

Superoscillations of prescribed amplitude and derivative

Dae Gwan Lee, Paulo J. S. G. Ferreira

Abstract—Superoscillations occur when a bandlimited signal oscillates at a rate higher than its maximum frequency. We show that it is possible to construct superoscillations by constraining not only the value of the signal but also that of its derivative. This allows a better control of the shape of the superoscillations. We find that for any given bandwidth, no matter how small, there exists a unique signal of minimum energy that satisfies any combination of amplitude and derivative constraints, on a sampling grid as fine as desired. We determine the energy of the signal, for any grid, regular or irregular. When the set of derivative constraints is empty the results reduce to minimum energy interpolation. In the absence of amplitude constraints, we obtain pure derivative-constrained extremals. The flexibility gained by having two different types of constraints makes it possible to design superoscillations based only on amplitudes, based only on derivatives, or based on both. In the last case, the amplitude and derivative sampling grids can be interleaved or aligned. We explore this flexibility to build superoscillations that cost less energy. Illustrating examples are given.

I. INTRODUCTION

A finite-energy band-limited function may oscillate faster than its maximum frequency, and it may do so over arbitrarily long finite intervals. This counter-intuitive phenomenon is called “superoscillation”.

Superoscillations come at a price: large amplitudes outside the superoscillatory regions. Bandwidth determines the average zero density of a signal, but superoscillatory regions may exhibit arbitrarily dense real zeros, which force large signal amplitudes away from those regions. One of the early works that discusses the connection between the zeros of a signal and its amplitude is [1], published in 1958. It solves the problem of synthesising a band-limited function with a prescribed set of zeros, but the authors observe that the distribution of the zeros could affect the amplitude of the generated signal, a fact of obvious practical importance.

The amplitude problem was very clearly treated by Berry [2] in 1994, already in the context of superoscillations. Berry defines the term and attributes the idea to Aharonov, who had told him that functions “could oscillate faster than any of their Fourier components” and had constructed such “superoscillations” guided by quantum-mechanical arguments [3]. Using integral representations, Berry investigated the cost of

superoscillations and found that the value of the function containing them would become exponentially larger in the region of normal oscillation. He gave a striking example: the reproduction of Beethoven’s ninth symphony as superoscillations with a 1 Hz bandwidth would require a signal $\exp(10^{19})$ stronger than with conventional oscillations.

Amplitude, energy and bandwidth are related quantities. It is easy to see that the amplitude of a signal with unit energy and band-limit $\mu/2$ is bounded by $\sqrt{\mu}$. The cost of superoscillations of a certain bandwidth can be studied in terms of amplitude or in terms of energy. The energy cost was discussed in detail in [4] and especially [5]. The latter paper investigates the dynamical range and energy required by superoscillating signals as a function of the superoscillation’s frequency, number, and maximum derivative. It discusses some of the implications of superoscillating signals, in the context of information theory and time-frequency analysis. It shows, among other things, that the required energy grows exponentially with the number of superoscillations, and polynomially with the reciprocal of the bandwidth or the reciprocal of the period of superoscillation. It also shows that there is no contradiction between Shannon’s capacity formula and superoscillating signals, and in the role that the amplitude and energy of such signals play in the matter.

A more recent article [6] considers the problem of optimising superoscillatory signals. The authors maximise the superoscillation yield, that is, the ratio of the energy in the superoscillations to the total energy of the signal, given the range and frequency of the solved superoscillations. The constrained optimisation leads to a generalised eigenvalue problem, which is solved numerically. The article [7] studies the mathematical properties of superoscillations and in particular their approximation properties. It studies, among other things, the problem of approximating a function by superoscillating functions, and constructs sequences converging (non-uniformly) to sinusoids. The article [8] investigates the extent to which superoscillations are affected by small deviations in the Fourier coefficients and [9] defines a periodicity measure and applies it to yield-optimised superoscillating signals.

The interest in superoscillating functions is relatively recent but a number of applications have already been described. A recent overview can be found in [10]. Here we mention the article [11], which discusses ways of beating the diffraction limit, and [12], which uses an optical mask to create superoscillations by constructive interference of waves, leading to a subwavelength focus. The authors also demonstrate that the mask can be used as a superresolution imaging device. The method introduced in [13], [14] is based on the observation that superdirectivity and superoscillation are related phenom-

Dae Gwan Lee is with the Department of Mathematical Sciences, KAIST, S. Korea. He is partially supported by the National Research Foundation of Korea (NRF) (2012R1A1A2038650).

Paulo J. S. G. Ferreira is with DETI/IEETA, Universidade de Aveiro, Portugal. E-mail: pjf@ua.pt. Work partially supported by the European Fund for Regional Development (FEDER) through the Operational Program Competitiveness Factors (COMPETE) and by the Portuguese Foundation for Science and Technology (FCT), in the context of projects FCOMP-01-0124-FEDER-022682 (FCT reference PEst-C/EEI/UI0127/2011) and Incentivo/EEI/UI0127/2013.

ena and leads to subwavelength focusing schemes in free space and within a waveguide. The simulations reported by the authors demonstrate subwavelength focusing down to 0.6 times the diffraction limit, five wavelengths away from the source. The work [15] demonstrates a superoscillatory subwavelength focus within a waveguide. The authors claim the formation of a focus at 75% the spatial width of the diffraction limited sinc pulse, 4.8 wavelengths away from the source. The work [16] tries to get around the limitations of the uncertainty principle. The authors seek to arbitrarily compress a temporal pulse and report the design of a class of superoscillatory electromagnetic waveforms for which the sideband amplitudes, and hence the sensitivity, can be regulated. They claim a pulse compression improvement of 47% beyond the Fourier limit.

