Elements of Logic 2024/25
Intuitive Set Theory

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Outline

Sets

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- $\emptyset$ is the set with no elements.
- A set, itself, may be an element of some other set
- **Actually, for mathematical purposes no other elements need ever be considered!**

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$$\{1, 2, 3\} = \{2, 3, 1\} = \{3, 1, 2, 1, 2, 3\} = \ldots.$$

To see that $A = B$

check that

- for any $a \in A$, we have $a \in B$; and
- for any $b \in B$, we have $b \in A$

Defining Sets

Definition by extension

- $A = \{a_1, a_2, \ldots, a_n\}$, for sets with finite number of elements
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The set A is constituted by the elements x such that satisfies the property φ

Exercise 1

Define by comprehension the sets:

- $A = \{2, 4, 6, \dots\}$,
- $B = \{a\}$, *and*
- $C = \{\}$

Subsets of a set

Definition 3 (Subset)

If every element of a set A is also an element of B, then we say that A is a **subset** of B, and write $A \subseteq B$. If A is not a subset of B we write $A \nsubseteq B$. If $A \subseteq B$ but $A \neq B$, we write $A \subsetneq B$ and say that A is a **proper subset** of B.

Example 4

- $\{a, b\} \subseteq \{a, b\}$, and $\{a, b\} \subsetneq \{a, b, c\}$
- $\emptyset \subseteq A$, for any set A

Hence:

$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Exercise 2

Prove that there is at most one empty set, i.e., show that if A and B are sets without elements, then $A = B$.

Some well known sets

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

the set of natural numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

the set of integers

$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$

the set of rationals

$\mathbb{R} = (-\infty, \infty)$

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Example 5

Some well known sets

Finite Strings on $\mathbb{B}$

$$\mathbb{B}^* = \{\epsilon, 0, 1, 00, 01, 10, 11,$$
$$000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\}.$$

Infinite Strings on a set $\mathbb{B}$

$$\mathbb{B}^\omega = \{b_1 b_2 \cdots \mid b_i \in \mathbb{B}, i \in \mathbb{N}\}$$

Basic Operations on Sets

If A is a set of sets, then $\bigcup A$ is the set of elements of elements of A:

$$\bigcup A = \{x \mid \text{there is a } B \in A \text{ so that } x \in B\}$$
$$\bigcap A = \{x \mid \text{for all } B \in A, x \in B\}$$

For a sequence of sets A_1, A_2, ...

$$\bigcup_i A_i = \{x \mid x \text{ belongs to one of the } A_i\}$$
$$\bigcap_i A_i = \{x \mid x \text{ belongs to every } A_i\}.$$

For a property φ,

$$\bigcup_{\varphi(x)} x = \bigcup\{x \mid \varphi(x)\}$$
$$\bigcap_{\varphi(x)} x = \bigcap\{x \mid \varphi(x)\}$$

Basic Operations on Sets

Exercise 3

Let us consider the well known opertions of **binary union** *of two sets A and B:*

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

and of **binary intersection** *of two sets A and B*

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Show that they can be expressed by the union and intersection operators of a set.

Exercises

Exercise 4

Show that:

1. if $A \subseteq B$, then $A \cup B = B$.
2. if $A \subseteq B$, then $A \cap B = A$.
3. if $A \subsetneq B$, then $B \backslash A \neq \emptyset$.
4. if A is a set and $A \in B$, then $A \subseteq \bigcup B$.

Exercises

Exercise 4

Show that:

1. *if* $A \subseteq B$*, then* $A \cup B = B$.
2. *if* $A \subseteq B$*, then* $A \cap B = A$.
3. *if* $A \subsetneq B$*, then* $B \backslash A \neq \emptyset$.
4. *if* A *is a set and* $A \in B$*, then* $A \subseteq \bigcup B$.

