Computational Logic 2024/25
Proof System for Modal Logic

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Motivations for the Section

- This section introduces the **proof theory** for modal logic
- To that end, we introduce a **general notion of logic**;
- We study the **completeness** of modal logic through the construction of the **canonical model**
- For this construction, we carefully examine the notions of **sets of consistent formulas** and **maximally consistent formulas**
- We also study the **decidability** of modal logic using the technique of **filtrations**

Outline

Uniform Substitution

Substitution of a propositional variable by a formula

Let $p \in \mathrm{Prop}$ and $A, B \in \mathrm{MFm}(\mathrm{Prop})$:

- if $A \in \mathrm{Prop}$, then $S^p_B A = \begin{cases} B, & \text{if } A = p \\ A, & \text{if } A \neq p \end{cases}$
- $S^p_B \perp = \perp$
- If $A = C \to D$, then $S^p_B A = S^p_B C \to S^p_B D$
- If $A = \Box C$, then $S^p_B A = \Box S^p_B C$

Instance

A is said to be an instance of B if A is obtained by uniform substitution from B

e.g. $\Box A \vee \neg \Box A$ is an instance of the propositional formula $p \vee \neg p$

Notion of Logic

Definition.

A **logic** is a set $\Lambda \subseteq \mathrm{MFm}(\mathrm{Prop})$ such that:

- Λ contains all (modal) instances of classical tautologies, i.e., **all formulas obtained from a classical propositional tautology by uniform substitution**
- Λ is closed under Modus Ponens i.e., **if A and $A \rightarrow B$ are in Λ, then B is in Λ**

Notion of Logic

Proposition (Examples of Logics)

1. Let M be a model. Define $\Lambda_M = \{A \mid M \models A\}$. Then Λ_M is a logic.
2. Let $\mathcal{C}$ be a class of models. Define $\Lambda_\mathcal{C} = \{A \mid M \models A, M \in \mathcal{C}\}$. Then $\Lambda_\mathcal{C}$ is a logic.
3. Let $(\Lambda_i)_{i \in I}$ be a family of logics. Then $\bigcap_{i \in I} \Lambda_i$ is a logic.
4. ...

Exercise

Verify the above proposition.

Notion of Logic

Definitions.

Let Λ be a logic.

- The elements of Λ are called **Λ-theorems**.
  We write $\vdash_\Lambda A$ when A is a Λ-theorem, i.e., when $A \in \Lambda$.
- A is said to be **Λ-deducible from Γ** and we write $\Gamma \vdash_\Lambda A$, if
  - $\vdash_\Lambda A$ or
  - there exist $B_i \in \Gamma$, $i = 1, \dots, n$, such that

$$(B_1 \to (B_2 \to (\cdots \to (B_n \to A) \dots)) \in \Lambda$$

- A set of formulas $\Gamma \subseteq \mathrm{MFm}(\mathrm{Prop})$ is said to be **Λ-consistent** if

$$\Gamma \nvdash_\Lambda \bot$$

Notion of Logic

Definition

Let C be a class of structures or models and Λ a logic.

- Λ is said to be **sound with respect to** C if for every formula A,

$$\vdash_\Lambda A \Rightarrow C \models A$$

- Λ is said to be **complete with respect to** C if for every formula A,

$$C \models A \Rightarrow \vdash_\Lambda A$$

- Λ is said to be **characterized by** C if for every formula A,

$$C \models A \Leftrightarrow \vdash_\Lambda A$$

Exercise: Prove the following properties

Let Λ be a logic. Show that:

