Elements of Logic 2024/25
Propositional Logic

EL 2024/25
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January 21, 2026

Outline

Formulæ

Definition 1 (Set of propositional formulas)

Let $\mathrm{Prop} = \{p_0, p_1, p_2, \dots\}$ be a countable set of propositions (or propositional variables). The **set of propositional formulas** is the smallest set $\mathrm{Fm}(\mathrm{Prop})$ with the following properties:

- $\bot \in \mathrm{Fm}(\mathrm{Prop})$
- for any $p_i \in \mathrm{Prop}$, $p_i \in \mathrm{Fm}(\mathrm{Prop})$
- for any $\varphi \in \mathrm{Fm}(\mathrm{Prop})$, $(\neg\varphi) \in \mathrm{Fm}(\mathrm{Prop})$
- for any $\varphi, \varphi' \in \mathrm{Fm}(\mathrm{Prop})$, $(\varphi \wedge \varphi') \in \mathrm{Fm}(\mathrm{Prop})$
- for any $\varphi, \varphi' \in \mathrm{Fm}(\mathrm{Prop})$, $(\varphi \vee \varphi') \in \mathrm{Fm}(\mathrm{Prop})$
- for any $\varphi, \varphi' \in \mathrm{Fm}(\mathrm{Prop})$, $(\varphi \rightarrow \varphi') \in \mathrm{Fm}(\mathrm{Prop})$
- for any $\varphi, \varphi' \in \mathrm{Fm}(\mathrm{Prop})$, $(\varphi \leftrightarrow \varphi') \in \mathrm{Fm}(\mathrm{Prop})$

Induction over formulas

Using the structure of formulas we can do inductive proofs

Let A be a property, then $A(\varphi)$ holds for all $\varphi \in \mathrm{Fm}(\mathrm{Prop})$ if

1. $A(p_i)$, for any $p_i \in \mathrm{Prop}$ and $A(\bot)$
2. $A(\varphi)$ implies $A((\neg\varphi))$
3. $A(\varphi)$ and $A(\varphi')$ implies that $A((\varphi \star \varphi'))$, $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Exercise 1

The number of brackets in any $\varphi \in \mathrm{Fm}(\mathrm{Prop})$ *is even.*

Recursion over formulas

Using the structure of formulas we can make recursive defs

Let mapping $H_{\text{Prop}} : \text{Prop} \cup \{\bot\} \to A$, $H_\neg : A \to A$ and $H_\star : A \times A \to A$ with $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$. Then, there exists exactly one mapping $F : \text{Fm}(\text{Prop}) \to A$ such that:

1. $F(p_i) = H_{\text{Prop}}(p_i)$, for any $p_i \in \text{Prop}$ and $F(\bot) = H_{\text{Prop}}(\bot)$
2. $F((\neg\varphi)) = H_\neg(F(\varphi))$ and
3. $F((\varphi \star \psi)) = H_\star(F(\varphi), F(\psi))$

Exercise 2

Recursively define functions to:

1. *the number of bracket occuring in a formula*
2. *determine the number of propositions occurring in a formula*
3. *determine the set of propositions occurring in a formula*
4. *determine the number of connectives occurring in a formula*

Sub-formulas

Definition 2

Let $\varphi \in \mathrm{Fm}(\mathrm{Prop})$. The **set of sub-formulas of φ** is the smallest set defined as follows:

- if $\varphi = p_i \in \mathrm{Prop}$, $Sub(\varphi) = \{\varphi\}$
- if $\varphi = \neg\psi$, $Sub(\varphi) = \{\neg\psi\} \cup Sub(\psi)$
- if $\varphi = \psi \star \psi'$, $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
  $Sub(\varphi) = \{\psi \star \psi'\} \cup Sub(\psi) \cup Sub(\psi')$

We say that **ψ is a sub-formula of φ** if $\psi \in Sub(\varphi)$

Exercises

Exercise 3

Determine $Sub(\neg(r \to (p \wedge q)))$

Exercise 4

Show that the relation "is a subform of" is:

- *reflexive*
- *transitive*

Exercise 5

Show that any formula with n connectives has at most $2n + 1$ *sub-formulas.*

Outline

Let us recall...Truth Tables

How to interpret Boolean Operators

negation:

$\neg$	
0	1
1	0

disjunction:

