Automatic Semigroups: Constructions and Subsemigroups

Luis Descalço

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to Margarida

Contents

Declarations

I, Luis Descalço, hereby certify that this thesis, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Signature........................**Name** Luis Descalço **Date** 10/07/2002

I was admitted as a research student in October, 1999 and as a candidate for the degree of Doctor of Philosophy in October, 2000; the higher study for which this is a record was carried out in the University of St Andrews between 1999 and 2002.

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Abstract

In this thesis we start by considering conditions under which some standard semigroup constructions preserve automaticity. We first consider Rees matrix semigroups over a semigroup, which we call the base, and work on the following questions:

(i) If the base is automatic is the Rees matrix semigroup automatic?

(ii) If the Rees matrix semigroup is automatic must the base be automatic as well?

We also consider similar questions for Bruck-Reilly extensions of monoids and wreath products of semigroups.

Then we consider subsemigroups of free products of semigroups and we study conditions that guarantee them to be automatic.

Finally we obtain a description of the subsemigroups of the bicyclic monoid that allow us to study some of their properties, which include finite generation, automaticity and finite presentability.

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Chapter 1

Introduction

The notion of an automatic group appears in the 1980's, beginning with the paper [4] by G. Baumslag, S. M. Gersten, M. Shapiro and H. Short and the book [12] by J. W. Cannon, D. B. A. Epstein, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, and from then many results about automatic groups have been published; see for example [17, 18, 19, 36, 44, 47]. In the end of the 1990's, the notion was generalized for semigroups and, in the paper [11] by C. M. Campbell, E. F. Robertson, N. Ruškuc and R. M. Thomas, the authors established the basic properties and obtained the first results about automatic semigroups. More work about automatic semigroups was done since then; see for example [9, 10, 13, 16, 23, 33, 34, 35, 48, 49].

Automatic groups are characterized by a geometric property of their Cayley graph, which is intuitively the following: "there is a constant K such that, if two fellows travel at the same speed by two paths ending at most one edge apart, then the distance between them is always less then K". This property, called *the fellow traveller property*, does not characterize automatic semigroups and therefore, the geometric theory that holds for automatic groups does not hold for automatic semigroups. Hence we have to work directly with regular languages instead of Cayley graphs when dealing with automatic semigroups. Nevertheless, the definition of "automatic" for semigroups leads to an interesting class of semigroups, that contains many known semigroups, and where some properties of automatic groups naturally hold while others either require different proofs or do not hold. The idea of defining a class of semigroups using the concept of a regular language is quite natural and establishes an interesting connection between semigroups and formal languages that allows us in particular to use tools from formal languages to obtain results about semigroups; for example, in this thesis, we often use the concept of a generalized sequential machine to deal with semigroup constructions.

It is worth mentioning the work in [23] where four alternative definitions of "automatic semigroup" have been considered, that are equivalent when applied to groups, but that determine four different classes of semigroups. In this context we can say that our work is about one of those four classes, the one considered in [11].

In this thesis we start by studying conditions under which some standard semigroup constructions preserve automaticity. Then we consider automaticity of subsemigroups of free products and finally we obtain a description of the subsemigroups of the bicyclic monoid and study their properties, in particular their automaticity.

We first have a short introduction, containing the essentials about regular languages, semigroups and automatic semigroups that we will need in the thesis. We start, in Chapter 2, by considering an example of an automatic semigroup, the free group in n generators, and by studying in detail its automatic structure. In Chapter 3 we establish some results regarding regular languages, that will be useful when constructing automatic structures from known automatic structures. In Chapter 4 we consider Rees matrix semigroups over a semigroup, which we call the base, and work on the following questions:

(i) If the base is automatic is the Rees matrix semigroup automatic?

(ii) If the Rees matrix semigroup is automatic must the base be automatic as well?

In Chapter 5 we consider similar questions for Bruck-Reilly extensions of monoids and wreath products of semigroups. Subsemigroups of free products of semigroups are studied in Chapter 6, where we start by considering subsemigroups of free semigroups. Finally in Chapters 7 and 8 we obtain a description of the subsemigroups of the bicyclic monoid and we study some of their properties which include finite generation, automaticity and finite presentability.

1 Regular languages and automata

In this section we present the more relevant definitions and known results about regular languages and automata we will require in this thesis and we establish our notation. Further results, that are only used in a single chapter, will be stated in that chapter.

Let A be a finite set. We define

$$A^+ = \{a_1 \dots a_n : a_1, \dots, a_n \in A, n \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of the positive integers, to be the set of all finite sequences of elements of A (with at least one element). We say that A is an *alphabet* and the elements of A^+ are called *words*. Given $w \in A^+$ we denote by $|w|$ the *length* of w, which is the number of elements of A that form the word w. We define $A^* = A^+ \cup \{\epsilon\}$, where ϵ is not an element of A^+, and we call *language* any subset of A^*. We define the operation *concatenation* on A^* by

$$\begin{aligned} \cdot : & A^* \times A^* \to A^*; \\ & a_1 \dots a_n \cdot b_1 \dots b_m = a_1 \dots a_n b_1 \dots b_m \ (n, m \in \mathbb{N}), \\ & a_1 \dots a_n \cdot \epsilon = \epsilon \cdot a_1 \dots a_n = a_1 \dots a_n \ (n \in \mathbb{N}), \\ & \epsilon \cdot \epsilon = \epsilon, \end{aligned}$$

and we often write $w_1 w_2$ instead of $w_1 \cdot w_2$ for $w_1, w_2 \in A^*$. It is convenient to see ϵ as a sequence with no elements and so we call ϵ the *empty word* and we have $|\epsilon| = 0$, by convention. Hence we can write

$$A^* = \{a_1 \dots a_n : a_1, \dots, a_n \in A, n \in \mathbb{N}_0\}.$$

For $w \in A^*$ we define $w^0 = \epsilon$ and $w^{n+1} = w \cdot w^n$ $(n \in \mathbb{N}_0)$. We observe that the operation $\cdot$ is an *associative* operation, i.e. we have $(w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3)$ for any $w_1, w_2, w_3 \in A^*$.

Given two languages $L, K \subseteq A^*$ we define their concatenation $L \cdot K$ by

$$L \cdot K = \{w_1 \cdot w_2 : w_1 \in L, w_2 \in K\}$$

and we often write LK instead of $L \cdot K$. Given a language L we define

$$\begin{aligned} &L^0 = \{\epsilon\}, \\ &L^{n+1} = L \cdot L^n \ (n \in \mathbb{N}_0), \\ &L^* = \textstyle\bigcup_{n=0}^{\infty} L^n = \{w_1 \cdot w_2 \cdot \ldots \cdot w_n : w_1, w_2, \ldots, w_n \in L, n \in \mathbb{N}_0\} \end{aligned}$$

and we call $*$ the *Kleene star operation.*

We say that a language is *regular* if it can be obtained from finite subsets of A^* by finitely many applications of $\cup$ (union), $\cdot$ (concatenation) and $*$ (Kleene's star operation). For example, if we define $A = \{b, c\}$, then the language $M = \{c^n b^m : n, m \in \mathbb{N}_0\}$ is regular because we have $M = \{c\}^* \cdot \{b\}^*$.

A *finite state automaton* (or simply an *automaton*) is a quintuple

$$\mathcal{A} = (Q, A, \mu, q_0, T)$$

where Q is a finite set called the set of *states*, A is an alphabet called the *input alphabet*, μ is a function $\mu : Q \times A \to \mathcal{P}(Q)$ called the *transition* (we denote by $\mathcal{P}(Q)$ the set of all subsets of Q), $q_0 \in Q$ is called the *initial state* and $T \subseteq Q$ is the set of *terminal states.* The situation $q' \in (q, a)\mu$, for $q, q' \in Q$, $a \in A$, can be intuitively understood the following way: if $\mathcal{A}$ is in state q and reads input a then it can move to state q'.

An automaton can also be seen as a directed graph with vertices Q and an edge (q, a, q') for each $q, q' \in Q$, $a \in A$ such that $q' \in (q, a)\mu$. We represent an edge (q, a, q') by $q \xrightarrow{a} q'$. We define a *path* π in the automaton to be a sequence of edges

$$(q_1, a_1, q_2), (q_2, a_2, q_3), \ldots, (q_n, a_n, q_{n+1}),$$

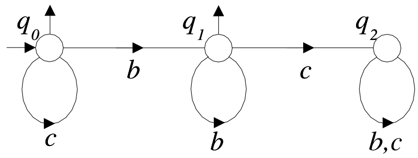

Figure 1.1: An automaton recognizing $\{c^n b^m : n, m \in \mathbb{N}_0\}$

where $q_1, \ldots, q_{n+1} \in Q$, $a_1, \ldots, a_n \in A$, and we represent it by

$$\pi : q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} \ldots \xrightarrow{a_n} q_{n+1}, \tag{1.1}$$

or a single triple (q, ϵ, q), with $q \in Q$, that we represent by

$$\pi : q \xrightarrow{\epsilon} q.$$

We say that a path is a *successful path* if it starts in the initial state and ends in a terminal state. We say that the path π above is *labeled* by $a_1 \ldots a_n$ (or by ϵ). We write simply

$$\pi : q \xrightarrow{w} q',$$

with $w \in A^*$, to mean that π is a path in $\mathcal{A}$ from state q to state q' labeled by w.

We say that a word $w \in A^*$ is *recognized* by the automaton $\mathcal{A}$ if there exists a successful path π in $\mathcal{A}$ labeled by w; we observe that the empty word ϵ is recognized if and only if the initial state is a terminal state. The *language recognized* by the automaton $\mathcal{A}$ is the set of all words that are recognized by $\mathcal{A}$; we denote it by $\mathcal{L}(\mathcal{A})$. A language is *recognizable* if there exists an automaton that recognizes it. For example, the language M defined above, is recognized by the automaton given by Figure 1.1; the figure also illustrates how an automaton can be defined by a picture: the incoming arrow marks the initial state q_0 and the two outgoing arrows mark the two terminal states q_0 and q_1. It is well known that the classes of regular and recognizable languages coincide (see for example [28]) and we will use both terms as synonyms.

We say that an automaton is *deterministic* if the set $(q, a)\mu$ has at most one element for any $q \in Q, a \in A$, and non deterministic otherwise. For a deterministic automaton we can write $q' = qw$, if there is a path from state q to state q' labeled by the word w. We say that an automaton is *complete* if for any $q \in Q$, $a \in A$ the set $(q, a)\mu$ has at least one element. We note that if an automaton is deterministic and complete then the transition μ can be seen as a function from $Q \times A$ to Q. It is known that if a language is regular then there exists a deterministic and complete automaton recognizing it; see [28]. Figure 1.1 is an example of a deterministic and complete automaton; we can remove state q_2 and the arrows arriving to it, to obtain an example of a non deterministic automaton recognizing the same language.

We say that a state q is *accessible* if there exists a path from the initial state to q and *co-accessible* if there exists a path from q to a terminal state. If q is a non co-accessible state and $(q, a)\mu \subseteq \{q\}$ for all $a \in A$ we say that q is a *fail state*. In Figure 1.1, the state q_2 is a fail state.

For a word $w = a_1 \dots a_n$ with $a_1, \dots, a_n \in A$, given $t \in \mathbb{N}_0$, we define $w(t) = a_1 \dots a_t$ for $t \leq n$ and $w(t) = w$ otherwise. For a language $L \subseteq A^*$ we define

$$\mathrm{Pref}(L) = \{w(t) : w \in L, t \in \mathbb{N}_0\} = \{w \in A^* : ww' \in L \text{ for some } w' \in A^*\},$$
$$\mathrm{Suff}(L) = \{w \in A^* : w'w \in L \text{ for some } w' \in A^*\},$$
$$\mathrm{Subw}(L) = \{w \in A^* : w'ww'' \in L \text{ for some words } w', w'' \in A^*\}.$$

We now present some known results about regular languages that we will need in the thesis. The proofs of these results can be found in [27], for example.

Proposition 1.1 *Let A be an alphabet. Then we have:*

(i) *$\emptyset, A^+$ and A^* are regular;*

(ii) *Any finite subset of A^* is regular;*

(iii) *If $L, K \subseteq A^*$ are regular, then $L \cup K$, $L \cap K$, $L - K$, LK, L^*, $\mathrm{Pref}(L)$, $\mathrm{Suff}(L)$ and $\mathrm{Subw}(L)$ are regular.*

We will use this proposition without explicitly referring to it, and we will say that a language is regular as soon as we can write it, for example, as a finite union of languages that we know are regular.

Lemma 1.2 (The Pumping Lemma) *Let L be an infinite recognizable language in A^*. Then there exists a positive integer N such that every word z in L of length exceeding N can be factorized as $z = uvw$ in such a way that:*

(i) $u, w \in A^*, v \in A^+$;

(ii) $|uv| \leq N$;

(iii) $uv^m w \in L$ *for every* $m \geq 0$.

This lemma is normally useful to prove that a language is not regular. For example, the language $L = \{c^n b^n : n \in \mathbb{N}\} \subseteq \{b, c\}^+$ is not regular. Suppose it is regular and let N by the constant given by the Pumping Lemma. If we take a word $c^n b^n \in L$ with $n > N$ then, by the lemma, there exist numbers $i, j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $i + j + k = n$ and the words of the form $c^i c^{mk} c^j b^n$ belong to L for every $m \geq 0$. In particular we would have $c^{i+j} b^n \in L$ with $i + j < n$ which is not true.

2 Semigroups

For an introduction about semigroup theory we refer the reader to [29]. In this section we introduce the more relevant definitions and results about semigroups for this thesis. In Appendix A we include the remaining semigroup theory we require.

Let S be a set and let $\cdot : S \times S \to S$ be an operation on S. We say that $(S, \cdot)$ is a *semigroup* if the operation is associative. If S has an *identity*, i.e. there exists $e \in S$ such that $s \cdot e = e \cdot s = s$ for all $s \in S$, then we say that $(S, \cdot, e)$ is a *monoid*. We say simply that S is a semigroup (monoid) when it is clear which operation on S (and which identity) we are considering. We write simply $s_1 s_2$ instead of $s_1 \cdot s_2$ for $s_1, s_2 \in S$. For example, A^+ is a semigroup under concatenation and A^* is a monoid with identity ϵ.