1. $\perp \notin \Lambda \Leftrightarrow \Lambda \neq \mathrm{MFm}(\mathrm{Prop})$
2. $\vdash_\Lambda A \Leftrightarrow \emptyset \vdash_\Lambda A$
3. If $\vdash_\Lambda A$ then $\Gamma \vdash_\Lambda A$
4. If $\Lambda \subseteq \Lambda'$ then $(\Gamma \vdash_\Lambda A \Rightarrow \Gamma \vdash_{\Lambda'} A)$
5. If $A \in \Gamma$ then $\Gamma \vdash_\Lambda A$
6. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash_\Lambda A$ then $\Delta \vdash_\Lambda A$
7. $\Gamma \cup \{A\} \vdash_\Lambda B \Leftrightarrow \Gamma \vdash_\Lambda A \to B$
8. If $\Gamma \vdash_\Lambda A$ and $\{A\} \vdash_\Lambda B$ then $\Gamma \vdash_\Lambda B$
9. If $\Gamma \vdash_\Lambda A$ and $\Gamma \vdash_\Lambda A \to B$ then $\Gamma \vdash_\Lambda B$
10. $\Gamma \vdash_\Lambda A$ if and only if there exists a finite sequence $A_0, A_1, \ldots, A_m = A$ such that for all $i \leq m$, each $A_i \in \Gamma \cup \Lambda$ or, otherwise, $A_k = (A_j \to A_i)$ for some $j, k < i$

Exercise: Prove the following properties

Let Λ be a logic. Show that:

11. If $M, w \models \Gamma \cup \Lambda$ and $\Gamma \vdash_\Lambda A$ then $M, w \models A$
[Note: If $\Delta \subseteq \mathrm{MFm}(\mathrm{Prop})$, then
$M, w \models \Delta \Leftrightarrow \forall A \in \Delta,\ (M, w \models A)$]
12. The set of classical tautologies is Λ-consistent, but $\mathrm{MFm}(\mathrm{Prop})$ is not Λ-consistent
13. Γ is Λ-consistent $\Leftrightarrow \exists A$ such that $\Gamma \nvdash_\Lambda A$
14. Γ is Λ-consistent $\Leftrightarrow \nexists A : (\Gamma \vdash_\Lambda A \ \&\ \Gamma \vdash_\Lambda \neg A)$
15. $\Gamma \vdash_\Lambda A \Leftrightarrow \Gamma \cup \{\neg A\}$ is not Λ-consistent
16. $\Gamma \cup \{A\}$ is Λ-consistent $\Leftrightarrow \Gamma \nvdash_\Lambda \neg A$
17. If Γ is Λ-consistent, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is Λ-consistent

Λ-Consistent Sets

Definition

Let $M = (W, R, V)$ be a model and $w \in W$. Define

$$\Gamma_w := \{A \in \mathrm{MFm}(\mathrm{Prop}) \mid M, w \models A\}$$

Proposition

The set Γ_w is

- Λ-consistent, and
- for each $A \in \mathrm{MFm}(\mathrm{Prop})$, either $A \in \Gamma_w$ or $\neg A \in \Gamma_w$

Maximally Consistent Sets

Definition

A set $\Gamma \subseteq \mathrm{MFm}(\mathrm{Prop})$ is said to be **maximally Λ-consistent** (or simply **Λ-maximal**) if:

- Γ is Λ-consistent, and
- for every $A \in \mathrm{MFm}(\mathrm{Prop})$, either $A \in \Gamma$ or $\neg A \in \Gamma$

Exercise – Properties

Let Γ be a Λ-maximal set. Show that:

19. $\Gamma \vdash_\Lambda A \Rightarrow A \in \Gamma$
20. $A \notin \Gamma \Rightarrow \Gamma \cup \{A\}$ is not Λ-consistent
21. For all $A \in \mathrm{MFm}(\mathrm{Prop})$, $A \notin \Gamma \Leftrightarrow \neg A \in \Gamma$
22. $\Lambda \subseteq \Gamma$
23. $\bot \notin \Gamma$
24. $(A \rightarrow B) \in \Gamma \Leftrightarrow (A \in \Gamma \Rightarrow B \in \Gamma)$
25. $(A \wedge B) \in \Gamma \Leftrightarrow A, B \in \Gamma$
26. $(A \vee B) \in \Gamma \Leftrightarrow (A \in \Gamma$ or $B \in \Gamma)$
27. $(A \leftrightarrow B) \in \Gamma \Leftrightarrow (A \in \Gamma \Leftrightarrow B \in \Gamma)$

Does Every Λ-Consistent Set Have a Λ-Maximal Extension?

