Superoscillations With Optimum Energy Concentration

Dae Gwan Lee and Paulo Jorge S. G. Ferreira

***Abstract*—Oscillations of a bandlimited signal at a rate faster than the bandlimit are called "superoscillations" and have applications e.g. in superresolution and superdirectivity. The synthesis of superoscillating signals is a numerically difficult problem. Minimum energy superoscillatory signals seem attractive for applications because (i) the minimum-energy solution is unique (ii) it has the smallest energy cost (iii) it may yield a signal of the smallest possible amplitude. On the negative side, superoscillating functions of minimum-energy depend heavily on cancellation and give rise to expressions that have very large coefficients. Furthermore, these coefficients have to be found by solving equations that are very ill-conditioned. Surprisingly, we show that by dropping the minimum energy requirement practicality can be gained rather than lost. We give a method of constructing superoscillating signals that leads to coefficients and condition numbers that are smaller by several orders of magnitude than the minimum-energy solution, yet yields energies close to the minimum. In contrast with the minimum-energy method, which builds superoscillations by linearly combining functions with an ill-conditioned Gram matrix, our method combines orthonormal functions, the Gram matrix of which is obviously the identity. Another feature of the method is that it yields the superoscillatory signal that maximises the energy concentration in a given set, which may or may not include the superoscillatory segment.**

***Index Terms*—Algorithms, Hilbert space, interpolation, matrices, nonuniform sampling, numerical stability, optimisation, sampling methods, signal design, superoscillations.**

I. Introduction

A bandlimited signal can oscillate at a rate much higher than its highest frequency throughout a finite interval $[a, b]$. Spectral analysis of the restriction of the signal to $[a, b]$ then yields a bandwidth much larger than the true value.

This counter-intuitive phenomenon, known as superoscillation, was first explored in the context of quantum mechanics (see Section II) and has already found several applications, e.g. in superdirectivity and superresolution. The paper [1], in which subwavelength resolutions down to $\lambda/6$ are reported, asserts that "super-oscillation-based imaging has unbeatable advantages over other technologies".

Superoscillations come at a cost. The amplitude and energy of the signal outside the interval that contains the superoscillations is very large compared with the amplitude and energy of the superoscillating part. In addition to this, the numerical procedures to generate superoscillatory signals tend to be difficult to control due to ill-conditioning. The difficulty increases rapidly with the frequency, number or duration of the superoscillations [2].

A natural step towards controlling the magnitude of a superoscillating signal is to control its total energy. Consider the set of finite-energy signals bandlimited to $\mu/2$ Hz, i.e.

$$f(t) = \int_{-\mu/2}^{\frac{\mu}{2}} \widehat{f}(\omega) e^{i2\pi\omega t}\, d\omega.$$

This set (the Paley-Wiener space PW) contains signals that superoscillate at any prescribed rate throughout a given interval. Among such signals, those of least energy appear to be the most convenient for applications for the following reasons: first, minimum-energy superoscillating solutions are uniquely determined by interpolatory constraints. Second, they obviously have the smallest possible energy cost. Third, since amplitude (or L^∞ norm) and energy (or squared L^2 norm) are related by

$$|f(t)|^2 \leq \mu \|f\|^2,$$

superoscillating signals of least energy also seem likely to exhibit the least amplitude. The minimum-energy requirement seems to be an essential step toward practicality.

Surprisingly, this is not always the case. We show that by dropping the minimum energy requirement, practicality can be gained rather than lost. Superoscillating functions of minimum-energy depend heavily on cancellation and give rise to expressions that have very large coefficients. Furthermore, the coefficients have to be found by solving equations that are very ill-conditioned.

We give a method of constructing superoscillating signals that leads to coefficients and condition numbers that are smaller by several orders of magnitude than in the minimum-energy solution, yet yields energies close to the minimum.

In fact, we design the superoscillations under two distinct constraints. We consider the interpolatory constraints needed to shape and control the superoscillatory segment but we also seek a solution that is maximally concentrated in a certain set. If the set is close to the superoscillatory segment, the solution is close to the minimum-energy solution. It progressively deviates from the minimum-energy solution as the set is moved away from the superoscillatory segment.

Manuscript received March 28, 2014; revised July 09, 2014; accepted July 09, 2014. Date of publication July 16, 2014; date of current version August 20, 2014. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Antonio Napolitano. This work was partially supported by the European Fund for Regional Development (FEDER) through the Operational Program Competitiveness Factors (COMPETE) and by the Portuguese Foundation for Science and Technology (FCT), in the context of projects FCOMP-01-0124-FEDER-022682 (FCT reference PEst-C/EEI/UI0127/2011) and Incentivo/EEI/UI0127/2013. D. G. Lee is partially supported by the National Research Foundation of Korea (NRF) (2012R1A1A2038650).

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Digital Object Identifier 10.1109/TSP.2014.2339794

We hope that the new results given in the paper help to further understand the construction of superoscillations and the numerical problems associated with it.

II. Related Work

Superoscillations were introduced in the context of quantum mechanics, as a part of Aharonov's weak measurement formalism [3], [4]. Consider the sequence of functions

$$f_n(t,a) = \left(\cos(2\pi t/n) + ia\sin(2\pi t/n)\right)^n,$$

where $a > 1$. In an interval sufficiently close to the origin, one has $f_n(t,a) \approx e^{i2\pi at}$, the frequency of which can greatly exceed the bandlimit [3]. More recently, Aharonov *et al.* [5] discussed some of the approximation properties of this sequence and show that for $|t| \leq c$, where c is a fixed real number, the sequence converges uniformly to $e^{i2\pi at}$.

Superoscillations can also be generated by zero manipulation. The effects of zero cancellation and replacement were mentioned or rediscovered a few times [6]–[9] and the minimum-energy interpolant was first described in [10] (see [11] for more references and details). However, there is no explicit mention of superoscillations until the cited work of Aharonov and co-workers. It was followed by works such as [12], which investigated the amplitude of superoscillating signals in the region of normal oscillation, or [13], in which superoscillations are discussed in the context of transplanckian frequencies in black hole radiation. Berry [14] studied superoscillations in the context of a quantum billiards problem and [15] discussed some associated phenomena of interest from the viewpoint of thermodynamics, information theory and measurement theory.

The energy cost of superoscillations was considered in [2], [16], which study the energy required by superoscillating signals as a function of the superoscillation's frequency, number, and maximum derivative, and discuss some of the implications of superoscillating signals in information theory and time-frequency analysis. The required energy was found to grow exponentially with the number of superoscillations, and polynomially with the reciprocal of the bandwidth or the reciprocal of the period of superoscillation.

The optimisation of superoscillations was considered in [17], which considers the maximisation of the yield, i.e. the ratio of the energy in the superoscillations to the total energy of the signal. The work [18] introduces a periodicity measure and applies it to yield-optimised superoscillating signals, and [19] investigates the impact of small deviations in the Fourier coefficients on the superoscillations.

The work [20] emphasises scale rather than frequency and discusses the approximation of an arbitrarily narrow pulse by linear combinations of arbitrarily wider pulses. Aharonov *et al.* [21, p. 2967] had pointed out long ago that a superposition of Gaussians centred between -1 and 1 could yield a Gaussian centred at 3. In [20] the pulses are of different width and the matter is treated from a different perspective.

A. Some Applications

Superoscillations have been successfully applied to a number of important problems. We point out a few in this section.

A function and its Fourier transform cannot both be sharply localised, but the authors of [22] seek to arbitrarily compress a temporal pulse and report the design of a class of superoscillatory electromagnetic waveforms for which the sideband amplitudes, and hence the sensitivity, can be regulated. They claim a pulse compression improvement of 47% beyond the Fourier transform limit.

The article [23] discusses optical superresolution without evanescent waves. The method introduced in [24], [25] explores the relation between superdirectivity and superoscillation and leads to subwavelength focusing schemes in free space and within a waveguide. The authors demonstrate subwavelength focusing down to 0.6 times the diffraction limit, five wavelengths away from the source. The work [26] demonstrates a superoscillatory sub-wavelength focus in a waveguide environment. The authors claim the formation of a focus at 75% the spatial width of the diffraction limited sinc pulse, 4.8 wavelengths away from the source distributions.