Exercise 5

Suppose $s = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$. *Determine* $\bigcup s$ *and* $\bigcap s$

Exercises

Exercise 6

1. $\bigcup_{x \in A} x = \bigcup A$
2. $\bigcap_{x \in A} x = \bigcap A$
3. $\bigcup \{x\} = x$
4. $\bigcap \{x\} = x$
5. $\bigcup_{x \in \{s\}} x = s$
6. $\bigcap_{x \in \{s\}} x = s$
7. $\bigcup \varnothing = \varnothing$
8. $\bigcup_{x \in \varnothing} x = \varnothing$

Powerset of a set

Definition 6 (Power Set)

The set consisting of all subsets of a set A is called the **power set of** A, written $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Example 7

$\mathcal{P}(\{a, b, c\})$

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Example 7

$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Powerset of a set

Exercise 7

1. *Determine* $\mathcal{P}(\{a, b, c, d\})$
2. *Determine* $\mathcal{P}(\emptyset)$
3. *Determine* $\mathcal{P}(\{\emptyset\})$
4. *Let* X *and* Y *be two sets. Prove or refute the following statements:*
    - $X \subseteq Y$ *iff* $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$
    - $\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y)$
    - $\mathcal{P}(X) \cup \mathcal{P}(Y) = \mathcal{P}(X \cup Y)$
5. *Show that if* A *has* n *elements, then* $\mathcal{P}(A)$ *has* 2^n

Ordered pairs and Cartesian products

In sets the order on the enumeration of members is irrelevant:

$$\{x, y\} = \{y, x\}$$

However, we need also to lead with ordered pairs, where:

$$\textbf{if } x \neq y \textbf{ then } (x, y) \neq (y, x)$$

How should we deal with ordered pairs in set theory?

Ordered pairs and Cartesian products

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Identity of pairs

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Definition 8 (Ordered pair)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Definition 9 (Cartesian product)

Given sets A and B, their **Cartesian product** $A \times B$ is defined by

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Ordered pairs and Cartesian products

Exercise 8

1. *Prove that*
   - $(a, b) = (c, d)$ *iff* $a = c$ *and* $b = d$.
   - $\bigcup(x, y) = \{x, y\}$ *and* $\bigcap(x, y) = \{x\}$
2. *How can we deal with tuples* $(a_1, \ldots, a_n)$*?*
3. *Let* A *be a set. List the elements of the set* $A \times \emptyset$
4. *Consider the following recursive definition:*

$$A^1 = A$$
$$A^{k+1} = A^k \times A$$

   List the elements of the set $\{1, 2, 3\}^3$

5. *Prove: "If* A *has* n *elements and* B *has* m *elements, then* $A \times B$ *has* $n \cdot m$ *elements"*

Ordered pairs and Cartesian products

Exercise 9

Consider the following operator:

$$<< a, b, c >> = \{\{a\}, \{a, b\}, \{a, b, c\}\}$$

Comment the statement:

$$<< a, b, c >> = << x, y, z >> \text{ iff } a = x, b = y \text{ and } c = z$$

Exercise 10

Consider the following operator:

$$((a, b)) = \{\{b\}, \{a, b\}\}$$

Comment the statement:

$$((a, b)) = ((x, y)) \text{ iff } a = x \text{ and } b = y$$

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Do all the properties φ define a set?
i.e. $\{x \mid \varphi(x)\}$ is a set, for any property φ?

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Namely, it shall be able to answer the questions:

- What are sets?
- What sets exist?

Zermelo-Fraenkel (ZF) Axiomatics

Extensionality

$$\forall x.\forall y.[x = y \leftrightarrow \forall z.(z \in x \leftrightarrow z \in y)]$$

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replacement for $\varphi(x, y)$ such that $\forall x.\forall y.\forall z.[\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z]$, we have

$$\forall z.\exists x.\forall y.[y \in x \leftrightarrow \exists u \in z.\varphi(u, y)]$$

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$$\forall z.\exists x.\forall y.[y \in x \leftrightarrow \exists u \in z.\varphi(u, y)]$$

union

$$\forall z.\exists x.[\forall y.(y \in x \leftrightarrow \exists u \in z.(y \in u))]$$

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infinite

$$\exists x.[\varnothing \in x \wedge \forall y \in x.(y \cup \{y\} \in x)]$$

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infinite

$$\exists x.[\emptyset \in x \wedge \forall y \in x.(y \cup \{y\} \in x)]$$

regularity

$$\forall x.[x \neq \emptyset \rightarrow \exists y \in x.(y \cap x = \emptyset)]$$

Observations on ZF Axiomatics

- The **axiomatic theory of sets** is a complex issue usually object of a complete UC
- there are in the literature **other equivalent presentations of ZF axiomatics**
- it follows some observations about the introduced axioms

