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Interpolation and Sampling: E.T. Whittaker, K. Ogura and Their Followers

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Abstract The classical sampling theorem has often been attributed to E.T. Whittaker, but this attribution is not strictly valid. One must carefully distinguish, for example, between the concepts of sampling and of interpolation, and we find that Whittaker worked in interpolation theory, not sampling theory. Again, it has been said that K. Ogura was the first to give a properly rigorous proof of the sampling theorem. We find that he only indicated where the method of proof could be found; we identify what is, in all probability, the proof he had in mind. Ogura states his sampling the-

In Memory of Gen-ichirô Sunouchi (1911–2008), a Great Mathematician and Human Being

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orem as a "converse of Whittaker's theorem", but identifies an error in Whittaker's work.

In order to study these matters in detail we find it necessary to make a complete review of the famous 1915 paper of E.T. Whittaker, and two not so well known papers of Ogura dating from 1920. Since the life and work of Ogura is practically unknown outside Japan, and there he is usually regarded only as an educationalist, we present a detailed overview together with a list of some 70 papers of his which we had to compile. K. Ogura is presented in the setting of mathematics in Japan of the early 20th century.

Finally, because many engineering textbooks refer to Whittaker as a source for the sampling theorem, we make a very brief review of some early introductions of sampling methods in the engineering context, mentioning H. Nyquist, K. Küpfmüller, V. Kotel'nikov, H. Raabe, C.E. Shannon and I. Someya.

Keywords Sampling theorem · Sampling techniques in engineering · Interpolation · Japanese mathematics history

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1 Introduction

Several major questions concerning the early history of sampling theory remain unresolved. One of these is: Where does one find the first rigorous proof of the sampling theorem? By this we mean the representation of a function f in the *cardinal* or *classical* sampling series:

$$f(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} f(n) \frac{(-1)^n}{z-n}, \qquad (1)$$

valid for all functions f belonging to some given function class. It has sometimes been stated that Ogura was the first to give a proof of the classical sampling theorem in 1920 [134]. One of our purposes here is to subject this statement to further review, and, we hope, clarification. In pursuit of this we are led to a study of the paper [81] by E.T. Whittaker and among the many ideas that emerge from this work we find the answer to another fundamental question: where does the notion of frequency content, in particular, band-limitation, first appear in the context of sampling theory? It seems that neither Whittaker's nor Ogura's contributions to sampling theory have been reviewed in depth before.

Sampling theory is known to have emerged from many independent beginnings [16, 17, 30–33, 56, 65], and in preparing a preliminary chapter for the book [18] it was felt necessary to come to a better understanding of its roots than has been achieved up to now. The present study is part of this larger design.

It is necessary to recall the difference between *interpolation* and *sampling*.

Interpolation: Points (n, a_n), $n \in \mathbb{Z}$, are given; one asks for an interpolant, that is, a function with good properties that passes through these points.

Sampling: A class of functions is given (e.g., the Bernstein space B_σ^p); one asks for a representation of its members in sampling series.

Interpolation and Sampling have been called ‘dual concepts’. Roughly speaking, interpolation means that one constructs something; sampling means that one represents something.[1]

There is a sense in which Interpolation and Sampling can be considered *converses* of each other. Indeed, suppose a theorem asserts that if a sequence of points $\{(n, a_n)\}$ is given, and the series

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} a_n \frac{(-1)^n}{z-n}$$

converges, then its sum F is an interpolant with some ‘good’ properties, it can be called an *interpolation theorem*.

A converse situation could be a theorem asserting that if members f of some function class have ‘good’ properties then the series

$$\frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} f(n) \frac{(-1)^n}{z-n}$$

converges and represents $f(z)$. This could be called a *sampling theorem*.

It will be in this sense that we refer to the notion of *converse-type theorems* in what follows.

2 Edmund Taylor Whittaker and His Cardinal Function; Interpolation

2.1 E.T. Whittaker’s Paper of 1915

In [81], his only published paper on the subject, Whittaker introduced what came to be called the *cardinal series* in the English literature:

$$C(x) := \sum_{r=-\infty}^{\infty} \frac{f(a+rw)\sin \frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)}, \tag{2}$$

in the context of interpolation theory.

There follows a section-by-section review of Whittaker’s paper. His methods include both real and complex function theory. He did not state explicitly any theorem in the paper [81]. However, he did set several passages in italic which serve to indicate his intentions, but sometimes these are vague or incomplete; hypotheses are often scattered throughout the paper. He includes no sources in the paper, except for an unreferenced attribution to Poisson, pp. 185–186.

[1] From a conversation that one of us (JRH) had with Wayne Walker many years ago.

2.1.1 Introduction

Whittaker starts by supposing that one knows the values taken by a single valued analytic function f at an arithmetic progression of points $\{a + rw\}$, $r \in \mathbb{Z}$, $a \geq 0$, $w > 0$. These values will not generally determine the function f uniquely since one could add to f any function that vanishes at the points. Whittaker calls the set of all analytic functions that agree on the set $\{a + rw\}$, $r \in \mathbb{Z}$, the *cotabular set*, which we shall denote by $\mathfrak{C}$.

He shows that the sampling series (2) converges to a particular member of this cotabular set which deserves a special name, the *cardinal function*, due to having two important properties; it *has no singularities in the finite part of the plane* (that is, it is an entire function; furthermore, Whittaker shows it to be bounded in strips parallel to the real axis) and it is *free of rapid oscillations* (that is, it is band-limited). To demonstrate these properties he uses a method of removing singularities and high frequency terms from a function, which will be reviewed in Sects. 2.1.2 and 2.1.4.

Whittaker considers the Gauß interpolation formula (but does not call it that).

Let $f(a + rw) = f_r$; denote $f_1 - f_0$ by $\Delta f_{\frac{1}{2}}$, $f_0 - f_{-1}$ by $\Delta f_{-\frac{1}{2}}$, $\Delta f_{\frac{1}{2}} - \Delta f_{-\frac{1}{2}}$ by $\Delta^2 f_0$, etc. Then the Gauß series is

$$f_0 + z\Delta f_{\frac{1}{2}} + \frac{z(z-1)}{2!}\Delta^2 f_0 + \frac{(z+1)z(z-1)}{3!}\Delta^3 f_{\frac{1}{2}} + \cdots . \tag{3}$$

He poses the following two questions:

(1) *Which one of the functions of the cotabular set is represented by the expansion* (3) (*that is, the Gauß formula*)?
(2) *Given any one function $f(x)$ belonging to the cotabular set, is it possible to construct from $f(x)$, by analytic processes, that function of the cotabular set which is represented by the expansion* (3) (*the Gauß formula*)?

Both questions are a little vague. It seems that Whittaker is asking for some properties of the sum of the Gauß expansion. Answers are found in Sect. 2.1.7.

2.1.2 Removal of Singularities from a Function, by Substituting a Cotabular Function for It

Whittaker's procedure for 'removing singularities' from a function F is as follows. He first supposes that F has a pole of order 1; let this be at $z = z_0$ so that near z_0 we can take F to be of the form $F = A(z)/(z - z_0)$, where $A(z)$ is analytic in a neighbourhood of z_0. If F is replaced with f, say, where

$$f(z) := F(z) - A(z)\frac{\sin[\pi(z-a)/w]}{(z-z_0)\sin[\pi(z_0-a)/w]}, \tag{4}$$

then clearly f is cotabular with F and has the effect of subtracting the singular part of F, leaving a function that is analytic in a neighbourhood of z_0. In fact, the process introduces no new singularity, and for all sufficiently large $|z|$ it remains bounded in strips parallel to the real axis.

Whittaker gives a more general construction for cases where the singularity is a pole of higher order or an essential singularity. He now supposes that an entire function f has been obtained by applying such replacement procedures as are necessary to remove its singularities.

2.1.3 Removal of Rapid Oscillations from a Function, by Substituting a Cotabular Function for It

Whittaker shows that if a term has period less than $2w$ an expression can be found which is cotabular with it but has period greater than $2w$. The actual procedure is found in the next subsection.

2.1.4 Introduction of the Cardinal Function

Here Whittaker starts with a function f which is assumed already to be an entire function bounded in strips parallel to the real axis, such that $\{f(n)\}$ is a bounded sequence, and synthesises a new function, the cardinal function, by replacing short-period components with long-period components which are cotabular with them.

In order to remove high frequency terms from f by replacing them with terms whose frequencies[2] lie in the range $[-\frac{\pi}{w}, \frac{\pi}{w}]$ he starts with the representation[3]

$$f(x) = \lim_{k\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \int_0^{\infty} e^{-\lambda k} \cos\lambda(x-t)\, d\lambda\, dt \quad (t \in \mathbb{R}). \tag{5}$$

He breaks up the inner integral into a sum of partial integrals, a typical one (apart from the first) being of the form

$$\int_{(2n-1)\pi/w}^{(2n+1)\pi/w} e^{-\lambda k} \cos\lambda(x-t)\, d\lambda \quad (n \in \mathbb{N}). \tag{6}$$

In this term the periodic factor $\cos\lambda(x-t)$ has frequency λ satisfying

$$(2n-1)\frac{\pi}{w} \le \lambda \le (2n+1)\frac{\pi}{w}.$$

This periodic factor is replaced with

$$\cos\left[\lambda(x-t) - \frac{2\pi n}{w}(x-a)\right], \tag{7}$$

which is cotabular with $x \mapsto \cos\lambda(x-t)$, and has frequency $\lambda - \frac{2n\pi}{w}$ satisfying

$$(2n-1)\frac{\pi}{w} - \frac{2n\pi}{w} \le \lambda - \frac{2n\pi}{w} \le (2n+1)\frac{\pi}{w} - \frac{2n\pi}{w},$$

or

[2] Whittaker uses the scale factor $\frac{1}{w}$ whereas most other writers use w.

