Elements of Logic 2024/25
First-order Logic

EL 2024/25
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Outline

Signatures

Definition 1

First-order signature A **first-order signature** is a pair

$$\Sigma = (P, F)$$

where

- P is an $\mathbb{N}$-family of sets of **predicate symbols**
- F is an $\mathbb{N}$-family of sets of **operation symbols**

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We use

- $f : s \times \cdots \times s \rightarrow s \in F$ to denote that $f \in F_n$ and
- $p : s \times \cdots \times s$ to denote that $p \in P_n$

SIGNATURES

Two representations for the same signature Σ:

REPRESENTATION 1

Σ is a first-order signature with

- **constant** symbols c_1 and c_2
- a **unary function** symbol f
- a **binary function symbol** g
- a **binary predicate symbol** r

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Two representations for the same signature Σ:

Representation 1

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- **constant** symbols c_1 and c_2
- a **unary function** symbol f
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Representation 2

$\Sigma = (P, F)$, where

- $F_0 = \{c_1, c_2\}$
- $F_1 = \{f\}$, $F_2 = \{g\}$
- $F_k = \emptyset$ for any $k > 2$
- $P_2 = \{r\}$ e $P_k = \emptyset$ for any $k \neq 2$

Examples of signatures

Exercise 1

Formalise suitable first-order signatures to specify

- *monoids*
- *ordered sets*
- *algebra of relations (worked on the previous chapter)*
- *natural numbers*
- *graphs*
- ...

First-order structures

Definition 2

Σ-structures Let $\Sigma = (P, F)$ be a first-order signature. A **Σ-structure** A consists of

- a non-empty set $|A|$, called universe.
- for each predicate symbol $r \in P_n$, a set $r^A \subseteq |A|^n$.
- for each operation symbol $f \in F_n$, a function $f^A : |A|^n \to |A|$.

Examples of Σ-structures

Let us consider the first-order signature $\Sigma = (P, F)$, with

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- $F_1 = \{f\}$, $F_2 = \{g\}$
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Two examples of Σ-structures:

$|A| = \{a, b\}$

$c_1^A = a \; c_2^A = b$

$f^A(a) = a$, $f^A(b) = a$

$g^A = \{(a, a) \mapsto a, (b, b) \mapsto$
$a, (a, b) \mapsto b, (b, a) \mapsto b\}$

$r^A = \{(a, b), (b, a)\}$

Examples of Σ-structures

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- $F_1 = \{f\}$, $F_2 = \{g\}$
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Two examples of Σ-structures:

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$f^A(a) = a$, $f^A(b) = a$

$g^A = \{(a, a) \mapsto a, (b, b) \mapsto a, (a, b) \mapsto b, (b, a) \mapsto b\}$

$r^A = \{(a, b), (b, a)\}$

$|B| = \{\heartsuit, \spadesuit\}$

$c_1^B = \heartsuit$, $c_2^B = \spadesuit$

$f^B(\heartsuit) = \heartsuit$ and $f^B(\spadesuit) = \heartsuit$

$$g^B(x, y) = \begin{cases} \heartsuit & \text{if } x = y \\ \spadesuit & \text{if } x \neq y \end{cases}$$

$r^B = \{(x, y) \mid x \neq y, x, y \in |B|\}$

Exercises

Exercise 2

1. *Define two different structures for each one of the signatures introduced in Exercise 1*

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2. *What about singleton structures?*

Morphisms between Σ-structures

Definition 3

Let $\Sigma = (P, F)$ be a first-order signature and A and B Σ-two structures. A **morphism between A and B** is a function

$$h : |A| \to |B|$$

such that:

- for any $r \in r_n$, and for any $a_1, \ldots, a_n \in |A|$,

$$r^A(a_1, \ldots, a_n) \text{ implies that } r^B(h(a_1), \ldots, h(a_n))$$

- for any $f \in F_n$, and for any $a_1, \ldots, a_n \in |A|$,

$$h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$$

Exercises

Exercise 3

1. *Revisit the examples of Exercise 2 a introduce pairs of structures that are related by a morphism.*

Outline

Formulæ

Σ-terms

Let Σ be a signature and X a set of variable. The **set of Σ-terms in X** is the smallest set $T(\Sigma, X)$ such that:

- for any $x \in X$, $x \in T(\Sigma, X)$; (variables are terms)

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- for any operation symbol $c \in F_0$, $c \in T(\Sigma, X)$; (constants are terms)
- for any $f \in F_n$, if $t_1, \ldots, t_n \in T(\Sigma, X)$, then $f(t_1, \ldots, t_n) \in T(\Sigma, X)$;

Exercise 4

List the terms of the signatures introduced in the previous exercises

Σ-formulæ

Definition 4

Let $\Sigma = (P, F)$ be a signature. The **set of formulas** $\mathrm{Fm}(\Sigma)$ is the smallest set with the such that:

1. for any $t_1, \ldots, t_n \in T(\Sigma, X)$ and for any $p \in P_n$, $p(t_1, \ldots, t_n) \in \mathrm{Fm}(\Sigma)$

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5. for any $\varphi \in \mathrm{Fm}(\Sigma)$, and $x \in X$, $\forall x.\varphi \in \mathrm{Fm}(\Sigma)$ and $\exists x.\varphi \in \mathrm{Fm}(\Sigma)$

Exercise 5

List the formulas of the signatures introduced in the previous exercises

Free and Bounded variables

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- $FV(\exists x \varphi) = FV(\varphi) \backslash \{x\}$

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Exercise 6

For each formula indicates whose variables are free, are bounded, or free and bounded:

1. $(y < x) \vee (x^2 + x - y = 0)$
2. $\exists x((y < x) \vee (x^2 + x - 2 = 0))$
3. $x > 0 \wedge \exists x(5 < x)$

Substitution operator

Definition 6 (Substitution in terms)

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2. $c[t/x] = c$, for any constant $c \in F_0$
3. $f(t_1, \ldots, t_n)[t/x] = f(t_1[t/x], \ldots, t_n[t/x])$

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Exercise 7

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Substitution operator

Definition 7 (Substitution in formulas)

Let $\varphi \in \mathrm{Fm}(\Sigma)$ be a formula, $x \in X$ a variable and $t \in T(\Sigma, X)$ a term. The substitution $\varphi[t/x]$ is defined by

1. $R(t_1, \ldots, t_n)[t/x] = R(t_1[t/x], \ldots, t_n[t/x])$
2. $(t_1 = t_2)[t/x] = (t_1[t/x] = t_2[t/x])$
3. $(\neg\varphi)[t/x] = \neg\varphi[t/x]$
4. $(\varphi \star \psi)[t/x] = \varphi[t/x] \star \psi[t/x]$, for any $\star \in \{\vee, \wedge, \rightarrow\}$
5. $(\forall y\varphi)[t/x] = \begin{cases} \forall y(\varphi[t/x]) & \text{if } x \neq y \\ \forall y\varphi & \text{if } x = y \end{cases}$
6. $(\exists y\varphi)[t/x] = \begin{cases} \exists y(\varphi[t/x]) & \text{if } x \neq y \\ \exists y\varphi & \text{if } x = y \end{cases}$

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Exercise 8

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Substitution process

On the substitution of x by a term t in a formula φ

we have to assure that:

- substitutions for bounded variables are forbid
  (e.g. we can not replace x by y in $\exists x \neg (x = y)$)
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  (e.g. we can not replace x by y in $\exists x \neg(x = y)$)
  this is assured by points 4 and 5 in previous definition
- we need to assure that free variable in t does not becomes bounded after substitution in φ, i.e., that **"t is free for x in φ"**

Substitution process

Definition 8

The term **t is free for x in φ** if

- φ is atomic
- φ is of form $\varphi_1 \star \varphi_2$, $\star \in \{\vee, \wedge, \rightarrow\}$ and t is free for x in φ_1 and in φ_2
- φ is of form $\neg\psi$ and t is free for x in ψ
- φ is of form $\forall y \psi$: if $x \in FV(\psi)$ then $y \notin FV(t)$ and t is free for x in ψ
- φ is of form $\exists y \psi$: if $x \in FV(\psi)$ then $y \notin FV(t)$ and t is free for x in ψ

Substitution Process

Exercise 9

Check which terms are free in the following cases:

1. x is free for x in $x = x$
2. y is free for x in $x = x$
3. $x + y$ is free for y in $z = \bar{0}$
4. $\bar{0} + y$ is free for y in $\exists x(y = x)$
5. $x + y$ is free for z in $\exists w(w + x = \bar{0})$
6. $x + w$ is free for z in $\forall w(x + z = \bar{0})$
7. $x + y$ is free for z in $\forall w(x + z = \bar{0}) \wedge \exists y(z = y)$
8. $x + y$ is free for z in $\forall u(u = v) \rightarrow \forall z(z = y)$

Substitution process

Assumption

From now we assume that all our substitutions are "free for"

Extended Signature for $|A|$

Definition 9

Let $\Sigma = (P, F)$ be a signature A be a Σ-structure. The **extended signature** $\Sigma^A = (F^A, P)$ **of** A is the signature that enriches Σ with a constant symbol for each value in $|A|$, i.e. such that $F_0^A = F_0 \cup \{\bar{a} \mid a \in |A|\}$ and $F_n^A = F_n$, for any $n \neq 0$.

Outline

Interpretation of terms

Definition 10

An interpretation of the closed terms of Σ^A in A is a mapping

$$[\![-]\!] : T(\Sigma, X)_c \to |A|$$

such that

- $[\![\bar{c}]\!] = c$
- $[\![f(t_1, \ldots, t_n)]\!] = f^A([\![t_1]\!], \ldots, [\![t_n]\!])$

Interpretation of formulas

Definition 11

Let Σ be a signature and A a Σ-Structure. An interpretation of sentences is a mapping

$$[\![-]\!]_A : \mathrm{Sen}(\Sigma) \to \{0, 1\}$$

where

- $[\![R(t_1, \ldots, t_n)]\!]_A = \begin{cases} 1 & \text{if } ([\![t_1]\!], \ldots, [\![t_n]\!]) \in R^A \\ 0 & \text{otherwise} \end{cases}$

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- $[\![t_1 = t_2]\!]_A = \begin{cases} 1 & \text{if } [\![t_1]\!]_A = [\![t_2]\!]_A \\ 0 & \text{otherwise} \end{cases}$
- $[\![\varphi \wedge \psi]\!]_A = min([\![\varphi]\!]_A, [\![\psi]\!]_A)$

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- $[\![t_1 = t_2]\!]_A = \begin{cases} 1 & \text{if } [\![t_1]\!]_A = [\![t_2]\!]_A \\ 0 & \text{otherwise} \end{cases}$
- $[\![\varphi \wedge \psi]\!]_A = min([\![\varphi]\!]_A, [\![\psi]\!]_A)$
- $[\![\varphi \vee \psi]\!]_A = max([\![\varphi]\!]_A, [\![\psi]\!]_A)$

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- $[\![R(t_1, \ldots, t_n)]\!]_A = \begin{cases} 1 & \text{if } ([\![t_1]\!], \ldots, [\![t_n]\!]) \in R^A \\ 0 & \text{otherwise} \end{cases}$
- $[\![t_1 = t_2]\!]_A = \begin{cases} 1 & \text{if } [\![t_1]\!]_A = [\![t_2]\!]_A \\ 0 & \text{otherwise} \end{cases}$
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Interpretation of formulas

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Let Σ be a signature and A a Σ-Structure. An interpretation of sentences is a mapping

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- Note that, until now, we just see how to interpret sentences $\varphi \in \mathrm{Sen}(\Sigma)$, i.e. formulas without free variables. The goal now is to know how to interpret any formula $\varphi \in \mathrm{Fm}(\Sigma)$.

Interpretation of formulas

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Universal closure Let $FV(\varphi) = \{z_1, \ldots, z_k\}$. The **universal closure of φ** $Cl(\varphi)$ is the formula

$$\forall z_1, \ldots, z_k \, \varphi$$

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4. $\Gamma \models \varphi$ iff
   for any Σ-structure A, if $A \models \Gamma$, then $A \models \varphi$

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4. $\Gamma \models \varphi$ iff
   for any Σ-structure A, if $A \models \Gamma$, then $A \models \varphi$ – we say that **φ is a semantic consequence of Γ**

Interpretation of formulas

Some more notions

Let φ a formula and $FV(\varphi) = \{z_1, \ldots, z_k\}$.