Observations on ZF Axiomatics

Axiom of Extensionality

Two sets are equal iff they have the same elements

$$\forall x. \forall y. [x = y \leftrightarrow \forall z. (z \in x \leftrightarrow z \in y)]$$

Obs: Hence, $A = B$ when for all x, $x \in A$ iff $x \in B$

Observations on ZF Axiomatics

Axiom of Replacement

As a consequence of this axiom (with the other ones) we can get the **axiom of separation**:
For any property φ,

$$\forall z.\exists x.\forall y.[y \in x \leftrightarrow y \in z \wedge \varphi(y)]$$

that is on the basis of **definitions by comprehension**

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Exercise 11

Show that this axiom assure the existence of

- *an emptyset*
- *the intersection set of a set*
- *the difference set between two sets, i.e. the set* $A \backslash B = \{a \in A \mid a \notin B\}$

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Axiom of powerset

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Axiom of powerset

$$\forall z.\exists x.[\forall y.(y \in x \leftrightarrow y \subseteq z)]$$

Obs: for any z, we have $x = \mathcal{P}(z) = \{y \mid y \subseteq z\}$

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Axiom of Infinity

there exists a set X having infinitely many members

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Natural numbers (**Von Neumann Ordinals**):

$$0 = \varnothing \qquad (1)$$

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$$\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}$$

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Axiom of Regularity

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Exercise 14

Check that this axiom implies that there is no closed sequences $x_1 \in x_2 \in \cdots \in x_1$

Exercises

Exercise 15

Check if the following sets are regular:

1. $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \dots\}$
2. $\{a, \{a\}, \{\{a\}\}, \dots\}$

Exercises

Exercise 15

Check if the following sets are regular:

1. $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots\}$
2. $\{a, \{a\}, \{\{a\}\}, \ldots\}$

Exercise 16

Given a set a let us consider the following construction:

- $a^0 = a$
- $a^{k+1} = a^k \cup \{a^k\}$

For a given $k \in \mathbb{N}$, *check the regularity of:*

1. a^k
2. $\varnothing^k$

Exercises

Exercise 17

Let A be a set. Then $A \cap \{A\} = \varnothing$.

Exercise 18

Consider the following operator:

$$[a, b] = \{a, \{a, b\}\}$$

Comment the statement:

$$[a, b] = [x, y] \text{ iff } a = x \text{ and } b = y$$

Axiom of Choice – Axiomatic ZFC

The Axiom of Choice (Zermelo 1904)

"Given any (non-empty set) whose elements are pairwise disjoint non-empty sets, there is a set which contains precisely one element from each set belonging to it"

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More formally:

$$\begin{aligned}\forall y \quad \big(\forall u \in y \quad & \\ & (u \neq \emptyset \wedge (\forall v \in y(v \neq u \rightarrow (u \cap v = \emptyset)) \\ & \rightarrow \\ & (\exists z \forall u \in y \exists w(z \cap y = \{w\})\big)\end{aligned}$$

Axiom of Choice

The Axiom of Choice (Alternative formalisation)

"Given any (non-empty) set x whose elements are non-empty sets, there is a function f such that $f(a) \in a$, for each $a \in x$"

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Exercise 19

Let $A = \{\{a\}, \{b\}, \{a, b\}\}$. Enumerate the choice functions for A.

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Analogy of Bertrand Russell

- for an infinite set of **pairs of shoes**, one can pick out the left shoe from each pair to obtain an appropriate set of shoes; **this makes it possible to define a choice function directly**
- For an infinite set of **pairs of socks**, there is no obvious way to make a function that forms a set out of **selecting one sock from each pair without invoking the axiom of choice**

Axiom of Choice

This is a controversial axiom

Apologists:

- Hilbert and Russel,... Poincaré (late)

Resistant:

- Borel, Lebesgue, ... Poincaré (ealier)

