Computational Logic 2024/25
Basic notions in Modal Logic

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Outline

Syntax of Modal Logic

Symbols

- $\bot$, $\rightarrow$, $\Box$ and
- a set Prop of propositional symbols (we will use $p, q, r, \ldots$ to refer to elements of Prop)

- $\bot$ – called the element **falsum**
- $\rightarrow$ – called the **implication**

Modal Formulas

Definition: Modal Formulas

Modal Formulas ($\mathrm{MFm}(\mathrm{Prop})$):
$F_0 = \mathrm{Prop} \cup \{\bot\}$
$F_{n+1} = F_n \cup \{\Box A : A \in F_n\} \cup \{A \to B : A, B \in F_n\}, n \geq 0$
$$\mathrm{MFm}(\mathrm{Prop}) = \bigcup_{\mathrm{n} \in \mathrm{N}} \mathrm{F_n}$$

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$\mathrm{MFm(Prop)} = \bigcup_{\mathrm{n} \in \mathrm{N}} \mathrm{F_n}$

Using BNF notation:

$$\varphi := \bot \mid p \mid \varphi \to \varphi \mid \Box\varphi, p \in \mathrm{Prop}$$

Abbreviations

- **Negation:** $\neg A := A \to \bot$
- **True:** $\top := \neg\bot$
- **Disjunction**: $A \vee B := \neg A \to B$
- **Conjunction:** $A \wedge B := \neg(A \to \neg B)$
- **Equivalence:** $A \leftrightarrow B := (A \to B) \wedge (B \to A)$
- **Diamond:** $\Diamond A := \neg\Box\neg A$

Uniform Substitution and Instance

Definition

Let A and B be two formulas and $p \in \mathrm{Prop}$.

- The **uniform substitution** of p by B in A, denoted $A[B/p]$, is the formula obtained by replacing all occurrences of p in A by B.
- A' is an **instance** of A if A' is obtained from A by a finite number of simultaneous uniform substitutions.

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Example

- $((r \vee p) \wedge q) \to (r \vee p)$ is an instance of $(x \wedge y) \to x$, since

$$\big(((r \vee p) \wedge q) \to (r \vee p)\big) \equiv \big((x \wedge y) \to x\big)[(r \vee p)/x,\ q/y]$$

Sub-formula

Definition.

Given $A \in \mathrm{MFm}(\mathrm{Prop})$, the **set of subformulas of A**, $Sub(A)$, is defined recursively as follows:

- If $A \in \mathrm{Prop} \cup \{\bot\}$, then $Sub(A) = \{A\}$;
- If $A = B \to C$, then $Sub(A) = \{A\} \cup Sub(B) \cup Sub(C)$;
- If $A = \Box B$, then $Sub(A) = \{A\} \cup Sub(B)$.

Exercise:

Determine the set of subformulas of $\Box((\Diamond p) \to (r \to \Box(r \to q)))$

Outline

Structures and Models

A **modal structure** (or simply **structure**) for a set of propositions Prop is a pair

$$F = (W, R)$$

where

- W is a non-empty set, called the **set of states** (or worlds);
- $R \subseteq W \times W$ is a binary relation called the **accessibility relation**.

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A **model** is a triple

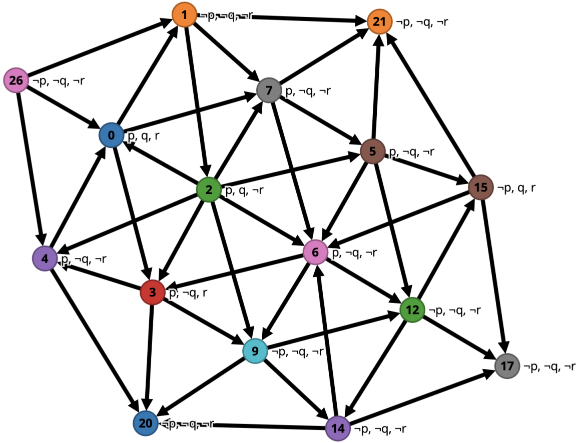
$$M = (W, R, V)$$

where

- (W, R) is a modal structure
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a function (called **valuation**).