$\vee$	0	1
0	0	1
1	1	1

conjunction:

$\wedge$	0	1
0	0	0
1	0	1

implication:

$\rightarrow$	0	1
0	1	1
1	0	1

equivalence:

$\leftrightarrow$	0	1
0	1	0
1	0	1

This is enough to interpret any $\varphi \in \mathrm{Fm}(\mathrm{Prop})$:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Truth tables

Exercise 6

Develop the truth table of the following formulas

- $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
- $\neg((p \vee q) \vee r) \rightarrow (q \vee r)$
- $\ldots$

Valuations and Interpretations

Definition 3

A **valuation** is a function $v : \mathrm{Prop} \to \{0, 1\}$. Given a valuation v, the **v-interpretation** is the function

$$[\![\cdot]\!]_v : \mathrm{Fm}(\mathrm{Prop}) \to \{0, 1\}$$

such that:

- $[\![\bot]\!]_v = 0$
- for any $p \in \mathrm{Prop}$, $[\![p]\!]_v = v(p)$
- $[\![\neg\varphi]\!]_v = 1 - [\![\varphi]\!]_v$
- $[\![\varphi \vee \psi]\!]_v = \max([\![\varphi]\!]_v, [\![\psi]\!]_v)$
- $[\![\varphi \wedge \psi]\!]_v = \min([\![\varphi]\!]_v, [\![\psi]\!]_v)$
- $[\![\varphi \to \psi]\!]_v = 0$ iff $[\![\varphi]\!]_v = 1$ and $[\![\psi]\!]_v = 0$
- $[\![\varphi \leftrightarrow \psi]\!]_v = 1$ iff $[\![\varphi]\!]_v = [\![\psi]\!]_v$

Valuations and Interpretations

Back to the truth tables

	p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
v_1	1	1	1	1
v_2	1	0	0	1
v_3	0	1	0	1
v_4	0	0	0	1

- Each line of the table corresponds to a valuation v_i.
- For instance, for line 1 we have the valuation v_1 such that $v_1(p) = v_1(q) = 1$. Hence, in the last column $[\![(p \wedge q) \rightarrow p]\!]_{v_1} = 1$

Valuations and Interpretations

Theorem 4

Let v and v' two valuations and $\varphi \in \mathrm{Fm}(\mathrm{Prop})$. If, $v(p) = v'(p)$ for any proposition p occurring φ, we have that $[\![\varphi]\!]_v = [\![\varphi]\!]_{v'}$.

For instance:

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

p	q	$\neg p$	$p \wedge \neg p$
1	1	0	0
0	1	1	0
1	0	0	0
0	0	1	0

Tautologies and contradictions

Definition 5

A formula φ is a:

- **tautology** if, for any valuation v, $[\![\varphi]\!]_v = 1$
- **contradiction** if, for any valuation v, $[\![\varphi]\!]_v = 0$

A tautology and a contradiction

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Semantic consequence

Definition 6

Let $\Gamma \subseteq \mathrm{Fm}(\mathrm{Prop})$ be a set of propositional formulas and $\varphi \in \mathrm{Fm}(\mathrm{Prop})$ be a formula. We say that **φ is a semantic consequence of Γ**, in symbols

$$\Gamma \models \varphi$$

if for any valuation v:
if for all $\gamma \in \Gamma$, $[\![\gamma]\!]_v = 1$ implies that $[\![\varphi]\!]_v = 1$

Notation

- we write $\varphi_0, \ldots \varphi_n \models \varphi$ to denote $\{\varphi_0, \ldots \varphi_n\} \models \varphi$
- we write $\models \varphi$ to denote $\varnothing \models \varphi$ hence, **$\models \varphi$ iff φ is a tautology**

Semantic Consequence

Exercise 7

Check that

- $\varphi, \psi \models \varphi \wedge \psi$
- $\varphi, \varphi \rightarrow \psi \models \psi$
- $\varphi \rightarrow \psi, \neg\psi \models \neg\varphi$
- $\models \varphi \rightarrow \varphi$

Exercise 8

Show that

$$\varphi \models \varphi$$

Semantic Consequence

Exercise 9

Show that:
if $\varphi \models \psi$ *and* $\psi \models \gamma$, *then* $\varphi \models \gamma$