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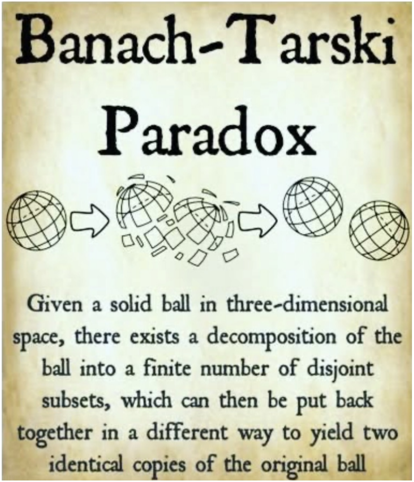
Axiom of Choice is quite useful

- Equivalent to the **Zorn Lemma**
- we use it to prove that any vectorial Space has a basis
- it is need to prove
  - **Compacity Theorem**
  - **Completeness Theorem for First-order Logic**

Axiom o Choice

However, it entails some "weird" effects

- Assuming the Axiom of Choice we can prove:



Outline

Binary relations

Definition 10 (Binary relation)

A **binary relation** on a set A is a set

$$B \subseteq A \times A$$

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Examples

1. $R = \{(n, m) \mid n, m \in \mathbb{N} \text{ and } n < m\}$
2. $E = \{(n, m) \mid n > 5 \text{ or } m \times n \geqslant 34\}$
3. $S = \{(A, B) \mid A, B \subseteq U \text{ and } A \subseteq B\}$

Binary relations

Exercise 20

List the elements of the relation $\subseteq$ on the set $\mathcal{P}(\{a, b, c\})$.

Properties of Binary relations

Definition 11

Let A be a set. A relation $R \subseteq A \times A$ is

- **reflexive**

Properties of Binary relations

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- **irreflexive** if, for all $x \in A$, not $(x, x) \in R$
- **asymmetric** if for no pair $x, y \in A$ we have both $(x, y) \in R$ and $(y, x) \in R$.

Properties of Binary relations

Exercise 21

Give examples of relations that are

1. *reflexive and symmetric but not transitive*
2. *reflexive and anti-symmetric,*
3. *anti-symmetric, transitive but not reflexive, and*
4. *reflexive, symmetric, and transitive.*

Properties of Binary relations

Definition 12

A relation $R \subseteq A \times A$ is a

- **preorder** if is both reflexive and transitive.
- **partial order** is a A preorder which is also anti-symmetric
- **linear order** is a partial order which is also connected
- **strict order** if it is irreflexive, asymmetric,
- **equivalence** if is is reflexive, symmetric and transitive

Properties of Binary relations

Exercise 22

Classify the following relations wrt the properties of Definition 12:

- *the identity relation* $Id_A \subseteq A \times A$*, i.e, the relation* $Id_A = \{(a, a) \mid a \in A\}$
- *the relation* $\subseteq$ *on sets*
- *the relation* $\leqslant$ *in integers*
- $\mathbb{B}^*$: $x \preccurlyeq y$ *iff* $len(x) \leqslant len(y)$
- $n \mid m$ *iff there is some integer* k *so that* $m = kn$

Equivalence Relations and Quotients

Definition 13

Let $R \subseteq A \times A$ be an equivalence relation. For each $x \in A$, the **equivalence class** of x in A is the set $[x]_R = \{y \in A \mid (x, y) \in R\}$. The *quotient* of A under R is $A/R = \{[x]_R \mid x \in A\}$, i.e., the set of these equivalence classes.

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Proposition 2.1

If $R \subseteq A \times A$ is an equivalence relation, then $(x, y) \in R$ iff $[x]_R = [y]_R$.

Proof.

Exercise! □

Equivalence Relations

Exercise 23

For any a, b, and $n \in \mathbb{N}$*, say that* $a \equiv_n b$ *iff dividing a by n gives the same remainder as dividing b by n.*
Show that $\equiv_n$ *is an equivalence relation, for any* $n \in \mathbb{Z}^+$*, and that* $\mathbb{N}/\equiv_n$ *has exactly n members.*

The Algebra of Relations

Definition 14

- Let A be a set. The **identity in A** is the relation $id_A \subseteq A \times A$ defined by

$$id_A = \{(a, a) \mid a \in A\}$$

- Let $R \subseteq A \times B$ a relation. The **converse of R** is the relation

$$R^\circ = \{(b, a) \mid (a, b) \in R\}$$

- Let $R \subseteq A \times B$ and $S \subseteq B \times C$ two relations. The **composition of R with S** is the relation $R \cdot S$ defined as follows

$$R \cdot S = \{(a, c) \mid \exists b \in B.(a, b) \in R \textbf{ and } (b, c) \in S\}$$