Let S, T be two semigroups. A function $\psi : S \to T$ is a (*semigroup*) *homomorphism* if it satisfies $(s_1 s_2)\psi = (s_1\psi)(s_2\psi)$ for all $s_1, s_2 \in S$. If S, T are monoids we say that $\psi : S \to T$ is a *monoid homomorphism* if it is a semigroup homomorphism and preserves the identity, i.e., the image of the identity of S by ψ is the identity of T.

We say that a semigroup F is *free* on a finite set A if: *(i)* there is a function $\alpha : A \to F$; *(ii)* for every semigroup S and every function $\phi : A \to S$ there exists a unique homomorphism $\psi : F \to S$ such that $\alpha\psi = \phi$ (we observe that most of the times we write function symbols on the right and by $\alpha\psi$ we mean the function $\alpha\psi : A \to S; a \mapsto (a\alpha)\phi$). The semigroup A^+ satisfies this definition (taking α to be the identity function $A \to A^+;\ a \mapsto a$) and we often refer to A^+ as the free semigroup on A. A free monoid can be defined similarly, just replacing "semigroup" by "monoid" and "homomorphism" by "monoid homomorphism" in the definition of free semigroup, and we refer to A^* as the free monoid on A.

Let S be a semigroup, let A be a finite set and let $\theta : A \to S$ be a function. If the unique extension of θ to a homomorphism $\psi : A^+ \to S$ is surjective, we say that A is a (*semigroup*) *generating set* for S (*with respect to* ψ). If S is a monoid and the unique extension of θ to a monoid homomorphism $\psi : A^* \to S$ is surjective then we say that A is a (*monoid*) *generating set* for S (*with respect to* ψ). When it is clear which homomorphism is associated with the generating set A we say simply that A is a generating set for S and we write $S = \langle A \rangle$.

We observe that, since we consider languages on A^*, it is not convenient to see the generating set A as a subset of the semigroup S, and in fact we do not even require the function $\theta : A \to S$ to be injective. Nevertheless, whenever possible, we will not make explicit use of the homomorphisms associated with the generating sets, in order to simplify notation. For two words $w_1, w_2 \in A^*$ we write $w_1 = w_2$ to mean that $w_1\psi = w_2\psi$ and we write $w_1 \equiv w_2$ to mean that w_1 and w_2 are equal as words in A^*. We also write $w = s$ and $s = w$ with $w \in A^*$ and $s \in S$ to mean that $w\psi = s$. Finally, a product of the form $x_1 \ldots x_n$ where

$x_i \in A \cup S$ $(i = 1, \ldots, n)$ is considered as a product in A^* if all factors belong to A and as a product in S otherwise.

Given a semigroup S and a generating set A for S with respect to the homomorphism $\psi : A^+ \to S$ we say that $L \subseteq A^+$ is a *set of normal forms* for S if $L\psi = S$. If the restriction $\psi \restriction_L$ is injective we say that L is a *set of unique normal forms* for S. We observe that the multiplication in the semigroup is defined when we know how the normal forms multiply. A similar definition applies for monoids and monoid homomorphisms.

3 Automatic semigroups

In this section we give the definition of automatic semigroup and some examples, we give further results about regular languages and we list known results about automatic semigroups we will need in the thesis.

To be able to deal with automata that accept pairs of words and to define automatic semigroups, given an alphabet A, we need to define the set

$$A(2, \$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) \backslash \{(\$, \$)\}$$

where \$ is a symbol not in A (called the *padding symbol*) and the function $\delta_A : A^* \times A^* \to A(2, \$)^*$ defined by

$$(a_1 \ldots a_m, b_1 \ldots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n \\ (a_1, b_1) \ldots (a_m, b_m) & \text{if } 0 < m = n \\ (a_1, b_1) \ldots (a_m, b_m)(\$, b_{m+1}) \ldots (\$, b_n) & \text{if } 0 \le m < n \\ (a_1, b_1) \ldots (a_n, b_n)(a_{n+1}, \$) \ldots (a_m, \$) & \text{if } m > n \ge 0. \end{cases}$$

We often omit the subscript A and write δ instead of δ_A.

Definition 1.3 Let S be a semigroup and A a finite generating set for S with respect to the homomorphism $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S (with respect to ψ)* if:

(i) L is a regular language on A^+ and $L\psi = S$;

(ii) $L_= = \{(\alpha,\beta)\delta_A : \alpha,\beta \in L, \alpha = \beta\}$ is a regular language in $A(2,\$)^+$;

(iii) For each $a \in A$, the language $L_a = \{(\alpha,\beta)\delta_A : \alpha,\beta \in L, \alpha a = \beta\}$ is regular in $A(2,\$)^+$.

If a semigroup S has an automatic structure (A, L) for some A and L then we say that S is *automatic*.

We observe that the definition of "automatic" in [4] uses a set of monoid generators instead of a set of semigroup generators. This distinction does not make any difference as to whether a group (or monoid) is automatic (see [11]) and we will use the definition with monoid generators whenever it is more convenient, which is normally the case when we are working with monoids.

We should say however that, as shown in [16], if we consider the definition with semigroup generators, then an automatic monoid has an automatic structure (A, L) for any generating set A, and this is not true if we consider the definition with monoid generators. Hence, working with semigroup generators, the existence of an automatic structure for a monoid does not depend on the generating set, as it is well known to happen for automatic groups; see [4].

Example 1.4 Let A be an alphabet. Then A^+, the free semigroup on A, is automatic. We can consider the regular language $L = A^+$ and the pair (A, L) is an automatic structure for the semigroup A^+. In fact we have

$$L_= = \{(a_1, a_1)\dots(a_k, a_k) : k \in \mathbb{N}, a_1, \dots, a_k \in A\}$$

which is a regular language and, for each generator $a \in A$, we have

$$L_a = \{(a_1, a_1)\dots(a_k, a_k)(\$, a) : k \in \mathbb{N}, a_1 \dots, a_k \in A\}$$

which is a regular language as well.

Our next example is the *bicyclic monoid* $\mathbf{B}$, which is defined by the monoid presentation $\langle b, c \mid bc = 1\rangle$ (see Appendix A). A natural set of unique normal forms for $\mathbf{B}$ is $\{c^i b^j : i, j \geq 0\}$ and we shall identify $\mathbf{B}$ with this set. The normal forms multiply according to the following rule:

$$c^i b^j c^k b^l = \begin{cases} c^{i-j+k} b^l \text{ if } j \leq k \\ c^i b^{j-k+l} \text{ if } j > k. \end{cases}$$

Example 1.5 Let $A = \{b, c\}$ and $L = \{c^i b^j : i, j \geq 0\}$. Since the regular language $L = \{c\}^*\{b\}^*$ is a set of unique normal forms for $\mathbf{B}$, we have

$$L_= = \{(w, w)\delta_A : w \in L\},$$

which is a regular language, by Proposition 1.6. Since we have

$$L_b = \{(c, c)\}^*\{(b, b)\}^*\{(\$, b)\}$$

and

$$L_c = \{(c, c)\}^*\{(b, b)\}^*\{(b, \$)\} \cup \{(c, c)\}^*\{(\$, c)\},$$

the languages L_b and L_c are regular and therefore (A, L) is an automatic structure for $\mathbf{B}$.

We have the following further results about regular languages.

Proposition 1.6 *Let A and B be two alphabets. We have the following:*

(i) *If $L \subseteq A^*$ and $K \subseteq B^*$ are a regular languages and $\psi : A^* \to B^*$ is a monoid homomorphism, then $L\psi$ and $K\psi^{-1}$ are regular languages;*

(ii) *If $L, K \subseteq A^*$ are regular languages, then $(L \times K)\delta_A$ is a regular language;*

(iii) *If L is a regular language in A^*, then $\{(w, w)\delta_A : w \in L\}$ is a regular language.*

Proof. See for example [27] and [4]. ∎

We need the following result about finite semigroups:

Theorem 1.7 *Let S be a finite semigroup, X be a finite set and $\psi : X^+ \to S$ be a surjective homomorphism. For any $s \in S$ the set $s\psi^{-1}$ is a regular language.*

Proof. For an arbitrary $s \in S$ we can define the automaton

$$\mathcal{A}_s = (Q, X, \mu, q_0, \{s\}),$$

where $Q = S \cup \{q_0\}$ is the set of states and the transition μ is defined by

$$\begin{aligned}(q_0, x)\mu &= s_1 \text{ if } x\psi = s_1 && (s_1 \in S, x \in X)\\ (s_1, x)\mu &= s_2 \text{ if } s_1(x\psi) = s_2 && (s_1, s_2 \in S, x \in X).\end{aligned}$$

As we will see, this automaton is in fact the Cayley graph of the semigroup S with respect to X, with the added initial state q_0 and with terminal state s. Given $w \in X^+$ there exists a successful path $q_0 \xrightarrow{w} s$ if and only if $w\psi = s$. So we have $\mathcal{L}(\mathcal{A}_s) = s\psi^{-1}$ and therefore $s\psi^{-1}$ is a regular language. ∎

We say that (A, L) is an *automatic structure with uniqueness* for a semigroup S if (A, L) is an automatic structure for S and L is a set of unique normal forms.

The results from [11] we will need follow.

Proposition 1.8 *If (A, L) is an automatic structure for a semigroup S then there is an automatic structure (A, K) with uniqueness for S.*

Proof. Let us choose an ordering on the finite set A. Then we can define the *shortlex ordering* on A^* by:

$\alpha < \beta$ if and only if either (i) $|\alpha| < |\beta|$ or else (ii) $|\alpha| = |\beta|$ and
α precedes β lexicographically (with respect to the ordering on A)

and define

$$K = \{\alpha \in L : (\forall \beta \in L)(\alpha, \beta)\delta_A \in L_= \implies \alpha \leq \beta\}.$$

The language K is regular by [12, Theorem 2.5.1], and the result follows from [11, Propostion 5.3]. ■

Proposition 1.9 *Let S be a semigroup and let A be a generating set for S with respect to the homomorphism $\psi : A^+ \to S$. If there exists an automatic structure (A, L) for S then for any $\gamma \in A^+$ the language*

$$L_\gamma = L_{\gamma\psi} = \{(\alpha, \beta)\delta_A : \alpha, \beta \in L, \alpha\gamma = \beta\}$$

is regular.

Proof. Let $\gamma \equiv a_1 \dots a_n$. The languages $L_{a_1}, \dots, L_{a_n}$ are all regular. So the languages

$$\begin{aligned} L_{a_1a_2} &= \{\, (\alpha, \alpha_2) \in L \times L : \text{ there exists } \alpha_1 \in L \text{ such that} \\ &\qquad (\alpha, \alpha_1)\delta_A \in L_{a_1}, (\alpha_1, \alpha_2)\delta_A \in L_{a_2}\}\delta_A, \\ L_{a_1a_2a_3} &= \{\, (\alpha, \alpha_3) \in L \times L : \text{ there exists } \alpha_2 \in L \text{ such that} \\ &\qquad (\alpha, \alpha_2)\delta_A \in L_{a_1a_2}, (\alpha_2, \alpha_3)\delta_A \in L_{a_3}\}\delta_A, \\ &\qquad \vdots \\ L_{a_1a_2\dots a_n} &= \{\, (\alpha, \beta) \in L \times L : \text{ there exists } \alpha_{n-1} \in L \text{ such that} \\ &\qquad (\alpha, \alpha_{n-1})\delta_A \in L_{a_1\dots a_{n-1}}, (\alpha_{n-1}, \beta)\delta_A \in L_{a_n}\}\delta_A \end{aligned}$$

are regular by Proposition 1.13; in particular, L_γ is regular as required. ■

Proposition 1.10 *Let S be a semigroup with an automatic structure (A, L) and let $\alpha \in A^+$. If $K = L \cup \{\alpha\}$, then (A, K) is an automatic structure for S.*

Proposition 1.11 *If S is a semigroup with an automatic structure (A, L), if $B \subseteq A$, and if $L \cap B^+$ maps onto S, then $(B, L \cap B^+)$ is an automatic structure for S.*

Proposition 1.12 *Suppose that S is a semigroup with an automatic structure (A, L) and let $B = A \cup \{b\}$ where $b \notin A$. For any word $\alpha \in A^+$, we have an automatic structure (B, K) for S, where $K = L$ and b is mapped to the element of S represented by α.*

We have stated these results for semigroups and semigroup homomorphisms but we note that the same results apply for monoids and monoid homomorphisms.

Proposition 1.13 *Let A be an alphabet and let U and V be subsets of $A^* \times A^*$ such that the languages $U\delta_A$ and $V\delta_A$ are regular. Let*

$$\begin{aligned} W = \{\, (\alpha, \gamma) \in A^* \times A^* :\ & \textit{there exists } \beta \in A^* \textit{ such that} \\ & (\alpha, \beta) \in U \textit{ and } (\beta, \gamma) \in V \}. \end{aligned}$$

Then $W\delta_A$ is regular.

Proposition 1.14 *Let S be a semigroup. Then S^1 is automatic if and only if S is automatic.*

We also need the following more general result, from [24]:

Proposition 1.15 *Let S be a semigroup and T be a subsemigroup of S such that the set $S \backslash T$ is finite. Then S is automatic if and only if T is automatic.*

We will now prove another useful result:

Theorem 1.16 *Let S be an automatic semigroup such that $S^2 = S$. Then S has an automatic structure with uniqueness (A, K) such that $K \cap A = \emptyset$.*

Proof. Let (A, L) be an automatic structure with uniqueness for S, where $L \subseteq A^+$. Suppose that there exists a word $w \in L \cap A$. Let s be the element of S represented by w. Since $S^2 = S$ there exist $s_1, s_2 \in S$ such that $s = s_1 s_2$. Let $w_1, w_2 \in L$ be words representing the elements s_1 and s_2, respectively. Defining $w' = w_1 w_2$ we have $w' = s$ and $w \notin A$. Letting $L' = L \cup \{w'\}$ we know by Proposition 1.10 that (A, L') is an automatic structure for S. We can now consider the regular language $L'' = L' \backslash \{w\}$. Since $L''_= = L'_= \cap (L'' \times L'')\delta$ and, for any $a \in A$, $L''_a = L'_a \cap (L'' \times L'')\delta$, the pair (A, L'') is also an automatic structure for S (with uniqueness). We can repeat this process until there are no more elements of the finite set A in our language. ■

The direct product of semigroups was considered in [10] where the authors have proved the following:

Proposition 1.17 *Let S and T be automatic semigroups.*

(i) *If S and T are infinite, then $S \times T$ is automatic if and only if $S^2 = S$ and $T^2 = T$.*

(ii) *If S is finite and T is infinite, then $S \times T$ is automatic if and only if $S^2 = S$.*

In [41], the authors have established necessary and sufficient conditions for the direct product of semigroups to be finitely generated:

Proposition 1.18 *Let S and T be two semigroups. If both S and T are infinite then $S \times T$ is finitely generated if and only if both S and T are finitely generated, $S^2 = S$ and $T^2 = T$. If S is finite and T is infinite then $S \times T$ is finitely generated if and only if $S^2 = S$ and T is finitely generated.*

Using this proposition, Proposition 1.17 has the following equivalent formulation:

Proposition 1.19 *The direct product of automatic semigroups is automatic if and only if it is finitely generated.*

The answer to the following converse question is not known even for groups: If the direct product $G_1 \times G_2$ is automatic are both factors G_1 and G_2 necessarily automatic?