Examples

Example of a Model for Prop $= \{p, q, r\}$



Satisfaction

Definition:

Let M be a model, $w \in W$, $p \in \mathrm{Prop}$ and $A, B \in \mathrm{MFm}(\mathrm{Prop})$. We define:

1. $M, w \models p$ iff $w \in V(p)$
2. $M, w \not\models \perp$
3. $M, w \models A \to B$ iff, $M, w \models A$ implies $M, w \models B$
4. $M, w \models \Box A$ iff, $\forall v \in W$, $wRv \Rightarrow M, v \models A$

Properties

Proposition:

1. $M, w \models \neg A$ iff $M, w \not\models A$
2. $M, w \models A \vee B$ iff $M, w \models A$ or $M, w \models B$
3. $M, w \models A \wedge B$ iff $M, w \models A$ and $M, w \models B$
4. $M, w \models A \leftrightarrow B$ iff $M, w \models A \Leftrightarrow M, w \models B$
5. $M, w \models \Diamond A$ iff there exists $v \in W$ such that wRv and $M, v \models A$

Exercise:

Prove the previous proposition.

Different Meanings for $\Box A$ and $\Diamond A$:

- **Necessity** - "*A* is necessarily true"; "*A* is possibly true".
- **Deontic Logic** - *A* must necessarily be true ("*A* ought to be true")
- **Temporal Logic** - "*A* is always true in the future"; "*A* is eventually true (at some future time)".
  Normally, here a structure (S, R) is such that S is $\mathbb{N}, \mathbb{R}$ or $\mathbb{Q}$ and R is $<, \leq, >$ or $\geq$.
- **Dynamic Logic** - "every execution of the program makes *A* true"; "there exists an execution of the program that makes *A* true"

Validity

Definition:

Let $F = (W, R)$ be a structure, $M = (W, R, V)$ a model, and $A \in \mathrm{MFm}(\mathrm{Prop})$ a formula.

- A is said to be **true** in M, in symbols $M \models A$, if for all states $w \in W$, we have $M, w \models A$.
- A is said to be **valid** in a structure F, in symbols $F \models A$, if it is true in all models M in F; i.e., for the valuation $V : \mathrm{Prop} \to \mathcal{P}(W)$, $M \models A$.

Exercises

1. Consider the model $M = (W, R, V)$, where:
   - $W = \{1, 2, 3, 4, 5, 6\}$
   - for any $a, b \in W$, $a\ R\ b$ iff $a + b$ is even;
   - $V(p) = \{1, 3, 5\}$ and $V(q) = \{2, 4, 6\}$.

   Verify whether

   a) $M, 3 \models \Diamond(p \wedge \Diamond q)$
   b) $M, 2 \models \Box(p \wedge \Diamond\Box q)$
   c) $M \models \Diamond p$
   d) $M, 6 \models p \rightarrow \Diamond p$
   e) $M \models q \rightarrow \Diamond q$
2. Show that neither the formula $\Box\Diamond p \rightarrow \Diamond\Box p$ nor its negation is valid in all modal structures.

Exercises

3. Explain in your own words the following transition patterns:
    - $\Box\top$
    - $\Diamond\bot$
    - $\Diamond\top$
    - $\Box\bot$

Properties

Proposition.

The following modal formulas are true in all models and therefore valid in all structures.

1. $\Box\top$
2. $\Box(A \to B) \to (\Box A \to \Box B)$
3. $\Diamond(A \to B) \to (\Box A \to \Diamond B)$
4. $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
5. $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$
6. $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$

Properties

Exercise

Prove that the following modal formulas do not have the property of being valid in all structures:

1. $\Box A \to A$;
2. $\Box A \to \Box\Box A$;
3. $\Box(A \to B) \to (\Box A \to \Diamond B)$
4. $\Diamond\top$
5. $\Diamond A \to \Box A$;
6. $\Box(\Box A \to B) \vee \Box(\Box B \to A)$
7. $\Box(A \vee B) \to (\Box A \vee \Box B)$;
8. $\Box(\Box A \to A) \to \Box A$;

Outline

Alternative Approach

Definition [Value of a formula in the model]

Let $A \in \mathrm{MFm(Prop)}$ and $M = (W, R, V)$ be a model. We define $[\![\_]\!]_M : \mathrm{MFm(Prop)} \to \mathcal{P}(\mathrm{W})$ as follows:

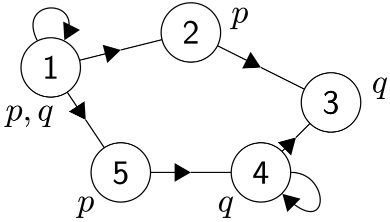
- If $A \in \mathrm{Prop}$, $[\![A]\!]_M = V(A)$
- If $A = \bot$, $[\![A]\!]_M = \emptyset$
- If $A = B \to C$, $[\![A]\!]_M = [\![B]\!]_M^C \cup [\![C]\!]_M$, where $[\![B]\!]_M^C$ denotes the set $W \setminus [\![B]\!]_M$
- If $A = \Box B$, $[\![A]\!]_M = \{w \in W : R[w] \subseteq [\![B]\!]_M\}$, where $R[w] = \{v | (w, v) \in R\}$

Properties

Proposition.