The Algebra of Relations

Exercise 24

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ two relations. Show that:

1. $R^\circ \subseteq S$ *iff* $R \subseteq S^\circ$
2. $R \subseteq S$ *iff* $R^\circ \subseteq S^\circ$
3. $R^{\circ\circ} = R$
4. $(R \cap S)^\circ = R^\circ \cap S^\circ$
5. $(R \cdot S)^\circ = S^\circ \cdot R^\circ$

The Algebra of Relations

Notions of Definition 14 can be used to characterize properties on relations:

Exercise 25

Note that for any $R \subseteq A \times A$ is reflexive iff $id_A \subseteq R$.

- *Analogously, for any relation $R \subseteq A \times A$, characterize the notions of*
  1. *transitivity,*
  2. *symmetry,*
  3. *anti-symmetry,*
  4. *connectivity and*
  5. *irreflexivity*

The Algebra of Relations

Some Special Relations

A relation $R \subseteq A \times B$ is

- **Entire** if

$$id_A \subseteq R \cdot R^{\circ}$$

The Algebra of Relations

Some Special Relations

A relation $R \subseteq A \times B$ is

- **Entire** if

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- **Simple** if

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The Algebra of Relations

Some Special Relations

A relation $R \subseteq A \times B$ is

- **Entire** if

$$id_A \subseteq R \cdot R^{\circ}$$

- **Simple** if

$$R^{\circ} \cdot R \subseteq id_B$$

Exercise 26

Using this definition, characterize the relations that are **functions**.

The Algebra of Relations

Exercise 27 (Monotonicity)

Prove that for any relations R, S and T:

- $S \subseteq T$ **implies that** $S \cdot R \subseteq T \cdot R$
- $S \subseteq T$ **implies that** $R \cdot S \subseteq R \cdot T$

The Algebra of Relations

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Exercise 28 (Identity preservation)

Prove that for any relation $R \subseteq A \times B$,

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Exercise 28 (Identity preservation)

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Exercise 29 (Galois connection)

Let R, f and S binary relations. Prove that, if f is a function,

$$R \cdot f \subseteq S \textbf{ iff } R \subseteq S \cdot f^\circ$$

and

$$f^\circ \cdot R \subseteq S \textbf{ iff } R \subseteq f \cdot S$$

The Algebra of Relations

Definition 15

Kernel and Images of a relation Let $R \subseteq A \times B$ a relation. The

- **Kernel of R** is the relation $\mathrm{Ker}(R) \subseteq A \times A$ that relates the elements in A that share the same images under R

The Algebra of Relations

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The Algebra of Relations

Exercise 30

Prove that

- $R \subseteq S \Rightarrow Ker(R) \subseteq Ker(S)$
- $R \subseteq S \Rightarrow Img(R) \subseteq Img(S)$

The Algebra of Relations

Exercise 31

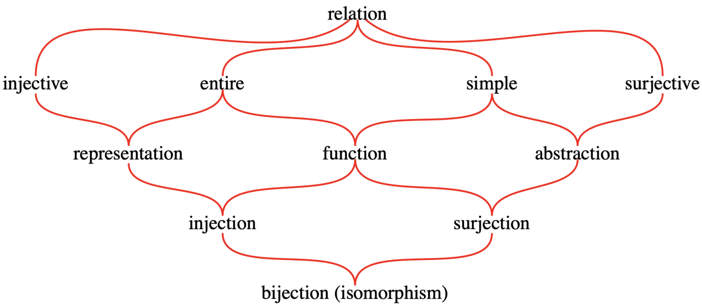
Using the Kernel and Image relations, characterize the relations that are

- *injective*
- *surjective*
- *entire and injective*
- *simple and surjective*
- *injective functions*
- *surjective functions*
- *bijections*

The Algebra of Relations

Binary relation taxonomy [a]

[a]from *First Steps in Pointfree Functional Dependency Theory. José Nuno Oliveira. 2005*



Equivalence Relations

Exercise 32

Prove or refute the following sentences:

1. *The intersection of two equivalence relations is an equivalence relation*
2. *The union of two equivalence relations is an equivalence relation*
3. *The composition of two equivalence relations is an equivalence relation*
4. *The converse of an equivalence relations is an equivalence relation*

Outline

Finite and infinite sets

Definition 16 (Finite and Infinite sets)

A non-empty **set A is finite** if there is a positive integer n and a bijection from $\{0, \ldots, n\}$ to A. Otherwise it is **infinite**.
The empty set is by convention taken to be finite.