- φ **is satisfied by** $a_1, \ldots, a_k \in |A|$ iff $A \models \varphi[\bar{a}_1, \ldots, \bar{a}_k / z_1, \ldots, z_k]$
- φ **is satisfiable** iff φ **is satisfied for some** $a_1, \ldots, a_k \in |A|$

Interpretation of formulas

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- **φ is satisfiable** iff **φ is satisfied for some** $a_1, \ldots, a_k \in |A|$

Exercise 10

Prove that

- *φ is satisfiable in A if $FV(\varphi) = \{z_1, \ldots, z_k\}$ and $A \models \exists z_1, \ldots, z_k\, \varphi$*

Exercises

Exercise 11

Let $\varphi \in \mathrm{Sen}(\Sigma)$. *Prove that:*

1. $A \models \varphi \wedge \psi$ *iff* $A \models \varphi$ *and* $A \models \psi$

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5. $A \models \exists x \varphi$ *iff* $A \models \varphi[\bar{a}/x]$*, for some* $a \in |A|$

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Exercise 12

Which cases from the previous exercise remain correct if we consider formulas in general?

Exercises

Exercise 13

Let $\Sigma = (P, F)$ *where* $P = \varnothing$ *and* $F_0 = \{zero\}$, $F_1 = \{succ\}$, $F_2 = \{add, mult\}$. *Let Nat be the* Σ*-structure with* $|Nat| = \mathbb{N}$ *and with the* $zero^{Nat} = 0$, $succ^{NAt}(n) = n + 1$, $add^{Nat}(n, m) = n + m$ *and* $mult^{Nat}(n, m) = n * m$.

1.
   1. *Give two distinct terms* $t, s \in T(X, \Sigma_{nat})$ *such that* $[\![t]\!]_{Nat} = [\![s]\!]_{Nat} = 3$
   2. *Show that for any* $n \in \mathbb{N}$, *there is a term* t *such that* $[\![t]\!]_{Nat} = n$
   3. *Show that for any* $n \in \mathbb{N}$, *there are infinitely many terms* t *such that* $t^{Nat} = n$
2. *Consider now the extended signature* Σ^{Nat}. *Determine*
   - $[\![((\bar{1} \to \bar{0}) \to (\neg zero)) \land (\neg\bar{0} \to (\bar{1} \to zero))]\!]$

Exercises

Exercise 14

For any $\varphi \in \mathrm{Sen}(\Sigma)$, and for any Σ-structure A, we have that $A \models \varphi$ or $A \models \neg\varphi$.
Prove, or refute the statement:
"For any $\varphi \in \mathrm{Fm}(\Sigma)$, and for any Σ-structure A, we have that $A \models \varphi$ or $A \models \neg\varphi$."

Exercises

Exercise 14

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Prove, or refute the statement:
"For any $\varphi \in \mathrm{Fm}(\Sigma)$, and for any Σ-structure A, we have that $A \models \varphi$ or $A \models \neg\varphi$."

Exercise 15

Show that, for any term $t \in T(\Sigma^A, X)_c$

- $A \models t = \overline{[\![t]\!]_A}$
- $A \models \varphi(t) \leftrightarrow \varphi(\overline{[\![t]\!]_A})$

Properties of first-order-logic

Some useful validities

1. $\models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$

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4. $\models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$
5. $\models \forall x\forall y\varphi \leftrightarrow \forall y\forall x\varphi$

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5. $\models \forall x\forall y\varphi \leftrightarrow \forall y\forall x\varphi$
6. $\models \exists x\exists y\varphi \leftrightarrow \exists y\exists x\varphi$
7. $\models \forall x(\varphi \wedge \psi) \leftrightarrow \forall x\varphi \wedge \forall x\psi$

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7. $\models \forall x(\varphi \wedge \psi) \leftrightarrow \forall x\varphi \wedge \forall x\psi$
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Exercise 16

Prove it!

Properties of first-order logic

Some useful validities

1. $\models \forall x \varphi(x) \leftrightarrow \varphi$, if $x \notin FV(\varphi)$

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Exercise 17

Prove it!