An array of nanoholes is used in [27] to focus light into subwavelength spots in the far-field. The article [28] discusses approaches capable of beating the diffraction limit and [29] proposes a solution based on an optical mask, which is used to create superoscillations by constructive interference of waves, leading to a subwavelength focus. The authors also demonstrate that the mask can be used also as a superresolution imaging device.

Subwavelength resolution down to $\lambda/6$ was reported in [1], using a mask divided into concentric annuli, each of which had either unit or zero transmittance, and then optimised using particle swarm optimisation. The focal spot waveform obtained resembles those in [30], [31], which are among the earliest works on the deterministic subwavelength focusing of superoscillatory optical waves.

The ambiguity present in ideal diffraction-limited imaging systems, which causes different objects to produce identical images, can be removed by using *a priori* information about the object (e.g., finite size). This has long been noted [32]–[35]. Recently, super-resolution post-processing schemes based on superoscillation have also been studied and demonstrated. See [36] and [37], where superoscillation-based numerical post-processing is used to restore lost image resolution in microwave and optical systems.

III. Motivation

Superoscillations must be discussed in reference to a fixed bandlimit, which can be normalised without loss of generality. From now on we restrict ourselves to $\mu = 1/2$ Hz, that is, to the Paley-Wiener space PW (which can be thought of as the Hilbert space of square-integrable functions bandlimited to π radians/second or 1/2 Hz).

To explain the motivation for the present paper we use a simple example. Consider the signals in PW that satisfy

$$f(t_i) = y_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where $t_1 < t_2 < \cdots < t_n$. The arrangement of the t_i in a sufficiently dense grid combined with adequately chosen y_i can result in superoscillations at any desired rate. The unique signal of minimum-energy that satisfies these constraints [2] is given by

$$f(t) = \sum_{i=1}^{n} c_i \,\mathrm{sinc}(t - t_i),$$

where the coefficients c_i are obtained by solving the linear equations

$$S\,\mathbf{c} = \mathbf{y},$$

with

$$S = [\mathrm{sinc}(t_i - t_j)]_{1\le i,j\le n}\,,$$
$$\mathbf{c} = \{c_i\}_{i=1}^{n}, \quad \mathbf{y} = \{y_i\}_{i=1}^{n}.$$

Fast superoscillations require small separations $\Delta t_i := t_i - t_{i-1}$. However, the similarity between the elements of S increases as the Δt_i decrease, causing the matrix to be ill-conditioned. The problem can also be understood from the following perspective: superoscillations of minimum energy are created by linear combinations of the functions

$$\phi_i(t) := \mathrm{sinc}(t - t_i),$$

which are linearly independent as long as the t_i are distinct. In fact, S is the (invertible) Gram matrix $[\langle\phi_i, \phi_j\rangle]_{1\le i,j\le n}$ associated with the $\{\phi_i\}_{i=1}^{n}$. However, as the density of the t_i increases, the inner products $\langle\phi_i, \phi_j\rangle$ become close to the unity and the matrix S close to singular. As a result, minimum-energy superoscillations can only be achieved by cancellation of terms with very large coefficients c_i.

In this paper we drop the minimum-energy requirement in order to build superoscillations with better numerical stability and coefficients of smaller magnitude. Instead of the functions $\mathrm{sinc}(t - t_i)$ associated with the minimum-energy solution, we consider the integer translates $\mathrm{sinc}(t - i)$, which are mutually orthonormal regardless of how dense the t_i are. The main advantage is that the associated Gram matrix is always the (perfectly conditioned) identity matrix.

To illustrate this, consider the following problem: given a fixed set $S \subseteq \mathbb{Z}$ (finite or infinite), find a function of the form

$$f(t) = \sum_{i\in S} f(i)\mathrm{sinc}(t - i)$$

that meets the constraints expressed by (1), i.e.,

$$f(t_j) = \sum_{i\in S} f(i)\mathrm{sinc}(t_j - i), \quad j = 1, 2, \ldots, n.$$

This leads to an equation of the form $A\mathbf{f} = \mathbf{c}$, with

$$A = [\mathrm{sinc}(t_j - i)]_{1\le j\le n, i\in S}\,.$$

We refer to this method as the direct approach to superoscillations (see Section IV-A for some background on the matrix A).

By the sampling theorem, when $S = \mathbb{Z}$ the set of possible f coincides with PW. Otherwise, it forms a subspace of PW.

Figs. 1 and 2 show superoscillations generated by the minimum energy approach and the direct approach. The spacing between the t_j was set to one-fifth of the standard sampling period. Inside the superoscillating interval, the difference between the two signals is small. Outside the interval, the minimum-energy method leads to smaller amplitudes, as expected.

Although the energy and amplitude of the two solutions are comparable, the size of the largest coefficient and the condition number of the two problems differ by several orders of magnitude (cf. Table I).

This behaviour is typical rather than exceptional. Figs. 3–5 show the results of 1000 experiments in which the number of

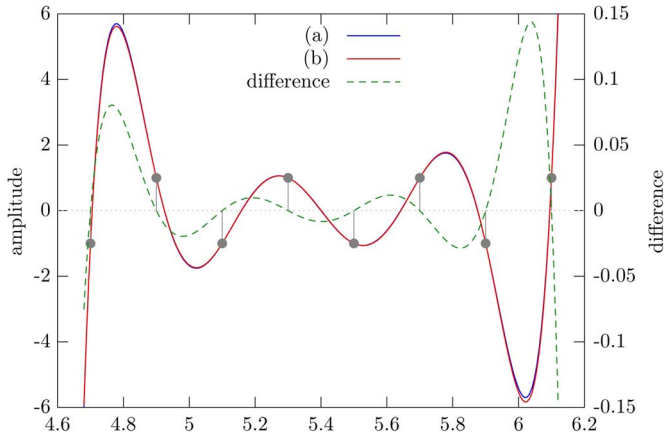

Fig. 1. Superoscillations at 10 times the Nyquist rate obtained using (a) the minimum energy method and (b) the direct method. The difference between the two solutions is also shown.

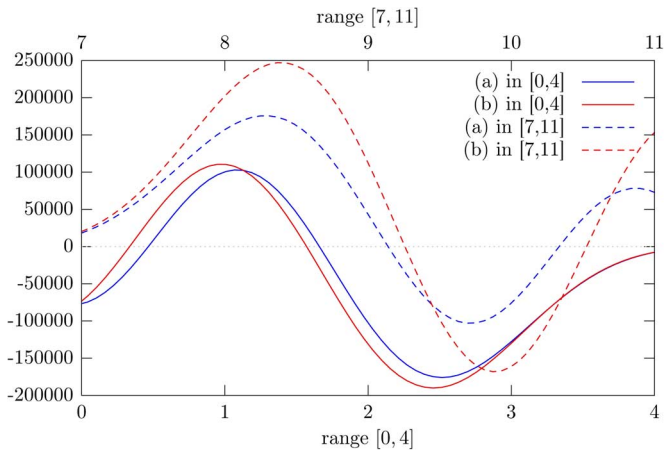

Fig. 2. Outside the superoscillatory interval, the minimum energy solution (a) shows oscillations of slightly smaller amplitude than those of solution (b).

TABLE I
ENERGY, LARGEST COEFFICIENT AND CONDITION NUMBER OF THE PROBLEMS OF FIG. 1 AND FIG. 2

	$E = \|f\|^2$	$\max_k \lvert c_k\rvert$	ρ
Minimum energy:	$1.1\ 10^{11}$	$2.9\ 10^{10}$	$1.0\ 10^{11}$
Direct method:	$1.6\ 10^{11}$	$2.0\ 10^{5}$	$3.9\ 10^{5}$

superoscillations and their rate were randomly chosen. The density of the t_k varied between 10 and 20 times the Nyquist density. Fig. 3 shows the energy ratios $\|f_a\|^2/\|f_b\|^2$ where f_a and f_b denote the minimum energy and the direct method solutions, respectively. As in the example discussed before, the energies of the two solutions are comparable but the coefficient magnitude and condition numbers differ by several orders of magnitude. More extreme problems lead to even larger differences.

This shows that by abandoning the minimum-energy condition practicality can be gained rather than lost. The next sections address an optimisation problem inspired by this observation. Our goal is not merely to construct superoscillations using translates $\mathrm{sinc}(t - i)$ in a stable way: we also try to concentrate the energy of the signal as much as possible in a given interval, which need not be close to the superoscillating segment.