[3] Formula (5) seems to be Fourier's integral formula modified by incorporating a summability factor. Whittaker attributes it to Poisson, p. 186.

$$-\frac{\pi}{w} \leq \lambda - \frac{2n\pi}{w} \leq \frac{\pi}{w}.$$

Thus all high frequency terms have been replaced with low frequency terms; that is, terms whose frequencies lie in the band $[-\frac{\pi}{w}, \frac{\pi}{w}]$.

Now the right-hand side of (5) has been replaced with $\lim_{k\to 0} G(x,k)$, where

$$G(x,k) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left\{\int_0^{\pi/w} e^{-\lambda k}\cos[\lambda(x-t)]\,d\lambda \right.$$
$$\left. + \sum_{n=1}^{\infty}\int_{(2n-1)\pi/w}^{(2n+1)\pi/w} e^{-\lambda k}\cos[\lambda(x-t) - 2\pi n(x-a)/w]\,d\lambda\right\} dt.$$

By means of special summations and integrations, Whittaker is able to simplify this expression, and finally taking $\lim_{k\to 0} G(x,k) = G(x)$, say, he obtains

$$G(x) = \sum_{r=-\infty}^{\infty} \frac{f(a+rw)\sin\frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)} \tag{8}$$

as the required function cotabular with f.

Whittaker achieves his purpose, then, of finding a function G with good properties (a Paley-Wiener function in current terminology) which, he says in his conclusion, could take the place of a not-so-well-behaved function f on the strength of being cotabular with it.

Remark 1 We recognise this as an interpolation theorem, not a sampling theorem, since Whittaker starts with points $(n, f(n))$ and constructs an interpolant G.

2.1.5 Examples

Whittaker gives two examples of constructing the cardinal function associated with a given cotabular set.

Example 1 Let $f(0) = 0$, $f(n) = \frac{(-1)^n}{n}$ and $f(-n) = \frac{(-1)^{n+1}}{n}$.

With these values Whittaker sums the series (8) (in which $a = 0$ and $w = 1$) by means of Gamma functions, obtaining

$$G(x) = \frac{\cos\pi x}{x} - \frac{\sin\pi x}{\pi x^2}.$$

Note that this is a Paley-Wiener function in current terminology (in fact it is the derivative of $\operatorname{sinc} x$).

Example 2 Let

$$f(a+rw) = \begin{cases} 1, & r = 1 \bmod 6,\ \text{or } 2 \bmod 6; \\ 0, & r = 0 \bmod 3; \\ -1, & r = 4 \bmod 6,\ \text{or } 5 \bmod 6. \end{cases}$$

This time Whittaker sums the series (8) by using the expansion of the cotangent function in partial fractions to obtain

$$G(x) = \frac{2}{\sqrt{3}} \sin[\pi(x-a)/3w].$$

This function is periodic and belongs to $B^{\infty}_{\pi/(3w)}$ in current terminology. This example was criticised by J.M. Whittaker [82, p. 41] because the sum cannot be analysed by Fourier's theorem. Perhaps J.M.W. overlooked the fact that E.T.W. had explicitly allowed for periodic functions as well, see [81, p. 184].

2.1.6 Direct Proof of the Properties of the Cardinal Function $C(x)$ (cf. (2))

1^0 *$C(x)$ is cotabular with $f(x)$.*
2^0 *$C(x)$ has no singularities in the finite part of the x-plane.*

Whittaker asserts that (8) cannot fail to converge since it is a sum of residues. It is interesting to note that because of this the sum must be understood as a Cauchy principal value.

3^0 *When $C(x)$ is analysed into periodic constituents by Fourier's integral-theorem, all constituents of period less than $2w$ are absent.*[4]

Whittaker's method is to use Fourier's integral-theorem to resolve

$$\frac{\sin\{\frac{\pi}{w}(x-c)\}}{\frac{\pi}{w}(x-c)} \tag{9}$$

into periodic constituents. Since this seems to be the first occurrence of the notion of frequency limitation in the context of sampling and interpolation, it is worth quoting Whittaker's analysis in full.

Note that he does not use the Fourier transform, but Fourier's integral theorem is used in the following form (see e.g., [78, p. 13]). Under suitable conditions on the function f (which in Whittaker's context includes continuity),

$$f(x) = \frac{1}{\pi} \int_0^{\to\infty} \int_{-\infty}^{\infty} f(t) \cos u(x-t)\, dt\, du, \tag{10}$$

the first integral being an improper Lebesgue integral. Then

$$\begin{aligned} \frac{\sin\{\frac{\pi}{w}(x-c)\}}{\frac{\pi}{w}(x-c)} &= \frac{1}{\pi} \int_0^{\to\infty} \int_{-\infty}^{\infty} \frac{\sin\{\frac{\pi}{w}(t-c)\}}{\frac{\pi}{w}(t-c)} \cos u(x-t)\, dt\, du, \\ &= \frac{w}{\pi^2} \int_0^{\to\infty} \int_{-\infty}^{\infty} \frac{\sin y}{y} \cos\left(u\left(x-c-\frac{w}{\pi}y\right)\right) dy\, du. \end{aligned} \tag{11}$$

[4]In more modern notation we have to understand this as meaning that when the Fourier transform of C is taken, none of its non-zero values lie outside the frequency range $[-\pi/w, -\pi/w]$.

Now Whittaker uses two special integrals:

$$\int_{-\infty}^{\infty} \frac{\sin y}{y} \cos \kappa y \, dy = 0 \quad (\kappa > 1);$$

and

$$\int_{-\infty}^{\infty} \frac{\sin y}{y} \sin \kappa y \, dy = 0 \quad (\text{all } \kappa).$$

Hence from (11) the inner integral vanishes if $uw > \pi$. Thus, (9) contains no constituent of the type $\cos u(x-t)$ where $uw > \pi$, that is, no constituent with period less than $2w$.

Whittaker concludes that "The theorem being thus seen to be true for every single term of the series (8), is consequently true for the cardinal function as a whole".

Next, Whittaker gives a formula by means of which a whole family of analytic functions that are cotabular with $C(x)$ can be constructed. It is:

$$\sum_{r=-\infty}^{\infty} e^{-c(x-a-rw)^{2m}} f(a+rw) \left\{ \frac{\sin \frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)} \right\}^n \tag{12}$$

for any positive constant c and any positive integers m and n.

2.1.7 Solution of the Questions Proposed in Sect. 2.1.1

Using an ingenious contour integration Whittaker shows that the Gauß series represents the cardinal function, but this is not strictly relevant to the present discussion.

2.1.8 Conclusion

Whittaker speculates on the place to be taken by his cardinal series in the general theory of expansion of functions, but again, there is nothing strictly relevant to the present discussion.

2.2 A Summary of the Relevant Parts of Whittaker's Paper

The main results in [81] appear to be the material in his Sect. 2.1.6 where he claims that if $f(z)$ is analytic and $\{f(n)\}$ is a bounded sequence, then the sum G of the series (8) has the properties Sect. 2.1.6, 1^0, 2^0 and 3^0. This is an interpolation result, but it is false in general as Ogura showed by the following counter-example [134, p. 65], repeated in [135, p. 138].

Example 3 Let φ be analytic and for $n \in \mathbb{Z}$ be such that

$$\varphi(n) = \begin{cases} (-1)^{n+1}, & n > 0; \\ 0, & n = 0; \\ (-1)^n, & n < 0. \end{cases}$$

Such functions can be constructed using (12). The series (8) (with $a = 0$, $w = 1$) is divergent, e.g., when $x = \frac{1}{2}$.

Whittaker did not state or prove the classical sampling theorem. However, he clearly understood the significance of frequency content and its restriction to what would now be called a "frequency band"; in fact he seems to have been the first to introduce this concept in the context of interpolation. Whittaker's work is not rigorous by current standards, but the *ideas* in his paper have historical value.

2.3 Edmund Taylor Whittaker (1873–1956); His Vita

Whittaker's family came from the north of England. His father, John Whittaker, was a man of independent means, whose family had been established in Lancashire since the late middle ages. His mother was Selina Taylor, whose father was a medical doctor with a practice near Manchester.

Whittaker did not go to school until he entered Manchester Grammar School at the age of eleven. Until then he had been taught solely by his mother. He entered Trinity College, Cambridge, with a scholarship in 1892 where his outstanding work in applied mathematics earned him several awards and scholarships. He graduated in 1895 and was elected fellow of Trinity College in 1896. At this time his interests turned more towards pure mathematics. Among those who attended his lectures at Cambridge were H.M. Bateman, A.S. Eddington, G.H. Hardy, J.H. Jeans, J.E. Littlewood and G.N. Watson.

In 1901 he married Mary Boyd. They had three sons and two daughters. The middle son was the mathematician John McNaghten Whittaker; the eldest daughter Beatrice Mary Whittaker married the mathematician E.T. Copson. Sampling and interpolation seems to have been quite a family affair; Whittaker and his son and his son-in-law all published research in this topic!

Whittaker had a strong interest in astronomy. He was a member of the Royal Astronomical Society and served as its secretary from 1901 to 1906. He was appointed Royal Astronomer of Ireland in 1906 and at the same time Professor of Astronomy at the University of Dublin, where his duties consisted of lecturing in mathematical physics.

In 1912 Whittaker was appointed professor at Edinburgh and remained there for the rest of his carrier. He would normally have retired in 1943 but because of World War II he agreed to extend his professorship for a further three years. His house in George Square was a great centre for social and intellectual gatherings. In this and in all his work he was strengthened and supported by the gracious presence of his wife Mary.

