Computational Logic 2024/25 Reviews – Propositional and First-Order Logic

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March 5, 2025

Outline

1. Review: Propositional Logic
2. Revisions: First-Order Logic

Syntax of Prop

Symbols

- $\bot$, $\rightarrow$
- A set Prop of propositional symbols (we will use $p, q, r, \ldots$ to refer to elements of Prop)

- $\bot$ – called **falsum**
- $\rightarrow$ – called **implication**

Formulas

Set of propositional formulas $\mathrm{Fm}(\mathrm{Prop})$

$F_0 = \mathrm{Prop} \cup \{\bot\}$
$F_{n+1} = F_n \cup \{A \to B : A, B \in F_n\}, n \geq 0$
$\mathrm{Fm}(\mathrm{Prop}) = \bigcup_{n \in N} F_n$

Or, using BNF syntax:

Set of propositional formulas $\mathrm{Fm}(\mathrm{Prop})$

$$\varphi := \bot \mid p \mid \varphi \to \varphi$$

where $p \in \mathrm{Prop}$

Formulas

Set of propositional formulas Fm(Prop)

$F_0 = \mathrm{Prop} \cup \{\bot\}$
$F_{n+1} = F_n \cup \{A \to B : A, B \in F_n\}, n \geq 0$
$\mathrm{Fm}(\mathrm{Prop}) = \bigcup_{n \in N} F_n$

Or, using BNF syntax:

Set of propositional formulas Fm(Prop)

$$\varphi := \bot \mid p \mid \varphi \to \varphi$$

where $p \in \mathrm{Prop}$

Exercise

- $\mathrm{Fm}(\emptyset)$
- $\mathrm{Fm}(\{p, q\})$

Interpretation of Formulas

Definition

A **valuation of** Prop is a function

$$f : \mathrm{Prop} \longrightarrow \{0, 1\}.$$

Definition

Let f be a valuation and A a formula. The **value** $\overline{f}$ **of a formula** $A \in \mathrm{Fm}(\mathrm{Prop})$ **under** f is defined by:

$$\overline{f} : \mathrm{Fm}(\mathrm{Prop}) \longrightarrow \{0, 1\}.$$

$$\begin{array}{lcll} \overline{f}(p) & = & f(p), & p \in \mathrm{Prop} \\ \overline{f}(\bot) & = & 0 & \\ \overline{f}(A \to B) & = & \overline{f}(A) \Rightarrow \overline{f}(B), & \end{array}$$

where

$\Rightarrow$	0	1
0	1	1
1	0	1

Satisfaction

Let $A \in \mathrm{Fm}(\mathrm{Prop})$, $\Gamma \subseteq \mathrm{Fm}(\mathrm{Prop})$, and $f : \mathrm{Prop} \to \{1, 0\}$ be a valuation. Then:

- f **satisfies** A, in symbols
  $$f \models A$$
  if $\bar{f}(A) = 1$;
- f **satisfies** Γ, in symbols
  $$f \models \Gamma$$
  if for every $A \in \Gamma$, $f \models A$;
- We write
  $$\models A$$
  if for every valuation $f : \mathrm{Prop} \to \{0, 1\}$, $f \models A$.

Formulas

Abbreviations

Negation:

$$\neg A := A \to \bot$$

Verum:

$$\top := \neg\bot$$

Disjunction:

$$A \vee B := \neg A \to B$$

Conjunction:

$$A \wedge B := \neg(A \to \neg B)$$

Equivalence:

$$A \leftrightarrow B := (A \to B) \wedge (B \to A)$$

Exercises

Verify that:

$f \models \neg A$ if and only if $f \not\models A$

$f \models A \vee B$ if and only if $f \models A$ or $f \models B$

$f \models A \wedge B$ if and only if $f \models A$ and $f \models B$

Tautology, Contradiction, and Contingency

Definition

We say that:

- A is a **tautology** if and only if $\models A$
- A is a **contradiction** if and only if for every valuation $f : \mathrm{Prop} \to \{0,1\}$, $f \not\models A$
- A is a **contingency** if and only if there exist valuations $f, g : \mathrm{Prop} \to \{0,1\}$ such that $f \models A$ and $g \not\models A$

Examples

Exercise: Identify the tautologies, contingencies, and contradictions:

- $(A \leftrightarrow (\neg B \vee C)) \rightarrow (\neg A \rightarrow B)$
- $\big(A \rightarrow (B \vee C)\big) \vee (A \rightarrow B)$
- $\neg(A \vee \neg A)$
- $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

SAT Problem

Boolean Satisfiability Problem (SAT Problem)

Given a propositional formula, decide whether there exists a valuation that satisfies it.

