane [7–9].

Remark 6. If the boundary data are of Hölder class, we consider the Riemann–Hilbert boundary value Problems 1, 1(α) with variable coefficients for functions of axial type defined over an axial domain of $\mathbb{R}^4$ which are null-solutions to the equation $\mathcal{D}_\alpha^k\phi = 0$, $k \geqslant 2(k \in \mathbb{N})$, $\alpha \in \mathbb{R}$. Note that Problem 1(α) reduces to Problem 1 when α equals 0 while Problem 1 reduces to Problem $(\star)$ when k equals to 1. Following the same argument, if boundary data belong to $L_p(1 < p < +\infty)$, all results obtained can be extended to the Riemann–Hilbert boundary value problems of monogenic functions of axial type in $\mathbb{R}^4$.

Remark 7. Although in this context all results for the Riemann–Hilbert boundary value problems considered are presented with restriction to the unit balls of $\mathbb{R}^4$, the methods constructed can be adapted to the Riemann–Hilbert boundary value problems with variable coefficients to null-solutions of the iterated generalized Cauchy–Riemann equations and their perturbed cases in other axially symmetric domains of $\mathbb{R}^4$.

Acknowledgments

The authors cordially thank two anonymous referees for their valuable comments which lead to the improvement of this paper. This work was supported in part by the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), through CIDMA-Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013, and by the postdoctoral grant from FCT (Portugal) under Grant No. SFRH/BPD/74581/2010.

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