Properties of first-order logic

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1. $\models \forall x\varphi(x) \leftrightarrow \varphi$, if $x \notin FV(\varphi)$
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4. $\models \exists x(\varphi(x) \wedge \psi) \leftrightarrow (\exists x(\varphi(x)) \wedge \psi)$, if $x \notin FV(\psi)$

Exercise 17

Prove it!

Exercise 18

Show that it is not true that:

- $\forall x(\varphi(x) \vee \psi(x)) \rightarrow \forall x\varphi(x) \vee \forall x\psi(x)$

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Prove it!

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Show that it is not true that:

- $\forall x(\varphi(x) \vee \psi(x)) \rightarrow \forall x \varphi(x) \vee \forall x \psi(x)$
- $\exists x \varphi(x) \wedge \exists x \psi(x) \rightarrow \exists x(\varphi(x) \wedge \psi(x))$

The identity

Characteristic properties of identity

1. $\forall x \, (x = x)$
2. $\forall xy(x = y \rightarrow y = x)$
3. $\forall xyz((x = y \wedge y = z) \rightarrow x = z)$
4. $\forall x_1 \dots x_n y_1 \dots y_n((\bigwedge_{i \leqslant n} x_i = y_i) \rightarrow (t(x_1, \dots, x_n) = t(y_1, \dots, y_n)))$
5. $\forall x_1 \dots x_n y_1 \dots y_n((\bigwedge_{i \leqslant n} x_i = y_i) \rightarrow (\varphi(x_1, \dots, x_n) \rightarrow \varphi(y_1, \dots, y_n))$

Examples – Groups

Exercise 19

- *Introduce a signature* Σ^{group} *to express the structure of a group*
- *Introduce a* $\Gamma \subseteq \mathrm{Fm}(\Sigma^{group})$ *the class of groups*
- *Introduce two structures* A *and* B *that are models of* Γ

Exercise 20

- *Introduce a signature* Σ^{ring} *to express the structure of a ring*
- *Introduce a* $\Gamma \subseteq \mathrm{Fm}(\Sigma^{ring})$ *the class of rings*
- *Introduce two structures* A *and* B *that are models of* Γ

Examples

Projective Geometry

- We consider the signature $\Sigma^{PG} = (P, F)$,
  - where $F_n = \emptyset$, $n \in \mathbb{N}$ and
  - $P_2 = \{I\}$ and $P_k = \emptyset$, $k \neq 2$
- and the abbreviations: $\Pi(x) \equiv \exists y(I(x, y))$ and $\Lambda(y) \equiv \exists x(I(x, y))$
- and the axiomatization:[a]
  - $\forall x(\Pi(x) \leftrightarrow \neg\Lambda(x))$
  - $\forall xy(\Pi(x) \wedge \Pi(y) \rightarrow \exists z(I(x, z) \wedge I(y, z)))$
  - $\forall uv(\Lambda(u) \wedge \Lambda(v) \rightarrow \exists z(I(x, u) \wedge I(x, v)))$
  - $\forall xyuv((I(x, u) \wedge I(y, u) \wedge I(y, v)) \rightarrow (x = y \vee u = v))$
  - $\exists x_0 x_1 x_2 x_3 u_0 u_1 u_2 u_3$ $\big(\bigwedge I(x_i, y_i) \wedge \bigwedge_{j=i-1(mod\,3)} I(x_i, u_j) \wedge \bigwedge_{j \neq i-1(mod\,3), j \neq i} \neg I(x_i, u_j)\big)$