The recent theoretical developments and the importance of the applications justifies further study of superoscillations. The goal of the present paper is to investigate the approximation properties of superoscillating functions by introducing a new element in the equation: controlled derivative. We ask if there exist signals of arbitrarily small bandwidth that interpolate an arbitrary set of amplitudes *and derivatives* prescribed on an arbitrarily dense and possibly nonuniform sampling grid. For example, given a collection of distinct real points t_i , one could look for functions of fixed bandwidth that meet finitely many conditions of the form

$$f(t_i) = \alpha_i, \quad f'(t_i) = \beta_i.$$

In fact, we consider the more general problem

$$f(t_i) = \alpha_i, \quad f'(\tau_j) = \beta_j, \quad (1)$$

in which the amplitudes are prescribed on one grid and the derivatives on another. The possibility of coincidence between a subset of the $\{t_i\}$ and $\{\tau_j\}$ is however left open.

All previous works have considered amplitude constraints only, and found the problem to be theoretically possible at a cost; the presence of the new derivative constraints appears to render the task even harder. However, solving the problem would mean that both the value *and the shape* of the superoscillations could be prescribed.

We solve the problem in this paper and show that shape-preserving interpolation beyond the Nyquist rate is indeed possible. For any bandwidth, no matter how small, there exists a unique signal of that bandwidth and of least energy that satisfies constraints such as (1), no matter how dense the grids are. We determine this signal and the required minimum energy.

Naturally, when the set of points τ_j on which the derivatives are prescribed is empty, the results reduce to those already known for amplitude-only interpolation. Conversely, in the absence of amplitude constraints, we obtain results for pure derivative-constrained interpolation.

The introduction of a second type of constraint in the problem leads to increased flexibility. One can design superoscillations using derivatives only, amplitudes only, or a combination of both. In the last case, the points t_i and τ_j on which the amplitudes and derivatives are prescribed can be exactly interleaved, aligned, or anything in between. We show

that the different arrangements lead to different minimum energy costs.

The new results given in the paper help to further understand the potential of superoscillations, their cost and properties.

II. MINIMUM ENERGY CONSTRAINED INTERPOLATION

The Fourier transform is normalised as follows:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt.$$

We are interested in finding real functions f of fixed bandwidth such that

$$\begin{aligned} f(t_k) &= \alpha_k, & 1 \leq k \leq N_1, \\ f'(\tau_k) &= \beta_k, & 1 \leq k \leq N_2, \end{aligned}$$

where $t_1 < t_2 < \dots < t_{N_1}$ and $\tau_1 < \tau_2 < \dots < \tau_{N_2}$. Here we may have $t_i = \tau_i$ for some or all pairs (t_i, τ_i) . Also, although $N_1 + N_2$ must be nonzero so that at least one constraint exists, we do not discard the possibility of having either $N_1 = 0$ or $N_2 = 0$.

We denote the bandwidth of the signals by μ , so that

$$f(t) = \int_{-\mu/2}^{\mu/2} \hat{f}(\omega) e^{i2\pi\omega t} d\omega.$$

We may tacitly assume a unitary bandwidth ($\mu = 1$), i.e. a maximum frequency of $1/2$. This assumption implies no loss of generality and is made only to simplify the notation.

The energy of the signal f can be expressed in terms of its Fourier transform as

$$E = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

and the $N_1 + N_2$ constraints can be expressed as

$$\begin{aligned} \int \hat{f}(\omega) e^{i2\pi\omega t_k} d\omega &= \alpha_k, & 1 \leq k \leq N_1, \\ 2\pi i \int \hat{f}(\omega) \omega e^{i2\pi\omega \tau_k} d\omega &= \beta_k, & 1 \leq k \leq N_2. \end{aligned}$$

The search for solutions of minimum energy thus becomes a variational problem under isoperimetric-type constraints [17], [18]. The functional to minimise is often denoted by

$$\int F(y, y', t) dt$$

and the constraints are defined by

$$\int G(y, y', t) dt = \text{constant}.$$

It is known [17], [18] that one has to solve the modified minimisation problem

$$\min \int (F + \lambda G),$$

where λ is a Lagrange multiplier. The Euler-Lagrange equation for the modified problem is

$$F_y + \lambda G_y - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} + \lambda \frac{\partial G}{\partial y'} \right) = 0,$$

and its generalisation to multiple constraints is easy to obtain:

$$\frac{\partial}{\partial y} \left(F + \sum_j \lambda_j G_j \right) - \frac{d}{dt} \left[\frac{\partial}{\partial y'} \left(F + \sum_j \lambda_j G_j \right) \right] = 0.$$