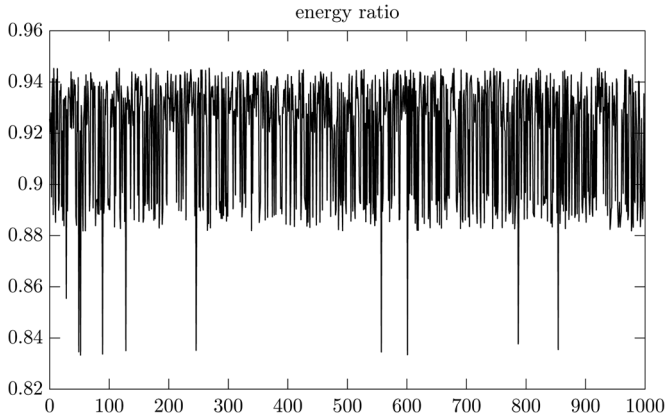

Fig. 3. The energy ratio $\|f_a\|^2/\|f_b\|^2$ for 1000 random superoscillation problems (frequencies between 10 and 20 times the Nyquist rate), obtained using the minimum energy approach (f_a) and the direct method (f_b).

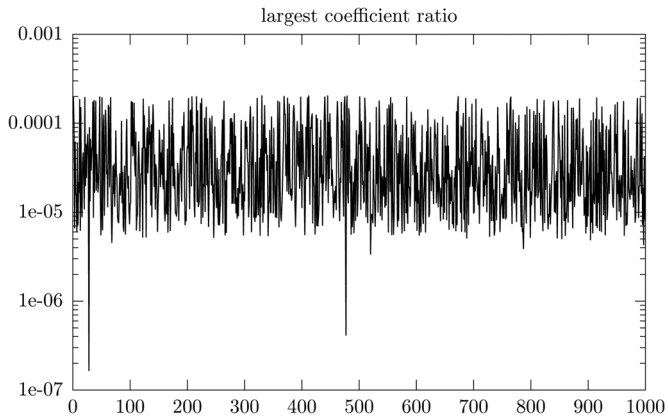

Fig. 4. Similar to Fig. 3, but shows the ratio of the largest coefficients of f_b (direct method) to f_a (minimum energy).

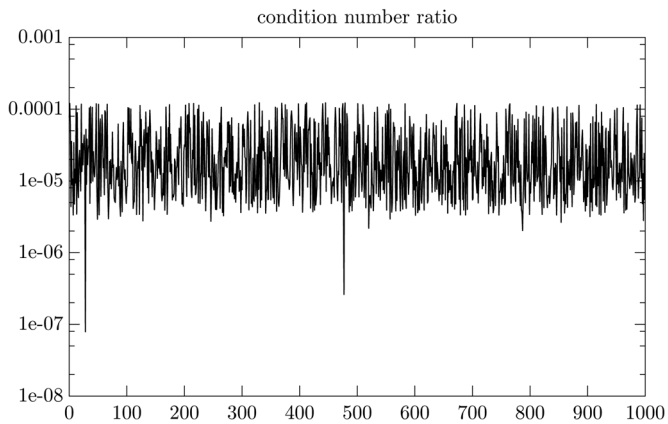

Fig. 5. Similar to Fig. 4, but shows the ratio of the condition numbers of the matrices of the direct method and the minimum energy approach.

IV. Results

A. Background

Let S be a finite or infinite set of integers (the case $S = \mathbb{Z}$ is not excluded). Let J be a set of M reals $t_1 < t_2 < \cdots < t_M$. Consider the set of signals of the form

$$f(t) = \sum_{k \in S} f(k)\, \mathrm{sinc}(t - k) \tag{2}$$

that satisfy the constraints

$$f(t_k) = c_k, \quad k = 1, 2, \ldots, M. \tag{3}$$

The constraints (3) determine the linear equations

$$A\,\mathbf{f} = \mathbf{c}$$

where $A = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in S}$, $\mathbf{f} = \{f(k)\}_{k \in S}$ and $\mathbf{c} = \{c_k\}_{t_k \in J}$. When $|S| = \infty$, A should be interpreted as the linear operator from $\ell^2(S)$ into $\mathbb{R}^M$ with the given matrix representation.

The following necessary and sufficient condition for A to have full rank [11] will be needed in the sequel.

Theorem 1: Let $A = [\mathrm{sinc}(t_m - n)]_{t_m \in J, n \in S}$ where $S \subseteq \mathbb{Z}$ and $|J| < \infty$. Let E denote the set of integer elements of J that are not in S.

(a) Assume that $|S| \geq |J|$. Then A is of full rank if and only if E is empty.

(b) Assume that $|S| \leq |J|$. Then A is of full rank if and only if $|E| \leq |J| - |S|$.

As a corollary, A has full rank if E is empty. For example, if J consists of numbers of the form mT, with $m \neq 0$ and T irrational, then E is necessarily empty. In general, the condition is extremely simple to verify (an examination of the sets S and J yields the answer).

The condition is also very natural. If the functions f under consideration are of the form (2), then by definition they satisfy $f(k) = 0$ for all integer k outside S. However, if the set E is nonempty, there exists at least one constraint of the form $f(k) = c_k$, where k is an integer outside S. This would establish a potential conflict.

For these reasons and to avoid repetitions, from this point onwards we will tacitly assume that the matrices of the form $A = [\mathrm{sinc}(t_m - n)]_{t_m \in J, n \in S}$ that will occur have full rank.

B. An Optimisation Problem

Ideally, an algorithm to construct superoscillations should be numerically stable and lead to a signal with as much energy inside the superoscillatory interval as possible. This suggests the problem

$$\max_{f \in PW} \int_{C(J)} |f(t)|^2 dt \tag{4}$$

where the integration is over the smallest interval $C(J)$ that contains J, the set that determines the superoscillations, under an additional constraint on the total signal energy.

We seek an alternative more flexible formulation closer to the motivating example in Section III: we select a finite set of integers I that covers J and use the energy inside the interval that contains I as a replacement for (4). This ensures two things: first, the superoscillations will be built by combining integer translates of the sinc function, leading to numerical procedures much more stable than those associated with the minimum-energy formulation; second, the fraction of energy transferred to the interval that contains I is maximised. By varying I, the energy concentration can be adjusted as needed.

Before proceeding we define more precisely what we mean by "I covers J".

Definition 1: Let J be a set of finitely many distinct real numbers. An integer set $I \subset \mathbb{Z}$ covers J if

$$\min I \leq \min J < \max J \leq \max I.$$

Returning to the problem of superoscillations, we may assume that the separation between the t_k in (3) satisfies $\Delta t_k < 1$, i.e. the constraints lie in a grid denser than the Nyquist density. It is then easy to select a set I with $|I| \leq |J|$ that covers J. The minimal set is obtained by taking $\lfloor t_1 \rfloor$ and $\lceil t_M \rceil$ as the smallest and largest element of I, respectively.

In fact, we will formulate a slightly more general problem. For greater flexibility, the finite set I is left unrestricted, i.e., it may or may not cover J; and the functions will be drawn from the entire PW space or from the subspace defined by

$$f(t) = \sum_{k \in S} f(k) \mathrm{sinc}(t-k),$$

where S is a fixed subset of $\mathbb{Z}$. By the sampling theorem, when $S = \mathbb{Z}$ we will be considering the entire PW space. In any case,

$$f(t) = f_1(t) + f_2(t),$$

where

$$f_1(t) = \sum_{k \in I} f(k)\, \mathrm{sinc}(t-k),$$
$$f_2(t) = \sum_{k \in S \setminus I} f(k)\, \mathrm{sinc}(t-k).$$

If S is finite, f_1 and f_2 are both given by finite sums. In any case,

$$E_1 = \|f_1\|^2 = \|\widehat{f_1}\|^2 = \int_{-1/2}^{1/2} \left| \sum_{k \in I} f_1(k) e^{-i2\pi k\omega} \right|^2 d\omega$$
$$= \sum_{k \in I} |f_1(k)|^2 = \|\mathbf{f_1}\|^2,$$

where $\mathbf{f_1} = \{f_1(k)\}_{k \in I}$. Similarly, $E_2 = \|\mathbf{f_2}\|^2$, with $\mathbf{f_2} = \{f_2(k)\}_{k \notin I}$, so that the total energy of f can always be written as $E = E_1 + E_2$, regardless of the nature of S.