[a] See [vanDalen], Sec 3.7

Outline

Back to Propositional Logic

	Introduction Rules	Elimination Rules
$\wedge$	$\dfrac{\psi \quad \varphi}{\psi \wedge \varphi}$	$\dfrac{\psi \wedge \varphi}{\psi} \qquad \dfrac{\psi \wedge \varphi}{\varphi}$
$\vee$	$\dfrac{\psi}{\psi \vee \varphi} \qquad \dfrac{\varphi}{\psi \vee \varphi}$	$\dfrac{\psi \vee \varphi \quad \begin{matrix}[\psi]\\ \mathcal{D}\\ \xi\end{matrix} \quad \begin{matrix}[\varphi]\\ \mathcal{D}\\ \xi\end{matrix}}{\xi}$
$\rightarrow$	$\dfrac{\begin{matrix}[\psi]\\ \mathcal{D}\\ \varphi\end{matrix}}{\psi \rightarrow \varphi}$	$\dfrac{\psi \quad \psi \rightarrow \varphi}{\varphi}$
$\neg$	$\dfrac{\begin{matrix}[\psi]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg\psi}$	$\dfrac{\begin{matrix}[\neg\psi]\\ \mathcal{D}\\ \bot\end{matrix}}{\psi}$
$\bot$	$\dfrac{\neg\varphi \quad \varphi}{\bot}$	$\dfrac{\bot}{\varphi}$

Natural Deduction for First-Order Logic

The calculus for Natural Deduction for First-order Logic is given by extending the Natural Deduction rules for Propositional Logic (in the previous slide) with

- **elimination and introduction rules for $\forall$**
- **elimination and introduction rules for $\exists$**

Introduction Rule for the universal quantifier

Introduction of ∀

$$\dfrac{\begin{array}{c}[x_0] \text{ fresh} \\ \vdots \\ \varphi[x_0/x]\end{array}}{\forall x \varphi}\ (I_\forall)$$

where fresh means that x_0 may not occur free in any hypothesis on which φ depends

Elimination Rule for the universal quantifier

Elimination of ∀

$$\frac{\forall x \varphi}{\varphi[t/x]} \; (E_\forall)$$

if x is free for t in φ

Natural deduction rules for universal quantifier

Exercise 21

Prove that:

1. $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x P(x) \to \forall x Q(x)$
2. $\forall x \forall y P(x, y) \vdash \forall z \forall w P(z, w)$
3. $\forall x \forall y P(x, y) \vdash \forall y \forall x P(x, y)$
4. $\forall x(\neg P(x) \to Q(x)), \neg Q(t) \vdash P(t)$

Natural deduction rules for universal quantifier

Exercise 22

Prove that:

1. $\forall x \forall y \varphi(x, y) \rightarrow \forall y \forall x \varphi(x, y)$
2. $\forall x(\varphi \wedge \psi) \rightarrow (\forall x \varphi \wedge \forall x \psi)$

Exercise 23

Prove that, if $x \notin FV(\varphi)$ *:*

1. $\vdash \forall x(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow \forall x \psi(x))$
2. $\vdash \varphi \rightarrow \forall \varphi(x)$
3. $\vdash \forall \varphi(x) \rightarrow \varphi$

Introduction Rule for the existential quantifier

Introduction of $\exists$

$$\frac{\varphi[t/x]}{\exists x \varphi}$$

if t is free for x in φ

Elimination Rule for the existential quantifier

Elimination of ∃

$$\dfrac{\exists x \varphi \qquad \begin{matrix} [x_0] \text{ fresh} \\ \varphi[x_0/x] \text{ (ass)} \\ \vdots \\ \chi \end{matrix}}{\chi} \; (E\vee)$$

Elimination Rule for the existential quantifier

Exercise 24

Show that

1. $\forall x(P(x) \wedge Q(x)) \vdash \forall x(P(x) \vee Q(x))$
2. $\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$
3. $\exists x P(x), \forall x \forall y(P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$
4. $\exists x P(x) \vdash \neg \forall x \neg P(x)$
5. $\forall P(a, x, x), \forall x \forall y \forall z(P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

Exercise 25

Show that

- $\neg \exists x P(x) \vdash \forall x \neg P(x)$
- $\forall x \neg P(x) \vdash \neg \exists x P(x)$

Natural deduction with identity

In the presence of identities we consider the following rules

$$\frac{}{x = x} \qquad \frac{x = y}{y = x} \qquad \frac{x = y \quad y = z}{x = z}$$

$$\frac{x_1 = y_1, \dots x_n = y_n}{t(x_1, \dots, x_n) = t(y_1, \dots, y_n)}$$

$$\frac{x_1 = y_1, \dots x_n = y_n \quad \varphi(x_1, \dots, x_n)}{\varphi(y_1, \dots, y_n)}$$