Exercise 10

Show that

$$\models \varphi \rightarrow \psi \text{ iff } \varphi \models \psi$$

Exercise 11

Show that

$$[\![\varphi \rightarrow \psi]\!]_v = 1 \text{ iff } [\![\varphi]\!]_v \leqslant [\![\psi]\!]_v$$

Semantic Consequence

Exercise 12 (Alternative presentation of interpretation of formulas)

Show that:

- $[\![\varphi \wedge \psi]\!]_v = [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$
- $[\![\varphi \vee \psi]\!]_v = [\![\varphi]\!]_v + [\![\psi]\!]_v - [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$
- $[\![\varphi \rightarrow \psi]\!]_v = 1 - [\![\varphi]\!]_v + [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$
- $[\![\varphi \leftrightarrow \psi]\!]_v = 1 - |[\![\varphi]\!]_v - [\![\psi]\!]_v|$

Semantic Consequence

Exercise 13 (Neller, Markov, Russel. Clue Deduction: Professor Plum Teaches Logic (2016))

Suppose that liars always speak what is false, and truth-tellers always speak what is true. Further suppose that Amy, Bob, and Cal are each either a liar or truth-teller. Amy says, "Bob is a liar." Bob says, "Cal is a liar." Cal says, "Amy and Bob are liars." Which, if any, of these people are truth-tellers?

Semantic Consequence

Exercise 14 (Logic in Action. van Benthem et al. 2016)

You want to throw a party, respecting people's incompatibilities. You know that:

- *John comes if Mary or Ann comes.*
- *Ann Comes if Mary does not come.*
- *If Ann comes, John does not.*

Can you invite people under these constraints?

Properties of Propositional Logic

Exercise 15

Prove or refute that the following formulas are tautologies:

- $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
- $(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$
- $p \vee q \leftrightarrow q \vee r$
- $p \wedge q \leftrightarrow q \wedge p$
- $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
- $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
- $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$
- $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
- $q \rightarrow p \wedge q$
- $p \wedge q \rightarrow q$

Substitution Lemma

Substitutions in propositional formulas

$$\varphi[\psi/p] = \begin{cases} \varphi & \text{if } \varphi \in \mathrm{Prop} \text{ and } \varphi \neq p \\ \psi & \text{if } \varphi \in \mathrm{Prop} \text{ and } \varphi = p \\ \neg\varphi_1[\psi/p] & \text{if } \varphi = (\neg\varphi_1) \\ \varphi_1[\psi/p] \star \varphi_2[\psi/p] & \text{if } \varphi = \varphi_1 \star \varphi_2 \end{cases}$$

Theorem 7 (Substitution Theorem)

Let $p \in \mathrm{Prop}$ *and* $\varphi_1, \varphi_2, \psi \in \mathrm{Fm}(\mathrm{Prop})$.
If $\models \varphi_1 \leftrightarrow \varphi_2$**, then** $\models \psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p]$

Properties of Propositional Logic

Exercise 16

Prove or refute the following formulas are tautologies:

- $(\varphi \vee \psi) \vee \gamma \leftrightarrow \varphi \vee (\psi \vee \gamma)$
- $(\varphi \wedge \psi) \wedge \gamma \leftrightarrow \varphi \wedge (\psi \wedge \gamma)$
- $\varphi \vee \psi \leftrightarrow \psi \vee \varphi$
- $\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$
- $\varphi \vee (\psi \wedge \gamma) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \gamma)$
- $\varphi \wedge (\psi \vee \gamma) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)$
- $\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$
- $\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$
- $\varphi \vee \varphi \leftrightarrow \varphi$
- $\varphi \wedge \psi \rightarrow \varphi$
- $\neg\neg\varphi \leftrightarrow \varphi$

Other connective systems

- As is well known, **the set of connectives $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$ is not minimal**,
  is not minimal, in the sense that a proper subset of these connectives suffices to define the logic without any loss of expressivity.
- For instance, using $\{\neg, \wedge\}$ we can introduced the other connectives by abbreviations:
  - $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$
  - $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$
  - ... (*Exercise*)

Other connective systems

Exercise 17

Show that the Sheaffer Stroke connective

$\_\vert\_$	0	1
0	1	1
1	1	0

is a enough to represent propositional logic.

Theorem 8

For each n-ary connective $\star$ defined by its valuation, there is a formula τ, containing only $p_1, \ldots, p_n$, $\vee$ and $\neg$, such that $\models \tau \leftrightarrow \star(p_1, \ldots, p_n)$

Proof.

