The energy expense of superoscillations

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ABSTRACT

For any fixed bandwidth, there are finite energy signals which oscillate arbitrarily fast on arbitrarily long time intervals. More precisely, for any fixed bandwidth, it is possible to find finite energy signals f which at arbitrarily chosen times $\{t_i\}_{i=1}^N$ possess arbitrarily prescribed amplitudes a_i, i.e. which obey $f(t_i) = a_i$ for $i = 1, 2, \ldots, N$ where N is arbitrarily large. This paper investigates to which extent such superoscillating signals could be used for the fast transmission of information through low bandwidth channels. The main result is that for fixed noise level, arbitrary amounts of information can be compressed into arbitrarily short segments of signals of arbitrarily low bandwidth, without having to increase the level of the amplitudes which encode the information. The price to be paid is that, for fixed message size, the energy expense grows polynomially with the compression and that, for fixed compression, the energy expense grows exponentially with the message size.

1 INTRODUCTION

It is commonly believed that a bandlimited signal cannot oscillate at a rate higher than the Nyquist rate. Also, it is well known that bandlimited functions are entire functions of exponential type, and that there is a connection between the growth (or bandwidth) of the function and the density of its zeros. Thus, it may come as a surprise that finite-energy bandlimited signal can indeed oscillate at rates arbitrarily higher than the Nyquist rate throughout intervals of arbitrary finite length. Fig. 1 depicts an example. Such so-called "superoscillations" have been known in various contexts in physics, see e.g. [1, 2].

Intuitively, superoscillations are possible because they are a merely *local* phenomenon. The *global* behavior of a signal (for example, the average zero density) is not affected by the occurrence of superoscillations (which occur over finite intervals): crucially, a bandlimited signal can oscillate at a rate higher than the Nyquist rate only on finite intervals, but *not* on infinite intervals. We will discuss the precise mathematical origin of superoscillations in Section 1.2.

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Let us now consider a channel bandlimited to $1/2$ Hz. Transmittable signals are of the form

$$f(t) = \int_{-1/2}^{1/2} \hat{f}(w)\, e^{2\pi i t \omega}\, d\omega$$

with Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t \omega}\, dt$$

and obeying:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \frac{\sin((t-n)\pi)}{(t-n)\pi}$$

Let us assume, for simplicity, that the channel's noise is such that amplitudes can be measured reliably only up to integers.

Now assume that we wish to transmit messages of b bits information content by sending suitable signals through this channel. One simple, conventional possibility is, of course, to encode the data as coefficients of a sampling series at the Nyquist rate. This possibility comes with a certain well-known trade-off between how large the signal amplitudes are allowed to be and how long it takes to transmit a given message.

Another possibility, however, is to use superoscillating signals of the same bandwidth. The use of superoscillating signals will allow one to compress messages into an arbitrarily short time interval without needing to increase the level of signal amplitudes which encode the data.

What price is there to be paid? Answering this question in quantitative terms is the main purpose of this paper.

To this end, let us now discuss the two ways of encoding bits into a bandlimited signal which we mentioned above. Both methods will be idealized, or "thought experiments". For example, while it is plausible that actual signals are bandlimited, it is also plausible that they are duration-limited, while both properties are known to excluded another, see, for example, [3]. We will need to discuss a related point at the end.

1.1 Conventional method

A well known, idealized method for transmitting messages of b bits through our type of channel is:

- Divide the b bits into N groups of k bits, i.e. $b = Nk$
- Consider the Nyquist rate grid of sampling times $t = 1, 2, 3, ..., N$
- Create a $1/2$ Hz bandlimited signal whose amplitudes a_n at those N sampling times encode k bits each, i.e. whose amplitudes a_n each take one of the 2^k possible values $\pm 1/2, \pm 3/2, \pm 5/2, ..., \pm \frac{2^k - 1}{2}$:

$$f(t) = \sum_{n=1}^{N} a_n \frac{\sin((t-n)\pi)}{(t-n)\pi} \quad (1)$$

- Transmit this $1/2$Hz bandlimited signal to the receiver, who then measures those same N discrete amplitudes which each carry k bits, to recover the full message of $b = Nk$ bits.

Obviously, choosing a large k means a need for fewer samples, which means that the message is "compressed" into a shorter signal.

1.2 Method using superoscillations

It is possible to transmit messages of b bits by sending a signal bandlimited to $1/2$Hz which encodes the b bits in b one-bit samples which are spaced more tightly than the Nyquist spacing. Such signals are called superoscillating because in order to encode a typical b bit message such $1/2$Hz signals must locally exhibit oscillations that are faster than $1/2$Hz.