We may now formulate our main problem.

Problem 1: Let $S \subseteq \mathbb{Z}$. Let I be a finite subset of S, with N elements. Let J be a set of M reals $t_1 < t_2 < \cdots < t_M$. Assume that I and J satisfy the condition of Theorem 1. Among all $f \in PW$ of the form

$$f(t) = \sum_{k \in S} f(k)\, \mathrm{sinc}(t-k), \tag{5}$$

that satisfy the constraints

$$f(t_k) = c_k, \quad k = 1, 2, \ldots, M,$$

find a signal f that maximises the energy ratio

$$\frac{E_1}{E_1 + E_2}$$

where $E_1 = \|\mathbf{f}_1\|^2$ with $\mathbf{f}_1 = \{f(k)\}_{k \in I}$ and $E_2 = \|\mathbf{f}_2\|^2$ with $\mathbf{f}_2 = \{f(k)\}_{k \in S \setminus I}$. In the rest of the paper we solve this problem.

C. An Equivalent Finite-Dimensional Problem

Rewriting the energy ratio in Problem 1 in the form

$$\frac{1}{1 + E_2/E_1}$$

we obtain the following formulation:

$$\mathbf{f} = \arg\min \frac{\|\mathbf{f}_2\|^2}{\|\mathbf{f}_1\|^2} \text{ subject to } \tilde{A}\mathbf{f} = \mathbf{c} \text{ and } \mathbf{f}_1 \neq 0, \tag{6}$$

where $\tilde{A} = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in S}$. We have replaced the function f with the vector $\mathbf{f}$ of its samples, formed by concatenating $\mathbf{f}_1$ and $\mathbf{f}_2$. The goal is to direct as much energy as possible to the interval I and to the vector $\mathbf{f}_1$. If the interval I covers J it contains the superoscillating segment of the signal, and so the fraction of the energy directed to the superoscillations will be maximised (or, alternatively, the fraction of energy directed to the complement of I will be minimised).

The constraint $\tilde{A}\mathbf{f} = \mathbf{c}$ can be written as

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \mathbf{c} \tag{7}$$

where

$$A_1 = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in I},$$
$$A_2 = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in S \setminus I}.$$

Here A_1 is $|J| \times |I|$ with $|I| \leq |J| < \infty$ and A_2 is $|J| \times |S \setminus I|$ with $|J| < |S \setminus I|$ where $S \setminus I$ is allowed to be infinite. These matrices are assumed to have full rank, as discussed immediately after Theorem 1 (the pairs (I, J) and $(S \setminus I, J)$ are assumed to satisfy the condition of the theorem).

Remark 2: The energy is concentrated in I but the interval that contains the superoscillations is determined by J. Our methods work for any set I, including sets that cover J. When I covers J, one may be interested in minimal covers, for reasons of efficiency. The tighter the covering of J, the more efficient the energy concentration is likely to be.

Remark 3: To generate superoscillations, J must be denser than the reference Nyquist density (in PW, the reference density is that of the integers). As a result, if I is a set of integers that tightly covers J, then the cardinal of I will naturally satisfy $|I| < |J|$. The contrary is in fact less interesting. For example, when $|I| = |J|$ the matrix A_1 is invertible, so we can simply put $\mathbf{f}_2 = 0$ and easily obtain the optimal solution for (6) (as studied in [11], [38], in a different context). Thus, in the following we assume that $|I| < |J|$, the condition under which the problem is simultaneously more interesting and more useful.

We may always express $\mathbf{c}$ as

$$\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$$

where $\mathbf{c}_1 \in R(A_1)$ and $\mathbf{c}_2$ is unrestricted. This turns (7) into two separate equations

$$A_1\, \mathbf{f}_1 = \mathbf{c}_1 \tag{8a}$$
$$A_2\, \mathbf{f}_2 = \mathbf{c} - \mathbf{c}_1\ (=: \mathbf{c}_2). \tag{8b}$$

By varying $\mathbf{c}_1$ in $R(A_1)$, we can attain all the solutions of (7). That is, the set $\{\mathbf{f} : \tilde{A}\mathbf{f} = \mathbf{c}\}$ is the same as

$$\bigcup_{\mathbf{c}_1 \in R(A_1)} \{\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) : A_1\, \mathbf{f}_1 = \mathbf{c}_1, A_2\, \mathbf{f}_2 = \mathbf{c}_2\}.$$

Fix any $\mathbf{c}_1 \in R(A_1)$, and consider the (8) with independent variables $\mathbf{f}_1$ and $\mathbf{f}_2$. The minimisation objective is $\|\mathbf{f}_2\|^2/\|\mathbf{f}_1\|^2$ and we want to minimise $\|\mathbf{f}_2\|^2$ while maximising $\|\mathbf{f}_1\|^2$. Notice that this can be done independently, since fixing $\mathbf{c}_1 \in R(A_1)$ has separated the constraint into two independent equations.

Consider the maximisation of $\|\mathbf{f}_1\|^2$ first. The equations involving $A_1 = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in I}$ are over-determined and there is no freedom to maximise $\mathbf{f}_1$. Since $\mathbf{c}_1 \in R(A_1)$, the unique solution is

$$\mathbf{f}_1 = \left(A_1^T A_1\right)^{-1} A_1^T \mathbf{c}_1.$$

This is the least-squares solution but it agrees with the exact solution because $\mathbf{c}_1 \in R(A_1)$.

Since to each vector $\mathbf{f}_1$ in $\mathbb{R}^N \setminus \{0\}$ there corresponds one and only one $\mathbf{c}_1$ in $R(A_1) \setminus \{0\}$, we may use $\mathbf{f}_1$ instead of $\mathbf{c}_1 \in R(A_1)$ as the variable.

Consider now the minimisation of $\|\mathbf{f}_2\|^2$. Since the equations involving $A_2 = [\mathrm{sinc}(t_j - k)]_{t_j \in J, k \in S \setminus I}$ are under-determined, we have the minimum norm solution

$$\mathbf{f}_2 = A_2^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}_2. \tag{9}$$

Substituting (9) in (6) yields

$$\begin{aligned}\mathbf{f}_1 &= \arg\min \left(\frac{\left(A_2^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}_2\right)^T A_2^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}_2}{\mathbf{f}_1^T \mathbf{f}_1} \right) \\ &= \arg\min \left(\frac{\mathbf{c}_2^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}_2}{\mathbf{f}_1^T \mathbf{f}_1} \right) \\ &\quad \text{subject to } \mathbf{c} = A_1 \mathbf{f}_1 + \mathbf{c}_2 \text{ and } \mathbf{f}_1 \neq 0.\end{aligned}$$

Taking the constraints inside the target function, we obtain the following equivalent formulation of Problem 1.

Problem 2: The signal $f \in PW$ that solves Problem 1 is determined by the minimisation problem

$$\mathbf{f}_1^* = \underset{0 \neq \mathbf{f}_1 \in \mathbb{R}^N}{\arg\min} \left(\frac{(\mathbf{c} - A_1 \mathbf{f}_1)^T \left(A_2 A_2^T\right)^{-1} (\mathbf{c} - A_1 \mathbf{f}_1)}{\mathbf{f}_1^T \mathbf{f}_1} \right) \tag{10}$$

where

$$\begin{aligned} A_1 &= \left[\mathrm{sinc}(t_j - k)\right]_{t_j \in J, k \in I}, \\ A_2 &= \left[\mathrm{sinc}(t_j - k)\right]_{t_j \in J, k \in S \setminus I}. \end{aligned}$$

Since $\left(A_2 A_2^T\right)^{-1}$ and A_1 are always finite-dimensional (see Section IV-H for implementation details), we have converted the original problem into a finite dimensional one, easier to solve.

D. Remarks

The following remarks help to appreciate the differences between the problem that we have formulated and some seemingly related problems.