Returning to the interpolation problem, we begin by splitting the Fourier transform $\hat{f}(\omega)$ into real and imaginary parts, $A(\omega)$ and $B(\omega)$. The fact that $f(t)$ is real-valued allows the energy and constraints to be written using only real functions of a real variable:

$$E = \int (A^2(\omega) + B^2(\omega)) d\omega,$$

and, with $\mu = 1$,

$$\begin{aligned} \int_{-1/2}^{1/2} [A(\omega) \cos(2\pi\omega t_k) - B(\omega) \sin(2\pi\omega t_k)] d\omega &= \alpha_k, \\ - \int_{-1/2}^{1/2} 2\pi\omega [B(\omega) \cos(2\pi\omega t_k) + A(\omega) \sin(2\pi\omega t_k)] d\omega &= \beta_k. \end{aligned}$$

The Euler-Lagrange equation can now be applied to the problem. Before dealing with the general case we consider a few illustrating examples. The first two are known but instructive to keep in mind, since they put the next ones in perspective. In the sequel, the symbol c will denote a constant, not necessarily the same from equation to equation. Also, the independent variable ω may be omitted if deemed convenient.

Example 1. Minimise the energy subject to $f(t_1) = \alpha_1$. This leads to the minimisation of

$$\int \{A^2 + B^2 + \lambda[A \cos(2\pi\omega t_1) - B \sin(2\pi\omega t_1)]\} d\omega.$$

The integrand does not depend on the derivatives of the unknown functions A and B , a characteristic shared by all problems considered here. Thus, the Euler-Lagrange equation leads to

$$2A + \lambda \cos(2\pi\omega t_1) = 0$$

$$2B - \lambda \sin(2\pi\omega t_1) = 0$$

hence $\hat{f}(\omega) = A(\omega) + iB(\omega) = c e^{-i2\pi\omega t_1}$ and so

$$f(t) = \int_{-1/2}^{1/2} \hat{f}(\omega) e^{i2\pi\omega t} d\omega = c \operatorname{sinc}(t - t_1).$$

The constraint now shows that $f(t) = \alpha_1 \operatorname{sinc}(t - t_1)$, that is, the sinc function is the minimum energy interpolant.

Example 2. Minimise the energy subject to multiple constraints $f(t_i) = \alpha_i$, $i = 1, 2, \dots, N$. In this case, f must be a linear combination of terms $\operatorname{sinc}(t - t_i)$ [5], [19]. The constraints lead to a set of equations for the coefficients of this linear combination. The underlying matrix is positive-definite, and the problem has a unique solution.

Example 3. Minimise the energy subject to the derivative constraint: $f'(t_1) = \beta_1$. The quantity to minimise is now

$$\int \{A^2 + B^2 - \lambda 2\pi\omega [B \cos(2\pi\omega t_1) + A \sin(2\pi\omega t_1)]\} d\omega.$$

The Euler-Lagrange equations yield

$$2A - \lambda 2\pi\omega \sin(2\pi\omega t_1) = 0$$

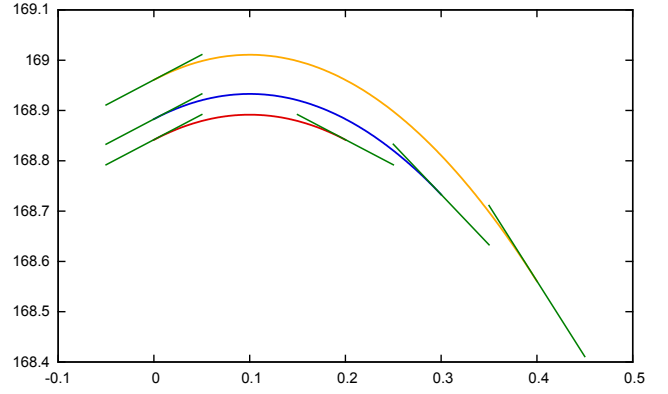


Figure 1. Functions of minimum energy and maximum frequency $\mu/2 = 0.05$ Hz subject to derivative constraints (example 4) in $[0, 0.2]$, $[0, 0.3]$ and $[0, 0.4]$. The required derivatives are shown as well.

$$2B - \lambda 2\pi\omega \cos(2\pi\omega t_1) = 0$$

hence $\hat{f}(\omega) = A(\omega) + iB(\omega) = i2\pi\omega c e^{-i2\pi\omega t_1}$ and so

$$f(t) = c \int_{-1/2}^{1/2} i2\pi\omega e^{-i2\pi\omega(t-t_1)} d\omega.$$

We rewrite this as follows:

$$f(t) = c \frac{d}{dt} \int_{-1/2}^{1/2} e^{i2\pi\omega(t-t_1)} d\omega.$$

Now, an integration shows that $f(t) = c \operatorname{sinc}'(t - t_1)$, where sinc' denotes the derivative of the sinc function. The constraint $f'(t_1) = \beta_1$ leads to $c \operatorname{sinc}''(0) = \beta_1$, and since $\operatorname{sinc}''(0) = -\pi^2/3$ we have

$$f(t) = -\frac{3\beta_1}{\pi^2} \operatorname{sinc}'(t - t_1).$$

For $f'(0) = 1$, a plot of $-3/\pi^2$ times the derivative of the sinc easily confirms the slope 1 at the origin.

Note that $f(t_1) = 0$. This suggests the possibility of accommodating an additional amplitude constraint such as $f(t_1) = \alpha_1$ by adding the term $\alpha_1 \operatorname{sinc}(t - t_1)$. This term will not disturb the derivative constraint since the derivative of the sinc at the origin is zero. We will see shortly that this guess is indeed correct.