The following is a method for constructing superoscillating signals bandlimited to $1/2$Hz which have prescribed amplitudes a_n (encoding k bits each) at sampling times t which are spaced tighter than the Nyquist rate, say:

$$t = T, 2T, 3T, ..., NT, \quad \text{where} \quad T < 1$$

- Define the $1/2$Hz bandlimited signal

$$f(t) = \sum_{r=1}^{N} x_r \frac{\sin((t-Tr)\pi)}{(t-Tr)\pi}$$

  with as yet undetermined coefficients $x_r,\ r = 1, ..., N$.

- The coefficients x_r are determined by requiring that the signal possesses the prescribed amplitudes $a_n = f(nT)$:

$$a_n = \sum_{r=1}^{N} x_r \frac{\sin((n-r)T\pi)}{(n-r)T\pi} \quad (2)$$

  This yields for the coefficients x_r:

$$x_r = \sum_{n=1}^{N} S^{-1}_{rn} a_n \quad (3)$$

  where S^{-1} is the inverse of the $N \times N$ matrix S with coefficients

$$S_{nr} = \frac{\sin((n-r)T\pi)}{(n-r)T\pi}$$

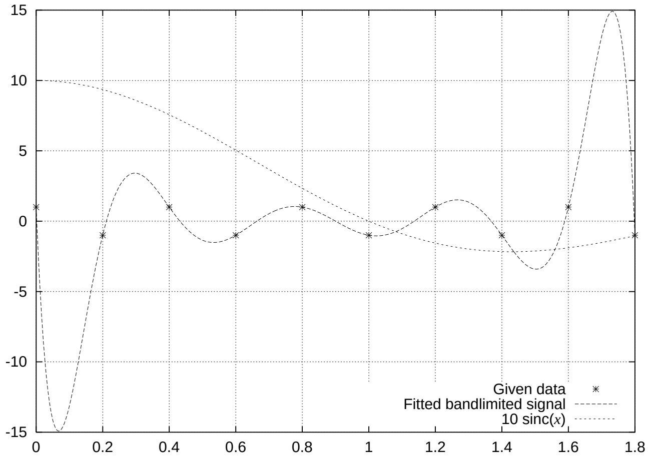

Figure 1: An example of a superoscillating signal. In this case, $N = 10$, and $T = 0.2$. The sinc signal is shown for comparison (both signals are bandlimited to $1/2$Hz).

Note: The matrix S is invertible because any finite set of functions

$$e_i(t) = \frac{\sin((t-\lambda_i)\pi)}{(t-\lambda_i)\pi}$$

(where the λ_i are N nonrepeated reals) and correspondingly of functions

$$\hat{e}_i(\omega) = e^{2\pi i \lambda_i \omega}$$

is linearly independent and consequently forms an in general nonorthogonal basis. The rows of the matrix S are the coefficients of those basis vectors in their own basis. The linear independence of the $e_i(t)$ thus implies that S is of full rank and therefore invertible.

An example of a superoscillating signal generated using this method is shown in Fig. 1. The highest frequency of the signal is $1/2$Hz, and the signal interpolates a set of alternating sign data points a_k, $k = 1, 2, ..., 10$, on a grid of mesh $T = 0.2$.

2 ENERGY EXPENSE

Let us consider the amount of energy that needs to be available in order to be able to transmit all possible b bit messages when using one of the two above mentioned methods respectively.

2.1 Energy need of conventional coding

The energy E of a $1/2$Hz signal is:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} f(t)^2\, dt \\ &= \sum_{n_1, n_2} f(n_1) f(n_2) \int_{-\infty}^{\infty} \frac{\sin((t-n_1)\pi)}{(t-n_1)\pi} \frac{\sin((t-n_2)\pi)}{(t-n_2)\pi}\, dt \\ &= \sum_{n=-\infty}^{\infty} f(n)^2 \end{aligned}$$

We recall that the amplitude range of the coding samples is

$$f(n) \in \{\pm 1/2, \pm 3/2, \pm 5/2 ... \pm \frac{2^k - 1}{2}\}$$

which implies that the maximum value for each of the $f(n)^2$ is $(2^{k-1})^2 = 2^{2k-2}$. Thus the amount of energy that needs to be available per signal in order to be able to transmit any arbitrary b bit message by the conventional method is:

$$\begin{aligned} E &= \sum_{n=1}^{N} 2^{2k-2} \\ &= N2^{2k-2} \end{aligned}$$

Using $b = Nk$ we find the maximum energy need per bit, E/b:

$$E/b = \frac{2^{2k-2}}{k} \tag{4}$$

There is an obvious tradeoff: For fixed message size b, the energy expense grows exponentially as one compresses the signal into a shorter time interval (by increasing k). Also, for fixed time compression (i.e. for fixed k), the energy expense grows linearly with the message size. For example, by sending one bit per sample, one clearly minimizes E/b while also minimizing the transmission speed.