Remark 4: The minimisation problem for the Rayleigh quotient for the matrix $\left(A_2 A_2^T\right)^{-1}$ can be written as

$$\begin{aligned} &\min \mathbf{z}^T \left(A_2 A_2^T\right)^{-1} \mathbf{z} \\ &\text{subject to } \mathbf{z} \in \left\{ \frac{\mathbf{f}_1}{\|\mathbf{f}_1\|} : \mathbf{f}_1 \neq 0 \right\}, \end{aligned}$$

or simply subject to $\mathbf{z}^T \mathbf{z} = 1$. Our problem can also be written as

$$\begin{aligned} &\min \mathbf{z}^T \left(A_2 A_2^T\right)^{-1} \mathbf{z} \\ &\text{subject to } \mathbf{z} \in \left\{ \frac{\mathbf{c} - A_1 \mathbf{f}_1}{\|\mathbf{f}_1\|} : \mathbf{f}_1 \neq 0 \right\}, \end{aligned}$$

The objective functions are the same, but the problems are very different due to their constraints.

Remark 5: As a particular case, the set I can be chosen to cover J in order to maximise the energy of the superoscillating part of the signal in reference to its total energy. This resembles the goal of [17], which considers the problem of yield maximisation, that is, the maximisation of the ratio of the energy in the superoscillations to the total energy of the signal.

However, the two approaches are very different. The work [17] considers periodic functions only (cosine series) whereas we consider finite-energy (non-periodic) PW functions. In fact, we consider the optimisation problem over the entire PW space (if $S = \mathbb{Z}$) or a subspace of PW (determined by S). Furthermore, we set no restrictions on I. As a result, by varying I we can control the energy concentration range. Finally, we set no restriction on the sets J (they need not be uniformly spaced).

E. Another Form of the Problem

The matrix $A_2 A_2^T$ is symmetric positive definite and so it has a unique Cholesky factorisation

$$A_2 A_2^T = L L^T,$$

where $L \in \mathbb{R}^{M \times M}$ is a lower triangular matrix with positive diagonal entries. In fact, such a decomposition will become irrelevant in the end, as long as L is an invertible square matrix. Then $\left(A_2 A_2^T\right)^{-1} = \left(L^{-1}\right)^T L^{-1}$ so that (10) can be rewritten as

$$\mathbf{f}_1^* = \underset{0 \neq \mathbf{f}_1 \in \mathbb{R}^N}{\arg\min} \left(\frac{\|L^{-1}(A_1 \mathbf{f}_1 - \mathbf{c})\|^2}{\|\mathbf{f}_1\|^2} \right).$$

We have obtained the following equivalent formulation of Problem 2.

Problem 3: The signal $f \in PW$ that solves Problem 1 is determined by the minimisation problem

$$\mathbf{f}_1^* = \underset{0 \neq \mathbf{f}_1 \in \mathbb{R}^N}{\arg\min} \left(\frac{\|B \mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} \right) \tag{11}$$

where $B := L^{-1} A_1 \in \mathbb{R}^{M \times N}$ $(M > N)$ is of full rank and $\mathbf{d} := L^{-1} \mathbf{c} \in \mathbb{R}^M$.

Since B is of full rank, $B^T B (\in \mathbb{R}^{N \times N})$ is positive definite and has positive eigenvalues (which are also the squared singular values of B). Recall that $\frac{1}{\|B^\dagger\|}$ (resp., $\|B\|$) is the square root of the smallest (resp., largest) eigenvalue of $B^T B$. (In fact, $\frac{1}{\|B^\dagger\|} \leq \frac{\|B \mathbf{f}_1\|}{\|\mathbf{f}_1\|} \leq \|B\|$ for any $\mathbf{f}_1 \neq 0$.) Let $\mathbf{v}$ denote a unit eigenvector of $B^T B$ associated with the smallest eigenvalue $\frac{1}{\|B^\dagger\|^2}$.

If $\mathbf{d} \in R(B)$ $(\Leftrightarrow \mathbf{c} \in R(A_1))$, then the minimum value of (11) is zero. This makes sense because having $\mathbf{c} \in R(A_1)$ is equivalent to saying that we can successfully build an interpolant with indices in I (therefore concentrating all energy in I, i.e., E_1 takes all energy and $E_2 = 0$). In this case, the corresponding solution is $\mathbf{f}_1 = \left(B^T B\right)^{-1} B^T \mathbf{d} = \left[A_1^T \left(A_2 A_2^T\right)^{-1} A_1\right]^{-1} A_1^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}$.

So our interest is in the case $\mathbf{d} \notin R(B)$ ($\Leftrightarrow \mathbf{c} \notin R(A_1)$), which is likely be the case when we consider superoscillation constraints. One should note that this trivially implies $\mathbf{d} \neq 0$. Then $B\mathbf{f}_1 - \mathbf{d} = B(\mathbf{f}_1 - \mathbf{f}_0) + \mathbf{d}_0$ where $\mathbf{f}_0 := (B^TB)^{-1}B^T\mathbf{d}$ is the least-squares solution to $B\mathbf{f} = \mathbf{d}$, and $0 \neq \mathbf{d}_0 \in R(B)^\perp$ ($\mathbf{d}_0 := [I - B(B^TB)^{-1}B^T]\mathbf{d}$). So,

$$\|B\mathbf{f}_1 - \mathbf{d}\|^2 = \|B(\mathbf{f}_1 - \mathbf{f}_0)\|^2 + \|\mathbf{d}_0\|^2.$$

If $\mathbf{f}_0 = 0$ ($\Leftrightarrow \mathbf{d} = \mathbf{d}_0 \in R(B)^\perp \Leftrightarrow B^T\mathbf{d} = 0 \Leftrightarrow A_1^T(A_2A_2^T)^{-1}\mathbf{c} = 0$), then the problem (11) becomes

$$\mathbf{f}_1^* = \underset{0\neq\mathbf{f}_1\in\mathbb{R}^N}{\arg\min} \left(\frac{\|B\mathbf{f}_1\|^2 + \|\mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} \right).$$

It is easy to see that any solution to this problem must be of the form $\mathbf{f}_1^* = \mu\mathbf{v}$ for some constant μ. However, since $\mathbf{d} \neq 0$, the minimum value occurs at $\mu = \pm\infty$.

Thus, in the following, we will assume that $\mathbf{d} \notin R(B)$, equivalent to

$$\mathbf{c} \notin R(A_1) \Rightarrow \mathbf{d} \neq 0,\ \mathbf{d}_0 \neq 0,$$

and $\mathbf{d} \notin R(B)^\perp$, equivalent to

$$\mathbf{f}_0 \neq 0 \Leftrightarrow B^T\mathbf{d} \neq 0 \ \Leftrightarrow\ \left(A_2^TA_2\right)^{-1}\mathbf{c} \notin R(A_1)^\perp.$$

F. Bounds for the Minimum Value

We find some bounds for the minimum value in (11). Note that for any nonzero real number α,

$$\frac{\|B(\alpha\mathbf{v}) - \mathbf{d}\|^2}{\|\alpha\mathbf{v}\|^2} = \frac{\|B(\alpha\mathbf{v})\|^2 + \|\mathbf{d}\|^2 - 2\alpha\langle B(\mathbf{v}), \mathbf{d}\rangle}{\|\alpha\mathbf{v}\|^2}.$$

where $\langle B(\mathbf{v}), \mathbf{d}\rangle \neq 0$ since $\mathbf{d} \notin R(B)^\perp$. Then for any nonzero α satisfying $\|\mathbf{d}\|^2 - 2\alpha\langle B(\mathbf{v}), \mathbf{d}\rangle < 0$,

$$\frac{\|B(\alpha\mathbf{v}) - \mathbf{d}\|^2}{\|\alpha\mathbf{v}\|^2} < \frac{\|B(\alpha\mathbf{v})\|^2}{\|\alpha\mathbf{v}\|^2} = \frac{1}{\|B^\dagger\|^2}.$$

Therefore

$$\min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \left(\frac{\|B\mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} \right) < \frac{1}{\|B^\dagger\|^2}.$$

On the other hand,

$$\begin{aligned}\min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \frac{\|B\mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} &= \min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \frac{\|B(\mathbf{f}_1 - \mathbf{f}_0)\|^2 + \|\mathbf{d}_0\|^2}{\|\mathbf{f}_1\|^2} \\ &\geq \min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \frac{\left(\frac{1}{\|B^\dagger\|}\right)^2\|\mathbf{f}_1 - \mathbf{f}_0\|^2 + \|\mathbf{d}_0\|^2}{\|\mathbf{f}_1\|^2}\end{aligned}$$

It is easy to see that the minimum of the last term is attained at $\mathbf{f}_1 = \beta\mathbf{f}_0$ for some constant β, and in fact, $\beta = 1 + \frac{\|B^\dagger\|^2\|\mathbf{d}_0\|^2}{\|\mathbf{f}_0\|^2}$ and the minimum value is

$$\frac{1}{\|B^\dagger\|^2 + \frac{\|\mathbf{f}_0\|^2}{\|\mathbf{d}_0\|^2}}.$$

Therefore,

$$\frac{1}{\|B^\dagger\|^2 + \frac{\|\mathbf{f}_0\|^2}{\|\mathbf{d}_0\|^2}} \leq \min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \left(\frac{\|B\mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} \right) < \frac{1}{\|B^\dagger\|^2}. \tag{12}$$

The strict inequality for the upper bound will become important in the following.