See van Dalen Theorem 1.3. □

Outline

Semantics Consequences Vs Derivations

Using the introduced **semantics** we have a way to interpret propositional formulas and to check consequences

$$\Gamma \models \varphi$$

Now, we will introduce a **set of rules** to make derivations:

$$\Gamma \vdash \varphi$$

At the end we will prove their **soundness**:

$$\Gamma \vdash \varphi \textbf{ implies } \Gamma \models \varphi$$

and their **completeness**

$$\Gamma \models \varphi \textbf{ implies } \Gamma \vdash \varphi$$

Natural Deduction CalculÆStyle

There are different styles of calculus to derive

$$\Gamma \vdash \varphi$$

In this course we will use a **Natural Deduction Calculus** constituted by a set of rules of form

$$\frac{\textbf{premisses}}{\textbf{conclusions}} \; \textbf{*conditions* (rule name)}$$

Hence, a proof of $\Gamma \vdash \varphi$ consists of a three rooted in φ and which leafs are $\gamma \in \Gamma$

Natural Deduction Rules

Definition 9

Let $\varphi, \varphi_1, \ldots, \varphi_n \in \mathrm{Fm}(\mathrm{Prop})$. A **(natural deduction) inference rule** is an expression

$$\frac{\psi_1 \qquad \ldots \qquad \psi_n}{\varphi} \ .$$

Formulas ψ_i are the **premisses** or **assumptions** and φ is the **conclusion**

Example: the **modus ponens** rule

$$\frac{\psi \qquad \psi \to \varphi}{\varphi} \ .$$

Hence, $\psi, \psi \to \varphi \vdash \varphi$

Natural Deduction Rules

Some assumptions may be discharged

If I have

$$\frac{\dfrac{\varphi}{\mathcal{D}}}{\psi}$$

we conclude that, assuming that φ is true, then ψ holds,
i.e. $\varphi \vdash \psi$.
But, hence

$$\dfrac{\dfrac{\dfrac{[\varphi]}{\mathcal{D}}}{\psi}}{\varphi \to \psi}\ I_{\to}$$

we can conclude $\varphi \to \psi$ without any assumption, i.e.
$\vdash \varphi \to \psi$

Natural Deduction Rules

Derivations

A **derivation** of a formula φ from assumptions Γ is a finite tree of formulas satisfying the following conditions:

1. The topmost formulas of the tree are either in Γ or are **discharged** by an inference in the tree.
2. The bottommost formula of the tree is φ.
3. Every formula in the tree except the sentence φ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that formula in the tree.

We then say that φ is the *conclusion* of the derivation and Γ its undischarged assumptions.

Inference Rules

Conjunction rules

- **Introduction**

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (I_\wedge)$$

- **Elimination**

$$\frac{\varphi \wedge \psi}{\varphi}\ (E_\wedge) \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\ (E_\wedge)$$

Disjunction rules

- **Introduction**

$$\frac{\varphi}{\varphi \vee \psi}\ (I_\vee) \qquad\qquad \frac{\varphi}{\psi \vee \varphi}\ (I_\vee)$$

- **Elimination**

$$\frac{\dfrac{\dfrac{[\theta]}{\mathcal{D}}}{\varphi} \qquad \dfrac{\dfrac{[\psi]}{\mathcal{D}}}{\varphi} \qquad \theta \vee \psi}{\varphi}\ (E\vee)$$

Inference Rules

Implication rules

- **Introduction**

$$\dfrac{\dfrac{\dfrac{[\varphi]}{\mathcal{D}}}{\psi}}{\varphi \rightarrow \psi}\;(I_{\rightarrow})$$