2.2 Energy need of superoscillation coding

By using superoscillating signals it is possible, for example, to transmit $N = b$ samples, whose amplitudes code for one bit each, in a time shorter than N seconds (in fact, in any arbitrarily short time interval). Therefore, the question arises, whether, in this way, fast transmission speed can be combined with low energy need.

To clarify this issue, let us determine the energy that needs to be available to be able to transmit any b bit message in a superoscillating signal by coding one bit each in N equidistantly spaced samples with a spacing $T < 1$.

Theorem: The lowest energy superoscillating signal $f(t)$ which at the times $\{nT\}_{n=1}^{N}$ has prescribed amplitudes $f(nT) = a_n$ is given by (2) and (3). Further, the energy of this signal is:

$$E = a^T S^{-1} a \tag{5}$$

where a is the vector with coefficients a_i.

Proof: The L_2 signal that satisfies

$$f(t_k) = a_k, \quad k = 1,2,\ldots,N,$$

and has minimum energy can be found by minimizing

$$E = \int_{-\infty}^{+\infty} |f(t)|^2\, dt = \int_{-1/2}^{+1/2} |\widehat{f}(\xi)|^2\, d\xi$$

subject to the N constraints

$$\int_{-1/2}^{+1/2} \widehat{f}(\xi)\, e^{i2\pi\xi t_k}\, d\xi = a_k.$$

This is a standard constrained minimization problem. Solving it and inserting the N constraints $f(t_i) = a_i$ we get

$$\sum_{k=1}^{N} x_k \int_{-1/2}^{+1/2} e^{i2\pi\xi(t_i - t_k)}\, d\xi = a_i, \quad i = 1,2,\ldots,N,$$

that is,

$$\sum_{j=1}^{N} x_j \operatorname{sinc}(t_i - t_j) = a_i, \quad i = 1,2,\ldots,N.$$

In matrix form, this is $Sx = a$, where x is the vector of coefficients x_i, and S is the symmetric matrix with elements

$$S_{ij} = \operatorname{sinc}(t_i - t_j),$$

and $\operatorname{sinc} x = \sin(\pi x)/(\pi x)$. The energy of the interpolating signal is

$$\begin{aligned} E &= \int_{-1/2}^{+1/2} \left| \sum_{k=1}^{N} x_k\, e^{-i2\pi\xi t_k} \right|^2 d\xi \\ &= \sum_{i=1}^{N} x_i \sum_{k=1}^{N} x_k \int_{-1/2}^{+1/2} e^{i2\pi\xi(t_i - t_k)} \\ &= \sum_{i=1}^{N} x_i \sum_{k=1}^{N} x_k S_{ik}, \end{aligned}$$

or

$$E = x^T S x = a^T S^{-1} a,$$

since $Sx = a$. This establishes (5).

2.3 Estimating the energy

Our aim is to find how the energy need E as given in (5) depends on the size of the message b and on the spacing $T < 1$, for small T.

For messages of $b = 2, 3$ and 4 alternating bits the energy of the carrying superoscillating signal can be calculated by machine to leading order in $1/T$:

$$E = \frac{3}{\pi^2 T^2} \quad \text{for } b = 2$$

$$E = \frac{45}{\pi^4 T^4} \quad \text{for } b = 3$$

$$E = \frac{700}{\pi^6 T^6} \quad \text{for } b = 4$$

This sequence suggests that, in general, the energy needed to be able to transmit by means of superoscillations any b bit message in bT seconds scales roughly as:

$$E_{sup.} \approx \frac{1}{15}\left(\frac{15}{\pi^2 T^2}\right)^{b-1} \tag{6}$$

This would mean that superoscillation coding is energetically exponentially expensive with respect to the message size, while being only polynomially expensive with respect to the time compression factor $1/T$.