We now present the generalization for semigroups, considered in [11], of the group concepts of the Cayley graph and the fellow traveller property. Let S be a semigroup generated by a finite set A. The (right) *Cayley graph* Γ of S with respect to A is the directed graph with vertex set S and an edge with label a from s to sa for every vertex $s \in S$ and every $a \in A$. If s and t are vertices of Γ, then an *(undirected) path* from s to t is just a sequence of edges from s to t (regardless of direction), and the *length* of the path is the number of edges it contains. We define the *distance* $d(s,t)$ from s to t to be the smallest length of a path from s to t if such path exists, and to be infinite otherwise. Let $L \subseteq A^+$ be a regular set of normal forms for S, then Γ is said to have the *fellow traveller property* with respect to L if there exists a constant K such that, whenever $\alpha, \beta \in L$ with $d(\alpha,\beta) \leq 1$, we have $d(\alpha(t),\beta(t)) \leq K$ for all $t \geq 1$. The fellow traveller property characterizes automatic groups (see [4]) and for semigroups we still have the following:

Proposition 1.20 *If S is a semigroup with an automatic structure (A, L) and if Γ is the Cayley graph of S with respect to A, then Γ has the fellow traveller property with respect to L.*

PROOF. See [11]. ■

As observed in [11], any non automatic semigroup with a zero, for example, has the fellow traveller property, and so the converse of this proposition is not true.

Finally, we say that a semigroup is *prefix-automatic* or *p-automatic* if it has an automatic structure (A, L) such that the set

$$L'_{=} = \{(w_1, w_2)\delta_A : w_1 \in L, w_2 \in \text{Pref}(L), w_1 = w_2\}$$

is also regular. It is known that the notions of "automatic" and "p-automatic" coincide for groups and more generally for right cancellative monoids (see [49]) but it is an open question if they coincide for semigroups.

For more details on automatic semigroups the reader is referred to [11] (introduction), [49] (geometric aspects and p-automaticity), [33, 34, 35] (computational and decidability aspects), [9, 10] (semigroup constructions) and [16] (invariance under change of generators).

Chapter 2

Syntactic monoids of an automatic structure for the free group

We will consider the free group in n generators, as an example of an automatic semigroup. We start by defining a natural automatic structure for the free group. Then we study the syntactic monoids associated with the regular languages that constitute this automatic structure.

1 Automatic structure

Let G be the free group in the n group generators $a_1, ..., a_n$. In order to define an automatic structure for G we consider the set of monoid generators $A = \{a_1, ..., a_{2n}\}$ and the monoid presentation

$$\langle a_1, \ldots, a_{2n} \mid a_i a_{2n+1-i} = a_{2n+1-i} a_i = 1 \ (i = 1, \ldots, n) \rangle.$$

The language we consider to represent the elements of G is then formed by the sequences of generators from A such that a generator does not precede its inverse which, according to our monoid presentation, can be simply written as

$$L = A^* - (\bigcup_{i=1,\ldots,2n} A^* a_i a_{2n+1-i} A^*).$$

We will prove that the pair (A, L) is an automatic structure for the free group G. This is a well known result (see [12]), but we include it for completeness. The language L is clearly regular and, since L is a set of unique normal forms for the free group, the language $L_=$ is regular as well. Let us fix a generator a_h in A and prove that the language

$$L_{a_h} = \{(w_1, w_2)\delta_A : w_1, w_2 \in L, w_1 a_h = w_2\}$$

is regular. We take an arbitrary word $w_1 \in L$ and find out which words $w_2 \in L$ we can obtain multiplying w_1 by the generator a_h on the right. If the last letter in the word w_1 is not the inverse, a_{2n+1-h}, of a_h then we have $w_2 \equiv w_1 a_h$. But if the last letter in w_1 is a_{2n+1-h} then we have $w_1 \equiv w_1' a_{2n+1-h}$, where the prefix w_1' of w_1 is a word in L that does not end with letter a_h, and so $w_2 \equiv w_1'$. Therefore we can write

$$\begin{aligned} L_{a_h} = & \bigcup_{\substack{1 \le j \le 2n \\ j \ne 2n+1-h}} \{(w, w)\delta_A \cdot (\$, a_h) : w \in \{\epsilon\} \cup (L \cap A^* a_j)\} \cup \\ & \bigcup_{\substack{1 \le j \le 2n \\ j \ne h}} \{(w, w)\delta_A \cdot (a_{2n+1-h}, \$) : w \in \{\epsilon\} \cup (L \cap A^* a_j)\}. \end{aligned}$$

To prove that L_{a_h} is regular it now suffices to prove that each set in this finite union is regular. The sets appearing in the union are obtained concatenating a set of the form $\{(w, w)\delta_A : w \in \{\epsilon\} \cup (L \cap A^* a_j)\}$, which is regular by Proposition 1.6, with a singleton set and therefore they are regular.

Instead of proving that the languages L and L_{a_h} are regular by building them using simpler languages that we know are regular we can define automata that recognize the languages. We will do that, to illustrate this alternative way of proving regularity but also because we will use the automata in the following section to obtain the syntactic monoids of our languages. Let

$$\mathcal{A} = (Q, A, \mu, q_0, T) \tag{2.1}$$

be an automaton, where $Q = \{q_0, \ldots, q_{2n+1}\}$ is the set of states, q_0 is the initial state, the alphabet A is our generating set, the transition $\mu : Q \times A \to Q$ is defined by

$$\begin{aligned}(q_i, a_j)\mu &= q_{2n+1} \text{ if } i \in \{2n+1, 2n+1-j\}\\ (q_i, a_j)\mu &= q_j \qquad \text{otherwise}\end{aligned}$$

where $i = 0, \ldots, 2n+1$; $j = 1, \ldots, 2n$, and the set of terminal states is $T = Q \backslash \{q_{2n+1}\}$. We observe that the automaton has only one non terminal state, which is the fail state q_{2n+1}. This automaton, for $n = 2$, is illustrated in Figure 2.1 (note that, for clarity, we did not represent the arrows ending in the fail state). The only way to enter the fail state is by an arrow labeled by a_j starting from a state q_{2n+1-j}, for some j, and the only way to enter state q_{2n+1-j} is by an arrow labeled by a_{2n+1-j}. Therefore, a word in $w \in A^*$ will take us from the initial state q_0 to the fail state q_{2n+1} if and only if w has the form $w' a_{2n+1-j} a_j w''$ for some words $w', w'' \in A^*$, which means that $w \notin L$. We conclude that $\mathcal{L}(\mathcal{A}) = L$.

We now define automata to recognize the languages L_{a_h} $(a_h \in A)$,

$$\mathcal{A}_h = (Q', A(2, \$), \mu_h, q_0, T'),$$

where $Q' = \{q_0, \ldots, q_{2n+2}\}$ is the set of states, q_0 is the initial state, A is again the monoid generating set for G, the transition μ_h is defined by

$$\begin{aligned}(q_i, (a_j, a_j))\mu_h &= q_j \qquad \text{if } i \notin \{2n+1-j, 2n+1, 2n+2\}\\ (q_i, (\$, a_h))\mu_h &= q_{2n+2} \text{ if } i \notin \{2n+1-h, 2n+1, 2n+2\}\\ (q_i, (a_{2n+1-h}, \$))\mu_h &= q_{2n+2} \text{ if } i \notin \{h, 2n+1, 2n+2\}\\ (q_i, p)\mu_h &= q_{2n+1} \text{ in all other cases}\end{aligned}$$

for $i = 0, \ldots, 2n+2$; $j = 1, \ldots, 2n$ and $p \in A(2, \$)$, and the set of terminal states is $T' = \{q_{2n+2}\}$.

Observing Figure 2.2, where the automaton $\mathcal{A}_1$ is represented for $n = 2$ (again the arrows ending in the fail state q_{2n+1} are not represented), we see that this automaton is very similar to $\mathcal{A}$. If $w \in L$, then the word $(w, w)\delta_A$ will move us from the initial state to some state $q_j \in \{q_0, \ldots, q_{2n}\}$, from where we can move to

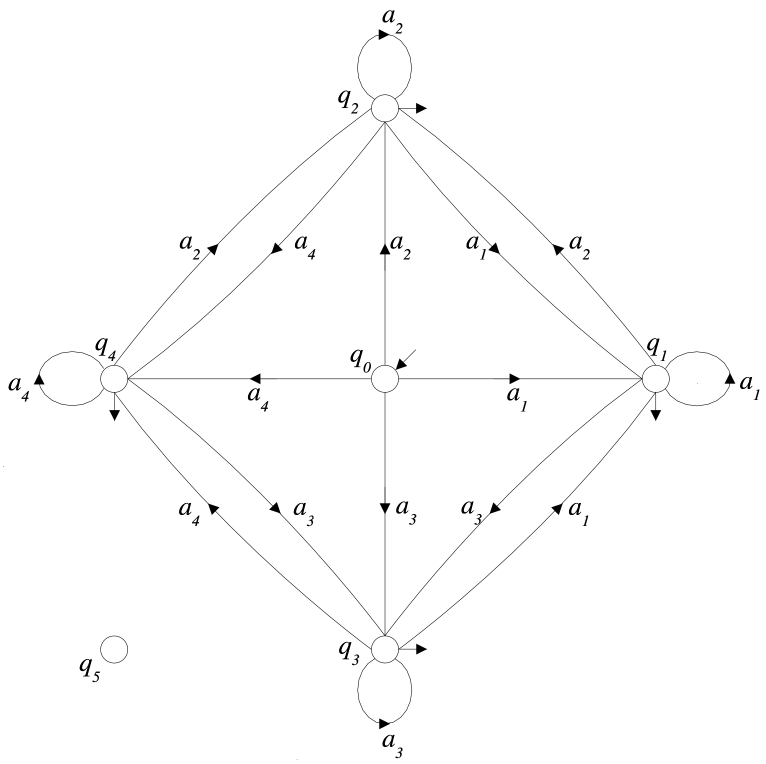

Figure 2.1: The minimal automaton recognizing L

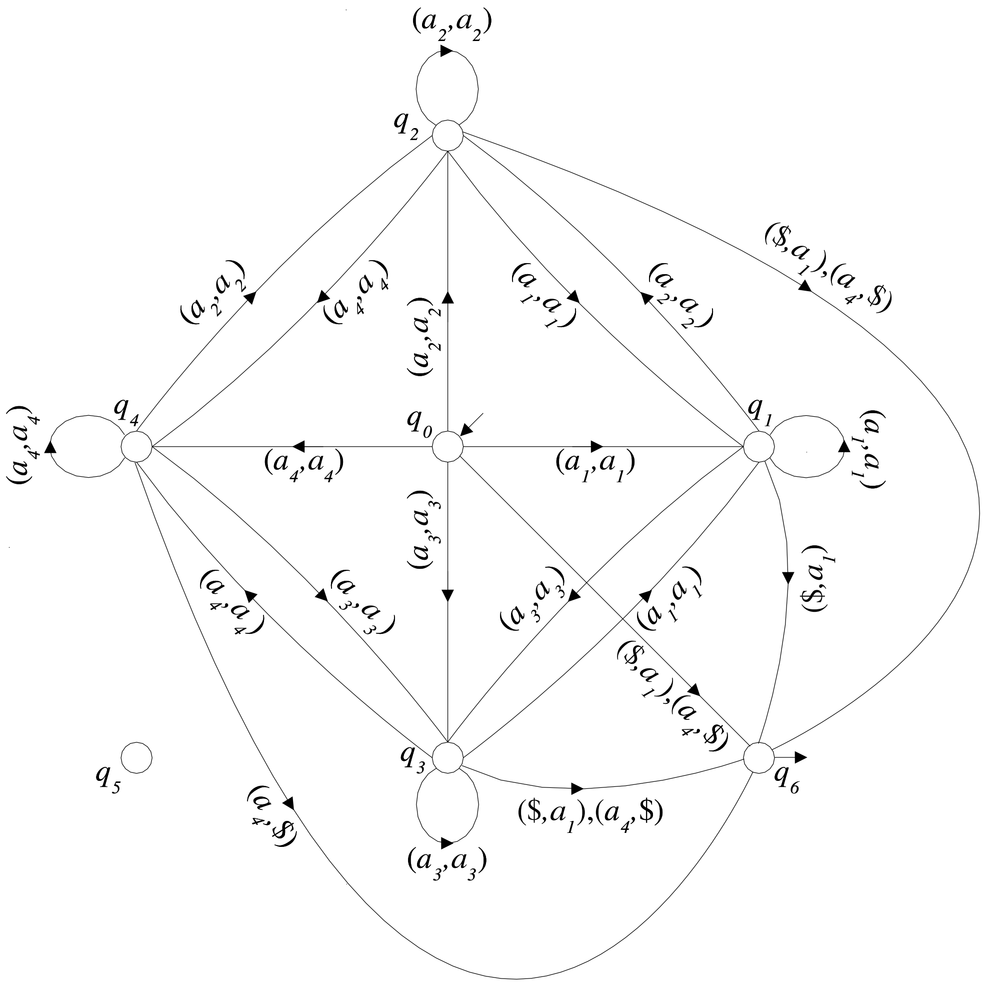

Figure 2.2: The minimal automaton recognizing L_{a_1}

the terminal state following an arrow labeled by $(a_{2n+1-h}, \$)$ if $j \neq h$ or following an arrow labeled by $(\$, a_h)$ if $j \neq 2n+1-h$. It is clear, from the definition of μ_h, that there is no other way to construct a successful path and so we have $\mathcal{L}(A_h) = L_{a_h}$.