- **Elimination**

$$\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\;(E_{\rightarrow})$$

Inference Rules

Negation Rules

- **Introduction**

$$\cfrac{\cfrac{\cfrac{[\varphi]}{\mathcal{D}}}{\bot}}{\neg\varphi}\ (I_\neg)$$

- **Elimination**

$$\cfrac{\cfrac{\cfrac{[\neg\varphi]}{\mathcal{D}}}{\bot}}{\varphi}\ (E_\neg)$$

Inference Rules

Bottom rules

Introduction

$$\frac{\varphi \qquad \neg\varphi}{\bot}\,(I_{\bot})$$

Elimination

$$\frac{\bot}{\varphi}\,(E_{\bot})$$

Natural Deduction Rules

	Introduction Rules	Elimination Rules
$\wedge$	$\dfrac{\psi \quad \varphi}{\psi \wedge \varphi}$	$\dfrac{\psi \wedge \varphi}{\psi}$ $\quad$ $\dfrac{\psi \wedge \varphi}{\varphi}$
$\vee$	$\dfrac{\psi}{\psi \vee \varphi}$ $\quad$ $\dfrac{\varphi}{\psi \vee \varphi}$	$\dfrac{\psi \vee \varphi \quad \begin{matrix}[\psi]\\ \mathcal{D}\\ \xi\end{matrix} \quad \begin{matrix}[\varphi]\\ \mathcal{D}\\ \xi\end{matrix}}{\xi}$
$\rightarrow$	$\dfrac{\begin{matrix}[\psi]\\ \mathcal{D}\\ \varphi\end{matrix}}{\psi \rightarrow \varphi}$	$\dfrac{\psi \quad \psi \rightarrow \varphi}{\varphi}$
$\neg$	$\dfrac{\begin{matrix}[\psi]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg\psi}$	$\dfrac{\begin{matrix}[\neg\psi]\\ \mathcal{D}\\ \bot\end{matrix}}{\psi}$
$\bot$	$\dfrac{\neg\varphi \quad \varphi}{\bot}$	$\dfrac{\bot}{\varphi}$

Examples

i) $\vdash \varphi \to \varphi$

$$(\to \text{int } 1)\ \frac{[\varphi]^1}{\varphi \to \varphi}$$

ii) $\vdash (\varphi \vee \varphi) \to \varphi$

$$(\to \text{int } 2)\ \frac{(\vee \text{ elim } 1)\ \dfrac{[\varphi]^1 \qquad [\varphi]^1 \qquad [\varphi \vee \varphi]^2}{\varphi}}{\varphi \vee \varphi \to \varphi}$$

Examples

iii) $\vdash (\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi))$

$$
(\to \text{int } 1)\ \dfrac{(\to \text{int } 3)\ \dfrac{(\to \text{int } 2)\ \dfrac{(\to \text{elim})\ \dfrac{(\to \text{elim})\ \dfrac{[\varphi \to (\psi \to \xi)]^1 \qquad [\varphi]^2}{\psi \to \xi} \qquad \dfrac{[\varphi]^2 \qquad [\varphi \to \psi]^3}{\psi}}{\xi}}{\varphi \to \xi}}{(\varphi \to \psi) \to (\varphi \to \xi)}}{(\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi))}
$$

Exercises

Exercise 18

Prove that:

1. $\vdash \varphi \to (\psi \to \varphi)$
2. $\vdash \varphi \to (\neg\varphi \to \psi)$
3. $\vdash (\varphi \to \psi) \to [(\psi \to \sigma) \to (\varphi \to \sigma)]$
4. $\vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$
5. $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
6. $\vdash \neg\neg\varphi \to \varphi$
7. $\vdash \varphi \to \neg\neg\varphi$
8. $\vdash \big((\varphi \wedge \psi) \to \sigma\big) \to \big(\varphi \to (\psi \to \sigma)\big)$

Exercises

Exercise 19

1. $\alpha, \theta \vdash \beta \rightarrow \big((\alpha \wedge \beta) \wedge (\alpha \wedge \theta)\big)$
2. $\theta \vdash \alpha \rightarrow \big(\beta \rightarrow \big((\alpha \wedge \beta) \wedge (\alpha \wedge \theta)\big)\big)$
3. $\vdash \theta \rightarrow \big(\alpha \rightarrow \big(\beta \rightarrow \big((\alpha \wedge \beta) \wedge (\alpha \wedge \theta)\big)\big)\big)$

Exercise 20

1. $\alpha \rightarrow \beta, \beta \rightarrow \theta \vdash \alpha \rightarrow \theta$
2. $\alpha \wedge \beta \vdash \beta \wedge \alpha$

Exercises

Exercise 21

Conclude the the following derivation justifying all the steps involved:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(A \vee B) \qquad \dfrac{A}{A \vee B}}{\bot}}{\neg A} \qquad\qquad\qquad\qquad}{\neg A \wedge \neg B}}{\neg(A \vee B) \to \neg A \wedge \neg B}$$