Whittaker founded the Edinburgh Mathematical Laboratory to strengthen the practical side of his interest in numerical analysis. Other interests included, on the applied side, interpolation, celestial mechanics, relativity, electromagnetic theory, actuarial mathematics and the history of applied mathematics and physics, and, on the pure side, algebraic functions, automorphic functions, special functions (especially Bessel functions) and partial differential equations.

He received many honours and prizes and held many lectureships at other universities throughout his career. He served on the Council of several learned societies, including the Royal Society to which he was elected Fellow in 1905. He received honorary degrees from the universities of California, Dublin, Manchester, St Andrews and from the National University of Ireland.

He was a Foreign Member of the Accademia dei Lincei, the Societa Reale di Napoli, the American Philosophical Society, the Académie Royal de Belgique, the Benares Mathematical Society, the Indian Mathematical Society and Corresponding Member of the Académie Française des Sciences.

Whittaker joined the Catholic Church in 1930. In 1935 Pope Pius XI conferred on him the Cross *Pro Ecclesia et Pontifice*, and appointed him to the Pontifical Academy of Sciences in 1936.

He was knighted in 1945. See also "The MacTutor History of Mathematics archive" http://www.gap-system.org/ history/.

3 Kinnosuke Ogura and the Sampling Theorem

To understand the work of Ogura (see Fig. 1) in regard to sampling, the authors find it appropriate to first present a scanned copy of Ogura's paper [134], "On a Certain Transcendental …", namely the page containing his first theorem, thus the result "I" below, together with the two pages of Lindelöf's book [51], cited by Ogura with respect to the proof of his theorem.

3.1 Ogura's Sampling Theorem; a Rigorous Proof Emerges

Ogura considers functions $f(z)$ with the properties:

- (i) f is a "transcendental integral function"; this is the older terminology for an entire function;
- (ii) "f does not become infinite even at $z=\infty$ so long as z is real", can be interpreted as f being bounded on $\mathbb{R}$;
- (iii) "$|f(z)|$ becomes infinite to a lower order than $e^{\pi r|\sin\theta|}$" means that $|f(z)| < e^{\sigma r|\sin\theta|}$, $z=re^{i\theta}$, $\sigma<\pi$, for all sufficiently large $|z|$.

He calls a function satisfying these three conditions a *cardinal function*. Observe that it is not the same as E.T. Whittaker's definition of cardinal function (see [81]).

Thus Ogura deals with entire functions of exponential type σ which are bounded on the real axis rather than being square integrable there, namely with B^∞_σ functions for $\sigma<\pi$. His first theorem [134, p. 64] can be stated as follows:

Theorem 1 *The cardinal function can be constructed analytically when its values* $\{f(n)\}$, $n\in\mathbb{Z}$, *are known. In fact,*

$$f(z)=\frac{\sin\pi z}{\pi}\sum_{k=-\infty}^{\infty}(-1)^k\frac{f(k)}{z-k}=\frac{\sin\pi z}{\pi}\lim_{m\to\infty}\sum_{k=-m}^{m}(-1)^k\frac{f(k)}{z-k}. \qquad (13)$$

On a Certain Transcendental Integral Function In the Theory of Interpolation,

by

KINNOSUKE OGURA, Ôsaka.

1. Consider a transcendental integral function $f(z)$ of the complex variable z ($z=re^{i\theta}$) such that it does not become infinite even at $z=\infty$ so long as z is real, and $|f(z)|$ becomes infinite to a lower order than

$$e^{\pi r|\sin\theta|}$$

when z approaches infinity. Such a function will be called the *cardinal function.*

The notion of cardinal functions is of fundamental importance in the theory of interpolation from the reason that:

I. It can be constructed analytically in a simple manner when the values

$$f(0),\ f(+1),\ f(-1),\ f(+2),\ f(-2),\ \cdots\cdots$$

are given: in fact,

$$f(z)=\frac{\sin\pi z}{\pi}\sum_{\nu=-\infty}^{+\infty}(-1)^{\nu}\frac{f(\nu)}{z-\nu}\ \ (^1);$$

II. It may be represented by the Gauss formula of interpolation

$$(1)\quad f(z)=f_0+\frac{z}{1!}\delta f_{\frac{1}{2}}+\frac{z(z-1)}{2!}\delta^2 f_0+\frac{(z+1)z(z-1)}{3!}\delta^3 f_{\frac{1}{2}} +\frac{(z+1)z(z-1)(z-2)}{4!}\delta^4 f_0+\cdots\cdots(^2)$$

Now we add the following remark: Prof. Whittaker stated that if

(1) E. T. Whittaker, "On the functions which are represented by the expansions of the interpolation-theory," Proc. Roy. Soc. Edinburgh (1915), pp. 181-194; especially pp. 182-187. Although his method is very interesting and instructive for practical work, it is not free from inaccuracies. A proof of this theorem, which is simple and rigorous, can be obtained in E. Lindelöf, Calcul des résidus (1905), p. 53.

(2) Whittaker, loc. cit., pp. 191-2. We use of the following scheme:

CHAPITRE III.

FORMULES SOMMATOIRES TIRÉES DU CALCUL DES RÉSIDUS.

I. — *Recherches de Cauchy. Transformations diverses des formules générales.*

26. Le calcul des résidus permet d'exprimer par une intégrale définie la somme des valeurs que prend une fonction analytique $f(z)$ pour des valeurs entières successives de la variable z. En effet, on a vu (p. 39) que la fonction $\pi \cot \pi z$ admet tout nombre entier ν comme pôle simple de résidu *un*, et il en résulte que la valeur $f(\nu)$ est égale au résidu de l'expression $\pi \cot \pi z\, f(z)$ relatif au point $z = \nu$, pourvu que $f(z)$ y soit holomorphe. Traçons donc un contour fermé simple C enveloppant les points $m, m+1, \ldots, n$, mais laissant à l'extérieur tout autre point dont l'affixe est un nombre entier, et supposons que, à l'intérieur de ce contour, la fonction $f(z)$ soit uniforme et ne présente qu'un nombre fini de points singuliers, distincts des points $m, m+1, \ldots, n$, en étant d'ailleurs holomorphe sur C. Dans ces conditions, on conclut du théorème général des résidus, en utilisant la notation qui a été expliquée page 38,

$$(1) \qquad \frac{1}{2\pi i}\int_C \pi \cot \pi z\, f(z)\, dz = \sum_m^n f(\nu) + \mathcal{E}_C\, \pi \cot \pi z (f(z)),$$

d'où

$$(2) \qquad \sum_m^n f(\nu) = \frac{1}{2\pi i}\int_C \pi \cot \pi z\, f(z)\, dz - \mathcal{E}_C\, \pi \cot \pi z (f(z)).$$

Si, en particulier, la fonction $f(z)$ est holomorphe à l'intérieur de C, le dernier terme de cette égalité disparaîtra.

En observant que le résidu de la fonction $\frac{\pi}{\sin \pi z}$ relatif au pôle $z = \nu$ est égal à $(-1)^\nu$, on arrive de même à la formule

$$(3) \qquad \sum_m^n (-1)^\nu f(\nu) = \frac{1}{2\pi i} \int_C \frac{\pi}{\sin \pi z} f(z)\, dz - \mathcal{E}_C \frac{\pi}{\sin \pi z} ((f(z))),$$

les hypothèses restant les mêmes que ci-dessus.

Les formules qu'on vient d'écrire conduisent facilement à la sommation de certaines séries particulières, renfermant un nombre infini de termes ([1]).

Soit, par exemple, $f(z)$ une fonction rationnelle ou méromorphe, telle que le résidu intégral de l'expression $\pi \cot \pi z\, f(z)$ s'annule (*voir* le n° 17). En prenant pour le contour C un cercle ayant l'origine comme centre et en faisant croître indéfiniment le rayon de ce cercle par des valeurs convenablement choisies, l'intégrale (1) tendra vers zéro, de sorte que l'égalité (2) deviendra

$$(4) \qquad \sum_{-\infty}^{+\infty} f(\nu) = -\mathcal{E}\, \pi \cot \pi z ((f(z))).$$

Si le résidu intégral de l'expression $\frac{\pi}{\sin \pi z} f(z)$ se réduit à zéro, on déduit de même de l'égalité (3) la formule

$$(5) \qquad \sum_{-\infty}^{+\infty} (-1)^\nu f(\nu) = -\mathcal{E} \frac{\pi}{\sin \pi z} ((f(z))).$$

On suppose, bien entendu, que les séries qui figurent dans les égalités (4) et (5) soient convergentes et que leurs termes soient réunis en groupes comme l'indique l'égalité (2), page 30.

Comme les modules $|\cot \pi z|$ et $\left|\frac{1}{\sin \pi z}\right|$ restent au-dessous d'une limite finie sur les cercles $|z| = \nu - \frac{1}{2}$, $\nu = 1, 2, \ldots$ (*voir* la note de la page 32), on peut conclure des résultats établis au n° 17 que les conditions énoncées ci-dessus sont vérifiées toutes les fois que $f(z)$ est une fonction rationnelle admettant le point à l'infini

([1]) Cf. *Œuvres de Cauchy,* série II. t. VII, 1827, p. 345-362.

Proof Lindelöf's general equation [51, p. 53, (3)], which he applies to different examples, is as follows:

$$\sum_{k=-m}^{m}(-1)^k g(k)=\frac{1}{2\pi i}\int_C \frac{\pi}{\sin\pi z}g(z)\,dz-\sum_C \frac{\pi}{\sin\pi z}\big(g(z)\big), \tag{14}$$

where C is a suitable contour and the last term indicates the sum of residues of the expression at the poles of g within C.