2 Syntactic monoids

The characterization of regular languages in terms of their syntactic monoids, is described in Chapter 3 of [28], and this reference constitutes our motivation to study syntactic monoids. We start by reproducing the definitions and results from [28] we will need. Let A be a finite alphabet and let $L \subseteq A^*$. The *syntactic congruence* σ_L on A^* is defined by

$$\sigma_L = \{(w, z) \in A^* \times A^* : (\forall u, v \in A^*) \; uwv \in L \text{ if and only if } uzv \in L\},$$

and the *syntactic monoid* of L, denoted by $\mathrm{Syn}(L)$, is A^*/σ_L. We say that a language L *is recognized by a monoid* M if there exists a morphism $\psi : A^* \to M$ and a subset P of M such that $\psi^{-1}(P) = L$. A language is always recognized by its syntactic monoid and we have

Proposition 2.1 *Let A be a finite alphabet and let $L \subseteq A^*$. The following statements are equivalent:*

(i) L is a rational subset of A^*;

(ii) $\mathrm{Syn}(L)$ is finite;

(iii) L is recognized by a finite monoid M.

Given a deterministic automaton $\mathcal{A} = (Q, A, \mu, q_0, T)$ we define the congruence σ on A^* by

$$w \sigma z \text{ if and only if } (\forall q \in Q) \; qw = qz.$$

The monoid A^*/σ is finite and denoting by $\bar{w}$ the σ-class of w we can define an action of A^*/σ on Q by the rule that

$$q\bar{w} = qw \; (q \in Q, w \in A^*).$$

Since we have the implication

$$(\forall q \in Q,\ q\bar{w} = q\bar{z}) \implies \bar{w} = \bar{z},$$

each element $\bar{w} \in A^*/\sigma$ can be seen as a different transformation in $\mathcal{T}_Q$ and we call the submonoid A^*/σ of $\mathcal{T}_Q$ the *transformation monoid of the automaton* $\mathcal{A}$ and denote it by $\mathrm{TM}(\mathcal{A})$. For each $q \in Q$ let us denote the subset $\{w \in A^* : qw \in T\}$ of A^* by $q^{-1}T$, and let us define the equivalence relation ρ on Q by $\rho = \{(q_1, q_2) \in Q \times Q : q_1^{-1}T = q_2^{-1}T\}$. We say that the automaton $\mathcal{A}$ is reduced if ρ is the identical relation on Q, i.e., if $q_1^{-1}T = q_2^{-1}T \implies q_1 = q_2$. Given a regular language L we call *minimal automaton for* L the complete, deterministic, accessible, reduced automaton recognizing L, which is unique up to an isomorphism (see [28, Theorem 3.3.10]), and by [28, Theorem 3.5.1] we have

Proposition 2.2 *Let A be a finite alphabet and let L be a regular language on A^*. The syntactic monoid of L coincides with the transformation monoid of the minimal automaton for L.*

The automata $\mathcal{A}$ and $\mathcal{A}_h$ ($a_h \in A$) defined in the previous section are complete, deterministic and accessible. We need to prove that they are also reduced. To see that $\mathcal{A}$ is reduced it suffices to observe that $1^{-1} \cdot T = L$, $q_{2n+1}^{-1} \cdot T = \emptyset$, and that for $i \in \{1, \ldots, 2n\}$ we have $a_j \in q_i^{-1} \cdot T$ for $j \neq 2n+1-i$ and $a_{2n+1-i} \notin q_i^{-1} \cdot T$, which implies that $q_1^{-1} \cdot T \neq q_2^{-1} \cdot T$ for any states $q_1 \neq q_2$. Similarly, the automata L_{a_h} are reduced because $1^{-1} \cdot T' = L_{a_h}$, $q_{2n+1}^{-1} \cdot T' = \emptyset$, $q_{2n+2}^{-1} \cdot T' = \{\epsilon\}$, and for $i \in \{1, \ldots, 2n\}$ we have $(q_i(a_{2n+1-i}, a_{2n+1-i}))^{-1} \cdot T' = \emptyset$ and for all $j \neq 2n+1-i$ we have $(q_i(a_j, a_j))^{-1} \cdot T' \neq \emptyset$ which implies $q_1^{-1} \cdot T' \neq q_2^{-1} \cdot T'$ for any states $q_1 \neq q_2$.

We will now obtain the transformation monoid $\mathrm{TM}(\mathcal{A})$ which, by the previous proposition, is the syntactic monoid of the language L. We define the transformations in $\mathcal{T}_Q$,

$$\begin{aligned}
\phi_{jk} :&\ q \mapsto q_j \text{ if } q \notin \{q_{2n+1-k}, q_{2n+1}\},\ q_{2n+1-k}, q_{2n+1} \mapsto q_{2n+1};\\
z :&\ \quad q \mapsto q_{2n+1},
\end{aligned}$$

for $j, k \in \{1, \ldots, 2n\}$ and we will prove that

$$\mathrm{TM}(\mathcal{A}) = \{1, \phi_{jk}, z : j, k \in \{1, \ldots, 2n\}\}.$$

By definition of $\mathrm{TM}(\mathcal{A})$ it is clear that this monoid is generated by the set

$$Y = \{\bar{a}_1, \ldots, \bar{a}_{2n}\} = \{\phi_{11}, \ldots, \phi_{2n,2n}\},$$

which is contained in $M = \{1, \phi_{jk}, z : j, k \in \{1, \ldots, 2n\}\}$. We have

$$\phi_{jk}\phi_{lm} = \phi_{lk} \text{ if } j \neq 2n + 1 - m,$$

because

$$q_{2n+1-k}\phi_{jk}\phi_{lm} = q_{2n+1}\phi_{jk}\phi_{lm} = q_{2n+1}\phi_{lm} = q_{2n+1}$$

and for $q \notin \{q_{2n+1-k}, q_{2n+1}\}$ we have

$$q\phi_{jk}\phi_{lm} = q_j\phi_{lm} = q_l.$$

Otherwise, if $j = 2n + 1 - m$, we have $\phi_{jk}\phi_{lm} = z$. Since z acts as a zero in M, it follows that M is a monoid and therefore $\mathrm{TM}(\mathcal{A}) \subseteq M$. To show that $M \subseteq \mathrm{TM}(\mathcal{A})$ it suffices to prove the following identities

$$\begin{aligned} \phi_{jk} &= \phi_{kk}\phi_{jj} && (k \neq 2n + 1 - j) \\ \phi_{j,2n+1-j} &= \phi_{2n+1-j,2n+1-j}\phi_{j,k} && (k \notin \{j, 2n + 1 - j\}) \\ z &= \phi_{kk}\phi_{jj} && (k = 2n + 1 - j). \end{aligned}$$

The first identity holds because: ϕ_{kk} maps any state $q \notin \{q_{2n+1-k}, q_{2n+1}\}$ to q_k and, since $k \neq 2n+1-j$, the state q_k is mapped by ϕ_{jj} to q_j; all the other states are mapped by ϕ_{kk}, and therefore by $\phi_{kk}\phi_{jj}$, to the fail state q_{2n+1}. To check the second identity we observe that both transformations $\phi_{j,2n+1-j}$ and $\phi_{2n+1-j,2n+1-j}$ map the states q_j and q_{2n+1} to the fail state. All the other states are mapped by $\phi_{j,2n+1-j}$ to q_j and by $\phi_{2n+1-j,2n+1-j}$ to q_{2n+1-j} and, since $k \neq j$, we have $2n + 1 - j \neq 2n + 1 - k$ and q_{2n+1-j} is mapped to q_j by $\phi_{j,k}$. The third identity holds because ϕ_{kk} maps all states to either the fail state or to q_{2n+1-j} and so $\phi_{kk}\phi_{jj}$ maps all states to the fail state.

We consider now the transformation monoid $\mathrm{TM}(\mathcal{A}_h)$ and, defining the transformations in $\mathcal{T}_{Q'}$,

$$\begin{aligned}
\psi_{jk} : & q \mapsto q_j \text{ if } q \notin \{q_{2n+1-k}, q_{2n+1}, q_{2n+2}\},\\
& q_{2n+2-k}, q_{2n+1}, q_{2n+2} \mapsto q_{2n+1};\\
\eta_j : \ & q \mapsto q_{2n+2} \text{ if } q \notin \{q_{2n+1-j}, q_{2n+1}, q_{2n+2}\},\\
& q_{2n+1-j}, q_{2n+1}, q_{2n+2} \mapsto q_{2n+1};\\
z : \ & q \mapsto q_{2n+1},
\end{aligned}$$

we will prove that

$$\mathrm{TM}(\mathcal{A}_h) = \{1, \psi_{jk}, \eta_j, z : j, k \in \{1, \dots, 2n\}\}.$$

The transformation monoid $\mathrm{TM}(\mathcal{A}_h)$ is generated by $Y = \{\bar{p} : p \in A(2, \$)\}$ and, observing the definition of μ_h, we have

$$\begin{aligned}
Y &= \{\overline{(a_1, a_1)}, \dots, \overline{(a_{2n}, a_{2n})}, \overline{(a_h, \$)}, \overline{(\$, a_{2n+1-h})}, \overline{(a_1, a_2)}\}\\
&= \{\psi_{11}, \dots, \psi_{2n\,2n}, \eta_h, \eta_{2n+1-h}, z\}.
\end{aligned}$$

The generating set Y is contained in the set $N = \{1, \psi_{jk}, \eta_j, z : j, k \in \{1, \dots, 2n\}\}$ and we will now show that N is a monoid. The product of two transformations of the form ψ_{jk} is either a transformation of this form or it is z, as is the case with the analogous transformations ϕ_{jk} in $\mathrm{TM}(\mathcal{A})$. A product of the form $\psi_{jk}\eta_l$ is in N because, if $j \neq 2n+1-l$ then $\psi_{jk}\eta_l = \eta_k$ and otherwise $\psi_{jk}\eta_l = z$. Moreover, since multiplying an element from $N \backslash \{1\}$ on the left by a transformation η_l we obtain z and since z acts as a zero in N, we conclude that N is closed for multiplication and therefore is a monoid, implying $\mathrm{TM}(\mathcal{A}_h) = \langle Y \rangle \subseteq N$.

In order to show that $N \subseteq \mathrm{TM}(\mathcal{A}_h)$ we will prove that, for any $j, k \in \{1, \dots, 2n\}$, we have the following identities

$$\begin{aligned}
\psi_{jk} &= \psi_{kk}\psi_{jj} && (k \neq 2n+1-j),\\
\psi_{j,2n+1-j} &= \psi_{2n+1-j,2n+1-j}\psi_{j,k} && (k \notin \{j, 2n+1-j\}),\\
z &= \psi_{kk}\psi_{jj} && (k = 2n+1-j),\\
\eta_j &= \psi_{kj}\eta_h && (k \neq 2n+1-h).
\end{aligned}$$

The first three identities are similar to those obtained for the transformations $\phi_{jk} \in \mathrm{TM}(\mathcal{A})$. To prove the fourth identity we observe that the transformation ψ_{kj} maps q to q_k if $q \notin \{q_{2n+1-j}, q_{2n+1}, q_{2n+2}\}$ and, since $k \neq 2n+1-h$, q_k is mapped to the terminal state q_{2n+2} by η_h, and the remaining states q_{2n+1-j}, q_{2n+1} and q_{2n+2} are mapped by ψ_{kj} (and therefore by $\psi_{kj}\eta_h$) to the fail state q_{2n+1}.

3 Green's relations

In this section we determine the Green's relations of the monoids $\mathrm{TM}(\mathcal{A})$ and $\mathrm{TM}(\mathcal{A}_h)$ obtained in the previous section.