- There are numerous computational problems (program verification, artificial intelligence, operations research, etc.) that can be encoded as a SAT problem.
- There are very efficient algorithms to handle this problem, even when involving a considerable number of variables.
- Extensions:
  - **MAX-SAT** – the maximum number of satisfiable clauses in a formula.
  - **SMT** – (Satisfiability Modulo Theories)

Exercise

Neller, Markov, Russel. Clue Deduction: Professor Plum Teaches Logic (2016)

Suppose that liars always speak what is false, and truth-tellers always speak what is true. Further suppose that Amy, Bob, and Cal are each either a liar or truth-teller. Amy says, "Bob is a liar." Bob says, "Cal is a liar." Cal says, "Amy and Bob are liars." Which, if any, of these people are truth-tellers?

Exercise

Logic in Action. van Benthem et al. 2016

You want to throw a party, respecting people's incompatibilities. You know that:

- John comes if Mary or Ann comes.
- Ann Comes if Mary does not come.
- If Ann comes, John does not.

Can you invite people under these constraints?

The Calculus of Prop

Axioms: All formulas of the form

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Inference Rule: (Modus Ponens)

$$\text{(MP)} \quad \frac{A \quad A \to B}{B}$$

Proof in Prop

Definition

Let $A_1, \ldots, A_n$ be a sequence of formulas of Prop.
$A_1, \ldots, A_n$ is a **demonstration** or **proof** (*in* Prop) if for each $i \in \{1, \ldots, n\}$:

(i) A_i is an axiom, or

(ii) A_i is inferred from two previous formulas in the sequence using the inference rule modus ponens.

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $$\frac{A \quad A \to B}{B}$$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (A2)

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (A2)
3. $((p \to (q \to p)) \to (p \to p))$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1.	$p \to ((q \to p) \to p)$	(A1)
2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	(A2)
3.	$((p \to (q \to p)) \to (p \to p))$	(MP)1.,2.

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $$\frac{A \quad A \to B}{B}$$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (A2)
3. $((p \to (q \to p)) \to (p \to p))$ (MP)1.,2.
4. $p \to (q \to p)$

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1. $p \to ((q \to p) \to p)$ (A1)
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (A2)
3. $((p \to (q \to p)) \to (p \to p))$ (MP)1.,2.
4. $p \to (q \to p)$ A1

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1.	$p \to ((q \to p) \to p)$	(A1)
2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	(A2)
3.	$((p \to (q \to p)) \to (p \to p))$	(MP)1.,2.
4.	$p \to (q \to p)$	A1
5.	$p \to p$	

Example

Propositional Calculus Prop

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

(MP) $\dfrac{A \quad A \to B}{B}$

Proving in Prop that $\vdash p \to p$

1.	$p \to ((q \to p) \to p)$	(A1)
2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	(A2)
3.	$((p \to (q \to p)) \to (p \to p))$	(MP)1.,2.
4.	$p \to (q \to p)$	A1
5.	$p \to p$	(MP)3.,4.

Exercise

Exercise

Prove in Prop that

$$B \to C, C \to D \vdash B \to D$$

Deduction Lemma (Herbrand, 1930)

Theorem:

Let $B, C \in \mathrm{Fm}(\mathrm{Prop})$, $\Gamma \subseteq \mathrm{Fm}(\mathrm{Prop})$. Then:

- If $\Gamma, B \vdash C$, then $\Gamma \vdash B \to C$.
- In particular, if $B \vdash C$, then $\vdash B \to C$.

Deduction Lemma (Herbrand, 1930)

Theorem:

Let $B, C \in \mathrm{Fm}(\mathrm{Prop})$, $\Gamma \subseteq \mathrm{Fm}(\mathrm{Prop})$. Then:

- If $\Gamma, B \vdash C$, then $\Gamma \vdash B \to C$.
- In particular, if $B \vdash C$, then $\vdash B \to C$.