Consider the following construction:

Let $A_1, A_2, A_3, \ldots$ be an enumeration of all formulas in $\mathrm{MFm}(\mathrm{Prop})$, and let Γ be a Λ-consistent set. Define the set

$$\Delta = \bigcup_{n \geq 0} \Delta_n$$

where:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{A_n\}, & \text{if } \Delta_n \cup \{A_n\} \text{ is } \Lambda\text{-consistent} \\ \Delta_n \cup \{\neg A_n\}, & \text{otherwise} \end{cases}$, for $n \geq 0$

Does Every Λ-Consistent Set Have a Λ-Maximal Extension?

Lemma

For every n, the set Δ_n is Λ-consistent.

Lemma

- For every formula A, exactly one of the formulas A or $\neg A$ is in Δ.
- If $\Delta \vdash_\Lambda B$, then $B \in \Delta$.

Lemma

Let $(\Sigma_i)_{i \in \mathbb{N}}$ be an increasing family of sets of formulas (i.e., $i < j \Rightarrow \Sigma_i \subseteq \Sigma_j$). If every Σ_i is Λ-consistent, then $\bigcup_{i \in \mathbb{N}} \Sigma_i$ is also Λ-consistent.

Does Every Λ-Consistent Set Have a Λ-Maximal Extension?

Lindenbaum Lemma

Every Λ-consistent set of formulas is contained in a Λ-maximal set.

Outline

Normal Logic

Definition

A logic Λ is called **normal** if it contains the axiom **K**:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

and is closed under the **Necessitation Rule**:

$$\vdash_\Lambda A \quad \Rightarrow \quad \vdash_\Lambda \Box A$$

The Logic K

Proposition.

Let $(\Lambda_i)_{i \in I}$ be a family of normal logics. Then $\bigcap_{i \in I} \Lambda_i$ is a normal logic.

The Logic K

The **Logic** K is called the smallest normal logic, i.e., the logic

$$K := \bigcap \{\Lambda \mid \Lambda \text{ is a normal logic}\}$$

Exercise

Show that

for any class C of models (or structures), $\Lambda_C = \{A \mid C \models A\}$ is a normal logic.

Properties

Exercise.

Let Λ be a normal logic. Then:

1. $\vdash_\Lambda A \rightarrow (B \rightarrow (A \wedge B))$
2. $\vdash_\Lambda \Box(A \rightarrow (B \rightarrow (A \wedge B)))$
3. $\vdash_\Lambda \Box(A \rightarrow (B \rightarrow (A \wedge B))) \rightarrow (\Box A \rightarrow \Box(B \rightarrow (A \wedge B)))$
4. $\vdash_\Lambda (\Box A \rightarrow \Box(B \rightarrow (A \wedge B)))$
5. $\vdash_\Lambda \Box(B \rightarrow (A \wedge B)) \rightarrow (\Box B \rightarrow \Box(A \wedge B))$
6. $\vdash_\Lambda \Box A \rightarrow (\Box B \rightarrow \Box(A \wedge B))$
7. $\vdash_\Lambda (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

Properties

Exercise.

Let Λ be a normal logic. Then:

1. $\vdash_\Lambda \Box\neg\neg A \leftrightarrow \Box A \quad \& \quad \vdash_\Lambda \Diamond\neg\neg A \leftrightarrow \Diamond A$
2. $\vdash_\Lambda A \rightarrow B \Rightarrow (\vdash_\Lambda \Box A \rightarrow \Box B \quad \& \quad \vdash_\Lambda \Diamond A \rightarrow \Diamond B)$
3. $\vdash_\Lambda A \leftrightarrow B \Rightarrow (\vdash_\Lambda \Box A \leftrightarrow \Box B \quad \& \quad \vdash_\Lambda \Diamond A \leftrightarrow \Diamond B)$
4. $\vdash_\Lambda \Diamond\neg A \leftrightarrow \neg\Box A$
5. $\vdash_\Lambda (\Box A \wedge \Box B) \leftrightarrow \Box(A \wedge B)$
6. $\vdash_\Lambda (\Diamond A \vee \Diamond B) \leftrightarrow \Diamond(A \vee B)$
7. $\vdash_\Lambda (\Box A \vee \Box B) \rightarrow \Box(A \vee B)$
8. $\vdash_\Lambda \Diamond(A \wedge B) \rightarrow (\Diamond A \wedge \Diamond B)$

Properties

Exercise.