Natural deduction with identity

Exercise 26

Check

1. $x = y, x^2 + y^2 > 5x \vdash 2y^2 > 5x$
2. $x = y, x^2 + y^2 > 5x \vdash x^2 + y^2 > 5y$
3. $\vdash \forall x(x = x)$
4. $\vdash \forall z(z = x \rightarrow z = y) \rightarrow x = y$

Outline

Soundness

The natural deduction rules for $\forall$ in terms of $\vdash$

- $\Gamma \vdash \varphi(x) \Rightarrow \Gamma \vdash \forall x \varphi(x)$, if $x \notin FV(\Gamma)$
- $\Gamma \vdash \forall x \varphi(x) \Rightarrow \Gamma \vdash \varphi(t)$, if t is free for x in φ

Soundness

The natural deduction rules for $\forall$ in terms of $\vdash$

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Definition 14

Let Γ be a set of formulae and let $\{x_1, x_2, \dots\} = \bigcup\{FV(\gamma) \mid \gamma \in \Gamma \cup \varphi\}$. If $\mathbf{a} = (a_1, a_2, \dots)$, $a_i \in |A|, i \leqslant 1$, then $\Gamma(\mathbf{a})$ denotes the set of formulas obtained from Γ by replacing simultaneously in all formulas in Γ x_i by $\bar{a}_i$. Hence,

1. $A \models \Gamma(\mathbf{a})$ if $A \models \gamma$, for all $\gamma \in \Gamma(\mathbf{a})$
2. $\Gamma \models \varphi$ if for any A and for any $\mathbf{a}$, $A \models \Gamma(\mathbf{a})$ implies $A \models \varphi(\mathbf{a})$

Soundness

Theorem 15 (Soundness)

Let $\varphi \in \mathrm{Fm}(\Sigma)$ *and* $\Gamma \subseteq \mathrm{Fm}(\Sigma)$.

$$\Gamma \vdash \varphi \text{ **implies** } \Gamma \models \varphi$$

Soundness

Theorem 15 (Soundness)

Let $\varphi \in \mathrm{Fm}(\Sigma)$ *and* $\Gamma \subseteq \mathrm{Fm}(\Sigma)$.

$$\Gamma \vdash \varphi \textbf{ implies } \Gamma \models \varphi$$

Proof.

Proof by induction on the structure of proof trees.
Exercise! □

Going to the completeness...

It is helpful to try to establish an analogy with completeness proof for Propositional Logic studied earlier

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Existence Lemma

Let $\Gamma \subseteq \mathrm{Fm}(\Sigma)$. **If Γ is consistent, then Γ has a model**

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Existence Lemma

Let $\Gamma \subseteq \mathrm{Fm}(\Sigma)$. **If Γ is consistent, then Γ has a model**

Proof.

This is the hard part of the completeness proof. Done later. □

Going to the completeness...

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Observation:

check the consistency characterization introduced during the presentation of propositional logic

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Existence Lemma

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Proof.

This is the hard part of the completeness proof. Done later. □

Observation:

check the consistency characterization introduced during the presentation of propositional logic

Exercise 27

Assuming the Existence Lemma and the previous observation, prove that

$$\Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi$$

Theory and Henkin theories

Definition 16

Let $T \subseteq \mathrm{Sen}(\Sigma)$

1. T is a theory if, for any $\varphi \in \mathrm{Sen}(\Sigma)$, $T \vdash \varphi$ **implies that** $\varphi \in T$

Theory and Henkin theories

Definition 16

Let $T \subseteq \mathrm{Sen}(\Sigma)$

1. T is a theory if, for any $\varphi \in \mathrm{Sen}(\Sigma)$, $T \vdash \varphi$ **implies that** $\varphi \in T$
2. Γ **is an axiom set for** T if $T = \{\varphi \mid \Gamma \vdash \varphi\}$

Theory and Henkin theories

Definition 16

Let $T \subseteq \mathrm{Sen}(\Sigma)$

1. T is a theory if, for any $\varphi \in \mathrm{Sen}(\Sigma)$, $T \vdash \varphi$ **implies that** $\varphi \in T$
2. Γ **is an axiom set for** T if $T = \{\varphi \mid \Gamma \vdash \varphi\}$
3. T **is an Henkin theory** if for each sentence $\exists x \varphi(x)$, there is a constant c such that $\exists x \varphi(x) \to \varphi(c) \in T$. The constant c is called a **witness** of $\exists x \varphi(x)$.