Exercise

Prove that

$$B \to C, C \to D \vdash B \to D$$

using the Deduction Theorem.

Soundness and Completeness of Propositional Calculus

Theorem (Soundness of Propositional Calculus)

$$\vdash A \implies \models A.$$

Theorem (Completeness of Propositional Calculus)

$$\models A \implies \vdash A.$$

Final Note

- Logical calculi characterized by a **large number of axioms and a small number of inference rules** are referred to as **Hilbert calculi** (such as the Prop Calculus presented earlier).
- There are other **equivalent representations** in the literature, e.g., those that include additional axioms based on conjunctions and negations, or on disjunctions and negations.
- Alternatively, we can consider logical calculi with a **small number of axioms and a large number of inference rules**. These are called **natural deduction calculi**.
- In these calculi, proofs are more similar to the natural way we reason.

Final Note

Natural Deduction Calculus for Propositional Logic

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge\mathrm{I} \qquad \frac{\varphi \wedge \psi}{\varphi}\wedge\mathrm{E}_1 \qquad \frac{\varphi \wedge \psi}{\psi}\wedge\mathrm{E}_2$$

$$\frac{\varphi}{\varphi \vee \psi}\vee\mathrm{I}_1 \qquad \frac{\psi}{\varphi \vee \psi}\vee\mathrm{I}_2 \qquad \frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{\theta}\vee\mathrm{E}$$

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \to \psi}\Rightarrow\mathrm{I} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\Rightarrow\mathrm{E}$$

$$\frac{\bot}{\varphi}\bot\mathrm{E} \qquad \frac{}{\varphi \vee \neg\varphi}\mathrm{EM}$$

Exercise

Using natural deduction, prove

- $\vdash p \to p$
- $B \to C, C \to D \vdash B \to D$
- $\neg\neg A \vdash A$

Outline

Context

- A multi-sorted version of first-order logic is presented
- Logic is widely used in the specification of abstract data structures, adopted in specification processes

Signature and Formulas

Signature

A signature in First-Order Logic consists of a triple

$$\Sigma = (S, F, P)$$

where

- S is a set of sorts
- F is an $S^* \times S$-family of function symbols
- P is an S-family of predicate symbols

We use $f : s_1 \times \cdots \times s_n \to s \in F$ to denote that $f \in F_{s_1 \cdots s_n, s}$ and $p : s_1 \times \cdots \times s_n$ to denote $p \in P_{s_1 \cdots s_n}$

Example of Signature $\Sigma = (S, F, P)$

Representation 1

- $S = \{s_1, s_2\}$
- $F_{\epsilon,s_1} = \{c_1\}$, $F_{\epsilon,s_2} = \{c_2\}$
- $F_{s_1,s_1} = \{f\}$, $F_{s_2 s_1,s_1} = \{g\}$
- $F_{\omega,s} = \emptyset$ for other $\omega \in S^*$, $s \in S$
- $P_{s_1 \times s_1} = \{r\}$ and $P_\omega = \emptyset$ for other $\omega \in S^*$

Example of Signature $\Sigma = (S, F, P)$

Representation 1

- $S = \{s_1, s_2\}$
- $F_{\epsilon,s_1} = \{c_1\}$, $F_{\epsilon,s_2} = \{c_2\}$
- $F_{s_1,s_1} = \{f\}$, $F_{s_2 s_1,s_1} = \{g\}$
- $F_{\omega,s} = \emptyset$ for other $\omega \in S^*$, $s \in S$
- $P_{s_1 \times s_1} = \{r\}$ and $P_\omega = \emptyset$ for other $\omega \in S^*$

Representation 2

ST	s_1
	s_2
OP	$c_1 :\rightarrow s_1$
	$c_2 :\rightarrow s_2$
	$f : s_1 \rightarrow s_1$
	$g : s_2 \times s_1 \rightarrow s_1$
RL	$r : s_1 \times s_1$

EXAMPLES

Monoids

ST s;

OP $\cdot : s \times s \to s$

$e :\to s$

Examples

Monoids

St s;

Op $\cdot : s \times s \to s$

$e : \to s$

Natural Numbers

St Nat;

Op $suc : Nat \to Nat$

$+ : Nat \times Nat \to Nat$

$\ldots$

Rl $\leq : Nat \times Nat$

$\ldots$

First-Order Structures

Σ-Structures

Let $\Sigma = (S, F, P)$ be a signature. A **Σ-structure** A consists of:

- an S-set $|A|$, i.e., for each $s \in S$, $|A|_s$ is a set;
- for each symbol $f : s_1 \times \cdots \times s_n \to s \in \Sigma$, a function $f^A : |A|_{s_1} \times \cdots \times |A|_{s_n} \to |A|_s$;
- for each symbol $r : s_1 \times \cdots \times s_n$, a set $r^A \subseteq |A|_{s_1} \times \cdots \times |A|_{s_n}$.