We start with the monoid $M = \mathrm{TM}(\mathcal{A})$. In order to determine relation $\mathcal{L}$ we first show that

$$M\phi_{jk} = \{\phi_{j1}, \dots, \phi_{j,2n}, z\} \ (j, k \in \{1, \dots, 2n\}). \tag{2.2}$$

For any $m = 1, \dots, 2n$ we can choose some $l \neq 2n+1-k$ and write $\phi_{jm} = \phi_{lm}\phi_{jk}$ and, with $l = 2n+1-k$ we obtain z, implying $\{\phi_{j1}, \dots, \phi_{j,2n}, z\} \subseteq M\phi_{jk}$. Since z acts as zero in the monoid, we also have the other inclusion. It is clear that $L_1 = \{1\}$, $L_z = \{z\}$ and so we have $L_{\phi_{jk}} = \{\phi_{j1}, \dots, \phi_{j,2n}\}$ for any $j, k \in \{1, \dots, 2n\}$. A similar argument shows that

$$\phi_{jk}M = \{\phi_{1k}, \dots, \phi_{2n,k}, z\} \ (j, k \in \{1, \dots, 2n\})$$

from which follows that $R_1 = \{1\}$, $R_z = \{z\}$ and $R_{\phi_{jk}} = \{\phi_{1k}, \dots, \phi_{2n,k}\}$ and relation $\mathcal{R}$ is determined as well. We have then three D-classes, $D_1 = \{1\}$, $D_2 = \{\phi_{jk} : j, k \in \{1, \dots, 2n\}\}$ and $D_3 = \{z\}$ and the eggbox of class D_2 is

$$\begin{array}{cccc}
\{\phi_{11}\} & \{\phi_{21}\} & \dots & \{\phi_{2n,1}\} \\
\{\phi_{12}\} & \{\phi_{22}\} & \dots & \{\phi_{2n,2}\} \\
\vdots & \vdots & \vdots & \vdots \\
\{\phi_{1,2n}\} & \{\phi_{2,2n}\} & \dots & \{\phi_{2n,2n}\}.
\end{array}$$

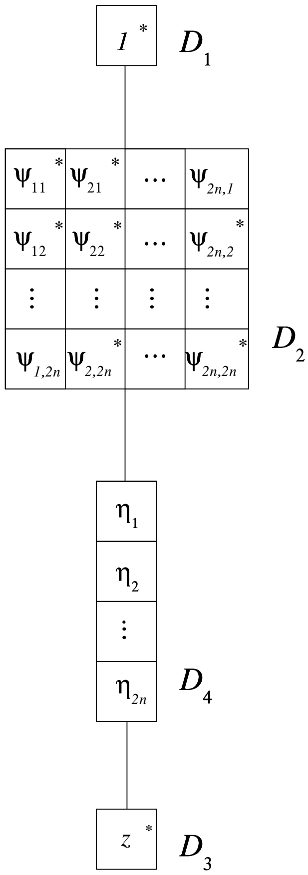

Figure 2.3: D-classes of the monoid $\mathrm{TM}(\mathcal{A}_h)$

Observing that any permutation ϕ_{jk} is idempotent for $j \neq 2n+1-k$ and $\phi_{jk}^2 = z$ otherwise, representing by 1 the trivial groups and z by 0, the eggbox of D-class D_2 is

$$\begin{array}{ccccc} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1. \end{array}$$

We will now consider the monoid $N = \mathrm{TM}(\mathcal{A}_h)$. Since multiplying an element in $N\backslash\{1\}$ on the left by a transformations of the form η_j we obtain z it is clear that, as in M, we have $L_1 = \{1\}$, $L_z = \{z\}$ and $L_{\psi_{jk}} = \{\psi_{j1}, \ldots, \psi_{j,2n}\}$. We will now prove that $N\eta_j = \{\eta_1, \ldots, \eta_{2n}, z\}$. We have already seen that the direct inclusion holds and, since for any $k \in \{1, \ldots, 2n\}$, we can write $\eta_k = \psi_{lk}\eta_j$ with $l \neq 2n+1-j$, the converse inclusion holds as well. So $L_{\eta_j} = \{\eta_1, \ldots, \eta_{2n}\}$. For the relation $\mathcal{R}$ we have $R_1 = \{1\}$, $R_z = \{z\}$ and $R_{\eta_j} = \{\eta_j\}$, and we will now show that $\psi_{jk}N = \{\psi_{1k}, \ldots, \psi_{2n,k}, \eta_k, z\}$ for any $j, k \in \{1, \ldots, 2n\}$. We have, as in M,

$$\psi_{jk}\{\psi_{jk}, z : j, k \in \{1, \ldots, 2n\}\} = \{\psi_{1k}, \ldots, \psi_{2n,k}, z\}$$

and a product of the form $\psi_{jk}\eta_l$ is either η_k if $j \neq 2n+1-l$ or z otherwise. We have the same D-classes $D_1 = \{1\}$, $D_2 = \{\psi_{j,k} : j, k \in \{1, \ldots, 2n\}\}$ and $D_3 = \{z\}$ as in M, and the class $D_4 = \{\eta_1, \ldots, \eta_{2n}\}$. The eggboxes of the D-classes of $\mathrm{TM}(\mathcal{A}_h)$ are shown in Figure 2.3.

Although, the syntactic monoids for an automatic structure for the free group, are the only syntactic monoids studied in this thesis, observing their Green's relations, it appears to be an interesting research subject, the study of the connection between automatic structures and their syntactic monoids in general.

Anticipating further research in this area we ask:

Question 2.3 What are the syntactic monoids of other (all) automatic structures for the free group?

Question 2.4 Investigate the syntactic monoids for the free abelian group.

Chapter 3

Generalized sequential machines and regular languages

It is known that the fellow-traveller property, which characterizes automatic groups, does not characterize automatic semigroups. So we have to use directly the definition and work with regular languages instead of the Cayley graph to prove that a semigroup is automatic. When working with semigroup constructions, we usually have to construct automatic structures from known automatic structures. For that purpose we use the concept of a generalized sequential machine.

1 Introduction

A *generalized sequential machine* (*gsm* for short) is a six-tuple

$$\mathcal{A} = (Q, A, B, \mu, q_0, T)$$

where Q, A and B are finite sets (called the *states*, the *input alphabet* and the *output alphabet* respectively), μ is a (partial) function from $Q \times A$ to finite subsets of $Q \times B^*$, $q_0 \in Q$ is the *initial state* and $T \subseteq Q$ is the *set of terminal states*. The inclusion $(q', u) \in (q, a)\mu$ corresponds to the following situation: if $\mathcal{A}$ is in state

q and reads input a, then it can move into state q' and output u.

We can interpret $\mathcal{A}$ as a directed labelled graph with vertices Q, and an edge $q \xrightarrow{(a,u)} q'$ for every pair $(q', u) \in (q, a)\mu$. For a path

$$\pi \ : \ q_1 \xrightarrow{(a_1,u_1)} q_2 \xrightarrow{(a_2,u_2)} q_3 \ldots \xrightarrow{(a_n,u_n)} q_{n+1}$$

we define

$$\Phi(\pi) = a_1 a_2 \ldots a_n, \ \Sigma(\pi) = u_1 u_2 \ldots u_n.$$

For $q, q' \in Q$, $u \in A^+$ and $v \in B^+$ we write $q \xrightarrow{(u,v)}_+ q'$ to mean that there exists a path π from q to q' such that $\Phi(\pi) \equiv u$ and $\Sigma(\pi) \equiv v$, and we say that (u, v) is the *label* of the path. We say that a path is *successful* if it has the form $q_0 \xrightarrow{(u,v)}_+ t$ with $t \in T$.

The gsm $\mathcal{A}$ induces a mapping $\eta_{\mathcal{A}} \ : \ \mathcal{P}(A^+) \longrightarrow \mathcal{P}(B^+)$ from subsets of A^+ into subsets of B^+ defined by

$$X\eta_{\mathcal{A}} = \{v \in B^+ \ : \ (\exists u \in X)(\exists t \in T)(q_0 \xrightarrow{(u,v)}_+ t)\}.$$

It is well known that if X is regular then so is $X\eta_{\mathcal{A}}$; see [27, Theorem 11.1 and Example 11.1]. Similarly, $\mathcal{A}$ induces a mapping $\zeta_{\mathcal{A}} : \mathcal{P}(A^+ \times A^+) \longrightarrow \mathcal{P}(B^+ \times B^+)$ defined by

$$Y\zeta_{\mathcal{A}} = \{(w, z) \in B^+ \times B^+ \ : \ (\exists (u, v) \in Y)(w \in u\eta_{\mathcal{A}} \ \& \ z \in v\eta_{\mathcal{A}})\}.$$

2 Preserving regularity

The next theorem asserts that, under certain conditions, the mapping $\zeta_{\mathcal{A}}$ also preserves regularity.

Theorem 3.1 *Let $\mathcal{A} = (Q, A, B, \mu, q_0, T)$ be a gsm, and let $\pi_A : (A^* \times A^*)\delta_A \longrightarrow A^* \times A^*$ be the inverse of δ_A. Suppose that there is a constant C such that for any two paths α_1, α_2 in $\mathcal{A}$, we have*

$$|\Phi(\alpha_1)| = |\Phi(\alpha_2)| \Longrightarrow ||\Sigma(\alpha_1)| - |\Sigma(\alpha_2)|| \leq C. \qquad (3.1)$$

If $M \subseteq (A^+ \times A^+)\delta_A$ is a regular language in $A(2,\$)^+$ then $N = M\pi_A\zeta_{\mathcal{A}}\delta_B$ is a regular language in $B(2,\$)^+$.

PROOF. To prove that N is regular we will define a gsm $\mathcal{B}$ such that $M\eta_{\mathcal{B}} = N$. First we define three functions with domain $B^* \times B^*$ that will be used in the definition of $\mathcal{B}$:

$$(a_1 \dots a_k, b_1 \dots b_l)\lambda = \begin{cases} a_{l+1} \dots a_k \text{ if } k > l \\ \epsilon \qquad\qquad \text{otherwise,} \end{cases}$$

$$(a_1 \dots a_k, b_1 \dots b_l)\rho = \begin{cases} b_{k+1} \dots b_l \text{ if } l > k \\ \epsilon \qquad\qquad \text{otherwise,} \end{cases}$$

$$(a_1 \dots a_k, b_1 \dots b_l)\kappa = (a_1, b_1) \dots (a_s, b_s), \;\; s = \min(k, l).$$

We now let $\mathcal{B} = (R, A(2,\$), B(2,\$), \nu, r_0, Z)$, where

$$R = Q \times Q \times W \times W, \;\; W = (\textstyle\bigcup_{k=0}^{C} B^k) \cup \{\$\},$$
$$r_0 = (q_0, q_0, \epsilon, \epsilon), \qquad Z = T \times T \times \{(\$,\$)\}.$$

In order to define the transition ν, we first extend the transition μ to allow input \$:

$$(q,a)\bar{\mu} = \begin{cases} (q,a)\mu \;\text{ if } a \in A \\ \{(q,\epsilon)\} \text{ if } a = \$. \end{cases}$$

Now the transition ν is defined by

$$((q,q',w,w'),(a,a'))\nu =$$
$$\bigcup_{\substack{(q_1,u)\in(q,a)\bar{\mu} \\ (q_1',u')\in(q',a')\bar{\mu}}} \{((q_1, q_1', (wu, w'u')\lambda, (wu, w'u')\rho), (wu, w'u')\kappa),$$
$$((q_1, q_1', \$, \$), (wu, w'u')\delta_B)\} \text{ for}$$
$$w, w' \in W\backslash\{\$\} \text{ and}$$
$$(a, a' \in A \text{ or } (a = \$ \text{ and } |w| > |w'u'|) \text{ or } (a' = \$ \text{ and } |w'| > |wu|))$$

provided $||wu| - |w'u'|| \leq C$,

$$((q,q',w,w'),(\$,a'))\nu =$$
$$\bigcup_{(q_1',u')\in(q',a')\mu} \{((q, q_1', \$, \epsilon), (w, w'u')\delta_B), ((q, q_1', \$, \$), (w, w'u')\delta_B)\} \text{ for}$$
$$a' \in A \text{ and } (w = \$ \text{ or } |w| \leq |w'u'|),$$

$$((q, q', w, w'), (a, \$))\nu = \bigcup_{(q_1,u)\in(q,a)\mu} \{((q_1, q', \epsilon, \$), (wu, w')\delta_B), ((q_1, q', \$, \$), (wu, w')\delta_B)\} \text{ for } a \in A \text{ and } (w' = \$ \text{ or } |w'| \le |wu|)$$

where $q, q' \in Q$.

Intuitively, the machine works the following way: when it receives a word from M, in each transition it outputs everything possible from the corresponding word in N, and memorizes in its states the remaining suffix of the component which is longer at that moment. This can be done because condition 3.1 holds.

We now prove that $N \subseteq M\eta_{\mathcal{B}}$. Let $(v, v')\delta_B \in N$. By definition of N there is $(u, u')\delta_A \in M$ such that $(v, v') \in \{(u, u')\}\zeta_{\mathcal{A}}$. So $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$. This means that in $\mathcal{A}$ there are paths of the form

$$q_{i-1} \xrightarrow{(a_i,w_i)} q_i \ (i = 1, \ldots, m, \ a_i \in A, \ w_i \in B^*)$$
$$q'_{i-1} \xrightarrow{(a'_i,w'_i)} q'_i \ (i = 1, \ldots, n, \ a'_i \in A, \ w'_i \in B^*),$$

with

$$u = a_1 \ldots a_m, \ v = w_1 \ldots w_m, \ u' = a'_1 \ldots a'_n, \ v' = w'_1 \ldots w'_n, \ q'_0 = q_0, \ q_m, q'_n \in T.$$

We now show that there is a successful path in $\mathcal{B}$ of the form

$$(q_{i-1}, q'_{i-1}, z_{i-1}, z'_{i-1}) \xrightarrow{((a_i,a'_i),\omega_i)} (q_i, q'_i, z_i, z'_i) \ (i = 1, \ldots, p),$$

where $p = \max(m, n)$ and

$$q_m = q_{m+1} = \ldots = q_p, \ q'_n = q'_{n+1} = \ldots = q'_p,$$
$$a_{m+1} = \ldots = a_p = a'_{n+1} = \ldots = a'_p = \$,$$

such that the output $\omega_1 \ldots \omega_p$ is equal to $(v, v')\delta_B$.

Let $r = \min(m, n)$. We begin in the initial state and follow a path visiting states from the set $Q\times Q\times(W\setminus\{\$\})\times(W\setminus\{\$\})$ while $i < r$. The output ω_i in these transitions is the longest prefix of $(z_{i-1}w_i, z'_{i-1}w'_i)\delta_B$ that belongs to $(B \times B)^*$, z_i is equal to the remaining letters in $z_{i-1}w_i$ if $|z_{i-1}w_i| \ge |z'_{i-1}w'_i|$ (otherwise it is ϵ)

and z_i' is equal to the remaining letters of $z_{i-1}'w_i'$ if $|z_{i-1}'w_i'| > |z_{i-1}w_i|$ (otherwise it is ϵ). We note that $|z_i|, |z_i'| \le C$ because of assumption (3.1). So after transition i, the complete output produced is the longest prefix of $(w_1 \dots w_i, w_1' \dots w_i')\delta_B$ that belongs to $(B \times B)^*$.

If $m = n$ then transition r is to the terminal state $(q_r, q_r', \$, \$)$ and produces output $\omega_r = (z_{r-1}w_r, z_{r-1}'w_r')\delta_B$, i.e., the output in this last transition is the remainder of $(v, v')\delta_B$. The machine ends in the terminal state $(q_r, q_r', \$, \$)$ and the complete output is $(v, v')\delta_B$.

