Whittaker differential equation associated to the initial heat problem†

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Abstract. In this paper, by using the theory of reproducing kernels, we investigate integral transforms with kernels related to the solutions of the initial Whittaker heat problem.

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1. Introduction

The Whittaker functions are closely related to the confluent hypergeometric functions which play an important role in various branches of applied mathematics and theoretical physics, for instance, fluid mechanics scalar and electromagnetic diffraction theory, atomic structure theory, input-output situations and storage consumption situations in economic problems, and so on. Moreover, they have acquired an ever increasing significance due to their frequent use in applications of mathematics to physical and technical problems [2] [3]. This justifies the continuous effort in studying properties of these functions and in gathering information about them.

Let consider the general method given in [5] for the existence and construction of the solution of the following initial problem

$$(\partial_t + L_x)u_f(t,x) = 0,\ t > 0 \tag{1.1}$$

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satisfying the initial value condition

$$u_f(0,x) = f(x), \tag{1.2}$$

for some general linear operator L_x on some function space, and on some domain, by using the theory of reproducing kernels.

We shall investigate the integral transforms with kernels related to the solutions of the equations by using the theory of reproducing kernels, for the Whittaker heat equations.

The Whittaker functions denoted by $M_{\mu,\nu}(x)$, $W_{\mu,\nu}(x)$ arise as solutions to the Whittaker differential equation, i.e., are solutions of the linear homogeneous ordinary differential equation of the second order

$$\frac{d^2W}{dx^2} + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4} - \nu^2}{x^2}\right) W = 0. \tag{1.3}$$

Here, we will deal with the Whittaker function $W_{\mu,\nu}$ which is an eigenfunction of a second order differential operator

$$A_x W_{\mu,\nu}(x) = 4\nu^2 W_{\mu,\nu}(x) \tag{1.4}$$

with a eigenvalue $4\nu^2$, and where

$$A_x = 4x^2 \frac{d^2}{dx^2} - x^2 + 4\mu x + 1. \tag{1.5}$$

Further, we consider the case that ν are reals, and the functions

$$\exp(-4\nu^2 t) W_{\mu,\nu}(x) \tag{1.6}$$

are the solutions of the operator equation

$$(\partial_t + A_x)\, u(t,x) = 0. \tag{1.7}$$

We shall consider some general solutions of (1.7) by a suitable sum of the solutions (1.6). In order to consider a fully general sum, we shall consider the kernel form for a nonnegative continuous function ρ,

$$\mathcal{K}_t(x,y;\rho) = \int_0^{+\infty} \exp\left\{-4\nu^2 t\right\} W_{i\tau,\nu}(x)\overline{W_{i\tau,\nu}}(y)\rho(\tau)d\tau. \tag{1.8}$$

Of course, here, we are considering the integral with absolutely convergence for the kernel form.

The fully general solutions of the equation (1.7) may be represented in the integral form

$$u(t,x) = \int_0^{+\infty} \exp\left\{-4\nu^2 t\right\} W_{i\tau,\nu}(x) F(\tau)\rho(\tau)d\tau. \tag{1.9}$$

for the functions F satisfying

$$\int_0^{+\infty} \exp\left\{-4\nu^2 t\right\} |F(\tau)|^2 \rho(\tau)d\tau < \infty. \tag{1.10}$$

Then, the solution $u(t,x)$ of (1.7) satisfying the initial condition

$$u(0,x) = F(x) \tag{1.11}$$

will be obtained by $t \to 0$ in (1.9) with a natural meaning. However, this point will be very delicate and we will need to consider some deep and beautiful structure. Here, (1.8) is a reproducing kernel and in order to analyze the logic above, we will need the theory of reproducing kernels, essentially and beautiful ways. Indeed, in order to construct natural solutions (1.8) we will need a new framework and function space.

2. Main results

In order to analyze the integral transform (1.11), we will need the essence of the theory of reproducing kernels. We are interesting in the integral transforms (1.11) in the framework of Hilbert spaces. Of course, we are interesting in the characterization of the image functions, the isometric identity like the Parseval identity and the inversion formula, basically. For these general and fundamental problems, we have a unified and fundamental method and concept in the general situation as follows in [6], [7] and [8], where we can found the general theory for linear mappings in the framework of Hilbert spaces.