Equinumerous sets

Definition 17

Equinumerous sets Two sets A and B are **equinumerous** if there is a bijection from A to B. We denote this by $A \sim B$.

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Exercise 33

For any sets A, B and C, show that:

1. $A \sim A$.
2. If $A \sim B$, then $B \sim A$.
3. If $A \sim B$ and $B \sim C$, then $A \sim C$.
4. if $A \sim C$ and $B \sim D$, and $A \cap B = C \cap D = \emptyset$, then $A \cup B \sim C \cup D$.

A less intuitive observation?

Exercise 34

Comment the following statement:

There exists sets A and B such that $A \sim B$ but $A \subsetneq B$.

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Supposing now that A and B are finite. What we can conclude?

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Exercise 34

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There exists sets A and B such that $A \sim B$ but $A \subsetneq B$.

Supposing now that A and B are finite. What we can conclude?

Exercise 35

Show that

- $\mathbb{N} \sim 2\mathbb{N} = \{0, 2, 4, \dots\}$
- $\mathbb{R} \sim \mathbb{R}^+$
- $\mathbb{R} \times \mathbb{R} \sim \mathbb{C}$

Sets domination

Definition 18

For sets A and B, A **is dominated by** B if there is an injection from A to B. We write $A \preceq B$. A **is strictly dominated by** B if $A \preceq B$ and A is not equinumerous with B.

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Exercise 36

Show that

1. *If A is a finite set, then $A \preceq \mathbb{N}$.*
2. *For any sets A and B, if $A \sim B$, then $A \preceq B$.*
3. *For any set A, $A \preceq A$.*
4. *For any sets A and B, if $A \subseteq B$, then $A \preceq B$.*
5. *If $A \preceq B$ and $A \sim C$, then $C \preceq B$*

Sets domination

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2. *For any sets A and B, if $A \sim B$, then $A \preceq B$.*
3. *For any set A, $A \preceq A$.*
4. *For any sets A and B, if $A \subseteq B$, then $A \preceq B$.*
5. *If $A \preceq B$ and $A \sim C$, then $C \preceq B$*

A is **smaller than** B, written $A \prec B$, iff there is an injection $f\colon A \rightarrow B$ but no a bijection $g\colon A \rightarrow B$, i.e., $A \prec B$ and $A \not\sim B$.

Countable Sets

An **enumeration** of a set A is a list (possibly infinite) of element of A such that every element of A appears on the list at some finite position. If A has an enumeration, then A is said to be **countable**.

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An **enumeration** of a set $A \neq \emptyset$ is a surjective function $f : \mathbb{N} \to A$.

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Definition 20

A set A is **countable** if either

1. it is finite, or
2. it is infinite and $\mathbb{N} \sim A$

Countable Sets

Exercise 37

Show that:

1. *Any subset of* $\mathbb{N}$ *is countable*
2. *If* A *is coutable and* $A \sim B$*, then* B *is countable*

Countable Sets

Exercise 37

Show that:

1. *Any subset of $\mathbb{N}$ is countable*
2. *If A is coutable and $A \sim B$, then B is countable*

Theorem 21 (Other characterizations for countable sets)

1. *A set A is countable iff there is an injection $f : A \to \mathbb{N}$ (i.e. $A \preceq \mathbb{N}$)*
2. *A non-empty set A is countable iff there is a surjection $f : \mathbb{N} \to A$*

Examples of enumerations

Example 22

- A function enumerating the naturals is simply the identity function given by $f(n) = n$.

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- The functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = 2n \text{ and}$$
$$g(n) = 2n + 1$$

enumerate the even positive integers and the odd positive integers, respectively.