Examples of Σ-Structures

Consider the signature Σ:

SORTS s_1, s_2

OP $c_1 :\to s_1,\ c_2 :\to s_2$

$f : s_1 \to s_1$

$g : s_2 \times s_1 \to s_2$

Two examples of Σ-algebras:

$|A|_{s_1} = \{a, b\},\ |A|_{s_2} = \{1, 2, 3\}$
$c_1^A = a,\ c_2^A = 3$
$f^A(a) = a,\ f^A(b) = a$
$g^A = \{(1, a) \mapsto 1, (1, b) \mapsto 1, (2, a) \mapsto 2, (2, b) \mapsto 2, (3, a) \mapsto 3, (3, b) \mapsto 3\}$

Examples of Σ-Structures

Consider the signature Σ:

SORTS s_1, s_2

OP $c_1 :\to s_1,\ c_2 :\to s_2$

$f : s_1 \to s_1$

$g : s_2 \times s_1 \to s_2$

Two examples of Σ-algebras:

$|A|_{s_1} = \{a, b\},\ |A|_{s_2} = \{1, 2, 3\}$
$c_1^A = a,\ c_2^A = 3$
$f^A(a) = a,\ f^A(b) = a$
$g^A = \{(1, a) \mapsto 1, (1, b) \mapsto 1, (2, a) \mapsto 2, (2, b) \mapsto 2, (3, a) \mapsto 3, (3, b) \mapsto 3\}$

$|B|_{s_1} = \{\bullet\},\ |B|_{s_2} = \{\heartsuit, \spadesuit\}$
$c_1^B = \bullet,\ c_2^B = \spadesuit$
$f^B(\bullet) = \bullet$
$g^B = \{(\heartsuit, \bullet) \mapsto \heartsuit, (\spadesuit, \bullet) \mapsto \spadesuit\}$

Exercise

Exercise

Define a Σ-structure for the signature of monoids and natural numbers defined earlier.

First-Order Formulas

Σ-terms

Let Σ be a signature and $X = (X_s)_{s \in S}$ an S-set of variables for Σ. The **set of Σ-terms in X** is the smallest S-set $T(\Sigma, X)_s$ such that:

- $X_s \subseteq T(\Sigma, X)_s$; (variables)
- $F_{\epsilon,s} \subseteq T(\Sigma, X)_s$; (constants)
- For any $f : s_1 \times \cdots \times s_n \to s \in \Sigma$ and $t_1 \in T(\Sigma, X)_{s_1}, \ldots, t_n \in T(\Sigma, X)_{s_n}$, $f(t_1, \ldots, t_n) \in T(\Sigma, X)_s$;

Exercise

Enumerate the terms of the signatures defined earlier.

Σ-formulas

Let $\Sigma = (S, F, P)$ be a signature. The **set $\mathrm{Fm}(\Sigma)$ of Σ-formulas** is defined by the following grammar:

$$\varphi ::= \bot \mid t_s^1 \approx t_s^2 \mid r(t_1, \ldots, t_n) \mid \varphi_1 \to \varphi_2 \mid \forall x : s \,.\, \varphi$$

where $t_i \in T(\Sigma, X)_{s_i}$, $i \in 1, \ldots, n$ and $x \in X$

Abbreviations

Negation:

$$\neg\varphi := \varphi \to \bot$$

Verum:

$$\top := \neg\bot$$

Disjunction:

$$\varphi \vee \varphi' := \neg\varphi \to \varphi'$$

Conjunction:

$$\varphi \wedge \varphi' := \neg(\varphi \to \neg\varphi')$$

Equivalence:

$$\varphi \leftrightarrow \varphi' := (\varphi \to \varphi') \wedge (\varphi \to \varphi')$$

