Lecture 4

<u>Absolutely</u> convergent series:

E.g. (1) $e^{-x} = 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$ $\quad R_c = \infty$

for $|x| < R_c$

(2) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ $\quad R_c = 1$

Note that coefficients do not grow.

I.e., if $S_n(x)$ is the sum of n first terms of the series for $f(x)$, then for any x : $\cancel{|x| < R_c}$ within the radius of convergence,

$$S_{n\to\infty}(x) \longrightarrow f(x)$$

In physics, most interesting series are <u>asymptotic</u>

Asymptotic series:

Example: $$f(x) = e^{x}\int_{x}^{\infty} dt\, \frac{e^{-t}}{t}$$

$\exists\, \cancel{t}\, \to \infty$

Integration by parts: $\Rightarrow$

$$f(x) = e^{x}\int_{x}^{\infty} dt\, \frac{1}{t}\left(-e^{-t}\right)' = \frac{1}{x} - e^{x}\int_{x}^{\infty} dt\, \frac{1}{t^2} e^{-t}$$

Repeating integration by parts again & again

$$\Rightarrow f(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-1)^{n-1}(n-1)!}{x^{n}} + O\left(\frac{1}{x^{n+1}}\right)$$

estimate for the last term

$$S_n(x) \equiv \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{x^{k}}$$

$S_n(x)$ diverges at any x.
$n\to\infty$

but if x is sufficiently large, first terms ~~give~~ of the series give better and better approximation to $f(x)$

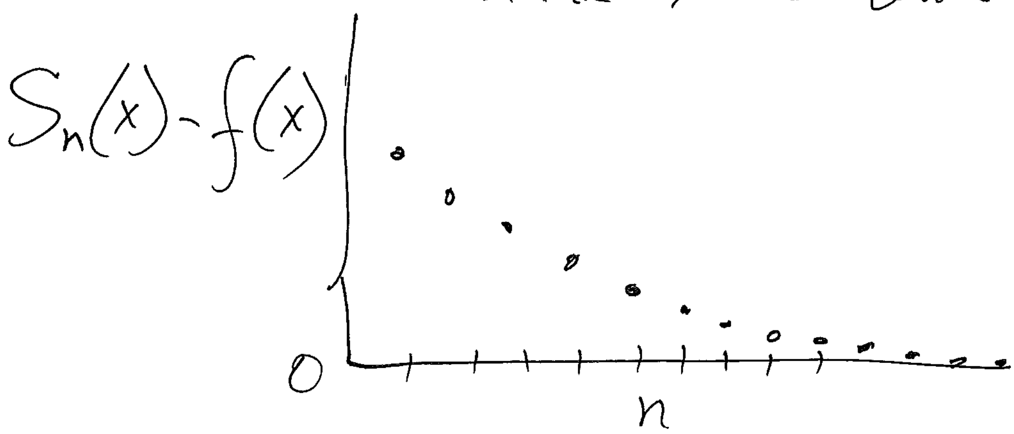
The ratio of two consequitive terms in the series is $\sim \frac{n}{x}$

$\Rightarrow$ for $n \gtrsim x$, the sum $S_n(x)$ begins to diverge from $f(x)$.

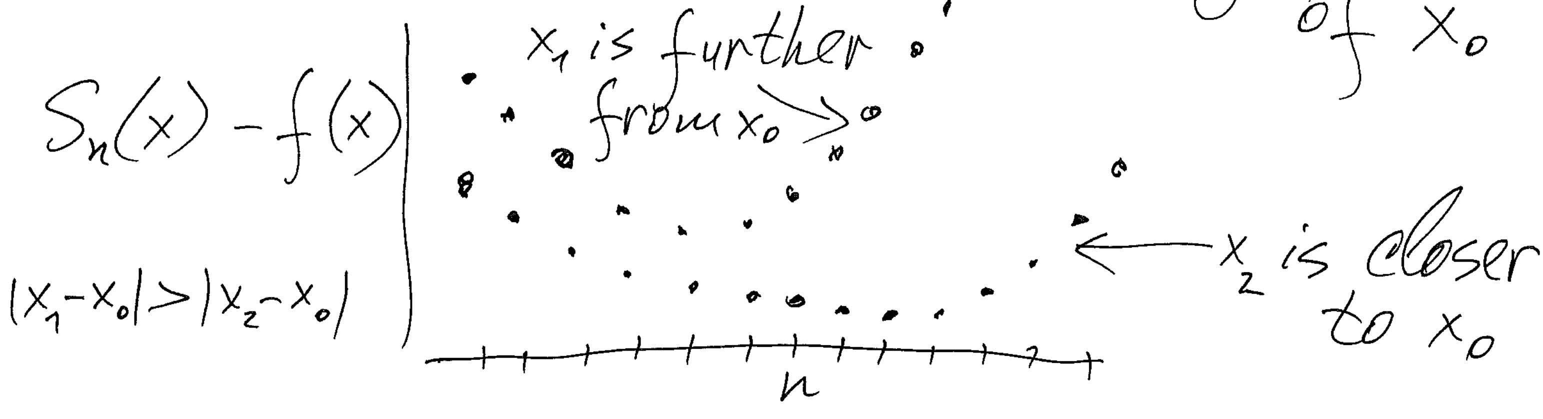
Here $x_0 = \infty$ is the limiting point of this asymptotic series

In general: Compare:

(1) Abs convergent series some x within R_c:



(2) As. series some x in the neighbourhood of x_0



So in the asymptotic series, (4)
first terms may provide the best description of the function $f(x)$.