G. Solution of the Problem

Assume that

$$\lambda^* := \min_{0\neq\mathbf{f}_1\in\mathbb{R}^N} \left(\frac{\|B\mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} \right)$$

is attained at $\mathbf{f}_1 = \mathbf{f}_1^*$. Then

$$h(\mathbf{f}_1) := \|B\mathbf{f}_1 - \mathbf{d}\|^2 - \lambda^*\|\mathbf{f}_1\|^2 \geq 0 \quad \text{for any } \mathbf{f}_1 \neq 0.$$

Since $h(\mathbf{x}) = \mathbf{x}^T(B^TB - \lambda^*I)\mathbf{x} - 2\mathbf{d}^TB\mathbf{x} + \mathbf{d}^T\mathbf{d}$ is a quadratic form, $(B^TB - \lambda^*I)$ is positive semi-definite, that is, $\frac{1}{\|B^\dagger\|^2} - \lambda^* \geq 0$. In fact, $(B^TB - \lambda^*I)$ is positive definite since $\frac{1}{\|B^\dagger\|^2} - \lambda^* > 0$ by (12). Then

$$\begin{aligned}h(\mathbf{f}_1) = &\left\|(B^TB - \lambda^*I)^{1/2}(\mathbf{f}_1 - (B^TB - \lambda^*I)^{-1}B^T\mathbf{d})\right\|^2 \\ &+ \mathbf{d}^T\left[I - B(B^TB - \lambda^*I)^{-1}B^T\right]\mathbf{d}, \quad \mathbf{f}_1 \neq 0,\end{aligned}$$

so that

$$\mathbf{f}_1^* = (B^TB - \lambda^*I)^{-1}B^T\mathbf{d} \tag{13}$$

is the *unique* solution to the problem and

$$0 = \mathbf{d}^T\left[I - B(B^TB - \lambda^*I)^{-1}B^T\right]\mathbf{d}.$$

The last equation is equivalent to

$$0 = \det(B^TB - \lambda^*I)\cdot\mathbf{d}^T\mathbf{d} - \mathbf{d}^TB\ \mathrm{adj}\ (B^TB - \lambda^*I)\ B^T\mathbf{d} \tag{14}$$

where $\mathrm{adj}(E)$ denotes the adjoint matrix of E, i.e., the transpose of the cofactor matrix of E. Since $B^TB \in \mathbb{R}^{N\times N}$, the right hand side of (14) is a polynomial of degree at most $N(< M)$ in λ^*. In fact, it is possible to show that the polynomial has only real zeros, as its connection with the quadratic form of a symmetric matrix suggests. The possible range of λ^* is given by (12) so λ^* can be obtained numerically. Once we find λ^*, $\mathbf{f}_1^*$ follows immediately from (13). Finally,

$$\mathbf{f}_2^* = A_2^T\left(A_2A_2^T\right)^{-1}(\mathbf{c} - A_1\mathbf{f}_1^*).$$

The solution resembles the elementary process of finding eigenvectors for a given square matrix (find λ from the characteristic polynomial and then use it to find the corresponding eigenvectors). Note that $\mathbf{f}_1^*$ depends on the value of λ^*. From the viewpoint of stability, the worst case scenario is when $\lambda^* \approx \frac{1}{\|B^\dagger\|^2}$, so that $(B^TB - \lambda^*I)$ becomes ill-conditioned.

H. Computational Aspects

For convenience, we collect and summarise here some of the results obtained, expressing them in a form suitable for practical usage.

As before, $N = |I|$ and $M = |J|$, and to get a non-trivial result it is assumed that $M > N$. All matrices are assumed to have full rank (see the conditions stated in Theorem 1).

The solution obtained can be drawn from the entire PW space or from the subspace defined by (5). When working with the PW space, the set S in (5) coincides with $\mathbb{Z}$ and $S \setminus I$ is of course infinite. When S is a finite subset of $\mathbb{Z}$, $S \setminus I$ is always finite.

The matrices to consider are

$$A_1 = \left[\operatorname{sinc}(t_j - k)\right]_{t_j \in J, k \in I},$$
$$A_2 = \left[\operatorname{sinc}(t_j - k)\right]_{t_j \in J, k \in S \setminus I}.$$

The vector of constraint values is $\mathbf{c} = \{c_j\}_{t_j \in J}$.

The equations (13) and (14) can be expressed in the following forms:

$$\mathbf{f}_1^* = (\mathcal{X} - \lambda^* I)^{-1} \mathcal{Y} \tag{15}$$
$$0 = \det(\mathcal{X} - \lambda^* I) \cdot \mathcal{Z} - \mathcal{Y}^T \operatorname{adj}(\mathcal{X} - \lambda^* I)\, \mathcal{Y} \tag{16}$$

where

$$\mathcal{X} := B^T B = A_1^T (L^{-1})^T L^{-1} A_1 = A_1^T \left(A_2 A_2^T\right)^{-1} A_1,$$
$$\mathcal{Y} := B^T \mathbf{d} = A_1^T (L^{-1})^T L^{-1} \mathbf{c} = A_1^T \left(A_2 A_2^T\right)^{-1} \mathbf{c},$$
$$\mathcal{Z} := \mathbf{d}^T \mathbf{d} = \mathbf{c}^T (L^{-1})^T L^{-1} \mathbf{c} = \mathbf{c}^T \left(A_2 A_2^T\right)^{-1} \mathbf{c}.$$

If necessary, the computation of $A_2 A_2^T$ should be approached in a manner adequate to the nature of the set S in (5). There are two cases to consider.

If S is a finite set, the set $S \setminus I$ is also finite and $A_2 A_2^T$ can be computed directly using a finite sum:

$$\left(A_2 A_2^T\right)_{mn} = \sum_{k \in S \setminus I} \operatorname{sinc}(t_m - k) \operatorname{sinc}(t_n - k).$$

If S^c is a finite set (e.g., $S = \mathbb{Z}$), then $S \setminus I$ is infinite so that A_2 has an infinite number of columns and the above summation is infinite. However, since $(S \setminus I)^c$ is finite, the computation can be reduced to the following finite sum:

$$\left(A_2 A_2^T\right)_{mn} = \operatorname{sinc}(t_m - t_n) - \sum_{k \in (S \setminus I)^c} \operatorname{sinc}(t_m - k) \operatorname{sinc}(t_n - k) \tag{17}$$

where $t_m, t_n \in J$. To verify it, note that the application of the sampling theorem to the function $\operatorname{sinc}(\cdot - t_m)$ yields

$$\operatorname{sinc}(t - t_m) = \sum_{k \in \mathbb{Z}} \operatorname{sinc}(k - t_m) \operatorname{sinc}(t - k). \tag{18}$$

Separate summation over $S \setminus I$ and $(S \setminus I)^c$, followed by the substitution $t = t_n$, leads to (17).