Moreover, recently, we obtained a very general inversion formula based on the Aveiro Discretization Method in Mathematics ([4]) by using the ultimate realization of reproducing kernel Hilbert spaces. Following that the general theory, we shall build our result.

For this purpose, we need to recall that for $k \neq \frac{1}{2}$ the Whittaker function $W_{i\tau,k-\frac{1}{2}}$ is an eigenfunction of a second order differential, i.e.

$$A_x W_{i\tau,k-\frac{1}{2}}(x) = 4\left(k-\frac{1}{2}\right)^2 W_{i\tau,k-\frac{1}{2}}(x), \; x>0 \tag{2.1}$$

where

$$A_x = 4x^2\frac{d^2}{dx^2} - x^2 + 4i\tau x + 1. \tag{2.2}$$

Then, we form the reproducing kernel

$$\mathcal{K}(x,y;\rho) = \int_0^{+\infty} W_{i\tau,k-\frac{1}{2}}(x)\overline{W_{i\tau,k-\frac{1}{2}}(y)}\rho(\tau)d\tau, t>0, \tag{2.3}$$

and consider the reproducing kernel Hilbert space $H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$ admitting the kernel $\mathcal{K}(x,y;\rho)$. In particular, note that

$$\mathcal{K}_t(x,y;\rho) \in H_{\mathcal{K}(\rho)}(\mathbb{R}^+), \; y>0.$$

Then, we obtain the main theorem in this paper:

Theorem 2.1 (Main Theorem). *For any member $f \in H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$, the solution $u_f(t,x)$ of the initial value problem, for $t>0$*

$$(\partial_t - A_x)u_f(t,x) = 0 \tag{2.4}$$

satisfying the initial value condition

$$u_f(0,x) = f(x), \tag{2.5}$$

exists and it is given by

$$u_f(t,x) = (f(\cdot), \mathcal{K}_t(\cdot, x; \rho))_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}. \tag{2.6}$$

Here, the meaning of the initial value (2.5) is given by

$$\begin{aligned}\lim_{t\to+0} u_f(t,x) &= \lim_{t\to+0} (f(\cdot), \mathcal{K}_t(\cdot, x\rho))_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\ &= (f(\cdot), \mathcal{K}(\cdot, x\rho))_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\ &= f(x), \end{aligned} \tag{2.7}$$

whose existence is, in general, ensured and the limit is the uniformly convergence on any subset of $\mathbb{R}^+$ *such that* $\mathcal{K}(x, x; \rho)$ *is bounded.*

The uniqueness property of the initial value problem is depending on the completeness of the family of functions

$$\{\mathcal{K}_t(\cdot, x; \rho) : x \in \mathbb{R}^+\} \tag{2.8}$$

in $H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$.

In our theorem, the complete property of the solutions $u_f(t,x)$ of (2.4) and (2.5) satisfying the initial value f may be derived by the reproducing kernel Hilbert space admitting the kernel

$$k(x, t; y, \tau; \rho) := (\mathcal{K}_\tau(\cdot, y; \rho), \mathcal{K}_t(\cdot, x; \rho))_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}. \tag{2.9}$$

In our method, we see that the existence problem of the initial value problem is based on the eigenfunctions and we are constructing the desired solution satisfying the desired initial condition. For a larger knowledge for the eigenfunctions we can consider a more general initial value problem.

Furthermore, by considering the linear mapping of (2.6) with various situations, we will be able to obtain various inverse problems looking for the initial values f from the various out put data of $u_f(t,x)$.

3. Proof of the Theorem

The first, note that the kernel $\mathcal{K}_t(x, y; \rho)$ satisfies the operator equation (2.4) for any fixed y, because the functions

$$\exp\left\{-4\left(k - \frac{1}{2}\right)^2 t\right\} \overline{W_{-i\tau, k-\frac{1}{2}}(x)}$$

satisfy the operator equation and it is the summation. Similarly, the function $u_f(t,x)$ defined by (2.6) is the solution of the operator equation (2.4).