Let Λ be a normal logic. Then:

1. $\vdash_\Lambda \Box\neg\neg A \leftrightarrow \Box A$
2. $\vdash_\Lambda \Diamond\neg\neg A \leftrightarrow \Diamond A$
3. $\vdash_\Lambda A \to B \Rightarrow \vdash_\Lambda \Diamond A \to \Diamond B$

Necessary Condition for the Normality of a Logic

Proposition.

If Λ is a normal logic, then $\Gamma \vdash_{\Lambda} A \Rightarrow \{\Box B | B \in \Gamma\} \vdash_{\Lambda} \Box A$.

Outline

Introduction

It is assumed in this section that Λ is a **consistent normal logic** (i.e., Λ is Λ-consistent)

Canonical Model

Definition

The **canonical model** of a consistent normal logic Λ is the structure

$$M^{\Lambda} = (W^{\Lambda}, R^{\Lambda}, V^{\Lambda})$$

where:

- $W^{\Lambda} = \{w \subseteq \mathrm{MFm}(\mathrm{Prop}) | w \text{ is } \Lambda\text{-maximal}\}$
- $wR^{\Lambda}v$ if and only if $\{A \in \mathrm{MFm}(\mathrm{Prop}) : \Box A \in w\} \subseteq v$
- $V^{\Lambda}(p) = \{w \in W^{\Lambda} | p \in w\}$

The **canonical structure** of Λ is defined as the pair $F^{\Lambda} = (W^{\Lambda}, R^{\Lambda})$.

Truth Lemma

Lemma.

For any $w \in W^\Lambda$, $A \in \mathrm{MFm}(\mathrm{Prop})$,

$$w \in [\![A]\!]_{M^\Lambda} \Leftrightarrow A \in w$$

Proof.

"$\Rightarrow$" **Case $A = \Box B$:**

$w \in [\![\Box B]\!]_{M^\Lambda}$

$\Leftrightarrow w \in \{w \in W^\Lambda \mid R^\Lambda[w] \subseteq [\![B]\!]_{M^\Lambda}\}$	(defn. of $[\![\_]\!]_{M^\Lambda}$)
$\Leftrightarrow w \in \{w \in W^\Lambda \mid \forall v \in W^\Lambda.\ wR^\Lambda v \Rightarrow v \in [\![B]\!]_{M^\Lambda}\}$	(simpl.)
$\Leftrightarrow w \in \{w \in W^\Lambda \mid \forall v \in W^\Lambda.\ wR^\Lambda v \Rightarrow B \in v)\}$	(Inductive Hypothesis)
$\Leftrightarrow w \in \{w \in W^\Lambda \mid \forall v \in W^\Lambda.\ \{C \mid \Box C \in w\} \subseteq v \Rightarrow B \in v\}$	(defn. R^Λ)
$\Leftrightarrow \forall v \in W^\Lambda.\ (\{C \mid \Box C \in w\} \subseteq v \Rightarrow B \in v)$	(simpl.)
$\Leftrightarrow B \in \cap\{v \in W^\Lambda \mid \{C \mid \Box C \in w\} \subseteq v\}$	(set theory)
$\Leftrightarrow \{C \mid \Box C \in w\} \vdash_\Lambda B$	(Corollary of Lindenbaum's Lemn
$\Rightarrow \{\Box C \mid \Box C \in w\} \vdash_\Lambda \Box B$	($\Gamma \vdash_\Lambda A \Rightarrow \{\Box B \mid B \in \Gamma\} \vdash_\Lambda \Box A$)
$\Rightarrow w \vdash_\Lambda \Box B$	($\Gamma \subseteq \Delta, \Gamma \vdash_\Lambda A \Rightarrow \Delta \vdash_\Lambda A$)
$\Leftrightarrow \Box B \in w$	(defn of Λ)

Other cases: Exercise □

Truth Lemma

Lemma.

For any $w \in W^\Lambda$ and $A \in \mathrm{MFm}(\mathrm{Prop})$,

$$w \in [\![A]\!]_{M^\Lambda} \Leftrightarrow A \in w$$

Proof.