Note that $\frac{1}{\|B^\dagger\|^2} = \lambda_{\min}(B^T B) = \lambda_{\min}(\mathcal{X})$ where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue. Moreover, since $\mathbf{f}_0 := (B^T B)^{-1} B^T \mathbf{d} = \mathcal{X}^{-1} \mathcal{Y}$ and $\mathbf{d}_0 := [I - B(B^T B)^{-1} B^T]\mathbf{d} = \mathbf{d} - B\mathcal{X}^{-1}\mathcal{Y}$, it follows that

$$\|\mathbf{f}_0\|^2 = \|\mathcal{X}^{-1}\mathcal{Y}\|^2$$

and

$$\begin{aligned}\|\mathbf{d}_0\|^2 &= (\mathbf{d} - B\mathcal{X}^{-1}\mathcal{Y})^T (\mathbf{d} - B\mathcal{X}^{-1}\mathcal{Y}) \\ &= \mathbf{d}^T \mathbf{d} - 2\mathbf{d}^T B \mathcal{X}^{-1} \mathcal{Y} + \mathcal{Y}^T (\mathcal{X}^{-1})^T B^T B \mathcal{X}^{-1} \mathcal{Y} \\ &= \mathcal{Z} - \mathcal{Y}^T \mathcal{X}^{-1} \mathcal{Y}.\end{aligned}$$

Then the bounds (12) for λ^* can be rewritten in terms of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ as

$$\frac{1}{\frac{1}{\lambda_{\min}(\mathcal{X})} + \frac{\|\mathcal{X}^{-1}\mathcal{Y}\|^2}{\mathcal{Z} - \mathcal{Y}^T \mathcal{X}^{-1} \mathcal{Y}}} \le \lambda^* < \lambda_{\min}(\mathcal{X}). \tag{19}$$

I. An Iterative Solution

We now present an approach to Problem 3 that does not require the computation of the zeros of a polynomial. Instead, it explores the quadratic nature of the minimisation objective function to obtain an iterative solution.

The derivative with respect to $\mathbf{f}_1$ of the function

$$L(\mathbf{f}_1) := \frac{\|B\mathbf{f}_1 - \mathbf{d}\|^2}{\|\mathbf{f}_1\|^2} = \frac{\mathbf{f}_1^T B^T B \mathbf{f}_1 - 2\mathbf{f}_1^T B^T \mathbf{d} + \mathbf{d}^T \mathbf{d}}{\mathbf{f}_1^T \mathbf{f}}$$

is given by

$$\frac{dL}{d\mathbf{f}_1} = \frac{\|\mathbf{f}_1\|^2 (2B^T B \mathbf{f}_1 - 2B^T \mathbf{d}) - \|B\mathbf{f}_1 - \mathbf{d}\|^2 2\mathbf{f}_1}{\|\mathbf{f}_1\|^4},$$

and so the necessary condition for an extremum $\mathbf{f}_1^*$ is

$$\|\mathbf{f}_1^*\|^2 \left(B^T B \mathbf{f}_1^* - B^T \mathbf{d}\right) - \|B\mathbf{f}_1^* - \mathbf{d}\|^2 \mathbf{f}_1^* = 0.$$

This is equivalent to

$$\left(B^T B - \frac{\|B\mathbf{f}_1^* - \mathbf{d}\|^2}{\|\mathbf{f}_1^*\|^2} I\right) \mathbf{f}_1^* = B^T \mathbf{d},$$

that is,

$$\left(B^T B - \lambda^* I\right) \mathbf{f}_1^* = B^T \mathbf{d},$$
$$\lambda^* = \frac{\|B\mathbf{f}_1^* - \mathbf{d}\|^2}{\|\mathbf{f}_1^*\|^2}.$$

This can be written separately as

$$\mathbf{f}_1^* = (\mathcal{X} - \lambda^* I)^{-1} \mathcal{Y},$$
$$\lambda^* = \frac{\mathbf{f}_1^T \mathcal{X} \mathbf{f}_1 - 2\mathbf{f}_1^T \mathcal{Y} + \mathcal{Z}}{\mathbf{f}_1^T \mathbf{f}_1},$$

suggesting the iteration

$$\mathbf{x}_k = (\mathcal{X} - \lambda_{k-1} I)^{-1} \mathcal{Y},$$
$$\lambda_k = \frac{\mathbf{x}_k^T \mathcal{X} \mathbf{x}_k - 2\mathbf{x}_k^T \mathcal{Y} + \mathcal{Z}}{\mathbf{x}_k^T \mathbf{x}_k}, \quad k = 1, 2, \ldots.$$

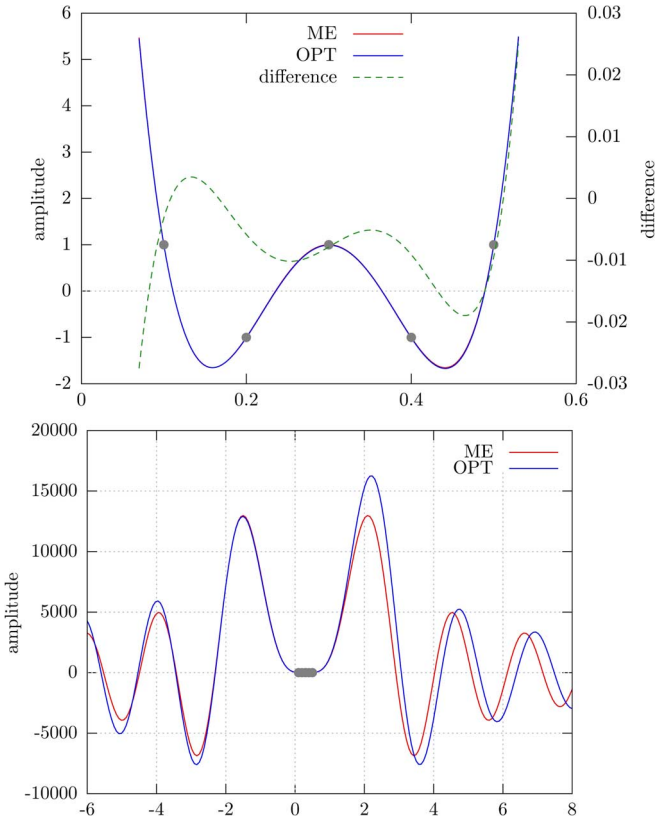

Fig. 6. Superoscillations at 10 times the Nyquist rate obtained using the ME and OPT methods. The set where the energy is optimised corresponds to $I = \{0, 1\}$.

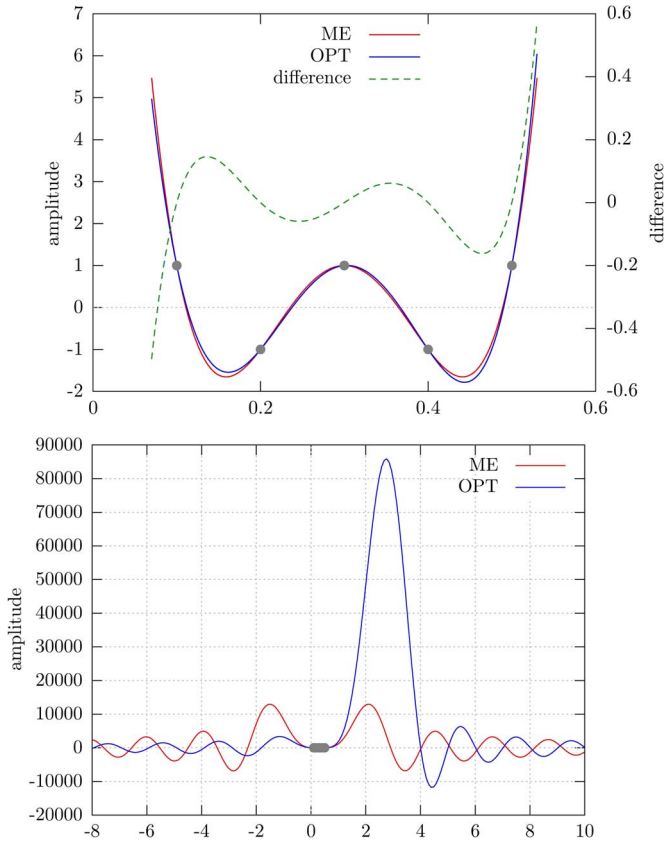

Fig. 7. Superoscillations at 10 times the Nyquist rate obtained using the ME and OPT methods. The set where the energy is optimised corresponds to $I = \{2, 3\}$.

We have used the difference between successive λ_k as the stopping criterion.

V. Examples

We now present some numerical examples that illustrate our approach to the construction of superoscillations and how it compares with the minimum-energy solution. The minimum-energy solution will be denoted by ME and the solution to Problem 1 (reformulated as Problem 2 and Problem 3) will be denoted by OPT.

Example 1: The set J (shown in Fig. 6 top) is uniformly spaced with $T = 0.1$, i.e. 10 times the Nyquist rate. The figure shows the superoscillations obtained with the ME and OPT methods. The set I where the energy concentration is maximised is $I = \{0, 1\}$, which corresponds to the minimal cover of J. The difference between the two solutions inside J cannot be seen at the scale of the plot. Outside J, the value $f(1)$ is larger in the optimised solution than in the minimum-energy (as expected, given the optimisation with respect to I).