Example 4. Minimise the energy subject to multiple constraints $f'(\tau_k) = \beta_k$. Proceeding as in the previous case, we obtain a linear combination of $\operatorname{sinc}'(t - \tau_k)$, the coefficients of which can be obtained by using the constraints. If there are only two constraints, we obtain

$$f(t) = a \operatorname{sinc}'(t - \tau_1) + b \operatorname{sinc}'(t - \tau_2)$$

hence

$$f'(t) = a \operatorname{sinc}''(t - \tau_1) + b \operatorname{sinc}''(t - \tau_2)$$

and so

$$\begin{bmatrix} \operatorname{sinc}''(0) & \operatorname{sinc}''(\tau_1 - \tau_2) \\ \operatorname{sinc}''(\tau_2 - \tau_1) & \operatorname{sinc}''(0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Fig. 1 illustrates this.

Example 5. Minimise the energy subject to $f(t_1) = \alpha_1$ and $f'(\tau_1) = \beta_1$. The functional to minimise is

$$\int \{A^2 + B^2 + \lambda_1(A \cos(2\pi\omega t_1) - B \sin(2\pi\omega t_1)) - \lambda_2 2\pi\omega(A \sin(2\pi\omega \tau_1) + B \cos(2\pi\omega \tau_1))\}.$$

It follows from the Euler-Lagrange equation that

$$f(t) = a \operatorname{sinc}(t - t_1) + b \operatorname{sinc}'(t - \tau_1).$$

The constraints can now be used:

$$f(t_1) = a \operatorname{sinc}(0) + b \operatorname{sinc}'(t_1 - \tau_1) = \alpha_1$$

$$f'(\tau_1) = a \operatorname{sinc}'(\tau_1 - t_1) + b \operatorname{sinc}''(0) = \beta_1.$$

Thus,

$$\begin{bmatrix} 1 & \operatorname{sinc}'(t_1 - \tau_1) \\ \operatorname{sinc}'(\tau_1 - t_1) & -\frac{\pi^2}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

or, with $\delta = \tau_1 - t_1$,

$$\begin{bmatrix} 1 & -\operatorname{sinc}'(\delta) \\ \operatorname{sinc}'(\delta) & -\frac{\pi^2}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}.$$

This separates into

$$f(t) = \alpha_1 \operatorname{sinc}(t - t_1) - \frac{3\beta_1}{\pi^2} \operatorname{sinc}'(t - t_1)$$

as $\delta \rightarrow 0$, as suggested before (see Example 3).

The following theorem gives the solution in the general case (cf. Theorem 2 in [5]).

Theorem 1: Let N_1 and N_2 be non-negative integers. Consider two sequences of real numbers satisfying $t_1 < t_2 < \dots < t_{N_1}$ and $\tau_1 < \tau_2 < \dots < \tau_{N_2}$. Let $\alpha = \{\alpha_k\}_{k=1}^{N_1}$ and $\beta = \{\beta_k\}_{k=1}^{N_2}$ be sets of arbitrary reals. Among all signals of finite energy bandlimited to $\mu/2$ Hz, there is one that has minimum energy and satisfies

$$f(t_k) = \alpha_k, \quad 1 \leq k \leq N_1,$$

and

$$f'(\tau_k) = \beta_k, \quad 1 \leq k \leq N_2.$$

The signal is given by

$$f(t) = \mu \sum_{k=1}^{N_1} x_k \operatorname{sinc}[\mu(t - t_k)] + \mu^2 \sum_{k=1}^{N_2} y_k \operatorname{sinc}'[\mu(t - \tau_k)], \quad (2)$$

where $\mathbf{x} = \{x_k\}_{k=1}^{N_1}$ and $\mathbf{y} = \{y_k\}_{k=1}^{N_2}$ satisfy

$$S \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

with

$$S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where A and C are symmetric matrices of order N_1 and N_2 respectively and B is an $N_1 \times N_2$ skew-symmetric matrix. Their elements are given by

$$\begin{aligned} A &= [\mu \operatorname{sinc}(\mu(t_j - t_k))]_{j,k}, \\ B &= [\mu^2 \operatorname{sinc}'(\mu(t_j - \tau_k))]_{j,k}, \\ C &= [-\mu^3 \operatorname{sinc}''(\mu(\tau_j - \tau_k))]_{j,k}. \end{aligned}$$

The energy of f is given by

$$E = \|f\|^2 = [\alpha^T \quad -\beta^T] S^{-1} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}.$$