How to obtain the asymptotic series:
Methods:

$$f(x) \equiv \int_0^\infty dt \frac{e^{-t}}{x+t} \overset{x\to\infty}{\cong} ?$$

(1)

$$\frac{1}{x}\int_0^\infty dt\, e^{-t} \frac{1}{1+\frac{t}{x}} = \frac{1}{x}\int_0^\infty dt\, e^{-t} \sum_{i=0}^\infty (-1)^i \frac{t^i}{x^i} =$$

$$= \frac{1}{x}\sum_{i=0}^\infty \frac{(-1)^i}{x^i}\int_0^\infty dt\, t^i e^{-t} = \frac{1}{x}\sum_{i=0} (-1)^i \frac{i!}{x^i}$$

(2) Consequitive Integration by parts:

$$\int_0^\infty dt \frac{1}{x+t}\left(-e^{-t}\right)' = -e^{-t}\frac{1}{x+t}\Big|_0^\infty - \int_0^\infty dt \frac{1}{(x+t)^2} e^{-t} =$$

$$= \ldots = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n! \int_0^\infty dt \frac{e^{t}}{(x+t)^{n-}}$$

Stirling formula

$$n! \cong \underline{\underline{\sqrt{2\pi n}\; n^n e^{-n}}}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \dots\right)$$

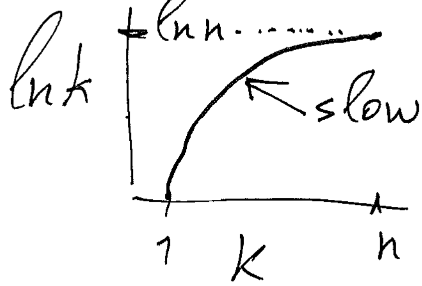
$(n \to \infty)$

Asymptotic series

Let us try to derive it in the simplest way:

$$\ln n! = \sum_{k=1}^{n} \ln k \approx \int_1^n dk \ln k \cong n \ln n$$

asymptotic estimate



compare with

$$\int_1^n dk \ln k \underset{\text{exact}}{=} (k \ln k - k)\Big|_1^n =$$

$$= n \ln n - n + 1 \cong n \ln n - n$$

improves

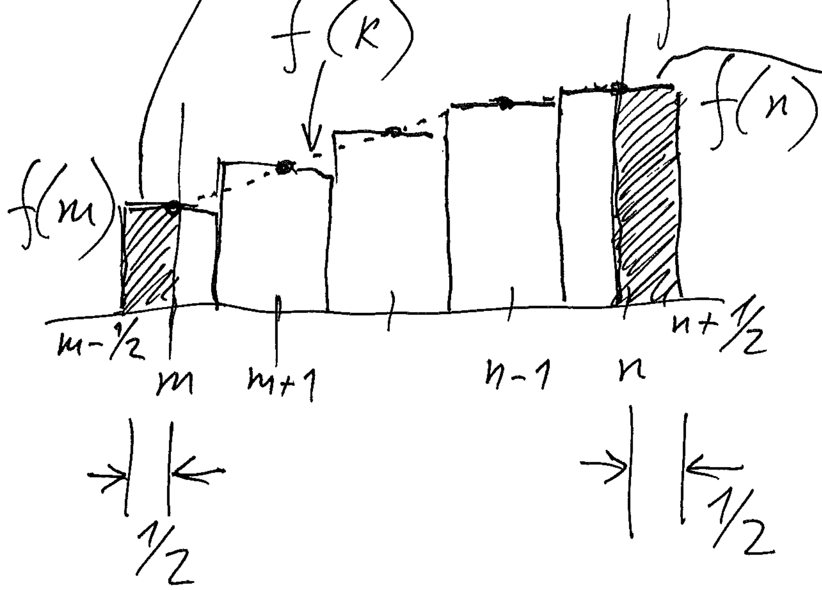
So we have the main term:

$$\ln n! \cong \ln(n^n e^{-n})$$

How to improve:

$$\sum_{k=m}^{n} f(k) = \int_m^n dk\, f(k) + \frac{1}{2} f(m) + \frac{1}{2} f(n) + \dots$$

first terms of the Euler-Maclaurin formula



$$\Rightarrow \sum_{k=1}^{n} \ln k \simeq \underbrace{\int_1^n dk \ln k}_{= n\ln n - n} + \frac{1}{2} \underbrace{\ln 1}_{= 0} + \frac{1}{2} \ln n$$

improved again

$$\Rightarrow \quad \ln n! \simeq \ln\left(\sqrt{n}\, n^n e^{-n}\right)$$