In order to see the initial value problem, we note the important general properties

$$\mathcal{K}_t(x, y; \rho) \ll \mathcal{K}(x, y; \rho); \tag{3.1}$$

that is, $\mathcal{K}(x, y; \rho) - \mathcal{K}_t(x, y; \rho)$ is a positive definite quadratic form function and we have

$$H_{\mathcal{K}_t(\rho)} \subset H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$$

and for any function $f \in H_{\mathcal{K}_t(\rho)}$

$$\|f\|_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} = \lim_{t\to+0} \|f\|_{H_{\mathcal{K}_t(\rho)}}$$

in the sense of non-decreasing norm convergence ([1]) . In order to see the crucial point in (2.7), note that

$$\begin{aligned}
&\|\mathcal{K}(x,y;\rho) - \mathcal{K}_t(x,y;\rho)\|^2_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\
&= \mathcal{K}(y,y;\rho) - 2\mathcal{K}_t(y,y;\rho) + \|\mathcal{K}_t(x,y;\rho)\|^2_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\
&\leq \mathcal{K}(y,y;\rho) - 2\mathcal{K}_t(y,y;\rho) + \|\mathcal{K}_t(x,y;\rho)\|^2_{H_{\mathcal{K}_t(\rho)}} \\
&= \mathcal{K}(y,y;\rho) - \mathcal{K}_t(y,y;\rho),
\end{aligned}$$

that converses to zero as $t \to +0$. We thus obtain the desired limit property in the theorem.

The uniqueness property of the initial value problem follows from (2.6) easily.

In the main theorem, the realization of the reproducing kernel Hilbert space $H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$ is a crucial point, for this purpose, we are interested in the calculation of the kernel $\mathcal{K}(x,y;\rho)$.

4. Concrete realization of the reproducing kernel Hilbert paces

As the theory of reproducing kernels, their realizations will give interesting research topics that are requested separate papers. So, here, we shall discuss the following concrete form of the reproducing kernels. From [2], we have

$$\begin{aligned}
&\int_0^{+\infty} |\Gamma(k+i\tau)|^2 W_{i\tau,k-\frac{1}{2}}(x)\overline{W_{i\tau,k-\frac{1}{2}}}(y)d\tau \\
&= \sqrt{\pi}\Gamma(2k)(xy)^k(x+y)^{-2k+1}K_{2k-\frac{1}{2}}\left(\frac{x+y}{2}\right), \qquad (4.1)
\end{aligned}$$

where $K_{2k-\frac{1}{2}}(z)$ denotes the modified Bessel function. Furthermore, since k, x are reals then $\overline{W_{i\tau,k-\frac{1}{2}}(x)} = W_{-i\tau,k-\frac{1}{2}}(x)$.

Moreover, note that:

$$W_{\mu,\nu}(0) = 0,\ -\frac{1}{2} < Re(\nu) < \frac{1}{2},$$

$$W_{\mu,\nu}(z) \sim z^\mu e^{-\frac{z}{2}}\left(1 + O\left(\frac{1}{z}\right)\right),\ |z| \to +\infty$$

We were able to realize the important reproducing kernel Hilbert space concretely and analytically. Meanwhile, we are also interested in the kernel forms $\mathcal{K}_t$ and K. There calculations will create a new and large field in integral formulas.

We have to analyze and realize the corresponding reproducing kernel Hilbert spaces, however, we can apply quite general formula by the Aveiro discretetization method in ([4]).

Proposition 4.1. *(***Ultimate realization of reproducing kernel Hilbert spaces***). In our general situation and for a uniqueness set* $\{p_j\}$ *for the reproducing kernel Hilbert space* H_K *of the set* E *satisfying the linearly independence of* $K(\cdot, p_j)$ *for any finite number of the points* p_j*, we obtain*

$$\|f\|_{H_K}^2 = \|\mathbf{f}^*\|_{\mathcal{H}}^2 = \lim_{n\to\infty} \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{4.2}$$

In this Proposition, for the uniqueness set of the space, if the reproducing kernel is analytical, then, the criteria will be very simple by the *identity theorem of analytic functions.* For the Sovolev space cases, we have to consider some dense subset of E for the uniqueness set. Meanwhile, the linearly independence will be easily derived from the integral representations of the kernels.

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