Let us now consider the case where $n > m$ (the other case is similar). In this case, transition m is similar to the previous ones and the remaining input is $(\$, a_{m+1}') \dots (\$, a_n')$. If $|z_m| > |z_m'w_{m+1}'|$ then, for $i > m$ and while $|z_{i-1}| > |z_{i-1}'w_i'|$ $(i < n)$, we can continue visiting the states from the set $Q \times Q \times (W \backslash \{\$\}) \times (W \backslash \{\$\})$, the output ω_i in these transitions is the longest prefix of $(z_{i-1}, z_{i-1}'w_i')\delta_B$ that belongs to $(B \times B)^*$, z_i is equal to the remaining letters in z_{i-1} and $z_i' = \epsilon$. We observe that in these transitions we have $|z_i|, |z_i'| \le |z_r| \le C$. Intuitively, in these transitions, the memory for the left hand side encoded in the states has more letters then those to be output on the right and so after producing the output there will still be letters memorized for the left. If for $i = n$ we still have $|z_{i-1}| > |z_{i-1}'w_i'|$ then last transition is to the terminal state $(\$, \$, q_m, q_n')$ and produces output $(z_{n-1}, z_{n-1}'w_n')\delta_B$ being $(v, v')\delta_B$ the total output produced. If we have $|z_{j-1}| \le |z_{j-1}'w_j'|$ for some j, this means intuitively that the memory for the left hand side can be now be emptied and it is safe to output \$ symbols on the left since the following inputs are of the form $(\$, a)$. So the path will now visit states from the set $\{q_m\} \times Q \times \{\$\} \times \{\epsilon\}$ and each transition produces output $(\epsilon, z_{i-1}'w_i')\delta_B$. Last transition is to the terminal state $(q_m, q_n', \$, \$)$ and the total output produced is again $(v, v')\delta_B$.

We now show the converse inclusion $M\eta_{\mathcal{B}} \subseteq N$. First we note that for a successful path α in $\mathcal{B}$, such that $\Phi(\alpha) \in (A^* \times A^*)\delta_A$ we have $\Sigma(\alpha) \in (B^* \times B^*)\delta_B$. In fact, from the definition of ν, we see that a \$ can only be output on one side if one of the two situations occurs: a \$ has appeared as input on the same side and the corresponding "memory" is empty; in the last transition. So there is no way to output a letter after a \$ on the same side, being the input from $(A^* \times A^*)\delta_A$. Hence, let $(v, v')\delta_B$ be an arbitrary element from $M\eta_{\mathcal{B}}$. So there is a successful path in $\mathcal{B}$ with label $((u, u')\delta_A, (v, v')\delta_B)$ with $(u, u')\delta_A \in M$. We will prove that $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$ to conclude that

$$(v, v')\delta_B \in \{(u, u')\delta_A\}\pi_A\zeta_{\mathcal{A}}\delta_B \subseteq M\pi_A\zeta_{\mathcal{A}}\delta_B = N,$$

as required. Let

$$u = a_1 \dots a_m,\ u' = a'_1 \dots a'_n,\ p = \max(m, n).$$

By definition of ν a successful path in $\mathcal{B}$ labeled by $((u, u')\delta_A, (v, v')\delta_B)$ has the form

$$(q_{i-1}, q'_{i-1}, w_{i-1}, w'_{i-1}) \xrightarrow{((a_i,a'_i),\omega_i)} (q_i, q'_i, w_i, w'_i)\ (i = 1, \dots, p),$$

where $q_0 = q'_0$, $q_p, q'_p \in T$, $a_{m+1} = \dots = a_p = a'_{n+1} = \dots = a'_p = \$$, $q_m = \dots = q_p$, $q'_n = \dots = q'_p$, and $\omega_1 \dots \omega_p = (v, v')\delta_B$. This yields successful paths

$$q_{i-1} \xrightarrow{(a_i,z_i)} q_i\ (i = 1, \dots, m)$$
$$q'_{i-1} \xrightarrow{(a'_i,z'_i)} q'_i\ (i = 1, \dots, n)$$

in $\mathcal{A}$, with $v = z_1 \dots z_m$ and $v' = z'_1 \dots z'_n$. So $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$, completing the proof of the theorem. ■

3 Concatenation of padded languages

In this section we fix an alphabet A, consider two regular languages M, N in $(A^* \times A^*)\delta$, and give a sufficient conditions for the *padded product of languages*

$$M \odot N = \{(w_1w'_1, w_2w'_2)\delta : (w_1, w_2)\delta \in M, (w'_1, w'_2)\delta \in N\}$$

to be regular, which we will prove by using a *gsm*. We start by observing that the set $M \odot N$ is not regular in general, when M, N are regular, as the following example shows.

Example 3.2 Let $A = \{a, b\}$, $M = \{(a, \$)^n : n \in \mathbb{N}\}$ and $N = \{(b, b)^n : n \in \mathbb{N}\}$. Then $M \odot N = \{(a^n b^m, b^m)\delta : n, m \in \mathbb{N}_0\}$ is not regular. To see that we can assume that $M \odot N$ is regular, and fix K to be the number of states of an automaton accepting $M \odot N$. If we take a word $(a, b)^n (b, \$)^n \in M \odot N$ with $n > K$ then, by using the Pumping Lemma, we know that there exist $i, j, k \in \mathbb{N}_0$ with $k > 0$ and $i + j + k = n$ such that the words $(a, b)^i (a, b)^{lj} (a, b)^k (b, \$)^n$ also belong to $M \odot N$, for any $l \in \mathbb{N}_0$. In particular we have $(a, b)^i (a, b)^k (b, \$)^n = (a^{i+k} b^n, b^{i+k})\delta \in M \odot N$ with $i + k < n$ which is impossible, and therefore $M \odot N$ is not regular.

The following theorem gives a sufficient condition for $M \odot N$ to be regular.

Theorem 3.3 *Let A be an alphabet and let M, N be regular languages on $(A^* \times A^*)\delta$. If there exists a constant C such that, for any two words $w_1, w_2 \in A^*$ we have*

$$(w_1, w_2)\delta \in M \implies ||w_1| - |w_2|| \leq C,$$

then the language $M \odot N$ is regular.

Proof. We assume, without loss of generality, that the languages M and N do not contain the empty word ϵ. Defining

$$M_k = \{(w_1, w_2)\delta \in M : |w_1| - |w_2| = k\},$$

for each $k = -C, \ldots, C$, we can write $M = \bigcup_{k=-C}^{C} M_k$ and we observe that, each set

$$M_k = M \cap ((A \times A)^+ \cdot \{(a_1, \$) \ldots (a_k, \$) : a_1, \ldots, a_k \in A\})$$

is regular. Therefore, defining

$$O_k = \{(w_1 w_1', w_2 w_2')\delta : (w_1, w_2)\delta \in M_k, (w_1', w_2')\delta \in N\}$$

we can write

$$N \odot M = \bigcup_{k=-C}^{C} O_k$$

and we only need to prove that, for each $k = -C, \dots, C$, the set O_k is regular. We have $O_0 = M_0 \cdot N$ and so O_0 is regular, and the two cases where k is positive and negative are similar. We will then consider only the case where k is positive and, to prove that O_k is regular, we will define a *gsm* $\mathcal{A}$ such that $(M_k \cdot N)\eta_{\mathcal{A}} = O_k$. We will define a machine such that an input of the form

$$(a_1, b_1) \dots (a_n, b_n)(c_1, \$) \dots (c_k, \$)(d_1 \dots d_p, e_1 \dots e_q)\delta$$

will determine a unique successful path and produce output

$$(a_1 \dots a_n c_1 \dots c_k d_1 \dots d_p, b_1 \dots b_n e_1 \dots e_q)\delta,$$

where $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_k, d_1, \dots, d_p, e_1, \dots, e_q \in A$ and $n, p, q \in \mathbb{N}_0$. Intuitively, our machine will receive input $(a_1, b_1) \dots (a_n, b_n)$ and reproduce it as the output, then it will receive input $(c_1, \$) \dots (c_k, \$)$, produce no output and memorize the word $c_1 \dots c_k$ encoded on its states, and finally the input $(d_1 \dots d_p, e_1 \dots e_q)\delta$ will produce the remaining output $(c_1 \dots c_k d_1 \dots d_p, e_1 \dots e_q)\delta$ with only last transition being to the unique terminal state. Formally, let

$$W_k = \{w \in A^* : |w| \leq k\}$$

and let

$$\mathcal{A} = (Q, A(2,\$), A(2,\$), \lambda, q_0, \{\chi\})$$

be a *gsm* where

$$Q = \{q_0\} \cup W_k \cup (W_k \times \{l, r\}) \cup \{\chi\}$$

is the set of states, χ is the unique terminal state and λ is a partial function from $Q \times A(2,\$)$ to finite subsets of $Q \times A(2,\$)^*$ defined by the following equations

$$
\begin{array}{lrl}
(i) & (q_0,(a_1,a_2))\lambda = & \{(q_0,(a_1,a_2))\}\ (a_1,a_2 \in A),\\
(ii) & (q_0,(a_1,\$)\lambda = & \{(a_1,\epsilon)\}\ (a_1 \in A),\\
(iii) & (w,(a_1,\$))\lambda = & \{(wa_1,\epsilon)\}\ (a_1 \in A, w \in W_k, |w| < k),\\
(iv) & (bw,(a_1,a_2))\lambda = & \{(wa_1,(b,a_2)),(\chi,(bwa_1,a_2)\delta)\}\\
 & & (b,a_1,a_2 \in A, w \in W_k, |bw| = k),\\
(v) & (bw,(a_1,\$))\lambda = & \{((wa_1,r),(b,\$)),(\chi,(bwa_1,\epsilon)\delta)\}\\
 & & (b,a_1 \in A, w \in W_k, |bw| = k),\\
(vi) & (bw,(\$,a_2))\lambda = & \{((w,l),(b,a_2)),(\chi,(bw,a_2)\delta)\}\\
 & & (b,a_2 \in A, w \in W_k, |bw| = k),\\
(vii) & ((bw,r),(a_1,\$))\lambda = & \{((wa_1,r),(b,\$)),(\chi,(bwa_1,\epsilon)\delta)\}\\
 & & (b,a_1 \in A, w \in W_k, |bw| = k),\\
(viii) & ((bw,l),(\$,a_2))\lambda = & \{((w,l),(b,a_2)),(\chi,(bw,a_2)\delta)\}\\
 & & (b,a_2 \in A, w \in W_k),\\
(ix) & ((\epsilon,l),(\$,a_2))\lambda = & \{((\epsilon,l),(\$,a_2)),(\chi,(\$,a_2))\}\ (a_2 \in A),\\
(x) & (q,(x_1,x_2))\lambda = & \emptyset \text{ in all other cases.}
\end{array}
$$

To prove that $(M_k \cdot N)\eta_{\mathcal{A}} \subseteq O_k$ let

$$(a_1,b_1)\ldots(a_n,b_n)(c_1,\$)\ldots(c_k,\$)(d_1\ldots d_p,e_1\ldots e_q)\delta$$

be an arbitrary element of $M_k \cdot N$. We will first consider the case where $p = q$. The input $(a_1,b_1)\ldots(a_n,b_n)$ will leave us on the initial state, by using equation (i), and produces output $(a_1,b_1)\ldots(a_n,b_n)$. Then, the input $(c_1,\$)\ldots(c_k,\$)$ will take us through the states $c_1, c_1c_2, \ldots, c_1\ldots c_k \in W_k$, by using equation (ii) once and equation (iii) $k-1$ times, and produces no output. Now, the input $(d_1,e_1)\cdots(d_q,e_q)$ forces us to use equation (iv) q times, but only the last transition is to state χ for the path to be successful, and produces output

$$(c_1\ldots c_k d_1\ldots d_q, e_1\ldots e_q)\delta$$

noting the way the states in W_k are used as a queue to delay the output for the left hand side and that last transition outputs all the remaining pairs. Hence there is a unique successful path determined by the input and the total output produced is

$$(a_1 \dots a_n c_1 \cdots c_k d_1 \dots d_q, b_1 \dots b_n e_1 \dots e_q)\delta \in O_k.$$

We will now consider the case where $p > q$. Again, the input

$$(a_1, b_1) \dots (a_n, b_n)(c_1, \$) \dots (c_k, \$)$$

takes us to state $c_1 \dots c_k$ and produces output $(a_1, b_1) \dots (a_n, b_n)$. Then we must use equation (iv) q times without entering state χ. Now, input $(d_{q+1}, \$) \dots (d_p, \$)$ will force us to use equation (v) once and equation (vii) $p - q - 1$ times, where again only the last transition can be to the terminal state χ. The total output produced is again

$$(a_1 \dots a_n c_1 \cdots c_k d_1 \dots d_q, b_1 \dots b_n e_1 \dots e_q)\delta \in O_k.$$

Finally we have the case where $p < q$. As in the previous case, the input

$$(a_1, b_1) \dots (a_n, b_n)(c_1, \$) \dots (c_k, \$)$$

takes us to state $c_1 \dots c_k$ and produces output $(a_1, b_1) \dots (a_n, b_n)$. Then we must use equation (iv) p times and the input $(\$, e_{p+1}) \dots (\$, e_q)$ will force us to use equation (vi) once and, if $q - p \leq k$ we now must use equation $(viii)$ $q - p$ times being last transition to χ, otherwise we use $(viii)$ k times and we enter state (ϵ, l) (our queue is empty), what force us to use equation (ix) $q - p - k$ times being last transition to the terminal state χ. Again the total output is

$$(a_1 \dots a_n c_1 \cdots c_k d_1 \dots d_q, b_1 \dots b_n e_1 \dots e_q)\delta \in O_k.$$

To prove the other inclusion let $(w_1, w_2)\delta \in O_k$. Then we have

$$w_1 \equiv a_1 \dots a_n \; c_1 \dots c_k d_1 \dots d_p,$$
$$w_2 \equiv b_1 \dots b_n \; e_1 \dots e_q$$

where $(a_1 \dots a_n c_1 \dots c_k, b_1 \dots b_n)\delta \in M_k$ and $(d_1 \dots d_p, e_1 \dots e_q)\delta \in N$. Hence we have

$$(a_1, b_1) \dots (a_n, b_n)(c_1, \$) \dots (c_k, \$)(d_1 \dots d_p, e_1 \dots e_q)\delta \in M_k \cdot N$$

and so $(w_1, w_2)\delta \in (M_k \cdot N)\eta_{\mathcal{A}}$. ■

Chapter 4

Rees matrix semigroups

It is well known that completely simple semigroups are Rees matrix semigroups over groups (see [29]) and that is the origin of the Rees matrix construction. Nevertheless, it is natural to consider Rees matrix semigroups over semigroups. The fundamental four-spiral semigroup, is an example of a known semigroup that is useful to define as a Rees matrix semigroup over a semigroup, as we will see.