We can show that this conjecture is in fact true, by arguing as follows: the matrix S is related to the prolate matrix $\rho(N,W)_{mn}$ [4], where N is the order of the matrix, and $W = T/2$. Using the results in [4, Section 2.5], we derive that, for small W, the smallest eigenvalue of $\rho(N,W)$ satisfies

$$\lambda_{N-1}(N,W) \approx \frac{G(N)}{\pi}\,(2\pi W)^{2N-1}$$

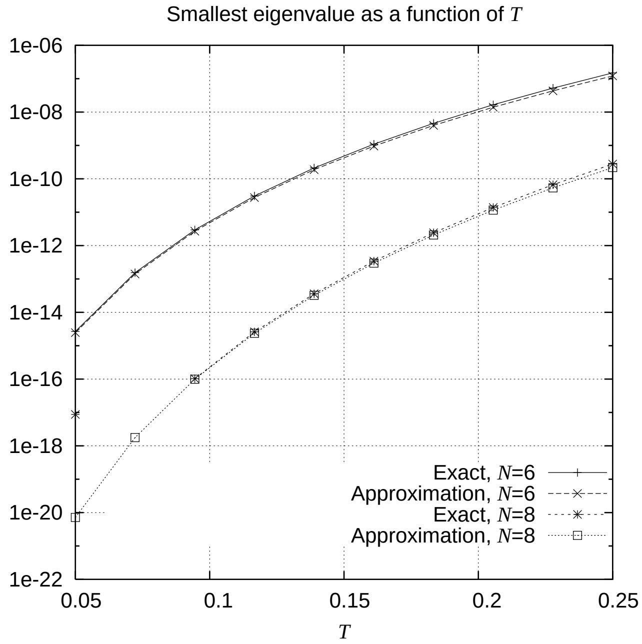

Figure 2: The smallest eigenvalue of S, as a function of T, for several N.

where

$$G(N) = \frac{2^{2N-2}}{(2N-1)\binom{2N-2}{N-1}^3}.$$

Expanding the binomial coefficient using Stirling's formula, we obtain the approximation

$$G(N) \sim \frac{\pi^{3/2}(N-1)^{3/2}}{2^{4N-4}(2N-1)},$$

and finally

$$\lambda_{N-1}(N,W) \sim \sqrt{\pi}\,(2\pi W)^{2N-1}\frac{(N-1)^{3/2}}{2^{4N-4}(2N-1)}.$$

The largest eigenvalue of S^{-1}, which is the reciprocal of the smallest eigenvalue of S (plotted in Fig. 2), therefore satisfies

$$\lambda_{\max} \sim \frac{2^{4N-4}(2N-1)}{\sqrt{\pi}\,(\pi T)^{2N-1}(N-1)^{3/2}}, \tag{7}$$

or

$$\lambda_{\max} = O\left(\frac{16^{N-1}}{T^{2N-1}N^{1/2}}\right). \tag{8}$$

3 REMARKS AND CONCLUSIONS

In general, the energy necessary for being able to send any b bit message by superoscillation coding satisfies

$$E = a^T S^{-1} a \le \lambda_{\max}\|a\|^2.$$

There is equality when the N dimensional vector a is proportional to the eigenvector of S to the eigenvalue $\lambda_{\max}$.

This defines the set of amplitudes that are "most difficult" to interpolate, in the sense of requiring the maximum energy among all other possible amplitude sets (of the same cardinality N, and for the same spacing T). In other words: the minimum energy of the signal f, band-limited to $1/2$Hz, and satisfying $f(kT) = a_k$, $k = 1, 2, \ldots N$, reaches a maximum when a is proportional to the eigenvector corresponding to $\lambda_{\max}$. Inspection of the eigenvectors of the prolate matrix shows that if the vector of amplitudes a has alternating signs, the component along the eigenvector associated with the eigenvalue $\lambda_{\max}$ is indeed large, implying that the upper bound is in fact a good approximation for E.

One can conclude, therefore, that indeed, for fixed time compression T the energy expense increases exponentially with the message size b

$$E = O(16^{b-1}),$$

(note that $b = N$), whereas for fixed message size b, the energy expense increases polynomially in the compression factor T,

$$E = O(T^{-2b+1}),$$

as $T \to 0$. This follows, of course, from (7) or (8).

We showed that arbitrary amounts of information can be encoded in an arbitrarily small segment of an arbitrarily low bandwidth signal, whereby, in particular, the energy need grows only polynomially with the compression into smaller and smaller segments — which is better than the exponential growth when coding conventionally. The caveat with superoscillation coding is, however, that it does not suffice to transmit merely the coding segment of the superoscillating signal, i.e. the benefit of time compression cannot be realized in transmissions. This is because the truncation makes the signal non-bandlimited, which must here cause large truncation distortions. In this way, there is no contradiction with the fundamental results of Landau [5] on the necessary conditions for stable data transmission.

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