Now Ogura probably had in mind to apply this formula to $g(z)=f(z)/(x-z)$ with the contour C_m, the circle centred at the origin, of radius $\rho_m=m+\frac{1}{2}$, $m\in\mathbb{N}$. The residue of $\pi f(z)/(x-z)\sin\pi z$ at $z=x$ is easily calculated to be $-\pi f(x)/\sin\pi x$. On taking the limit as $m\to\infty$, (14) becomes

$$\frac{\pi}{\sin\pi x}f(x)=\lim_{m\to\infty}\sum_{k=-m}^{m}(-1)^k\frac{f(k)}{x-k}-\lim_{m\to\infty}\frac{1}{2\pi i}\int_{C_m}\frac{\pi}{(x-\zeta)\sin\pi\zeta}f(\zeta)\,d\zeta. \tag{15}$$

This will give the required result if the limit as $m\to\infty$ of these integrals vanishes, i.e. if

$$\lim_{m\to\infty}\frac{1}{2\pi i}\int_0^{2\pi}\frac{\rho_m e^{i\theta}f(\rho_m e^{i\theta})}{(x-\rho_m e^{i\theta})\sin(\pi\rho_m e^{i\theta})}\,d\theta=0. \tag{16}$$

Basic now is an estimate from below for the modulus of the sine-function $|\sin\pi\zeta|$ on the circles C_m, where $\zeta=\xi+i\eta$ with $\xi^2+\eta^2=\rho_m^2$, i. e. $\zeta\in C_m$; see lemma below.

To establish (16) we now use a uniform convergence argument, one which Ogura could have had in mind. First note that $|x-\rho_m e^{i\theta}|\geq\rho_m-|x|$ for m sufficiently large. Thus, when condition (iii) and (19) (below) are taken into account, the integrands in (16) are no larger than

$$K\frac{\rho_m}{\rho_m-|x|}e^{-\pi\rho_m|\sin\theta|}f(\rho_m e^{i\theta})\leq K\frac{\rho_m}{\rho_m-|x|}e^{-\pi\rho_m|\sin\theta|}e^{\sigma\rho_m|\sin\theta|}\leq K' \tag{17}$$

where K, K' are suitable constants.

Of course these inequalities do not imply uniform convergence to zero of the integrands in (16) on the whole interval $[0,2\pi]$; but they do so on the subintervals $[\delta,\pi-\delta]$ and $[\pi+\delta,2\pi-\delta]$ for every $0<\delta<\pi$ in view of $\sigma<\pi$.

Let us split up the integral as follows:

$$\begin{aligned}&\int_0^{2\pi}\frac{\rho_m e^{i\theta}f(\rho_m e^{i\theta})}{(x-\rho_m e^{i\theta})\sin(\pi\rho_m e^{i\theta})}\,d\theta\\&\quad=\Big(\int_\delta^{\pi-\delta}+\int_{\pi+\delta}^{2\pi-\delta}+\int_0^{\delta}+\int_{\pi-\delta}^{\pi+\delta}+\int_{2\pi-\delta}^{2\pi}\Big)\frac{\rho_m e^{i\theta}f(\rho_m e^{i\theta})}{(x-\rho_m e^{i\theta})\sin(\pi\rho_m e^{i\theta})}\,d\theta.\end{aligned}$$

Then the first two integrals tend to zero for $m\to\infty$ in view of the uniform convergence of the integrands. The latter three integrals can be made smaller than any given

$\varepsilon > 0$ by choosing δ appropriately, e.g. $\delta = \varepsilon/4K'$; it works since the integrands are uniformly bounded by K'. □

Observe that another argument establishing (16) follows by integrating inequality (17), thus

$$\left|\int_0^{2\pi} \frac{\rho_m e^{i\theta} f(\rho_m e^{i\theta})}{(x-\rho_m e^{i\theta})\sin(\pi\rho_m e^{i\theta})}\,d\theta\right| \leq \frac{K\rho_m}{\rho_m - |x|}\int_0^{2\pi} e^{-\pi\rho_m|\sin\theta|}e^{\sigma\rho_m|\sin\theta|}\,d\theta. \tag{18}$$

Since $\sin\theta \geq 2\theta/\pi$ for $\theta\in[0,\pi/2]$, as follows by a convexity argument, this yields

$$\int_0^{2\pi} e^{-\pi\rho_m|\sin\theta|}e^{\sigma\rho_m|\sin\theta|}\,d\theta = 4\int_0^{\pi/2} e^{-(\pi-\sigma)\rho_m\sin\theta}\,d\theta$$
$$\leq 4\int_0^{\pi/2} e^{-2(1-\sigma/\pi)\rho_m\theta}\,d\theta \leq \frac{2}{(1-\sigma/\pi)\rho_m}.$$

This shows that the left-hand side of (18) approaches zero as $m\to\infty$. (It would even approach zero if instead of K we had a quantity satisfying $o(\rho_m)$ as $m\to\infty$.)

Even a third argument could have been used in this respect, namely Lebesgue's dominated convergence theorem. Indeed, the integrands of (16) tend to zero for all $\theta\in[0,2\pi]$ except for $\theta=0$ and $\theta=\pi$, and are dominated by K' in view of (17).

Lemma *There exists a constant $C>0$ such that*

$$\frac{1}{|\sin\pi z|} \leq Ce^{-\pi|y|} = Ce^{-\pi\rho_m|\sin\theta|} \leq C \quad (z = x+iy = \rho_m e^{i\theta}\in C_m). \tag{19}$$

Proof First consider those x for which $m+\frac{1}{4}\leq |x| \leq m+\frac{1}{2} = \rho_m$. Since $|\cos|$ is non increasing on $[\pi(m+\frac{1}{4}), \pi\rho_m]$,

$$|\cos\pi x| \leq \left|\cos\pi\left(m+\frac{1}{4}\right)\right| = \frac{1}{\sqrt{2}}.$$

Thus

$$|\sin\pi z|^2 = \cosh^2\pi|y| - \cos^2\pi|x| \geq \frac{1}{4}\{e^{2\pi|y|}+e^{-2\pi|y|}+2\} - \frac{1}{2} \geq \frac{1}{4}e^{2\pi|y|},$$

yielding that

$$|\sin\pi z| \geq \frac{1}{2}e^{\pi|y|} \quad \left(z = x+iy\in C_m;\ m+\frac{1}{4}\leq|x|\leq\rho_m\right).$$

Now to those x with $|x| \le m + \frac{1}{4}$. Then $|y|^2 = \rho_m^2 - |x|^2 \ge \rho_m^2 - (m + \frac{1}{4})^2 \ge \frac{1}{2}$, i.e. $|y| \ge \frac{1}{\sqrt{2}}$. So we obtain

$$|\sin \pi z|^2 = \cosh^2 \pi |y| - \cos^2 \pi |x| \ge \cosh^2 \pi |y| - 1 \ge \frac{1}{4} e^{2\pi |y|} - \frac{1}{2} \ge \frac{1}{8} e^{2\pi |y|},$$

which gives the inequality

$$|\sin \pi z| \ge \frac{1}{\sqrt{8}} e^{\pi |y|} \quad \left(z = x + iy \in C_m;\ |x| \le m + \frac{1}{4} \right).$$

Altogether we have established (19) with $C = \sqrt{8}$. □

Ogura's first theorem is in all respects the sampling theorem established in, e.g., Butzer-Ries-Stens [11], stating:

If $f \in B_{\pi\tau}^{\infty}$ for some $0 < \tau < w$, then

$$f(z) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc}(wz - k) = \frac{\sin \pi z}{\pi} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \frac{(-1)^k}{z - k} \quad (z \in \mathbb{C}),$$

where the two series, understood as Cauchy principal values, are uniformly convergent on each bounded domain of $\mathbb{C}$.

Here τ has to be strictly less than w as follows from the counterexample $f(z) = \cos \pi wz \in B_{\pi w}^{\infty}$. This corresponds to the condition $\sigma < \pi$ in Ogura's theorem. It should be pointed out that this particular sampling theorem can be proved, as far as we know, only by methods of complex analysis.

The proof in [11] was also based on Cauchy's integral formula and residue methods. However, instead of a circular contour a square of side length $2m + 1$ centred at the origin was used.

Note that Jagerman-Fogel [37] already applied such an approach in 1956 under somewhat different conditions.

Churkin-Jakowlev-Wunsch [22] and Wunsch [84] worked with a circular contour but their proofs are not correct since they made use of the inequality $|\sin \alpha z| > \frac{1}{2} \exp(\alpha r |\sin \theta|)$ for $z = re^{i\theta}$, which fails e.g. for $\theta = \frac{\pi}{2}$, $\alpha > 0$.

3.2 A "Converse" to Ogura's Sampling Theorem

Ogura also presents a "converse" to his Theorem 1, found in [136, p. 240]. It states

Theorem 2 *Let $\sum_{k=-\infty}^{\infty} |f(k)| < \infty$. Then*

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{f(k)}{z - k} \tag{20}$$

is cardinal such that F is cotablular with f, i.e. $F(k)=f(k)$, $k\in\mathbb{Z}$, and can be represented for some $g\in C[-\pi,\pi]$ as

$$F(z)=\int_{-\pi}^{\pi} g(u)e^{-izu}\,du \qquad (z\in\mathbb{C}).$$

Ogura calls it a "converse of Whittaker's Theorem", p. 240. Since the integral representation implies that F satisfies the conditions (i)–(iii) it is indeed a converse to Theorem 1. Whittaker, however, did not state a result of this type. As we have seen in Sect. 2, his result is an interpolation formula as is (20). So it is not clear why Ogura speaks of a "converse of Whittaker's Theorem".

As to the differences between Whittaker's and Ogura's interpolation formula, Whittaker assumes that f is an entire function bounded in strips parallel to the real axis, which implies, in particular, that $(f(k))_{k\in\mathbb{Z}}$ is a bounded sequence. Ogura in his Theorem 2 requires that this sequence is absolutely summable without any other assumptions upon f. He shows that Whittaker's boundedness assumption is indeed too weak and that his result is false by giving a counterexample.