1. $[\![\neg A]\!]_M = [\![A]\!]_M^C$
2. $[\![\top]\!]_M = W$
3. $[\![A \vee B]\!]_M = [\![A]\!]_M \cup [\![B]\!]_M$
4. $[\![A \wedge B]\!]_M = [\![A]\!]_M \cap [\![B]\!]_M$
5. $[\![\Diamond A]\!]_M = \{w \in W | R[w] \cap [\![A]\!]_M \neq \varnothing\}$
6. $[\![A \rightarrow B]\!]_M = W \Leftrightarrow [\![A]\!]_M \subseteq [\![B]\!]_M$

Exercise



For the model M represented above, compute:

1. $[\![\Diamond\top]\!]_M$, $[\![\Box\top]\!]_M$, $[\![\Diamond\bot]\!]_M$, $[\![\Box\bot]\!]_M$
2. $[\![\Box p]\!]_M$, $[\![\Box\neg p]\!]_M$, $[\![\neg\Box p]\!]_M$
3. $[\![\Diamond q]\!]_M$, $[\![\Box q]\!]_M$
4. $[\![\Diamond q \wedge \Diamond\neg q]\!]_M$, $[\![\Box q \wedge \Box\neg q]\!]_M$
5. $[\![\Diamond\top \to \Diamond q]\!]_M$

Equivalence of the Approaches

Proposition.

$$M, w \models A \Leftrightarrow w \in [\![A]\!]_M$$

Demonstração.

(**Exercise:** Prove the result by induction on the structure of formulas) □

Equivalence of Approaches

Corollary.

$$M \models A \Leftrightarrow W = [\![A]\!]_M$$

Demonstração.

$M \models A \Leftrightarrow \forall w \in W, M, w \models A \Leftrightarrow \forall w \in W, w \in [\![A]\!]_M \Leftrightarrow W = [\![A]\!]_M$. □

Exercise

Revisit the Exercise from Slide 17 and characterize the set of states where the properties are valid, now using the operator $[\![\_]\!]_M$.

Modal Equivalence

Modal Equivalence

Two modal formulas A and A' are **modally equivalent** if for any model M and any $w \in W$,

$$M, w \models A \text{ if and only if } M, w \models A'$$

Proposition.

1. $\Diamond\top$ and $\Box A \to \Diamond A$ are valid exactly in the same models.
2. $\Box\bot$ is valid only in structures where all of its points are final.

Outline

Modal Logic vs First-Order Logic

The following discusses some relationships between **Modal Logic and First-Order Logic**. Specifically:

- properties of binary relations that **can be** described in modal formulas
- properties of binary relations that **cannot be** described in modal formulas
- encoding Modal Logic in First-Order Logic

Exercise

1. Using **First-Order Logic**, express the following properties about a binary relation $R \subseteq W \times W$.
    - R is reflexive
    - R is symmetric
    - R is serial
    - R is transitive
    - R is Euclidean
    - R is partially functional
    - R is functional

Exercise

1. Using **First-Order Logic**, express the following properties about a binary relation $R \subseteq W \times W$.
   - R is reflexive
   - R is symmetric
   - R is serial
   - R is transitive
   - R is Euclidean
   - R is partially functional
   - R is functional
2. Express the same properties using **Modal Logic** on structures

Theorem.

Let $R \subseteq W \times W$. R satisfies the first-order formula on the left if and only if **all** models $M = (W, R, V)$ satisfy the formula scheme on the right.