Exercises

Exercise 22

Conclude the the following derivation justifying all the steps involved:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \land B}}{(A \land B) \land (A \land C)}}{B \to (A \land B) \land (A \land C)}}{A \to (B \to (A \land B) \land (A \land C))}}{C \to (A \to (B \to (A \land B) \land (A \land C)))}$$

Exercises

Exercise 23

Conclude the the following derivation justifying all the steps involved:

$$\dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{B \vee C} \qquad \dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{A}}{A \wedge B}}{(A \wedge B) \vee (A \wedge C)} \qquad \dfrac{}{(A \wedge B) \vee (A \wedge C)}}{(A \wedge B) \vee (A \wedge C)}}{(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))}$$

Outline

Back to the motivations

We studied two ways to infer in Propositional Calculus:

Using Semantics

$$\Gamma \models \varphi$$

Using Calculus

$$\Gamma \vdash \varphi$$

In this section we will prove

that $\models$ and $\vdash$ are equivalent, in the sense that

Soundness $\Gamma \vdash \varphi$ **implies that** $\Gamma \models \varphi$

Completeness $\Gamma \models \varphi$ **implies that** $\Gamma \vdash \varphi$

Soundness

Theorem 10

$\Gamma \vdash \varphi$ *implies that* $\Gamma \models \varphi$

Proof.

Exercise: use induction on the structure of derivations. □

Consistent set of formulas

Definition 11

Let $\Gamma \subseteq \mathrm{Fm}(\mathrm{Prop})$ a set of propositional formulas. The set **Γ is consistent** if $\Gamma \nvdash \bot$

Lemma 12

The following three conditions are equivalent:

1. *Γ is consistent*
2. *There is no φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$*
3. *There is at least a φ such that $\Gamma \nvdash \varphi$*

Consistent set of formulas

Exercise 24

If there is a valuation v such that $[\![\gamma]\!]_v = 1$, for any $\gamma \in \Gamma$, then Γ is consistent.

Exercise 25

- *If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \vdash \varphi$*
- *If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$*

Maximal consistent set of formulas

Definition 13

A set Γ is maximally consistent iff

(i) Γ is consistent

(ii) for any Γ' consistent, if $\Gamma \subseteq \Gamma'$ then $\Gamma = \Gamma'$

Maximal consistent set of formulas

Lemma 14

Each consistent set of formulas Γ *is contained in a maximally consistent set of formulas* Γ^*

Proof.

Hint: we recursively construct such set. In order to do that we use the fact that $\mathrm{Fm}(\mathrm{Prop})$ is countable (since Prop is countable as well). □

Lemma 15

If Γ *is maximally consistent, then*

$$\Gamma \vdash \varphi \textbf{ implies } \varphi \in \Gamma$$

Lemma 16

Let Γ *be a maximally consistent set of formulas. Then*

- *for all* φ*, either* $\varphi \in \Gamma$ **or** $\neg\varphi \in \Gamma$
- *for all* φ, ψ*,*

$$(\varphi \rightarrow \psi \in \Gamma) \textbf{ iff } (\varphi \in \Gamma \textbf{ implies that } \psi \in \Gamma)$$

Completeness

Corollary 17

If Γ is maximally consistent, then

- $\varphi \in \Gamma$ *iff* $\neg\varphi \notin \Gamma$ *and*
- $\neg\varphi \in \Gamma$ *iff* $\varphi \notin \Gamma$

Lemma 18

If Γ is consistent, then there exists a valuation v such that $[\![\gamma]\!]_v = 1$ for any $\gamma \in \Gamma$.

Corollary 19

$\Gamma \not\models \varphi$ *iff there is a valuation v such that $[\![\gamma]\!]_v = 1$, for any $\gamma \in \Gamma$ and $[\![\varphi]\!]_v = 0$.*

Completeness

Theorem 20

$\Gamma \models \varphi$ **implies that** $\Gamma \vdash \varphi$

Finally, the Natural Deduction Calculus for Propositional Logic is **sound and complete**

Corollary 21

$$\Gamma \models \varphi \text{ iff } \Gamma \vdash \varphi$$

Soundness and Completeness of Propositional Logic

Corollary 22

$$\Gamma \models \varphi \textbf{ iff } \Gamma \vdash \varphi$$