3.3 First Conclusions

Ogura's result "I" in his paper [134], given in our Theorem 1, is the first clear statement of the classical sampling theorem that the authors have ever met. Moreover, the hypotheses and the formulation are both correct. In his footnote (1) to his result Ogura cites Whittaker's paper [81], especially pp. 182–187, and adds: "Although his method is very interesting and instructive for practical work, it is not free from inaccuracies. A proof of this theorem, which is simple and rigorous, can be obtained in Lindelöf, Calcul des résidus [51], p. 53".

In trying to follow Ogura's "simple proof" the authors applied Lindelöf's equation (3) on his p. 53, to give (15) above. This immediately yields Theorem 1 provided the integral in (15) tends to zero. The authors have presented three proofs of this fact, the third of which is probably the simplest. But a rigorous approach was by no means the usual practice around 1920. Even as late as 1971 proofs of the convergence to zero of the integral in question given in [22, 84] were not fully accurate.

Ogura's second result in the sampling area, namely Theorem 2, although quite interesting, does not have the same historical value.

Unfortunately, Ogura's work in sampling remained unrecognised for several generations. Only in 1992 was his fundamental theorem brought to the attention of the mathematical community (see [12]), with the observation that Ogura referred erroneously to Whittaker as the initiator of the sampling theorem.

Sampling series are present in the work Cauchy, but only under suitable interpretation (see, e.g., [17, 32, 49]). It is interesting that Lindelöf mentions Cauchy as a source for his method (see the scan of his p. 53).

4 K. Ogura, His Life and Work; a Survey

4.1 Ogura's Vita

Kinnosuke Ogura was born in 1885 in Sakata Funaba-cho, Yamagata Prefecture, Japan, as the eldest son of Kinzou Ogura. He began his studies in 1902 at the Tokyo Butsuri Gakko[5], now the prestigious Tokyo University of Science.

After graduating from there in 1905, he studied chemistry at the Imperial University of Tokyo as a non-regular student, married in 1906, and then worked with his father for a while in his home town. He was not a regular student there perhaps because his father was a wealthy business man in a rural city and he should take over the business later. However, he also had been studying mathematics at the same time under the guidance of Tsuruichi Hayashi (1873–1935), even writing in 1910 a book on infinite series. Under Hayashi he received in 1911 an assistantship at Tôhoku University, and in 1916 the Doctoral Degree[6] from Tôhoku University with the thesis "Paths of a mass point in conservative fields". In the next year he received a position as researcher at Siomi Institute of Physical and Chemical Research[7] in Osaka.

In 1919 he went abroad to Europe for study purposes and stayed mainly in France. After returning to Japan 1922 he concerned himself with Einstein's visit to Japan in the Fall of 1922, published a book on the elements of mathematical education in 1924, another on statistics, and in 1925 became President of the Siomi Institute. Simultaneously he was professor at the College of medicine,[8] Osaka, from 1917 to 1926, when he had to retire from the university due to poor health; already in 1906 it is mentioned [1, 24] that he was not in good health, likewise in 1941, 1945 and again in 1951.

For the years 1927 to 1931 it is reported [1, 24, 159] that he read a book of Tolstoi, translated several books and became interested in history of mathematics, starting by gathering old Japanese mathematical books, and taught also at the University of Hiroshima between 1931 and 1934. In 1932 he became Lecturer in mathematics at Osaka Imperial University and taught there off and on till 1943, when he retired.

From 1940 to 1943 he was President of Tokyo Butsuri Gakko, his first Alma Mater, in 1948 President of the History of Science Society of Japan, and finally in 1962 President of the Society of Japanese Mathematical History. He passed away in 1962 at the age of 77 in Sakata Funaba-cho, his birthplace.

Ogura's first two papers, on differential geometry appeared 1908 [85, 86], and on its applications to mathematical physics in 1911 [95]. His majors were in fact differential geometry and applied analysis. Later his activities turned to mathematical

[5] A private institution founded 1881 as The Tokyo College of Physics by 21 graduates of the Department of Physics of the pre-University of Tokyo; see [40]. The founders had studied physics mainly in France.

[6] As Masaru Kamada has kindly pointed out, the Tokyo University of Science celebrated its 125th Anniversary in 2006, and Prof. Ogura is featured as the first D.Sc. among the graduates from private universities. D.Sc. in Japan at the time was a fame comparable with being a minister.

[7] The Institute was set up by Dr. Hantaro Nagaoka (1865–1950) (see [35]); it was the antecedent of the Faculty of Science of Osaka Imperial University, founded 1931.

[8] The Osaka Prefectural Medical College, with a long tradition, was a further antecedent of Osaka University.

Fig. 1 Photos of Ogura. The first one was probably taken in 1916 when he received the D.Sc. from Tôhoku University, the second from a web site of Yamagata prefecture, celebrating their local hero

education and history of both Western and Japanese mathematics, fields in which his interests were deep and wide; they extended over to social problems in economics (being influenced by Marxist historiography), also in the thirties to those against fascism. He was thus a man of very broad interests, including applied mathematics, statistics, philosophy and culture, and was able to make illuminating contributions from the viewpoint of a scientist and a humanist.

Ogura was the author of at least 70 papers in mathematics; 42 of these papers alone were published in the renowned Tôhoku Mathematical Journal in the short period between 1911 and 1923. Furthermore, he was the author of some 35 books in the areas described, especially ones relating to social, educational and scientific problems in connection with human beings (16 of these books were published by Keisoshobo and these books together with other related books on Ogura are in the Yamagata Prefecture Library); see also [1, 24, 71, 159] as well as Poggendorff [66], vol. V, pp. 920/921. We also find his contributions cited with admiration in Google and other internet search engines.

Considering that Ogura began studying mathematics by himself only after 1905 in his early twenties, Hayashi, who had a gift for nurturing young mathematicians, must have recognised his talents as he accepted him as his assistant when Tôhoku university opened in 1911. It is reported that since Ogura received his education at a private university in Tokyo and not at the Imperial one, he did not obtain a good initial academic position; in fact he was not able to become full professor, nor even lecturer or associate professor, for many years. However, he was finally the first to make the grade. On top, the private institution specialized not in mathematics but physics. But that this education was a solid one is revealed by his work in the area of relativity theory already in 1913. Thus was he a mathematician, physicist or a mathematical physicist at the time? Interdisciplinary scientists, although so important in research, may have their problems. Since Hayashi stood in best of terms with the Tôhoku physicists—world famous ones at the time being Hantaro Nagaoka and his student Kotaro Honda (1870–1954)—Ogura seems to have been lucky in this respect. A further prerequisite for being appointed to a top level mathematical position at an Imperial university was generally a period of further study abroad, such as Göttingen, Berlin or Paris (see [23, p. 430]).

On checking Ogura's bibliography, it seems that his preference was for France and that he had hoped to be there by 1913 at the latest. Ogura's academic uncles, Takagi and Fujiwara, had both studied in Europe, the former returning to Japan in 1901 and the latter in 1911. Fujiwara had in fact studied in all three cities (but only received his doctorate in 1916 at age the age of 31). With World War I beginning in 1914 Ogura probably decided to study for his doctorate at Tôhoku university—completing it in 1916—and it was not until 1919 that he was able to leave for France, at the age of 34.

4.2 Ogura in Europe

Ogura left for Europe in late 1919, arrived in Paris in January 1920 and stayed in France till December 1921. He is reported to have studied French until August; having translated in 1913 Rouché's two volume work "Traité de Géométrie" (first edition 1883, seventh 1900; a total of 1212 pages), he certainly was fully familiar with mathematical French. Having also read many French novels and books he was by no means a newcomer in written French either. Thus 1920 he attended the lectures of Émil Borel (1871–1956) and of Paul Langevin (1872–1946), the former, a student of Henri Poincaré, Professor of the Theory of Functions at the Sorbonne, the latter Professor of Physics there, both since 1909.

Especially on account of his monographs on entire functions of 1900 and on infinite series of 1901, Borel was no doubt one of Europe's great experts in mathematical analysis. In fact in 1899 he also studied the convergence of the cardinal series for all $t \in \mathbb{C}$. Since Ogura had written two books on infinite series, one together with Hayashi in 1912, both had at least this common interest. Langevin, the prominent physicist, a student of Pierre Curie, who developed Langevin dynamics and the Langevin equation [50], had all the tools needed for proposing the special theory of relativity before Einstein; but Einstein beat him to it and proposed it, as Einstein himself reported.

As to Borel, Ogura can be said to have been on par with him. In fact, in his paper [6] Borel noted with pleasure Ogura's adoption of the term "kinematic space" in Ogura's paper of 1913 [99] on the Lorentz transformation. But he regretted that Ogura had not yet "seen all the advantages" of the law of velocity addition in its original, non-commutative form adopted by Borel in his newer paper [7]. Borel, shortly after 1909 had indeed taken up the study of relativity theory, as he said "in the form given by the late Minkowski", which he communicated in the two papers in question in the Comptes Rendus.