Proof: Equation (2) is obtained from the Euler-Lagrange equation as discussed before. Therefore, it remains to prove the invertibility of S . We show that S is positive definite (and thus, invertible). The Fourier transform of (2) is

$$\hat{f}(\omega) = \sum_{k=1}^{N_1} x_k e^{-i2\pi t_k \omega} \chi(\omega) + \sum_{k=1}^{N_2} y_k (i2\pi\omega) e^{-i2\pi \tau_k \omega} \chi(\omega)$$

where $\chi(\omega)$ is the characteristic function of $[-\mu/2, \mu/2]$. The energy of f can then be expressed as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \\ &= \int_{-\mu/2}^{\mu/2} \left| \sum_{k=1}^{N_1} x_k e^{-i2\pi t_k \omega} + \sum_{k=1}^{N_2} y_k (i2\pi\omega) e^{-i2\pi \tau_k \omega} \right|^2 d\omega \\ &= \int_{-\mu/2}^{\mu/2} \left(\sum_{j=1}^{N_1} x_j e^{i2\pi t_j \omega} + \sum_{j=1}^{N_2} y_j (-i2\pi\omega) e^{i2\pi \tau_j \omega} \right) \\ &\quad \cdot \left(\sum_{k=1}^{N_1} x_k e^{-i2\pi t_k \omega} + \sum_{k=1}^{N_2} y_k (i2\pi\omega) e^{-i2\pi \tau_k \omega} \right) d\omega \\ &= \sum_{j,k=1}^{N_1} x_j x_k \int_{-\mu/2}^{\mu/2} e^{i2\pi(t_j - t_k)\omega} d\omega \\ &\quad + \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} x_j y_k \int_{-\mu/2}^{\mu/2} (i2\pi\omega) e^{i2\pi(t_j - \tau_k)\omega} d\omega \\ &\quad - \sum_{j=1}^{N_2} \sum_{k=1}^{N_1} x_k y_j \int_{-\mu/2}^{\mu/2} (i2\pi\omega) e^{i2\pi(\tau_j - t_k)\omega} d\omega \\ &\quad - \sum_{j,k=1}^{N_2} y_j y_k \int_{-\mu/2}^{\mu/2} (i2\pi\omega)^2 e^{i2\pi(\tau_j - \tau_k)\omega} d\omega. \end{aligned}$$

Performing the integrations, we find that

$$\begin{aligned} E &= \sum_{j,k=1}^{N_1} x_j x_k \mu \operatorname{sinc}(\mu(t_j - t_k)) \\ &\quad + \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} x_j y_k \mu^2 \operatorname{sinc}'(\mu(t_j - \tau_k)) \\ &\quad - \sum_{j=1}^{N_2} \sum_{k=1}^{N_1} x_k y_j \mu^2 \operatorname{sinc}'(\mu(\tau_j - t_k)) \\ &\quad - \sum_{j,k=1}^{N_2} y_j y_k \mu^3 \operatorname{sinc}''(\mu(\tau_j - \tau_k)). \end{aligned}$$

Using the fact that sinc' is odd finally yields

$$E = \sum_{j,k=1}^{N_1} x_j x_k \mu \text{sinc}(\mu(t_j - t_k)) + 2 \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} x_j y_k \mu^2 \text{sinc}'(\mu(t_j - \tau_k)) + \sum_{j,k=1}^{N_2} y_j y_k (-\mu^3 \text{sinc}''(\mu(\tau_j - \tau_k)))$$

or, in matrix form,

$$E = [\mathbf{x}^T \ \mathbf{y}^T] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}.$$

Note that since $\{e^{-i2\pi t_k \omega}\}_{1 \leq k \leq N_1} \cup \{\omega e^{-i2\pi \tau_k \omega}\}_{1 \leq k \leq N_2}$ is linearly independent in $L^2[-\mu/2, \mu/2]$, any signal of the form (2) with $[\mathbf{x}^T \ \mathbf{y}^T] \neq 0$ has positive energy E . This implies that S is positive definite. Finally,

$$E = [\alpha^T \ -\beta^T] S^{-1} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

since

$$S \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}.$$

If $N_2 = 0$ one obtains the equation $Ax = \alpha$, which is the same as the one discussed in [5]. On the other hand, the special case $N_1 = 0$ is new and worth mentioning by itself:

Corollary 1: Let N be a positive integer. Consider a sequence of real numbers

$$\tau_1 < \tau_2 < \dots < \tau_N.$$

Let $\beta = \{\beta_k\}_{k=1}^N$ be a set of arbitrary real numbers. Then, among all signals of finite energy bandlimited to $\mu/2$ Hz, there is one that has minimum energy and satisfies

$$f'(\tau_k) = \beta_k, \quad 1 \leq k \leq N.$$

The signal is given by

$$f(t) = \mu^2 \sum_{k=1}^N y_k \text{sinc}'[\mu(t - \tau_k)], \quad (3)$$

where $\mathbf{y} = \{y_k\}_{k=1}^N$ satisfies $C\mathbf{y} + \beta = 0$, for the matrix C given in Theorem 1. The energy of f is given by

$$E = \|f\|^2 = \beta^T C^{-1} \beta.$$

Proof: Set $N_1 = 0$ and $N_2 = N$ in Theorem 1. ■

III. EXAMPLES AND DISCUSSION

We fix $\mu/2 = 0.05$ Hz, for a bandwidth of $\mu = 0.1$. The corresponding Paley-Wiener class contains signals that oscillate at a mean rate of μ , just as its prototype sinc function, the real zeros of which occur at the rate $1/\mu = 10$. This determines a reference grid and a reference sampling rate (the Nyquist rate).

Consider now a grid of spacing 2, which corresponds to 5 times the reference or Nyquist density, and an irregular grid obtained by adding random values to the regular grid.