In this chapter we start with an automatic semigroup U, and prove that a Rees matrix semigroup $S = \mathcal{M}[U; I, J; P]$ over U is automatic whenever it is finitely generated. This implies that if a semigroup is finitely generated and can be described as a Rees matrix semigroup over an automatic semigroup then it is automatic.

We also consider the converse problem: does the automaticity of S imply that of U? We prove that this is the case when S is prefix-automatic or when there is an element p in the matrix P such that $pU^1 = U$. Finally, we prove the analogous results for Rees matrix semigroups with zero.

1 The Rees matrix construction

The Rees matrix semigroup

$$S = \mathcal{M}[U; I, J; P]$$

over the semigroup U, with $P = (p_{ji})_{j\in J, i\in I}$ a matrix with entries in U, is the semigroup with the support set

$$I \times U \times J$$

and multiplication defined by

$$(l_1, s_1, r_1)(l_2, s_2, r_2) = (l_1, s_1 p_{r_1 l_2} s_2, r_2)$$

where $(l_1, s_1, r_1), (l_2, s_2, r_2) \in I \times U \times J$. We say that U is the *base semigroup* of the Rees matrix semigroup S.

Necessary and sufficient conditions for the Rees matrix semigroup to be finitely generated are given in [3]:

Proposition 4.1 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup over a semigroup U. Then S is finitely generated if and only if both I and J are finite sets, U is finitely generated and the set $U \backslash V$ is finite, where V is the ideal of U generated by the entries in the matrix P.*

2 Automaticity of a Rees matrix semigroup

We start this section by stating our first main result.

Theorem 4.2 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If U is an automatic semigroup and if S is finitely generated then S is automatic.*

Let V be the ideal of U generated by the entries of the matrix P i.e. $V = \{s p_{ji} s' : s, s' \in U^1, i \in I, j \in J\}$, where U^1 is the monoid obtained by adding an identity to U regardless of whether or not U already has an identity. From Proposition 4.1 we know that $S = \mathcal{M}[U; I, J; P]$ is finitely generated if and only if U is finitely generated and I, J and $U \backslash V$ are finite. So the previous theorem has the following equivalent formulation:

Theorem 4.3 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup, where I, J and $U \backslash V$ are finite sets, where V is the ideal of U generated by the entries of the matrix P. If U is an automatic semigroup then S is automatic.*

PROOF. Since U is automatic and $U \backslash V$ is finite, by Proposition 1.15, V is an automatic semigroup. Let (B, K) be an automatic structure with uniqueness for V, where $B = \{b_1, \ldots, b_n\}$ is a set of semigroup generators for V. Since V is the ideal of U generated by the entries in the matrix P we can write each b_h ($h \in N = \{1, \ldots, n\}$) as $b_h = s_h p_{\rho_h \lambda_h} s'_h$ where $s_h, s'_h \in U^1, \rho_h \in J, \lambda_h \in I$. Let $S_1 = \mathcal{M}[U^1; I, J; P]$. Given $(l, s, r) \in I \times V \times J$ we can write

$$s = b_{\alpha_1} b_{\alpha_2} \ldots b_{\alpha_h}$$

where $b_{\alpha_1} b_{\alpha_2} \ldots b_{\alpha_h}$ is a word in K and so we have

$$s = s_{\alpha_1} p_{\rho_{\alpha_1} \lambda_{\alpha_1}} s'_{\alpha_1} s_{\alpha_2} p_{\rho_{\alpha_2} \lambda_{\alpha_2}} s'_{\alpha_2} \ldots s_{\alpha_h} p_{\rho_{\alpha_h} \lambda_{\alpha_h}} s'_{\alpha_h}.$$

Hence we can write

$$\begin{aligned}(l, s, r) &= (l, s_{\alpha_1} p_{\rho_{\alpha_1} \lambda_{\alpha_1}} s'_{\alpha_1} s_{\alpha_2} p_{\rho_{\alpha_2} \lambda_{\alpha_2}} s'_{\alpha_2} \ldots s_{\alpha_h} p_{\rho_{\alpha_h} \lambda_{\alpha_h}} s'_{\alpha_h}, r) \\ &= (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1} s_{\alpha_2}, \rho_{\alpha_2}) \ldots (\lambda_{\alpha_h}, s'_{\alpha_h}, r).\end{aligned}$$

We note that the elements in the above sequence are elements of S_1 but some of them can be outside S. Since $U^1 \backslash V$ is finite and non empty we can write $U^1 \backslash V = \{x_1, \ldots, x_m\}$ with $m \geq 1$. We define the set $A = C \cup D$ where

$$\begin{aligned}C &= \{c_{li} : l \in I, i \in N\} \cup \{d_{ij} : i, j \in N\} \cup \{e_{jr} : j \in N, r \in J\}, \\ D &= \{f_{lhr} : l \in I, h \in \{1, \ldots, m\}, r \in J\}\end{aligned}$$

and the homomorphism

$$\begin{aligned}\psi : A^+ \to S_1,\ c_{li} \mapsto (l, s_i, \rho_i),\ \ d_{ij} \mapsto (\lambda_i, s'_i s_j, \rho_j), \\ e_{jr} \mapsto (\lambda_j, s'_j, r), f_{lhr} \mapsto (l, x_h, r),\end{aligned}$$

and we will see that A is a set of semigroup generators for S_1 with respect to ψ.

We now define the language $L = L_1 \cup D$ where

$$L_1 = \{c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} : b_{\alpha_1} \dots b_{\alpha_h} \in K, h \geq 1, l \in I, r \in J\}$$

and we will now show that L is a set of normal forms for S_1 with respect to ψ (and in particular that A generates S_1). Let (l, s, r) be an arbitrary element of S_1. If $(l, s, r) \in I \times V \times J$ then there is a (unique) word $b_{\alpha_1} \dots b_{\alpha_h} \in K$ representing s and therefore the word $c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} \in L$ represents (l, s, r), since we have

$$\begin{aligned}(l, s, r) &= (l, s_{\alpha_1} p_{\rho_{\alpha_1}\lambda_{\alpha_1}} s'_{\alpha_1} s_{\alpha_2} p_{\rho_{\alpha_2}\lambda_{\alpha_2}} s'_{\alpha_2} \dots s_{\alpha_h} p_{\rho_{\alpha_h}\lambda_{\alpha_h}} s'_{\alpha_h}, r) \\ &= (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1} s_{\alpha_2}, \rho_{\alpha_2}) \dots (\lambda_{\alpha_h}, s'_{\alpha_h}, r) \\ &= c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r}.\end{aligned}$$

If we have $(l, s, r) = (l, x_h, r) \in I \times U^1 \backslash V \times J$ then there is a unique word in L representing (l, s, r), which is $f_{lhr} \in D$.

We are now going to prove that (A, L) is an automatic structure for S_1. First we need to prove that L is a regular language. To this end let

$$L^{(l,r)} = L \cap (\{c_{li} : i \in N\} \cdot A^* \cdot \{e_{jr} : j \in N\}).$$

Then we can write

$$L = (\bigcup_{l \in I, r \in J} L^{(l,r)}) \cup D$$

and it is sufficient to prove that for each $l \in I, r \in J$ the language $L^{(l,r)}$ is regular. To do that we define a gsm $\mathcal{A}$ such that $K\eta_{\mathcal{A}} = L^{(l,r)}$. Let

$$\mathcal{A} = (Q, B, A, \mu, q_0, \{\xi\})$$

with $Q = N \cup \{q_0, \xi\}$, where q_0 is the initial state, ξ is the only accept state and μ is a partial function from $Q \times B$ to finite subsets of $Q \times A^+$ defined by:

$$\begin{aligned}(q_0, b_i)\mu &= \{(i, c_{li}), (\xi, c_{li} e_{ir})\} \ (i \in N), \\ (i, b_j)\mu &= \{(j, d_{ij}), (\xi, d_{ij} e_{jr})\} \ (i, j \in N).\end{aligned}$$

Given $u \equiv b_{\alpha_1} \dots b_{\alpha_h} \in K$ it is clear that the only word v in A^+ such that (u, v) corresponds to the label of a successful path in $\mathcal{A}$ is the word

$$c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} \in L^{(l,r)}.$$

So $K\eta_{\mathcal{A}} = L^{(l,r)}$ and $L^{(l,r)}$ is a regular language.

If a word

$$u \equiv b_{\alpha_1} \dots b_{\alpha_h} \in K$$

represents the element $s \in V$ then the word

$$u\eta_{\mathcal{A}} = c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} \in L^{(l,r)}$$

represents the element

$$(l, s, r) \in \{l\} \times V \times \{r\}.$$

Given two different words $u_1, u_2 \in K$, they represent two different elements $s_1, s_2 \in V$ because we have assumed that (B, K) is an automatic structure with uniqueness for V. Since the words $u_1\eta_{\mathcal{A}}$ and $u_2\eta_{\mathcal{A}}$ in $L^{(l,r)}$ represent the elements (l, s_1, r) and (l, s_2, r) respectively, it follows that two different words in $L^{(l,r)}$ represent two different elements in $\{l\} \times V \times \{r\}$, and hence that two different words in L represent two different elements in S_1. Therefore L is a regular set of unique normal forms for S_1.

The language

$$L_= = \{(w, w)\delta_A : w \in L\}$$

is regular, by Proposition 1.6. To conclude that (A, L) is an automatic structure for S_1 (with uniqueness) it remains to show that, given $a \in A$, the language L_a is regular. Let

$$a\psi = (l_0, s_0, r_0) \in S_1.$$

Let us fix $l \in I$ and $r \in J$ and prove that the language

$$L_a^{(l,r)} = L_a \cap (L^{(l,r)} \times A^*)\delta_A$$

is regular. Let $\overline{w}$ be the only word in K that represents the element $p_{rl_0}s_0 \in V$. We now define a gsm

$$\mathcal{C} = (\{q_0\}, A(2,\$), A(2,\$), \rho, q_0, \{q_0\})$$

where q_0 is the unique state and the transition ρ is defined by:

$$\begin{aligned}(q_0,(a,b))\rho &= \{(q_0,(a,b))\} \ ((a,b) \in A(2,\$), b \notin \{e_{jk} : j \in N, k \in J\}),\\ (q_0,(a,e_{jr}))\rho &= \{(q_0,(a,e_{jr_0}))\} \ (a \in A \cup \{\$\}, j \in N).\end{aligned}$$

We know, from Proposition 1.9, that $K_{\overline{w}}$ is a regular language and we will now show that

$$K_{\overline{w}}\pi_B\zeta_{\mathcal{A}}\delta_A\eta_{\mathcal{C}} = L_a^{(l,r)},$$

where $\pi_B : (B^* \times B^*)\delta_B \to B^* \times B^*$ is the inverse of δ_B. For $b_{\alpha_1}\dots b_{\alpha_h}, b_{\beta_1}\dots b_{\beta_k} \in K$ we have

$$\begin{aligned}&(b_{\alpha_1}\dots b_{\alpha_h}, b_{\beta_1}\dots b_{\beta_k})\delta_B \in K_{\overline{w}}\\ \Longleftrightarrow\ & b_{\alpha_1}\dots b_{\alpha_h}\overline{w} = b_{\beta_1}\dots b_{\beta_k}\\ \Longleftrightarrow\ & (l, b_{\alpha_1}\dots b_{\alpha_h}\overline{w}, r_0) = (l, b_{\beta_1}\dots b_{\beta_k}, r_0)\\ \Longleftrightarrow\ & (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1}s_{\alpha_2}, \rho_{\alpha_2})\dots(\lambda_{\alpha_{h-1}}, s'_{\alpha_{h-1}}s_{\alpha_h}, \rho_{\alpha_h})(\lambda_{\alpha_h}, s'_{\alpha_h}, r)(l_0, s_0, r_0)\\ &= (l, s_{\beta_1}, \rho_{\beta_1})(\lambda_{\beta_1}, s'_{\beta_1}s_{\beta_2}, \rho_{\beta_2})\dots(\lambda_{\beta_{k-1}}, s'_{\beta_{k-1}}s_{\beta_k}, \rho_{\beta_k})(\lambda_{\beta_k}, s'_{\beta_k}, r_0)\\ \Longleftrightarrow\ & c_{l\alpha_1}d_{\alpha_1\alpha_2}\dots d_{\alpha_{h-1}\alpha_h}e_{\alpha_h r}a = c_{l\beta_1}d_{\beta_1\beta_2}\dots d_{\beta_{k-1}\beta_k}e_{\beta_k r_0}\\ \Longleftrightarrow\ & (b_{\alpha_1}\dots b_{\alpha_h}, b_{\beta_1}\dots b_{\beta_k})\zeta_{\mathcal{A}}\delta_A\eta_{\mathcal{C}} \in L_a^{(l,r)}.\end{aligned}$$

We note that in a path in $\mathcal{A}$ each transition outputs a word of length 1 except possibly the last that can output a word of length 1 or 2 and so condition (3.1) in Theorem 3.1 holds with $C = 1$. Applying Theorem 3.1 we conclude that $L_a^{(l,r)}$ is a regular language.

The language $\overline{L}_a^{(l,r)} = L_a \cap (\{f_{lhr} : h \in \{1,\dots,m\}\} \times A^*)\delta_A$ is regular because it is finite. But then

$$L_a = (\bigcup_{l\in I, r\in J} L_a^{(l,r)}) \cup (\bigcup_{l\in I, r\in J} \overline{L}_a^{(l,r)}),$$

and so L_a is regular.

We conclude that S_1 is an automatic semigroup. Since S is a subsemigroup of S_1 such that $S_1 \backslash S$ is finite we can use Proposition 1.15 to conclude that S is an automatic semigroup. ■

Note that Theorem 4.3 generalizes one of the implications of the main result of [9] where it is assumed that U is a group and also [48, Theorem 7.2], where it is assumed that U is a monoid and that P contains the identity of U. Another interesting application of our theorem arises when U is a simple semigroup:

Corollary 4.4 *If U is an automatic simple semigroup then every Rees matrix semigroup $\mathcal{M}[U; I, J; P]$ (I and J finite) is automatic.* ■

Example 4.5 The fundamental four spiral semigroup Sp_4 (see Section 8.6 in [20]) can be represented as a 2×2 Rees matrix semigroup over the bicyclic monoid B; $Sp_4 = \mathcal{M}[\mathbf{B}; \{1,2\}, \{1,2\}; P]$ with

$$P = \begin{pmatrix} 1 & c \\ 1 & 1 \end{pmatrix}$$

(see [29, Exercise 3.8.19] and [7]). Since B is simple and automatic ([11, Example 4.2]) it follows that Sp_4 is also automatic.