The early twentieth century witnessed the development of Einstein's special theory of relativity (1905) and the extension to his general theory (1915), as well as of non-Euclidean and n-dimensional geometries. Two lectures by Hermann Minkowski (1864–1909), Einstein's mathematical teacher, played a basic role in this development (Göttingen, Nov. 1907; Cologne, Sept. 1908). Borel first concerned his thoughts upon the principle of relativity before his discovery of the notion of proper time (Eigenzeit) with which he eventually elaborated the structure of space-time in terms of four-dimensional point trajectories (or "world lines") and a Lorentz-covariant mechanics. In fact he began by pointing out the differential equations—the Lorentz transformation—used by Hendrik A. Lorentz (1853–1928) as the foundation of his successful theory of electrons. After Minkowski's death [80] Arnold Sommerfeld[9] (1868–1951) took up the matter (Salzburg, Sept. 1909) and so did Vladimir Varicak (1865–1942) (Karlsruhe, 1911). Already in 1913 there was a priority dispute between Borel, Langevin, and Sommerfeld regarding the application of non-Euclidean geometry to relativity. It is in this respect that Borel [6] had commented on Ogura's paper. While in Paris, Ogura actually wrote five papers in the broad area of relativity theory, Borel communicated them to the Comptes Rendus (see papers [142–147]). Reviews of two of these were made by Philipp Frank (1884–1966), the inaugurator of the non-Euclidean style.

Ogura also attended the very exciting meetings of the universal Jacques Hadamard (1865–1963), Professor at the Collège de France and the École Polytechnique at the time, with Masazo Sono (1886–1969) and Yayotaro Abe (1883–1951).

Of interest is that Ogura also met Teiji Takagi, his academic uncle, while the latter lectured on his field equation at the ICM held at Strasbourg in 1920. Ogura himself also read a paper [141] there.

Moreover, he met Matsusaburô Fujiwara, a student of Fujisawa, when Fujiwara was visiting Paris in 1921. Fujiwara and Hayashi were the founding mathematics professors of Tôhoku University. It is also reported that he met Fréchet and the Americans L.P. Eisenhart and D.E. Smith while in Europe.

The European tradition of science and culture which Ogura observed especially in Paris did impress him. Indeed, the level of general culture of France was very high in comparison with the level of Japan, as he remarked.

4.3 Einstein in Japan; Ogura's Comments

Einstein visited Japan for six weeks, arriving at Kobe on November 19, 1922 and giving academic lectures on the theory of relativity at Tokyo and Kyoto, public ones in several other cities, see [25]. Of unusual interest is that 14 publications connected with Einstein's lecture tour appeared in Japan between 1921 and 1922; all but one were written by Japan's great physicists — three had studied in Germany and had met

[9] Sommerfeld, who was the academic father of four Nobel laureates (Debye, Heisenberg, Pauli, and Bethe), was Professor of Mechanics at Aachen 1900–1906 when he became Professor of Physics at Munich's University; he took his assistant Peter Debye, born 1884 in Maastricht, with him. In Munich he belonged together with Perron, Tietze, Caratheodory to the group of ten Munich professors of mathematics, all members of the Bavarian Academy of Sciences, who were neither Nazis nor (what is not the same) members of the Party, as Georg Faber reported. This contraposition to Nazism, headed by the great Oskar Perron (1880–1975), in a city dominated by Germany's highest Nazis, may have been unique for German universities at the time. See [57].

Einstein personally. One was Ayao Kuwaki (1878–1945), one of the first Japanese physicists to study in Europe, in Berlin between 1907 and 1909; another was Jun Ishiwara[10] (1881–1947). Ogura, who was in close contact with Borel in connection with relativity and who had attended Langevin's lectures—probably also in relativity theory—and who was also familiar with geometry, wrote the article "Interaction of Physics and Geometry" [149]. Einstein had lectured on "Theory of Relativity and Galileo" at Osaka's Public Hall on 11 December 1922, with an audience of 2500. Ogura taught at Osaka at the time.

Whereas the physicists expressed nothing more than admiration of Einstein's way of lecturing (see [25]), only Ogura touched on the content of the lectures. In fact he wrote:

> I have never seen such a lecturer always smiling and having great composure. He was not very good at calculations nor rigorous in logic, but he had strong intuition. He showed us how intuitive the great work of a genius is at its foundation.

Even though Ishiwara, who had studied under Sommerfeld and Planck in Germany, was the author of "Principle of Relativity" (Iwanami Shoten Publ., Tokyo, 1921), had accompanied Einstein during his travels over Japan, he did not express himself in such a form. In fact, as the prominent physicist Hiroshi Ezawa (1932–) [25] writes, "the direct impact of Einstein was not great among established scientists". This encounter speaks for Ogura's noted role in Japanese top level academic life at the time.

4.4 Ogura's Mathematical Supervisor and Academic Grandfathers

4.4.1 Tsuruichi Hayashi

Tsuruichi Hayashi was born on June 13, 1873, in Tokushima (Shikoku Island), Japan and died on October 4, 1935, in Matsue City, Shimane Prefecture. He came from a traditional family of Wazan and was educated at the Third Higher School in Kyoto. He studied mathematics at the Imperial University in Tokyo under Dairoku Kikuchi (1855–1917) and Rikitaro Fujisawa (1861–1933), graduating in 1897. Tokyo University was the only university in Japan at the time.

Hayashi first taught mathematics at the Higher Normal School in Tokyo and at the newly founded Kyoto Imperial University. When plans were made in 1907 to establish Tôhoku Imperial University, he became a founding professor of mathematics there together with Matsusaburô Fujiwara (1881–1946). When the university was officially inaugurated in 1911, Hayashi served as director of the Mathematical Institute,

[10] Ishiwara (or Ishihara), who graduated from Tokyo University in 1906, must have had Rikitaro Fujisawa as his mathematics professor. He was the author of ten papers on relativity theory, six of which were published in Germany, including the review [34] written upon the request of Johannes Stark (1874–1857); he was in Europe from April 1912 to May 1914, thereafter at Tôhoku University. Stark, Professor of Physics in Aachen 1908–1917, received the Physics Nobel Prize 1919. But he was of a dubious character and became a top Nazi, often called the "Physics-Führer", when Hitler came to power.

and both decided to make their department a rival to that at Tokyo Imperial University. In fact, until 1945 Tôhoku University was regarded as the Japanese replica of Göttingen and Tokyo as that of Berlin University.

First of all, Hayashi launched the Tôhoku Mathematical Journal with his own funds in 1911, the first international journal devoted to mathematics in Japan. As Hayashi's assistant, Ogura cooperated with Tadahiko Kubota (1885–1952), an associate professor at the time, and the physicist Jun Ishihara in editing the journal. Secondly, Hayashi and Fujiwara were the first to organize "colloquia" for professors to communicate their research results, and "meetings for students", to enlarge their mathematical knowledge by reading books and international journals. Thirdly, they established a regular collaboration between the departments of mathematics and physics so that mathematics students could attend the basic physics courses and vice versa. Thus the university became a world leader in differential geometry (with e.g. Shigeo Sasaki (1912–1987), real analysis and especially Fourier series.

Hayashi and Fujiwara were also known as erudite and prolific authors of high quality mathematical books as well as of text books for secondary schools. But Hayashi, who apparently did not study in Europe, also wrote a great number of mathematical research papers; one just needs to check the long list in Poggendorff [66], vol. V, pp. 509–511, vol. VI, p. 1055. For an obituary note on Hayashi see [28, 29], and on Fujiwara see [46].

Hayashi was also interested in the history of Japanese mathematics, stimulated by Kikuchi. His rivalry with Yoshio Mikami (see [58, 59]) over this history began in 1906 and ended only with Hayashi's sudden death in 1935. His papers on the history of mathematics were collected and published posthumously in two volumes in Tokyo in 1937 under the title "Collected Papers on Japanese Mathematics".

4.4.2 Dairoku Kikuchi

Of Ogura's two academic grandfathers, Dairoku Kikuchi was the elder. Kikuchi was sent to Great Britain in 1866, at age 11, to study for two years in London. He returned to England in 1870, first to University College School, London, and then to St. John's College, Cambridge University, where he studied under Isaac Todhunter (1820–1884).

Kikuchi was the first Japanese student to graduate from Cambridge University with the BA degree in 1877, being the 19th Wrangler in the Mathematical Tripos of that year. He received the MA (by proxy) in 1881. His specialization was physics and mathematics, in particular geometry. After returning home in 1877, Kikuchi became the first professor of mathematics at the Imperial University of Tokyo, founded 1877. He received his doctorate in science there in 1888. He had a remarkable career as an educator and administrator of science, serving as Dean of the College of Science, 1881–1898 when he was named President of the University. The title Baron was bestowed on him in 1904.

He was also the first President of the Science Research Institute of Japan; see [44, 45].

4.4.3 Rikitaro Fujisawa

Finally to Ogura's other academic grandfather, the eminent Fujisawa (or Fudzisawa). Born 1861 in Niigata as a son of a direct vassal of the Tokugawa shogunate, Fujisawa entered Tokyo University in 1878, studying mathematics and astronomy, and graduated in 1882.

Fujisawa was the first undergraduate student[11] of Kikuchi who sent him in 1882 to Europe to continue his studies, first to University College, London, to his friend Prof. Richard Rowe, for several months, and then to Germany, to the University of Berlin, where he attended the lectures of Karl Weierstraß and Leopold Kronecker, and in October 1884 to the Kaiser-Wilhelm University at Strasbourg where he obtained his doctorate, under the well-known Elwin Bruno Christoffel (1833–1900) in 1886; see [10].[12]

Returning home in 1887, he was appointed a professor at the newly founded Faculty of Mathematics of Tokyo University. Whereas this university was originally based on the English model, "Fujisawa steadily transferred the art of research in a German university to Japan" by introducing, for example, research seminars in the style initiated by Lejeune Dirichlet.

Kikuchi's teaching generally did not go beyond analytic geometry and differential and integral calculus, what he had received in England in the 1860s and 1870s. "Fujisawa went far beyond these, delivering lectures on the theory of real and complex variables, the theory of differential equations, and an introduction to the theory of elliptic functions"; see [69].

Fujisawa was Japan's speaker on the occasion of the third ICM held at Paris, in 1900.