"$\Leftarrow$" **Case $\Box B$**

Let us assume that $A = \Box B \in w$.

$$\begin{array}{llll} \Box B \in w & \Rightarrow & \forall v \in R^\Lambda[w].\ B \in v & (\text{defn } M) \\ & \Rightarrow & \forall v \in R^\Lambda[w].\ v \in [\![B]\!]_{M^\Lambda} & (\text{I.H.}) \\ & \Rightarrow & R^\Lambda[w] \subseteq [\![B]\!]_{M^\Lambda} & (\text{set theory}) \\ & \Leftrightarrow & w \in [\![\Box B]\!]_{M^\Lambda} & (\text{defn. } [\![\_]\!]_{M^\Lambda}) \end{array}$$

Other cases: Exercise □

Definition.

Let M be a model, F a structure, and C a class of models or structures. It is said that:

- **M determines Λ** if for every $A \in \mathrm{MFm}(\mathrm{Prop})$,

$$M \models A \Leftrightarrow \vdash_\Lambda A$$

- **F determines Λ** if for every $A \in \mathrm{MFm}(\mathrm{Prop})$,

$$F \models A \Leftrightarrow \vdash_\Lambda A$$

- **C determines Λ** if for every $A \in \mathrm{MFm}(\mathrm{Prop})$,

$$C \models A \Leftrightarrow \vdash_\Lambda A$$

Completeness

Corollary.

$$M^\Lambda \models A \Leftrightarrow \vdash_\Lambda A$$

Proof.

$$\begin{array}{lll}
 & M^\Lambda \models A & \\
\Leftrightarrow & [\![A]\!]_{M^\Lambda} = W^\Lambda & \text{(and defn. of } [\![_]\!]_{M^\Lambda}\text{)} \\
\Leftrightarrow & w \in [\![A]\!]_{M^\Lambda}, \text{ for all } w \in W^\Lambda & \text{(set equality)} \\
\Leftrightarrow & A \in w, \text{ for all } w \in W^\Lambda & \text{(Truth Lemma)} \\
\Leftrightarrow & A \in \bigcap W^\Lambda & \text{(set theory)} \\
\Leftrightarrow & A \in \Lambda & \text{(Lindenbaum's Lemma)}
\end{array}$$

□

Completeness

Theorem (Characterization of K)

$\vdash_K A \Leftrightarrow$ *A is valid in all structures.*

Examples of Normal Logics

To demonstrate that a normal logic Λ is complete with respect to a class of models (or modal structures) defined by certain properties, it is sufficient to show that M^{Λ} has that property.

Examples of Normal Logics

A few classes ago:

T :	$\Box A \to A$	valid in frames with reflexive relation
4 :	$\Box A \to \Box\Box A$	valid in frames with transitive relation
B :	$A \to \Box\Diamond A$	valid in frames with symmetric relation
$\vdots$		

Examples of Normal Logics

Theorem.

If a normal logic Λ contains one of the schemas from the previous slide, then R^Λ satisfies the corresponding property.

Proof.

Consider the case of transitivity:

- Suppose $\Box A \to \Box\Box A \in \Lambda$. Then, all the members of W^Λ contain all instances of this schema.
- Assuming $wR^\Lambda v$ and $vR^\Lambda u$. We have:

$$\Box A \in w \underset{MP}{\Rightarrow} \Box\Box A \in w \underset{(\text{def.} R^\Lambda)}{\Rightarrow} \Box A \in v \underset{(\text{def.} R^\Lambda)}{\Rightarrow} A \in u.$$

Thus, $wR^\Lambda u$.

Exercise: Prove the other cases. □

Examples of Normal Logics

$$\begin{aligned} T: &\quad \Box A \rightarrow A \\ 4: &\quad \Box A \rightarrow \Box\Box A \\ B: &\quad A \rightarrow \Box\Diamond A \\ W: &\quad \Box(\Box A \rightarrow A) \rightarrow \Box A \end{aligned}$$

Some well-known logics:

$$\begin{aligned} S4 = &\quad KT4 \\ S5 = &\quad KT4B \\ G = &\quad KW \end{aligned}$$