Examples of enumerations

A possible enumeration of the set $\mathbb{Z}$ is done by the function

$$f(n) = (-1)^n \lceil \frac{(n-1)}{2} \rceil$$

(where $\lceil x \rceil$ rounds x up to the nearest integer):

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$\ldots$
$-\lceil\frac{0}{2}\rceil$	$\lceil\frac{1}{2}\rceil$	$-\lceil\frac{2}{2}\rceil$	$\lceil\frac{3}{2}\rceil$	$-\lceil\frac{4}{2}\rceil$	$\lceil\frac{5}{2}\rceil$	$-\lceil\frac{6}{2}\rceil$	$\ldots$
0	1	-1	2	-2	3	$\ldots$	

Exercises

Exercise 38

Define an enumeration of the positive squares 1, 4, 9, 16, ...

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Show that if A *and* B *are countable, the set* $A \cup B$ *is countable.*

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Show that if A and B are countable, the set $A \cup B$ is countable.

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Show that if $B \subseteq A$ and A is countable, the set B is countable.

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Show that if A and B are countable, the set $A \cup B$ is countable.

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Show that if $B \subseteq A$ and A is countable, the set B is countable.

Exercise 41

Show by induction on n that if $A_1, A_2, \ldots, A_n$ are all coutable, then the set $A_1 \cup \cdots \cup A_n$ is countable.

Cantor's zig-zag method

How to enumerate the set $\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n, m \in \mathbb{N}\}$

	0	**1**	**2**	**3**	...
0	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	...
1	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	...
2	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	...
3	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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⋮	⋮	⋮	⋮	⋮	⋱

Cantor's zig-zag method

	0	**1**	**2**	**3**	**4**	...
0	0	1	3	6	10	...
1	2	4	7	11	...	...
2	5	8	12	...	...	...
3	9	13	...	...	...	...
4	14	...	...	...	...	...
⋮	⋮	⋮	⋮	⋮	...	⋱

Cantor's zig-zag method

Hence $\mathbb{N} \times \mathbb{N}$ is enumerated as follows:

$$(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2),(2,1),(3,0),\ldots$$

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Proposition 3.1

$\mathbb{N} \times \mathbb{N}$ *is contable.*

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Proof.

It is easy to see that $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ given by the Cantor's table is surjective. $\square$

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Pairing Function

Can we back?

$$g(n,m) = \frac{(n+m+1)(n+m)}{2} + n$$

An example of an uncontable set

Proposition 3.2

$\mathbb{B}^\omega$, the set of infinite $\{0,1\}$-strings, is **uncontable**

- **Cantor's diagonal method:** Suppose, that $\mathbb{B}^\omega$ is enumerable, i.e., suppose that there is a list s_1, s_2, s_3, s_4, ... of all elements of $\mathbb{B}^\omega$.

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	1	2	3	4	...
1	$\mathbf{s_1(1)}$	$s_1(2)$	$s_1(3)$	$s_1(4)$	...
2	$s_2(1)$	$\mathbf{s_2(2)}$	$s_2(3)$	$s_2(4)$	...
3	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	$s_3(4)$	...
4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$\mathbf{s_4(4)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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3	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	$s_3(4)$	...
4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$\mathbf{s_4(4)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- but the sequence $\bar{s}(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1. \end{cases}$ is not on the list.

Contradiction!

Countable and uncontable sets

Exercise 42

Prove or refute the following sentence:
"(i) The set $\mathbb{Q}$ is contable.
(ii) Moreover, it is also dense, i.e. for any $x, y \in \mathbb{Q}$ with $x < y$, there is a $z \in \mathbb{Q}$ such that and $x < z$ and $z < y$"

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Exercise 43

Prove or refute the following sentence:
"The set of reals in the interval $[0, 1)$ is uncontable"

The size of sets

Exercise 44

Prove the Cantor Theorem:
For any set A, $A \prec \mathcal{P}(A)$.

Exercise 45

Show that $\mathcal{P}(\mathbb{N})$ is not contable.

The size of sets

Theorem 23 (Schröder-Bernstein)

If $A \preceq B$ and $B \preceq A$, then $A \sim B$.

The size of sets

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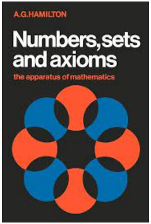
Some consequences:

- Let I be an interval in $\mathbb{R}$ which is not empty and not a singleton. Then $I \sim \mathbb{R}$
- $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$
- ...



Elements of Logic 2024/25
Intuitive Set Theory

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