He worked in elliptic functions and published 14 papers, one being in Japanese, the others in German. He was elected to the Japan Academy in 1932. His collected works were published in 1938 in three volumes [77]. One of his great practical credits was the establishment of the Japanese insurance industry; see [27, 70].

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[11] Fujisawa was an undergraduate student of Kikuchi, but a doctoral student of Christoffel. Hence, Kikuchi and Fujisawa, as joint teachers of Hayashi, can be considered as academic grandfathers of Ogura; see [77].

[12] As to the real genealogies of Christoffel and Dirichlet (1805–1059), Christoffel was born in Monschau, 35 km south of Aachen, Dirichlet comes from Düren, 28 km east of Aachen. However, both paternal grandfathers were born in Verviers, Belgium, 30 km south-west of Aachen, the one Charles Christophe in 1746, the other Antoine Lejeune Dirichlet in 1711.

Whereas the maternal ancestors of Christoffel came mostly from Monschau, Dirichlet's mother on the paternal side came from Annaberg (Saxony) and on the maternal side from the Rheinland. The ancestors of both mathematicians descended from important families of the Verviers, Malmedy, Liège, Visé, Aachen region; they have at least ten common ancestors between the eighth and tenth generation.

Of interest is that the genealogical tree of Christoffel has been traced back to the 25th generation, to a certain Knight Michel d'Awir, 10 km south-west of Liège, who was born ca. 1035, and that of Dirichlet in the 17th generation to a Knight Balduin of Waimes, 45 km south-west of Aachen, who is documented for 1166. This information was kindly supplied by the late Manfred Jansen of Kalterherberg, just south of Monschau, a friend of PLB. Jansen spent several years of his spare time in the excellent Archive of the City of Liège locating and working out these genealogical tables. His own lineage also goes back to Michel d'Awir; see [13, 38, 39]. This knight was a descendant of the Carolingian nobility according to E. Winkhaus: "Ahnen zu Karl dem Großen und Widukind", Selbstverlag, Ennetal (Hagen), Bd. I, 1950, Bd. II, 1953.

4.5 Further Conclusions

Of Ogura's circa 70 papers in mathematics, we have examined in great detail two papers in the area of sampling theory, dating to 1920, and, somewhat shorter, his paper [99] of 1913 on the Lorentz transformation in the area of relativity theory à la Minkowski, following up work of Borel. Being a graduate of the Tokyo College of Physics in 1905, lecturing there from 1910 to 1911, as well as his cooperation with physicists from 1911 to 1917 at Sendai, especially with Jun Ishiwara, explains Ogura's expertise in relativity, one of the extremely popular fields at the time.

In view of Ogura's interdisciplinary work in mathematical analysis—his two papers on sampling being just a part of his 42 papers which appeared in the Tôhoku Math. Jour. between 1911 and 1923, and his work in physics, differential geometry applied to mathematical physics, specifically relativity theory—five of his six papers in the area had been accepted for publication by Borel—one can certainly classify him as a first rate, pre-WW II mathematical analyst.

Hantaro Nagaoka was surely of this opinion when in 1917 he founded the Siomi Research Institute in Osaka, a part of the later Imperial University at Osaka, and offered Ogura the research position. Nagaoka was Professor of Physics at Tokyo University from 1901 to 1925. From 1893 to 1898 he had studied in Vienna, Berlin and Munich and was the first to present, in 1904, a Saturnian atomic model close to the presently accepted model.

The question may arise why Ogura picked France and not Germany for his European research stay; Germany had been the country practically all Japanese mathematicians (and physicists) had chosen before WW I. Well, after WW I the outside world was essentially closed to German mathematicians, basic examples being the ICM at Strasbourg and Toronto in 1920 and 1924, respectively. The Italians were the first to admit them again, to the ICM at Bologna in 1928. As to that at Strasbourg, Teiji Takagi reported that at his invited lecture on class field theory, in which he had generalized work of H. Weber and Hilbert, "I could find only a few people among the audience who could be interested in it", and "the German mathematicians were not invited". Therefore he visited Hecke and Blaschke in Hamburg.

At the latest by 1913 Ogura planned to go to France, on account of his scientific contacts with Borel and perhaps also because he had heard of *Revue du mois*, the rallying point of the young scientists at Paris: Perrin, Langevin, Pierre and Marie Curie and others, its chief protagonist being Borel. Of that group he attended Langevin's lectures on relativity theory.

4.6 Ogura's Academic Uncles and Cousins, Founders of the Japanese School of Modern Mathematics

Students of both Kikuchi and Fujisawa included Hayashi and Jittaro Kawai (1865–1945), the teacher of Masazo Sono with his offspring Yasuo Akizuki (1902–1984) and the Fields Medalist Heisuke Hironaka (1931–).

Students whose primary mentor and teacher was Fujisawa included:

- Teiji Takagi (1875–1960) [36, 60, 76], who studied at Berlin with Fuchs, H.A. Schwarz and Frobenius, and at Göttingen with Klein and Hilbert in 1898–1901,

who had at least 380 mathematical descendants, including Sigekatu Kuroda (1905–1972), Tadashi Nakayama (1912–1964), and especially of Shokichi Iyanaga (1906–2006, who studied in Paris and Hamburg under Artin), with his own offspring Makoto Abe (1914–1945), the differential geometer Kentaro Yano (1912–1993), the Fields Medalist Kunihiko Kodaira (1915–1997), Kiyosi Ito (1915–2008) and Kenkichi Iwasawa (1917–1998); Takagi's successor in office in 1936 was Zyoiti Suetuna (1898–1970), also a student of his;
- Yosie Takuji (1874–1947), who studied for three years with Klein and Hilbert in Göttingen, with his offspring Mitio Nagumo (1905–1995), Kinjiro Kunugi (1903–1975), Masuo Hukuhara (1905–2007),[13] Kosaku Yosida (1909–1990), all expert analysts;
- Senkichi Nakagawa (1876–1942), who studied for three years under H.A. Schwarz at Berlin and was the co-supervisor of Shokichi Iyanaga;
- Matsusaburô Fujiwara was Ogura's second teacher at Sendai as well as teacher of Yoshitomo Okada (1892–1957), Shin-ichi Izumi (1904–1990), of Tadao Tannaka (1908–1986), co-inspirer of Kodaira, and of Gen-ichirô Sunouchi (1911–2008);[14] Okada spent some time at Göttingen in 1928 and was co-mentor (with Tatsujiro Shimizu) of Shizuo Kakutani (1911–2004);
- Tadahiko Kubota [68], regarded as the founder of modern geometry in Japan;
- Soichi Kakeya (1886–1947), who was also inspired by Takagi. He is known for the "Kakeya needle problem" (related to Reuleaux's triangle);

As the late Prof. Gen-ichirô Sunouchi (1911–2008)[15] of Tôhoku University had written to one of the authors in 1979 (see [10, p. 6]), Fujisawa and his many outstanding students were responsible for raising the standard of mathematics in Japan to the European level. His student Teiji Takagi was regarded as the founder of the Japanese school of modern mathematics, according to Prof. Katsuya Miyake (1941–) of Waseda University (Tokyo) [60]; see also [69]. Kinnosuke Ogura was indeed an academic grandson of Fujisawa, his living model, as were also his academic uncles. However, Ogura's academic life was considerably harder than that of his shining examples—he had not begun his education at an Imperial university.

[13] It may be of interest that Prof. Hukuhara (or Fukuhara) accepted as chief-editor of Publications Research Institute of Mathematical Sciences for publication in 1968 the paper Butzer, H. Berens and S. Pawelke, "Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten", Publ. RIMS, Kyoto Univ., Ser. A, Vol. 4 (1968), 201–268. According to W. Kunyang and Li Luoqing, "Harmonic Analysis and Approximation on the Unit Sphere", Science Press, Beijing/New York, 2000, p. i, it is the first basic paper on approximation problems on the sphere, which became an active field of research from 1980 onwards.

[14] The doctoral students of Sunouchi at Tôhoku University included Chinami Watari (1932–199?), Y. Suzuki, Satoru Igari (1936–), Kōichi Saka (1944?–), Jun Tateoka (1944?–), Makoto Kaneko (1944?–), Takahiro Mizuhara (1945?–), Shigehiko Kuratsubo (1945?–).

[15] It was Prof. Sunouchi who during his three months research visit at PLB's chair in 1963 explained the structure of Japan's university education, of the seven Imperial universities at the time, their great (exclusive) role in its university life. Sunouchi's great respect for Japan's social customs, for his ancestors, truly impressed PLB. While in Aachen, a joint paper was written: Butzer-Sunouchi, "Approximation theorems for the solution of Fourier's problem and Dirichlet's problem", Math. Ann. 155:316–330 (1964). During his time in Aachen Sunouchi also participated in the first conference on approximation PLB conducted at the Mathematical Research Center, Lorenzenhof, Oberwolfach. See Butzer, J. Korevaar (eds.), "On Approximation Theory", Birkhäuser, Basel, 1964. See [14].

4.7 Ogura, Protagonist in the Modernization of Japanese Education, the Basis to Scientific Research

Whereas the Japanese descendants of E.B. Christoffel,[16] headed by Rikitaro Fujisawa and especially his academic son Teiji Takagi, raised the standard of mathematics in Japan to the European level, it was Fujisawa's academic grandson Ogura who was Japan's chief protagonist in raising the scientific level of the general public itself and that of mathematical education to the European level.

In fact, just back in Japan from France, he lectured in many cities during 1922–1923 on the need to popularize mathematics in Japan, it being "a treasure not to be monopolized by mathematicians" [150]. He criticised the feudal and closed character of the academic world in Japan and insisted on the need for modernization [152]. He lectured about the "esprit scientifique" as he found it in the École Polytechnique, particularly the synthesis of theory, applications and practice, and about the importance of spreading this scientific spirit in Japan.