Existential Quantification:

$$\exists x.\varphi := \neg\forall x.\neg\varphi$$

Interpretation of Terms

A valuation for an S-set of variables in a (S, F, P)-structure A

is an S-function $v : X \to |A|$, i.e., an S-family of functions $v_s : X_s \to |A|_s$

The interpretation of a (S, F, P)-term t

in a (S, F, P)-structure A and a valuation $v : X \to |A|$ is defined recursively:

- $t_A^v = c^A$, if $t = c$ (constant)
- $t_A^v = v(x)$, if $t = x$ (variable)
- $t_A^v = f^A({t_1}_A^v, \ldots, {t_n}_A^v)$, if $t = f(t_1, \ldots, t_n)$

Satisfaction

Definition

Let Σ be a signature and A a Σ-structure and $v : X \to |A|$ a valuation for A. The **satisfaction relation of a formula $\varphi \in \mathrm{Fm}(\Sigma)$ in a structure A** is defined recursively as follows:

- $A, v \models t \approx t'$ if $t_v^A = {t'}_v^A$;
- $A, v \models r(t_1, \ldots, t_n)$ if ${t_1}_v^A \times \cdots \times {t_n}_v^A \in r^A$
- $A, v \models \neg\varphi$ if it is false that $A, v \models \varphi$;
- $A, v \models \varphi_1 \to \varphi_2$ if $A, v \models \varphi_1$ implies $A, v \models \varphi_2$;
- $A, v \models \forall x : s \,.\, \varphi$ if $A, v\{x : s \mapsto a\} \models \varphi$ for all $a \in A_s$, where $v\{x : s \mapsto a\}(x) = a$ and $v\{x : s \mapsto a\}(y) = v(y)$ for $y \neq x$.

We write $A \models \varphi$ if for every valuation $v : X \to |A|$, $A, v \models \varphi$.

Examples of Specifications in FOL: Groups

```
Spec GRUPO  =
[S]
     elt;
[F]

0+: -> elt;

(-_):elt -> elt;

+:elt,elt -> elt;

[AX]

(∀a,b,c:elt).(a+b)+c=a+(b+c);

(∀a:elt).a+0+=0++a=a;

(∀a:elt).a+(-a)=0+;

(∀a:elt).(-a)+a=0+;
```

Examples of Specifications in FOL: Abelian Groups

```
Spec ABEL = enrich GRUPO by
[AX]
(∀a,b:elt).a+b=b+a;
```

Examples of Specifications in FOL: Abelian Groups

```
Spec ABEL = enrich GRUPO by
[AX]
```

(∀a,b:elt).a+b=b+a;

```
Spec ANEL  = enrich ABEL by
[F]
    * :elt,elt -> elt;
    1:         ->elt;
[AX]
```

(∀a:elt).(a*1)=a;

(∀a:elt).(1*a)=a;

(∀a,b,c:elt).(a*b)*c=a*(b*c);

(∀a,b,c:elt).a*(b+c)=a*b+a*c;

(∀a,b,c:elt).(a+b)*c=a*c+b*c;

Examples of Specifications in FOL: Lists

```
Spec LISTs =
[S]
    list, elt
[F]

     nil:  -> list;

     _._:elt,list -> list;

     _::_:list,list -> list;

[AX]
```

$(\forall$x:elt).($\forall$l:list)x.l $\neq$ l

$(\forall$x,x':elt)($\forall$l,l':list).x.l=x'.l' $\rightarrow$ x=x' $\wedge$ l=l'

$(\forall$l:list).nil::l=l

$(\forall$x:elt)($\forall$l,l':list).(x.l)::l'=x.(l::l')

Exercise

Define a specification to express the following properties about binary relations:

- Carlos is a parent of Luís
- Mothers are women
- Every person has at least a mother or a father
- x is the grandfather of y
- Carlos has a grandfather

Exercise: Cluedo 2

Consider the following sentences:

1. All the invited friends and family members arrived late.
2. There is at least one person who arrived on time.
3. There is at least one guest who is neither a family member nor a friend.

Formalize the following sentences and show that 3 is not a logical consequence of 1 and 2.

Exercise

Define a specification to express the following properties about binary relations:

- Symmetry
- Anti-symmetry
- Transitivity
- Reflexivity
- Determinism
- Connectivity
- ....