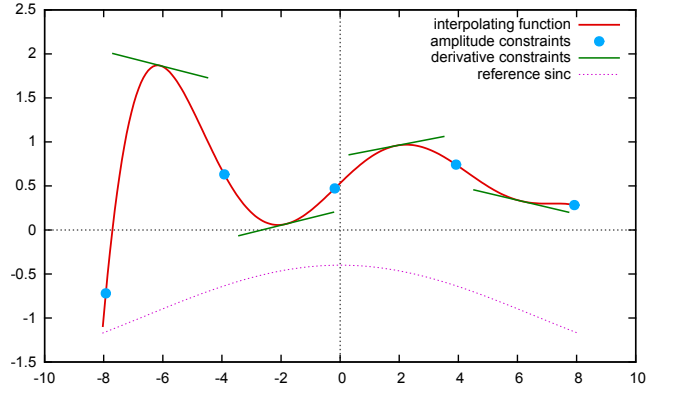


Figure 2. Function of minimum energy and maximum frequency $\mu/2 = 0.05$ Hz subject to by 5 amplitude and 4 derivative constraints, on a nonuniform grid, with a density of about 5 times the Nyquist density. A sinc function of the natural bandwidth μ is also shown for reference.

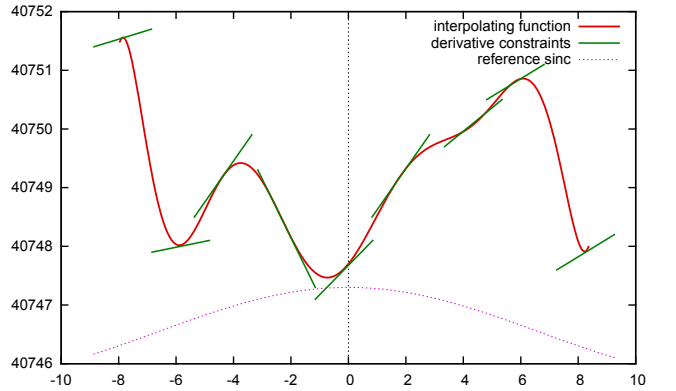


Figure 3. The signal of minimum energy and maximum frequency $\mu/2 = 0.05$ Hz that solves a problem with 9 derivative constraints, on a nonuniform grid of density 5 times the natural density. A sinc function of the natural bandwidth μ is also shown for reference.

The function of minimum energy that satisfies 5 amplitude constraints and 4 derivative ones is shown in Fig. 2. The derivative constraints are shown as tangents to the function.

The example shown in Fig. 3 demonstrates the possibility of interpolating derivatives only. The function was constrained to have a given derivative over 9 consecutive points, on a grid of density similar to the one used in the previous example. The derivative constraints are also illustrated (again, as tangents to the curve).

Fig. 4 shows two distinct ways of building superoscillations. In the first, the amplitudes are fixed at ± 1 , so that the interpolating function has to oscillate and change its sign. In the second, the derivative of the bandlimited function is prescribed to ± 1 . In general, this does not force the function to change sign, but it does force the function to oscillate, as shown in the figure.

Fig. 5 shows how the minimum energies required to satisfy amplitude constraints and derivative constraints compare. In both cases, the constraints are alternating, that is, ± 1 , which appear to be close to the worst case for amplitude interpolation [5]. When the grids are relatively coarse (less than twice the Nyquist rate) it takes less energy to meet the amplitude

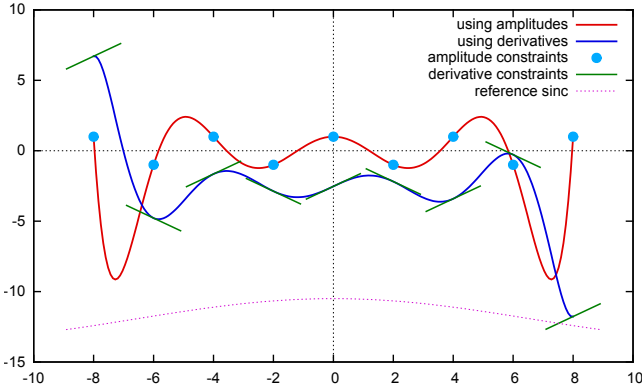


Figure 4. How to build superoscillations by prescribing the amplitude of the signal, its derivative, or both. The figure shows the function of minimum energy and maximum frequency $\mu/2 = 0.05$ Hz with amplitude constrained to alternating values ± 1 and another function obtained by fixing its derivative at ± 1 . The grids were the same in both cases. A sinc function of bandwidth $\mu = 0.1$ is also shown for reference.

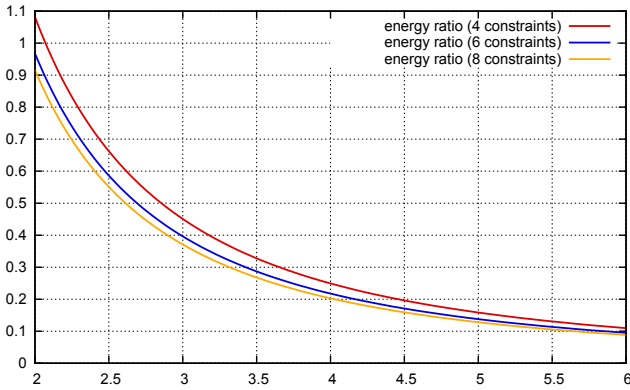


Figure 5. The ratio of the minimum energy required to interpolate ± 1 amplitude constraints and ± 1 derivative constraints. The horizontal axis shows the ratio between the grid separation and the natural separation, so that 1 corresponds to the Nyquist rate.

constraints. However, as the density of the grids increases, the energy necessary to satisfy the derivative constraints becomes smaller than the energy necessary for the amplitude constraints (in the example, about one order of magnitude smaller at 6 times the Nyquist rate).