To advance these aims he began to write pioneer articles on the history of Western mathematics, also from the point of view of social, cultural and economic history during 1929–1930, followed by his broad study of traditional Japanese as well as Chinese mathematics, even in part in collaboration with the Chinese historian Li Yan (1892–1963). For further details in this respect see [55].

5 Development of Sampling Techniques Among Engineers; a Survey

Although Whittaker and Ogura were working in interpolation theory, a purely mathematical topic, there were on the other hand engineers working independently on the applied side who introduced the sampling theorem or had a major influence on its development. It seems that E.T. Whittaker's paper is the link by which the two strands are connected. In fact, although Shannon cites the book of J.M. Whittaker (son of E.T.W.) [83], many engineering papers and text books from about 1952 onwards cite E.T. Whittaker [81] himself. Thus this section deals with the engineers who had a hand in the matter, and addresses the question who was perhaps the first engineer to give a proof of the sampling theorem (for bandlimited functions).

[16] Having gone into Japanese academic genealogies, a word is due to that of Christoffel, Fujisawa's doctoral father, as well as to those of his teachers Weierstraß and Kronecker in Berlin. Christoffel was first and foremost a student of J.P.G. Lejeune Dirichlet, although the referees of his dissertation at Berlin were M. Ohm and E. Kummer. Dirichlet in turn, who had no formal doctoral degree, considered Gauß as his teacher. He had studied from 1822 to 1826 in Paris where Fourier, Lacroix, Legendre and Poisson also had sponsored him.

During his stay in Germany Fujisawa had studied also under Weierstraß, an academic grandson of Gauß, as well as under Kronecker, who regarded Dirichlet as his teacher. Fujisawa's student Teiji Takagi and various other descendants as well as several Japanese physicists had studied under Fuchs, H.A. Schwarz, Frobenius, Klein, Hilbert and Sommerfeld. Klein was an academic grandson of Dirichlet, so that Hilbert, Minkowski and Sommerfeld (as students of Lindemann, in turn a student of Klein), were academic great-grandsons of Dirichlet. Fuchs, Schwarz and Frobenius, as students of Weierstraß, were academic great-grandsons of Gauß. Einstein had singled out A. Hurwitz (a student of Klein) and Minkowski as his teachers at Zürich. See [8, 19–21].

5.1 Harry Nyquist and Karl Küpfmüller

One person very often connected with the sampling theorem is the highly honoured Swedish-born communication engineer H. Nyquist (1889–1976) in view of his landmark papers of 1924 [61] and 1928 [62] on telegraphy; he is often cited in this respect together with Küpfmüller and Shannon. According to the recent [26], these two papers reveal a complete strikingly modern understanding of the connection between signalling speed, number of bits per symbol, and bandwidth; intersymbol interference and its use are fully understood. However, the sampling theorem cannot be found explicitly anywhere in these two papers. Nor do they contain a statement concerning the need to sample at twice the maximum frequency of the input signal. Nyquist did point out, however, the importance of the frequency $1/T$, where T is the duration of the "time unit", as well as its connection with the speed of transmission. The expressions "Nyquist interval" or "Nyquist rate", applied in relevance to band-limited signals and the sampling theorem very likely had their origin in Shannon's paper of 1949 [74]. For details see [26].

Concerning K. Küpfmüller (1897–1977), who is known for his seminal work in almost all fields of electrical engineering as well as for being the German initiator of early systems theory (see e.g. Bissel [4]), S. Verdú [79] writes: "In contrast to the 1920 papers of Nyquist and Küpfmüller, Shannon's crisp statement (see below) and proof [74] of the sampling theorem were instrumental in popularizing this result in engineering".

Küpfmüller is also known for the "Küpfmüller uncertainty principle" of 1924, the inverse relationship between frequency and time domains: the narrower the bandwidth the greater the rising/setting time of signals; see [47], [48, pp. 43–53, 149–167, 347–353].

5.2 Vladimir Kotel'nikov

The scientist who was honoured by the Eduard Rhein Foundation for being "the first to formulate in a mathematically exact manner and publish the sampling theorem within the context of problems in communication technology", was the Russian Vladimir A. Kotel'nikov (1908–2005); it was the award for the year 1999 (an amount of 150,000 DM); see http://www.uni-koblenz.de/~physik/ERS/html/hauptseite_e.html. His theorem in this respect is presented in his famous manuscript [43], prepared for a conference never held, published 1933, but only internationally accessible since 2001, in the English translation by V. E. Katsnelson in [2, p. 27–45]. The knowledge of its existence in the West is probably due to the appearance of two works of Kolmogorov and Tichomirov[17] of 1956 and 1960 [41, 42]. Kotel'nikov's main theorem reads:

[17] The authors doubt whether either Russian author ever saw Kotel'nikov's paper [43], their assertions regarding his theorem being rather vague. In fact, Prof. Tichomirov mentioned to PLB while in Aachen 1995 that he never saw the 1933 paper itself, it not being available in Moscow. But the late Prof. Lüke obtained a copy of it from a friend in Moscow (Dr. R. Rachev). It is also available in the British Library in Leeds.

Any function $F(t)$ which consists of frequencies between 0 and f_1 is representable by the series

$$F(t)=\sum_{-\infty}^{\infty} D_k \frac{\sin\omega_1(t-\frac{k}{2f_1})}{t-\frac{k}{2f_1}}, \tag{21}$$

where k is an integer number, $\omega_1 = 2\pi f_1$, D_k are constants depending on $F(t)$.

Conversely, any function $F(t)$, represented by the series (21), consists only of frequencies between 0 and f_1.

Kotel'nikov's proof is superbly intuitive but not rigorous in the mathematical sense. Also one does not notice an exact statement of the sampling theorem with full hypotheses. But it is very interesting that Kotel'nikov gives his theorem in both a direct and converse form.

5.3 Herbert Raabe

A further contribution to the sampling theorem is the doctoral dissertation "Untersuchungen an der wechselseitigen Mehrfachübertragung (Multiplexübertragung)", published in full 1939 in the journal "Elektrische Nachrichtentechnik" [67], which became known after the historical studies by H.D. Lüke [52–54]. Its author was Herbert Raabe (1909–2004), a student of Küpfmüller, W. Stäblein (1900–1945) and H. Fassbender (1884–1970) at the TH Berlin. In this milestone dissertation Raabe analyzed and built the first time-division multiplex system for telephony. This task required of him an overall complete understanding of sampling of finite duration and sampling of lowpass and bandpass signals. In comparison with the classical sampling theorem there are two main differences: The (non-$L(\mathbb{R})$-integrable) sinc-kernel, the Fourier transform of the ideal filter, has been replaced by (an integrable) $h(t)$, the impulse response of the filter used in the implementation; and there are only finitely many samples $f(k)$. But both differences must necessarily occur in any practical realization of sampling.

In more mathematical terms Raabe actually studied what are now called generalized sampling series, first investigated systematically at Aachen (see [15, 75]),

$$(S_w^{\varphi} f)(t)=\sum_{k=-\infty}^{\infty} f\Big(\frac{k}{w}\Big)\varphi(wt-k)$$

which have the properties that the kernel φ is (at least) integrable over $\mathbb{R}$ such that $S_w f$ exists for all $f \in C(\mathbb{R})$ (not just bandlimited f) but with $\lim_{w\to\infty}(S_w^{\varphi} f)(t) = f(t)$. Non-bandlimited kernels such as B-splines, or linear combinations of translates of these, are examples; in such cases the sampling sums also have finitely many samples, as with Raabe. But also the kernels of Fejér ($\mathrm{sinc}^2\, t/2$), Jackson, de La Vallée Poussin, etc. are possible. Thus generalized sampling series are in every respect easier to handle and more practical; for details see [9].

5.4 Claude Shannon

The person most closely connected with the nomenclature of the sampling theorem, whether mathematician or engineer, is no doubt Claude Shannon (1916–2001), although he never claimed any credit for originating it. For him it was simply a part of his original development of his mathematical theory of communication; it provided for him the means of converting continuous time signals to discrete time signals without loss of information. Shannon, who had been a student of Norbert Wiener, Vannevar Bush and Frank Hitchcock at MIT between 1936 and 1940, received the Eduard Rhein Award of 1991 (DM 200,000) for "his fundamental research on information theory"; see http://www.uni-koblenz.de/~physik/ERS/html/hauptseite_e.html. He was among the first recipients of the Kyoto Prize in 1985 (50 million yen). For the details see http://www.inamori-f.or.jp/laureates/K01_b_claude/prf_e.html and http://www.inamori-f.or.jp/e_kp_out_out.html.

As one of his sources of the sampling theorem Shannon cites J.M. Whittaker [83], as mentioned above, and, in particular, W.R. Bennett's work [3] of 1941, which in turn cited Raabe's paper [67]; it had appeared only two years earlier during World War II, then going on. Shannon's Theorem 13 in [72, 73] reads:

Let $f(t)$ contain no frequencies over W. Then

$$f(t) = \sum_{-\infty}^{\infty} X_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

where $X_n = f(n/2W)$.

5.5 Isao Someya

Independently and simultaneously to Shannon's significant work, the book "Waveform Transmission", written by the communication engineer Isao Someya (1915–2007) appeared in Japan. In Chap. 4 it contains the sampling theorem along with many interesting extensions and applications, treated in Chaps. 5 and 6. Someya derived his basic result, in the same classical form as Shannon, using Poisson's sum formula (as did R.P. Boas [5] in 1972); he presents no references. A feature is that topics are always discussed in both time and frequency domains. On the basis of this book Someya received his doctorate in Electrical Engineering from the University of Tokyo in 1950 (his bachelor's degree 1938). For details see Ogawa [63, 64].

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