The Relation R is	**Formula**
Reflexive: $\forall w(wRw)$	$\Box A \rightarrow A$
Symmetric: $\forall w, \forall v(wRv \rightarrow vRw)$	$A \rightarrow \Box\Diamond A$
Serial: $\forall w \exists v(wRv)$	$\Box A \rightarrow \Diamond A$
Transitive: $\forall w \forall v \forall z(wRv \wedge vRz \rightarrow wRz)$	$\Box A \rightarrow \Box\Box A$
Euclidean: $\forall w \forall v \forall z(wRv \wedge wRz \rightarrow vRz)$	$\Diamond A \rightarrow \Box\Diamond A$
Partially Functional: $\forall w, \forall v, \forall z(wRv \wedge wRz \rightarrow w = z)$	$\Diamond A \rightarrow \Box A$
Functional: $\forall w \exists! v(wRv)$	$\Diamond A \leftrightarrow \Box A$
Weakly Dense: $\forall w \forall v(wRv \rightarrow \exists z(wRz \wedge zRv))$	$\Box\Box A \rightarrow \Box A$
Weakly Connected: $\forall w \forall v \forall z(wRv \wedge wRz \rightarrow vRz \vee v = z \vee zRv)$	$\Box(A \wedge \Box A \rightarrow B) \vee \Box(B \wedge \Box B \rightarrow A)$
Weakly Directed: $\forall w \forall v \forall z(wRv \wedge wRz \rightarrow \exists u(vRu \wedge zRu))$	$\Diamond\Box A \rightarrow \Box\Diamond A$

There are properties that cannot be expressed with modal formulas:

Theorem.

The following properties of binary relations cannot be defined using Modal Logic:

- $\forall x \neg(xRx)$ (Irreflexivity)
- $\forall x \forall y \, xRy \vee yRx \vee x = y$ (Trichotomy)

Standard Translation

Standard Translation

A formula A can be translated into a First-Order Logic formula $ST_x(A)$ with at most one free variable x. $ST_x(A)$ is defined inductively on the structure of A, as follows:

- $ST_x(p) = P(x)$
- $ST_x(\bot) = \bot$
- $ST_x(A \to B) = ST_x(A) \to ST_x(B)$
- $ST_x(\Box A) = \forall y.(R(x,y) \to ST_y(A))$

Proposition.

- $ST_x(\top) = \top$
- $ST_x(A \vee B) = ST_x(A) \vee ST_x(B)$, $ST_x(A \wedge B) = ST_x(A) \wedge ST_x(B)$
- $ST_x(\Diamond A) = \exists y.(R(x,y) \wedge ST_y(A))$
- $ST_x(\neg A) = \neg ST_x(A)$

Exercises

Calculate:

- $ST_x(\neg\Box\neg p)$
- $ST_x(\Box p \to p)$
- $ST_x(\Box(p \vee \Diamond q))$

ST can also be applied (partially) to formula schemes.

Calculate:

- $ST_x(\Diamond A \to B)$

Standard Translation

Theorem.

Let M be a model. Then

$$M, w \models A \text{ iff } \bar{M} \models^{FOL} ST_x(A)[x \mapsto w]$$

where $\bar{M}$ is the corresponding first-order model to M, and $\models^{FOL}$ represents the satisfaction relation in First-Order Logic.

Corollary

Let M be a model. Then

$$M \models A \text{ iff } \bar{M} \models^{FOL} \forall x.ST_x(A)$$

where $\bar{M}$ is the corresponding first-order model to M.

Outline

Bisimulation

Bisimulation

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models for Prop. A **bisimulation between M and M'**, denoted $B : M \rightleftharpoons M'$, is a relation $B \subseteq W \times W'$ such that, for all $(w, w') \in B$:

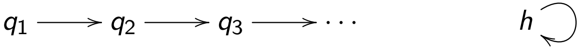
(Atom) $w \in V(p)$ iff $w' \in V'(p)$, for all $p \in \mathrm{Prop}$

(Zig) if $(w, v) \in R$ then there exists a $w' \in W'$ such that $(w', v') \in R'$ and $(v, v') \in B$

(Zag) if $(w', v') \in R'$ then there exists a $w \in W$ such that $(w, v) \in R$ and $(v, v') \in B$

Examples

- the relation $=$ is a bisimulation
- $\emptyset$ is a bisimulation



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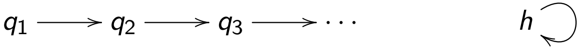
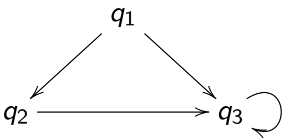
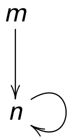


Exercise:

Consider the models M and M' for $\mathrm{Prop} = \{p\}$ represented below. Suppose that $V'(p) = \{n\}$.

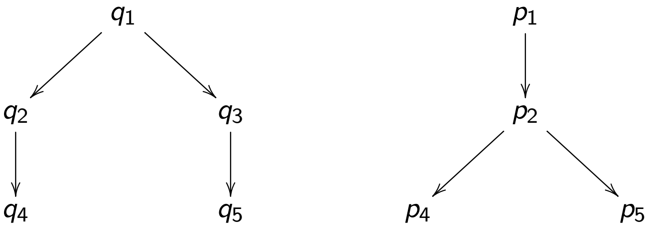




Determine V such that there exists a bisimulation $B : M \rightleftharpoons M'$ with $(q_1, n) \in B$.