The condition number of the ME matrix, the energy of ME solution and its largest coefficient in absolute value were all of the order of 10^8.

The OPT solution was found using the iterative method described in Section IV-I. The condition number of the matrix was 10^5. Three iterations were required to bring the difference in successive values of λ^* below 10^{-8}. The largest coefficient in absolute value had magnitude 10^4 and the energy of the OPT solution was close to that of the ME solution (also about 10^8).

Example 2: The set J (shown in Fig. 7 top) is uniformly spaced with $T = 0.1$, as in Example 1. Thus, the ME solution is the same as in Example 1. In particular, its condition number, energy and largest coefficient in absolute value are the same as in Example 1 (order of magnitude 10^8).

However, the set I where the energy concentration is maximised is now $I = \{2, 3\}$, which does not cover J. This obviously impacts the optimisation problem and as result the difference between the ME solution and the OPT solution becomes large outside J (see Fig. 7 bottom).

Inside J, however, the discrepancy is still small, proving that the superoscillations can still be induced in J, just as before, despite the apparently conflicting requirement of concentrating the energy in a different set I. The numerical stability of the OPT solution is not affected by this and, as we will see, it is even improved.

The OPT solution was found using the iterative method of Section IV-I. The condition number of the matrix was below 220. Two iterations were required to bring the difference in successive values of λ^* below 10^{-8}. The largest coefficient in absolute value is smaller than 10^5, or 3 orders of magnitude below the coefficients of the ME solution.

Example 3: To show how the energy, condition number and magnitude of the largest coefficient vary as functions of T we varied T up to 20 times the Nyquist rate (more precisely, in the interval [0.05,0.15]) and built the corresponding superoscillations by the ME and OPT methods (the latter solved using the iterative method of Section IV-I).

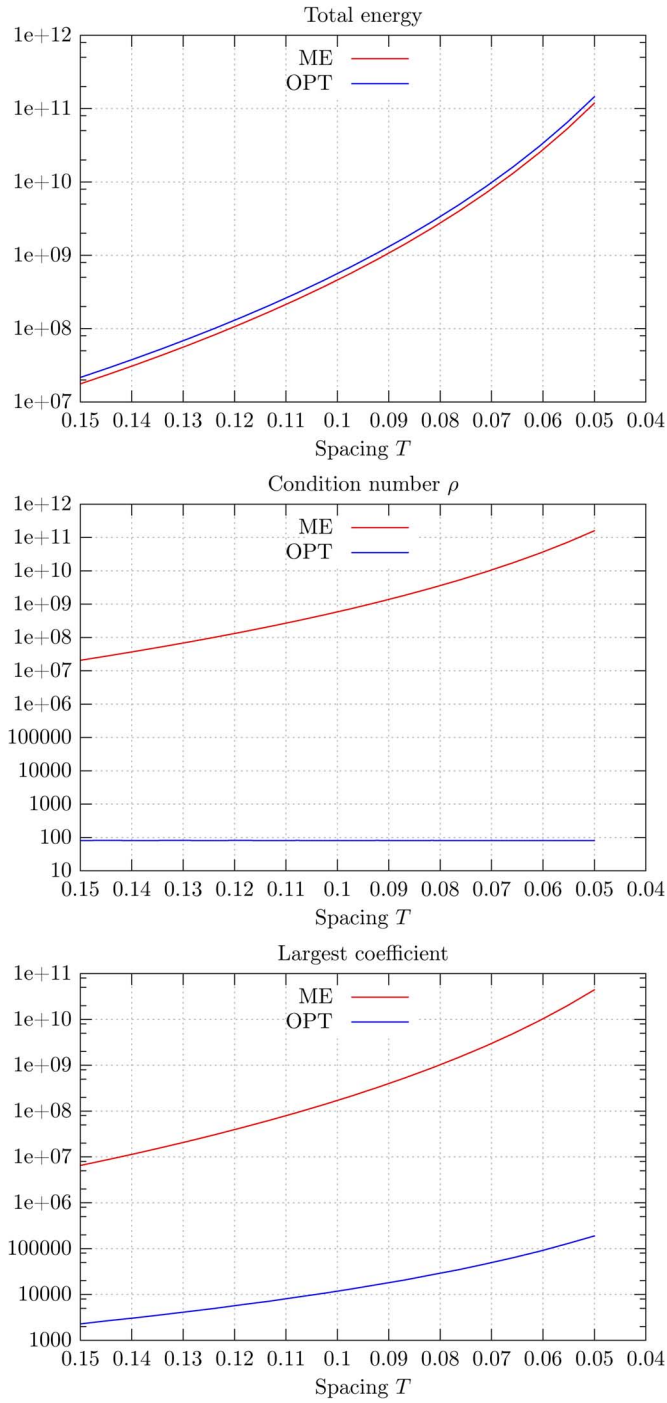

Fig. 8. The evolution with T of the energy, condition number and magnitude of the largest coefficient for the ME and OPT methods.

The energy of the two solutions, as shown in Fig. 8 (top), is comparable. However, the condition numbers of the ME and OPT matrices, the latter being given by

$$\mathcal{X} - \lambda^* I = (B^T B - \lambda^* I),$$

differ by orders of magnitude. Similarly, the largest coefficient in absolute value of the ME and OPT solutions also differ by orders of magnitude.

VI. Conclusion

To build superoscillations with respect to the bandlimit $\mu = 1/2$ Hz, one prescribes the value of a PW function on a sufficiently dense grid of points $J = \{t_1, t_2, \ldots, t_M\}$. Superoscillations of minimum-energy are attractive because they have minimum energy cost, but being linear combinations of the functions $\mathrm{sinc}(t - t_k)$ they are prone to ill-conditioning and require coefficients of very large magnitude. It is impossible to circumvent these disadvantages: the minimum energy interpolant is unique and necessarily given by a linear combination of the functions $\mathrm{sinc}(t - t_k)$ [2], and the points t_k must be near each other in order to build superoscillations.

Our goal was to build superoscillations by combining translates of the sinc function by integers, instead of translates by t_k, to improve the numerical conditioning and reduce the size of the expansion coefficients. This leads to Problem 1, in which we ask for the superoscillating signal with energy maximally concentrated in a given set I. The solution, which exhibits the required superoscillations, can be expressed as a linear combination of translates of the sinc function. It shows interesting numerical behaviour, with relatively small coefficients and condition numbers, and maximal energy concentration in I as desired.

We gave both noniterative and iterative methods to determine the solution. The noniterative solution requires the computation of the roots of a polynomial, a step that the iterative method does not require.

We gave examples that illustrate the behaviour of the new method. It leads to solutions that are linear combinations of orthonormal functions, in contrast with the minimum-energy approach, the solution of which is a linear combination of functions with an ill-conditioned Gram matrix. Because of this, the numerical behaviour of the new method can be much more favourable than the minimum-energy approach.

We also found out that its benefits can be obtained without a significantly increase in the energy cost. If I tightly covers J, the additional energy can be almost negligible. If I grows, or if I is moved away from J, the energy cost increases. Note that in the latter case the superoscillations must be constructed in a certain interval (determined by J) by combining sinc functions (determined by I) centred far from that interval.

If I is moved away from J, the difference between the optimised solution and the minimum-energy solution becomes large. This behaviour is expected since I is the set where the energy concentration is maximised. If I and J are not close to each other, the optimised solution will still exhibit the superoscillations imposed by the constraints on J. However, its energy will also get concentrated in I, a requirement that is not present in the minimum-energy formulation. As I is moved away from the superoscillations, the optimised solution tends to deviate more and more from the minimum-energy solution.

Superoscillations are difficult to build but have important applications, e.g. in antenna theory and subwavelength imaging, where the current limit has been brought down to one sixth of the wavelength [1]. The importance of the applications justify the development of new construction methods, with better numerical properties, capable of going further than the existing ones. The approach presented in this paper is a step towards that goal.

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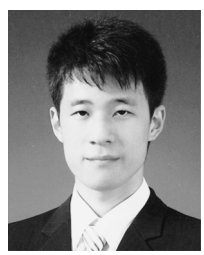

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