The introduction of a second type of constraint in the interpolation problem leads to additional flexibility, since it opens the possibility of designing superoscillations by using derivatives only, amplitudes only, or a combination of both. Furthermore, in the last case, the points t_i and τ_i on which the amplitudes and derivatives are prescribed can be exactly aligned, or interleaved.

Figure 6 shows how the different arrangements impact the minimum energy cost. The grids are comparable in the following sense: the number of measurements per unit of time is the same in all examples. Consider, for example, that 10 amplitude values are prescribed in an interval of duration T . For the equivalent problem on aligned function and derivative grids one would have to double the sampling period. Note that the required energy can vary by more than one order of

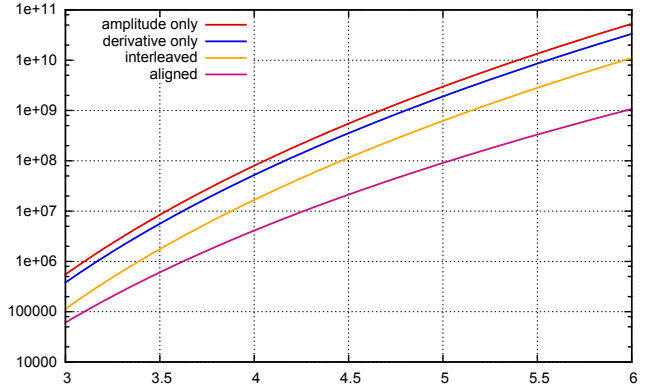


Figure 6. Minimum energy required to solve (a) an amplitude-only interpolation problem (b) a derivative-only problem (c) an amplitude/derivative problem in which the amplitude and derivative values are interleaved (d) another amplitude/derivative problem in which the amplitude and derivatives are prescribed on the same grid. In the three cases the grids were comparable in terms of density. The horizontal axis shows the rate, as a multiple of the Nyquist rate.

magnitude, with amplitude-only interpolation being the worst, and aligned amplitude/derivative interpolation the best. One should note that this is not always the case, and it is easy to find examples in which one of the possibilities is better than the other.

In the experiments made to build Fig. 6 the rate varied between 3 and 6 times the Nyquist rate. Fig. 7 shows individual results at 6 times the Nyquist rate.

IV. CONCLUSION

It is possible to design superoscillations by prescribing their amplitude on a grid arbitrarily denser than the Nyquist density, but one may also prescribe their derivatives, or both. When both derivatives and amplitude constraints are considered, the grids on which each is prescribed may coincide or not, i.e., both aligned grids or staggered grids are possible. The interpolation problem is always possible since the variational problem can be reduced to a set of equations with a nonsingular (indeed positive-definite) matrix (the matrix can become ill-conditioned, as discussed previously in [5]). The expression for the minimum energy is also obtained.

The flexibility gained due to the derivative constraints can be put to use in the design of superoscillations. There are four basic possibilities: amplitude constraints, without any restriction on the derivative; derivative constraints, without restrictions on the amplitude; both amplitude and derivative constraints on staggered grids; and amplitude and derivative constraints on aligned grids at one half the density. We showed that the different arrangements impacted the required minimum energy, by as much as an order of magnitude.

We hope that the results given help to further understand superoscillations, which have already proven their usefulness in a number of emerging applications.

REFERENCES

- [1] F. E. Bond and C. R. Cahn, "On sampling the zeros of bandwidth limited signals," *IRE Trans. Infor. Theory*, vol. 4, no. 3, pp. 110–113, Sep. 1958.