Examples

Consider the following models M and M'



where $V(p) = \{q_4\}$, $V(q) = \{q_5\}$, $V'(p) = \{p_4\}$, $V'(q) = \{p_5\}$. Is there a bisimulation B such that $(q_1, p_1) \in B$?

Bisimulation

Properties

1. If $B : M \rightleftharpoons M'$ and $S : M' \rightleftharpoons M''$, then $B \circ S : M \rightleftharpoons M''$, where $B \circ S = \{(w, w'') \mid$ there exists a w' such that $(w, w') \in B$ and $(w', w'') \in S\}$
2. If $B : M \rightleftharpoons M'$, then $B^\circ : M' \rightleftharpoons M$ where $B^\circ = \{(w', w) \mid (w, w') \in B\}$ is a bisimulation
3. If $B : M \rightleftharpoons M'$ and $S : M \rightleftharpoons M'$, then $B \cup S : M \rightleftharpoons M'$

Exercise

Prove the previous properties

Properties of Bisimulation

Exercise

"The intersection of bisimulations is a bisimulation."

Prove or disprove the above statement.

Bisimilarity

Bisimilarity

Let $M = (W, R, V)$, the **bisimilarity in M** is the relation

$$\sim := \{(w, v) \mid (w, v) \in B \text{ for some bisimulation } B \subseteq W \times W\}$$

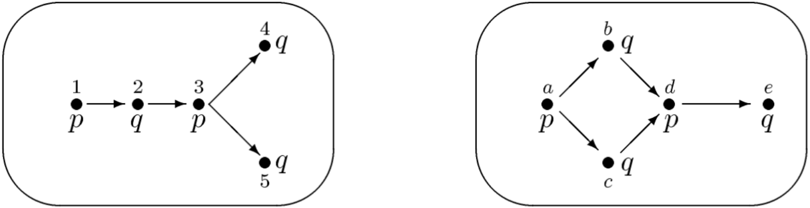
Exercises:

Prove that the relation $\sim \subseteq W \times W$

- is a bisimulation between M and M, i.e., $\sim : M \rightleftharpoons M$
- is an equivalence relation in W

Exercise

Say, justifying, if $1 \sim a$



[a]

[a]Example taken from the book "Modal Logic. P. Blackburn, M. Rijke, and Y. Venema"

Modal Invariance

Theorem (Modal Invariance)

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models for Prop, and $B : M \rightleftharpoons M'$ be a bisimulation. Then, for any states $w \in W$, $w' \in W'$ such that $(w, w') \in B$, and for any formula $\varphi \in \mathrm{MFm}(\mathrm{Prop})$,

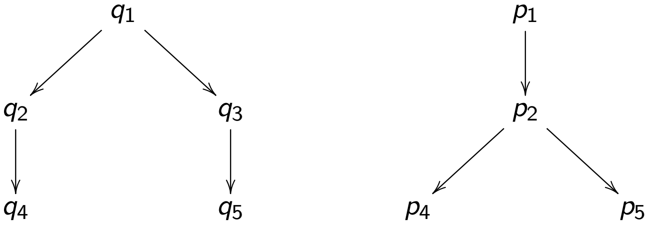
$$M, w \models \varphi \textbf{ iff } M', w' \models \varphi$$

Proof: Exercise (induction on the structure of formulas).

Consequences

- To prove that two states are not bisimilar, it suffices to find a modal formula that distinguishes them, i.e., that is true in one and false in the other.

Consider the following models M and M'



where $V(p) = \{q_4\}$, $V(q) = \{q_5\}$, $V'(p) = \{p_4\}$, $V'(q) = \{p_5\}$.
Distinguish the states q_1 and p_1 with modal formulas.

Hennessy-Milner Theorem

Finite Image Model

A model $M = (W, R, V)$ is said to have a **finite image** if for every $w \in W$, the set $R[w] = \{v \mid (w, v) \in R\}$ is finite.

Hennessy-Milner Theorem

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Hennessy-Milner Theorem

Let M and M' be two finite image models. Then, for any $w \in W$ and $w' \in W'$, the following conditions are equivalent:

1. There exists a bisimulation $B : M \rightleftharpoons M'$ such that $(w, w') \in B$
2. For all $\varphi \in \mathrm{MFm}(\mathrm{Prop})$,

$$M, w \models \varphi \text{ iff } M', w' \models \varphi$$