Theories extensions

Definition 17

Let T and T' be theories for Σ and Σ'.

1. **T is an extension of T'** if $T \subseteq T'$
2. **T is a conservative extension of T'** if $T' \cap \Sigma = T$, i.e. all theorem of T' in Σ are already theorems of T

Theories extensions

Definition 18

Let T be a theory for $\Sigma = (P, F)$.

- The signature Σ^* is obtained from Σ by
  - $P_0^* = P_0 \cup \{c_\varphi \mid \exists x \varphi(x)\}$, and $P_k^* = P_k$, $k > 0$
  - $F^* = F$
- $T^* = T \cup \{\exists x \varphi(x) \rightarrow \varphi(c_\varphi) \mid \exists x \varphi(x)$ closed, with witness $c_\varphi\}$

Theories extensions

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Let T be a theory for $\Sigma = (P, F)$.

- The signature Σ^* is obtained from Σ by
  - $P^*_0 = P_0 \cup \{c_\varphi \mid \exists x \varphi(x)\}$, and $P^*_k = P_k$, $k > 0$
  - $F^* = F$
- $T^* = T \cup \{\exists x \varphi(x) \rightarrow \varphi(c_\varphi) \mid \exists x \varphi(x)$ closed, with witness $c_\varphi\}$

Lemma 19

T^* is a conservative extension of T

Theories extensions

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- $T^* = T \cup \{\exists x \varphi(x) \rightarrow \varphi(c_\varphi) \mid \exists x \varphi(x)$ closed, with witness $c_\varphi\}$

Lemma 19

T^* is a conservative extension of T

Note that there is no evidence that T^* is still an Henkin theory

Theories extensions

The T_ω construction

T_ω is recursively defined as follows:

- $T_0 := T$
- $T_{n+1} = (T_n)^*$
- $T_\omega = \bigcup\{T_n \mid n \geqslant 0\}$

Theories extensions

The T_ω construction

T_ω is recursively defined as follows:

- $T_0 := T$
- $T_{n+1} = (T_n)^*$
- $T_\omega = \bigcup\{T_n \mid n \geqslant 0\}$

Lemma 20

T_ω is an Henkin theory

Theories extensions

Remember from the Propositional Logic completeness proof:

Lemma 21

Each consistent theory is contained in a maximally consistent theory

Theories extensions

Remember from the Propositional Logic completeness proof:

Lemma 21

Each consistent theory is contained in a maximally consistent theory

Lemma 22

An extension of a Henkin theory is a Henkin theory

Completeness Theorem

Lemma 23 (Model Existence Lemma)

If Γ is consistent, then Γ has a model

Proof.

Check the construction of the **standard model** in the proof of Lemma 4.1.11 in Van Dalen: Logic and Structure.

□

Completeness Theorem

Lemma 23 (Model Existence Lemma)

If Γ is consistent, then Γ has a model

Proof.

Check the construction of the **standard model** in the proof of Lemma 4.1.11 in Van Dalen: Logic and Structure.

□

Theorem 24 (Completeness Theorem)

$\Gamma \models \varphi$ **implies** $\Gamma \vdash \varphi$

Completeness Theorem

Lemma 23 (Model Existence Lemma)

If Γ is consistent, then Γ has a model

Proof.

Check the construction of the **standard model** in the proof of Lemma 4.1.11 in Van Dalen: Logic and Structure.

□

Theorem 24 (Completeness Theorem)

$\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$

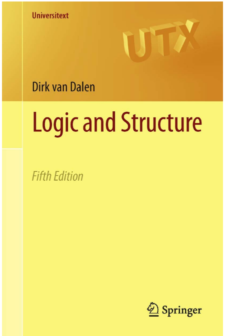
Proof.

Exercise 27.

□

References

The presentation of First-Order Logic done in this set of slides was based in the book



that is strongly recommended for the preparation of this course.