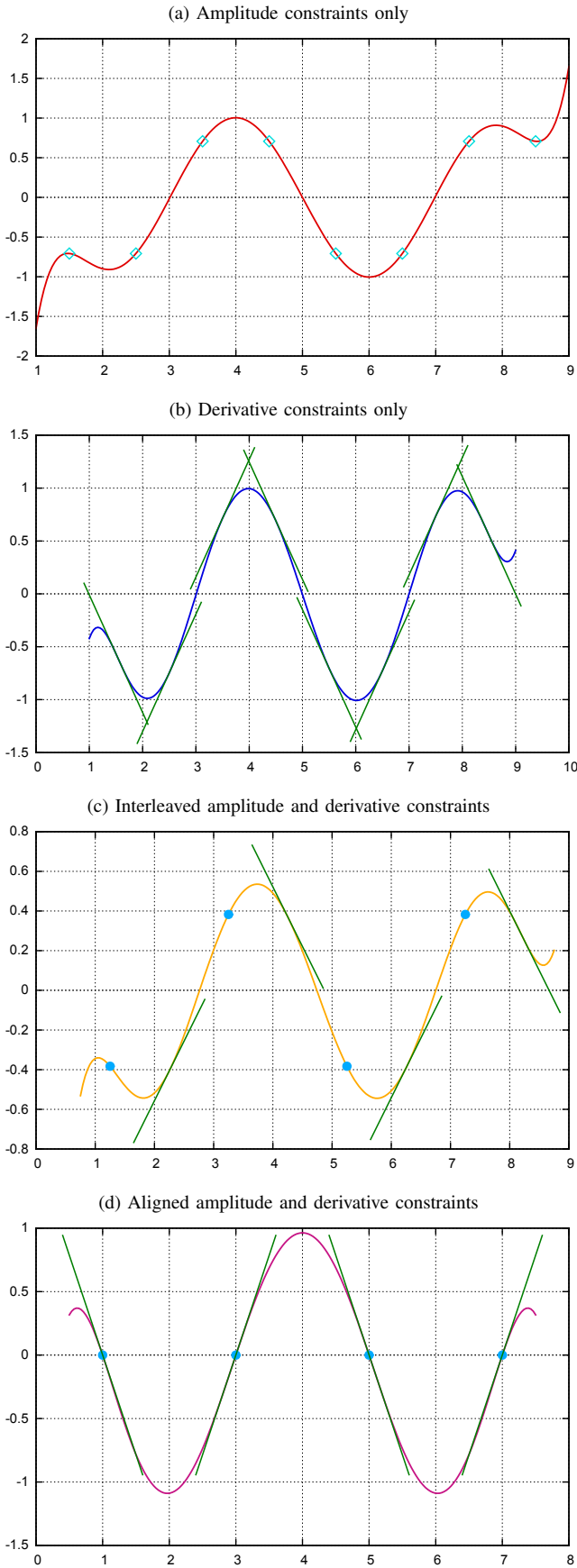


Figure 7. The problem in Fig. 6 that corresponds to 6 times the Nyquist density. a) Minimum energy interpolation with amplitude constraints only. b) Minimum energy interpolation with derivative constraints only. c) Minimum energy interpolation based on interleaved amplitudes and derivatives. d) Minimum energy interpolation with aligned amplitude and derivative constraints.

- [2] M. Berry, "Faster than Fourier," in *Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov*, J. S. Anandan and J. L. Safko, Eds. Singapore: World Scientific, 1994, pp. 55–65.
- [3] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, "Superpositions of time evolutions of a quantum system and a quantum time-translation machine," *Phys. Rev. Lett.*, vol. 64, no. 25, pp. 2965–2968, Jun. 1990.
- [4] P. J. S. G. Ferreira and A. Kempf, "The energy expense of superoscillations," in *Signal Processing XI — Theories and Applications. Proceedings of EUSIPCO-2002, XI European Signal Processing Conference*, vol. II, Toulouse, France, Sep. 2002, pp. 347–350.
- [5] —, "Superoscillations: Faster than the Nyquist rate," *IEEE Trans. Signal Processing*, vol. 54, no. 10, pp. 3732–3740, Oct. 2006.
- [6] E. Katzav and M. Schwartz, "Yield-optimized superoscillations," *IEEE Trans. Signal Processing*, vol. 61, no. 12, pp. 3113–3118, 2013.
- [7] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa, and J. Tollaksen, "Some mathematical properties of superoscillations," *J. Phys. A: Math. Theor.*, vol. 44, p. 365304 (16pp), 2011.
- [8] M. Schwartz and E. Perlsman, "Sensitivity of interpolated yield optimized superoscillations," arXiv:1402.3089 [physics.data-an], 2014.
- [9] N. Schwartz and M. Schwartz, "On a periodicity measure and superoscillations," arXiv:1402.4092 [physics.data-an], 2014.
- [10] P. J. S. G. Ferreira, "Superoscillations," in *New Perspectives on Approximation and Sampling Theory: Festschrift in honor of Paul Butzer's 85th birthday*, G. Schmeisser and A. Zayed, Eds. Springer, 2014, (in press).
- [11] N. I. Zheludev, "What diffraction limit?" *Nature Mater.*, vol. 7, pp. 420–422, Jun. 2008.
- [12] F. M. Huang and N. I. Zheludev, "Super-resolution without evanescent waves," *Nano Letters*, vol. 9, no. 3, pp. 1249–1254, 2009.
- [13] A. M. H. Wong and G. V. Eleftheriades, "Adaptation of Schelkunoff's superdirective antenna theory for the realization of superoscillatory antenna arrays," *IEEE Antennas Wireless Propag. Letters*, vol. 9, pp. 315–318, 2010.
- [14] —, "Superoscillatory antenna arrays for sub-diffraction focusing at the multi-wavelength range in a waveguide environment," in *2010 IEEE Antennas and Propagation Society International Workshop (APSURSI)*, Toronto, Canada, Jul. 2010, pp. 1–4.
- [15] —, "Sub-wavelength focusing at the multi-wavelength range using superoscillations: An experimental demonstration," *IEEE Trans. Antennas Propag.*, vol. 59, no. 12, pp. 4766–4776, Dec. 2011.
- [16] —, "Temporal pulse compression beyond the Fourier transform limit," *IEEE Trans. Microwave Theory Tech.*, vol. 59, no. 9, pp. 2173–2179, Sep. 2011.
- [17] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*. Englewood Cliffs, New Jersey: Prentice-Hall, 1963.
- [18] L. Komzsik, *Applied Calculus of Variations for Engineers*. Boca Raton: CRC Press, 2009.
- [19] L. Levi, "Fitting a bandlimited signal to given points," *IEEE Trans. Inform. Theory*, pp. 372–376, Jul. 1965.