3 On the automaticity of the base semigroup

If the Rees matrix semigroup $S = \mathcal{M}[U; I, J; P]$ is automatic then we know, by Proposition 4.1, that the base semigroup U must be finitely generated. It is an open question if U is automatic in general. We prove that U is automatic if we assume that there is an element p in the matrix such that the principal right ideal pU^1 generated by p is equal to U.

Theorem 4.6 *Let $S = \mathcal{M}[U; I, J; P]$ be a semigroup, and suppose that there is* *an entry p in the matrix P such that $pU^1 = U$. If S is automatic then U is automatic.*

Proof. Let $S_1 = \mathcal{M}[U^1; I, J; P]$. Then S is a subsemigroup of S_1 such that $S_1 \backslash S$ is finite. Since S is automatic, by Proposition 1.15, S_1 is also automatic. Let (A, L) be an automatic structure for S_1 with uniqueness, where $A = \{a_1, \ldots, a_n\}$ is a generating set for S_1 with respect to

$$\psi : A^+ \to S_1,\ a_h \mapsto (i_h, s_h, j_h)\ (h = 1, \ldots, n).$$

We will now show that

$$B = \{b_1, \ldots, b_n\} \cup \{c_{ji} : j \in J, i \in I\}$$

is a generating set for U^1 with respect to

$$\phi : B^+ \to U^1; b_h \mapsto s_h, c_{ji} \mapsto p_{ji}\ (h = 1, \ldots, n,\ j \in J, i \in I);$$

this was shown in [3] and we include our proof for completeness. Given $s \in U^1$ we can consider any element of the form $(i, s, j) \in S_1$. Since A is a generating set for S_1 we have

$$\begin{aligned}(i, s, j) &= a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_k}\\ &= (i_{\alpha_1}, s_{\alpha_1}, j_{\alpha_1})(i_{\alpha_2}, s_{\alpha_2}, j_{\alpha_2}) \ldots (i_{\alpha_k}, s_{\alpha_k}, j_{\alpha_k})\\ &= (i_{\alpha_1}, s_{\alpha_1} p_{j_{\alpha_1} i_{\alpha_2}} s_{\alpha_2} \ldots p_{j_{\alpha_{k-1}} i_{\alpha_k}} s_{\alpha_k}, j_{\alpha_k})\end{aligned}$$

and so we have

$$s = b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \ldots c_{j_{\alpha_{k-1}} i_{\alpha_k}} b_{\alpha_k} \in B^+.$$

Without loss of generality we can assume that $p_{11} = p$. Let

$$L_{11} = L \cap (\{1\} \times U^1 \times \{1\})\psi^{-1}.$$

This language is regular because

$$\begin{aligned}(\{1\} \times U^1 \times \{1\})\psi^{-1} = &\{a_h \in A : i_h = 1\} \cdot A^* \cdot \{a_h \in A : j_h = 1\} \cup\\ &\{a_h \in A : i_h = j_h = 1\}.\end{aligned}$$

Let

$$f : A^+ \to B^+; a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h} \mapsto b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \dots c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h}.$$

We define $K = L_{11}f$ and prove that (B, K) is an automatic structure with uniqueness for U^1 with respect to ϕ. We observe that $f : L_{11} \to K$ is a bijection and K is a set of unique normal forms for U^1. In fact, if a word $w \in L_{11}$ represents the element $(1, s, 1) \in \{1\} \times U^1 \times \{1\}$ then the corresponding word wf in K represents the element $s \in U^1$.

Next we show that K is a regular language by defining a gsm $\mathcal{A}$ such that $L_{11}\eta_{\mathcal{A}} = K$. Let $\mathcal{A} = (Q, A, B, \mu, q_0, \{\chi\})$ with $Q = \{q_0, \chi\} \cup J$, where q_0 is the initial state, χ is the only accept state and the transition μ is a partial function from $Q \times A$ to finite subsets of $Q \times B^+$ defined by:

$$\begin{aligned}
(q_0, a_h)\mu &= \{(j_h, b_h), (\chi, b_h)\} \ (h \in \{1, \dots, n\}), \\
(j, a_h)\mu &= \{(j_h, c_{ji_h} b_h), (\chi, c_{ji_h} b_h)\} \ (j \in J, h \in \{1, \dots, n\}).
\end{aligned}$$

Given a word

$$u \equiv a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h} \in L_{11}$$

there is a unique successful path α in $\mathcal{A}$ such that $\Phi(\alpha) = u$. This path is

$$q_0 \xrightarrow{(a_{\alpha_1}, b_{\alpha_1})} \chi$$

for $h = 1$ and

$$\begin{aligned}
&q_0 \xrightarrow{(a_{\alpha_1}, b_{\alpha_1})} j_{\alpha_1} \xrightarrow{(a_{\alpha_2}, c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2})} j_{\alpha_2} \to \dots \\
&\dots \xrightarrow{(a_{\alpha_{h-1}}, c_{j_{\alpha_{h-2}} i_{\alpha_{h-1}}} b_{\alpha_{h-1}})} j_{\alpha_{h-1}} \xrightarrow{(a_{\alpha_h}, c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h})} \chi
\end{aligned}$$

for $h > 1$, and its output is

$$\Sigma(\alpha) = b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \dots c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h} \equiv uf \in K.$$

We conclude that $K = L_{11}\eta_{\mathcal{A}}$ is regular, as claimed.

We now start proving that K_b is regular for $b \in B$. If $b\phi = 1$ then $K_b = K_=$ and it follows from the uniqueness of K that K_b is regular. If $b\phi \neq 1$ then $b\phi \in U = p_{11}U^1$ and we can write $b\phi = p_{11}s$ for some $s \in U^1$. Since $(1, s, 1)$ is an element of S_1 there is a word $\overline{w} \in L$ that represents the element $(1, s, 1)$. We know by Proposition 1.9 that $L_{\overline{w}}$ is a regular language. Let us consider the regular language $H = L_{\overline{w}} \cap (L_{11} \times L_{11})\delta_A$ and prove that $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b$. For $a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k} \in L_{11}$ we have

$$
\begin{aligned}
& (a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k})\delta_A \in H \\
\iff & a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}\overline{w} = a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k} \\
\iff & (1, s_{\alpha_1}, j_{\alpha_1})(i_{\alpha_2}, s_{\alpha_2}, j_{\alpha_2})\ldots(i_{\alpha_h}, s_{\alpha_h}, 1)(1, s, 1) \\
& = (1, s_{\beta_1}, j_{\beta_1})(i_{\beta_2}, s_{\beta_2}, j_{\beta_2})\ldots(i_{\beta_k}, s_{\beta_k}, 1) \\
\iff & (b_{\alpha_1}c_{j_{\alpha_1}i_{\alpha_2}}b_{\alpha_2}c_{j_{\alpha_2}i_{\alpha_3}}\ldots b_{\alpha_h}, b_{\beta_1}c_{j_{\beta_1}i_{\beta_2}}b_{\beta_2}c_{j_{\beta_2}i_{\beta_3}}\ldots b_{\beta_k})\delta_B \in K_b \\
\iff & (a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k})\zeta_{\mathcal{A}}\delta_B \in K_b.
\end{aligned}
$$

Since in any path in $\mathcal{A}$ only the first transition can output a word of length 1 and all the others output words of length 2 we can apply Theorem 3.1 with $C = 1$ and conclude that K_b is a regular language. So U^1 is an automatic semigroup and, by Proposition 1.14, U is automatic. ∎

Note that Theorem 4.6 generalizes [48, Theorem 7.4], where it is assumed that U is a monoid and that P has a row and a column consisting entirely of ones.

4 P-automaticity of the base semigroup

We prove in this section that if S is p-automatic then U is p-automatic. It is an open question if the definitions of p-automatic and automatic coincide for semigroups as they do for groups and, more generally, for right cancellative monoids (a monoid M is right cancellative if $(\forall s, t, u \in M)\ su = tu \implies s = t$); see [49, Theorem 8.1].

Theorem 4.7 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If S is prefix-automatic then U is prefix-automatic.*

PROOF. By [49, Corollary 5.4] we can fix a prefix-automatic structure with uniqueness (A, L) for S. We define $A, \psi, B, \phi, L_{11}, f, \mathcal{A}$ and K as in the proof of the previous theorem just replacing U^1 by U and S_1 by S in the definitions, and assume that $\psi \restriction_A$ is injective. We will prove that (B, K) is a prefix-automatic structure with uniqueness for U with respect to ϕ. We have proved that K is regular, that f is a bijection and that $\phi \restriction_K$ is injective, without using the fact that U^1 is a monoid. So we just have to prove that

$$K_b = \{(v_1, v_2)\delta_B : v_1, v_2 \in K, v_1 b = v_2\}$$

is a regular language for $b \in B$ to conclude that U is automatic. We start by writing K_b as a finite union of languages which we then prove are regular. We can write

$$K_b = \{(w_1 f, w_2 f)\delta_B : w_1, w_2 \in L_{11}, (w_1 f)b = w_2 f\}.$$

Let $A_1 = \{a_h \in A : j_h = 1\}$. We define

$$K_b^a = \{(w_1 f, w_2 f)\delta_B \in K_b : w_1 \in A^+ a\}$$

for $a \in A_1$. We also define $K_b^1 = K_b \cap (B \times B^*)\delta_B$. It is clear that

$$K_b = (\bigcup_{a \in A_1} K_b^a) \cup K_b^1.$$

The language K_b^1 is regular because it is finite. Let us fix an element $a \in A_1$, with $a\psi = (l, s, 1)$, and prove that K_b^a is a regular language. Let $\overline{w}$ be the only word in L representing $(l, sb, 1)$. The language

$$L'_{=} = \{(w_1, w_2)\delta_A : w_1 \in L, w_2 \in \mathrm{Pref}(L), w_1 = w_2\}$$

is regular by hypothesis and the language $L_{\overline{w}}$ is regular by Proposition 1.9. So the language

$$D = \{(w_1', w_2)\delta_A : (\exists w_1'' \in A^*)((w_1', w_1'')\delta_A \in L'_{=}\tau \ \& \ (w_1'', w_2)\delta_A \in L_{\overline{w}})\}$$

is regular by Proposition 1.13, where

$$\tau : A(2,\$)^* \to A(2,\$)^*;\ (a,b) \mapsto (b,a)$$

is the homomorphism that swaps coordinates. The language

$$E = \{(w_1'a, w_2)\delta_A : (w_1', w_2)\delta_A \in D\}$$

is also regular, since we can write

$$E = \{(w_1, w_2)\delta_A : (\exists w_1' \in A^*)((w_1, w_1')\delta_A \in F \ \&\ (w_1', w_2)\delta_A \in D)\},$$

where $F = \{(wa, w)\delta_A : w \in A^*\}$ is a regular language.

We now use the regular language

$$H = E \cap (L_{11} \times L_{11})\delta_A$$

to prove that L_b^a is regular by showing that $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b^a$. We note that we can write

$$H = \{(w_1, w_2)\delta_A : w_1, w_2 \in L_{11} \ \&\ w_1 \equiv w_1'a \ \&\ w_1'\overline{w} = w_2\}$$

and so for $a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\dots a_{\beta_k} \in L_{11}$ we have

$$\begin{aligned}
&(a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\dots a_{\beta_k})\delta_A \in H \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \&\ a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_{h-1}}\overline{w} = a_{\beta_1}a_{\beta_2}\dots a_{\beta_k} \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \&\ (1, s_{\alpha_1}, j_{\alpha_1})(i_{\alpha_2}, s_{\alpha_2}, j_{\alpha_2})\dots(i_{\alpha_{h-1}}, s_{\alpha_{h-1}}, j_{\alpha_{h-1}})(i_{\alpha_h}, s_{\alpha_h}b, 1) \\
& \qquad\qquad = (1, s_{\beta_1}, j_{\beta_1})(i_{\beta_2}, s_{\beta_2}, j_{\beta_2})\dots(i_{\beta_k}, s_{\beta_k}, 1) \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \&\ s_{\alpha_1}p_{j_{\alpha_1}i_{\alpha_2}}s_{\alpha_2}p_{j_{\alpha_2}i_{\alpha_3}}\dots s_{\alpha_h}b = s_{\beta_1}p_{j_{\beta_1}i_{\beta_2}}s_{\beta_2}p_{j_{\beta_2}i_{\beta_3}}\dots s_{\beta_k} \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \&\ ((a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_h})f)b = (a_{\beta_1}a_{\beta_2}\dots a_{\beta_k})f \\
\Longleftrightarrow\ & (a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\dots a_{\beta_k})\zeta_{\mathcal{A}}\delta_B \in K_b^a.
\end{aligned}$$

So we have $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b^a$ and we can use Theorem 3.1 to conclude that K_b^a is regular. Therefore K_b is regular and U is an automatic semigroup.

To prove that U is prefix-automatic we prove that $K'_{=}\tau$ is a regular language. We begin by writing it as a union of languages which we then prove are regular:

$$K'_{=}\tau = (\bigcup_{i,b,c} X_{(i,b,c)}) \cup Y \cup Z,$$

where

$$\begin{aligned}
X_{(i,b,c)} &= \{(w_1, w_2)\delta_B \in K'_{=}\tau : (\exists \alpha \in \{1, \ldots, n\})(w_1 \in B^+ b_\alpha c_{j_\alpha i} bc)\} \\
Y &= \{(w_1, w_2)\delta_B \in K'_{=}\tau : (\exists b \in \{b_1, \ldots, b_n\})(w_1 \in B^+ b)\} \\
Z &= \{(w_1, w_2)\delta_B \in K'_{=}\tau : |w_1| < 5\}
\end{aligned}$$

for $i \in I, b \in \{b_1, \ldots, b_n\}, c \in \{c_{ji} : j \in J, i \in I\}$. Let us fix i, b and c and let $\overline{w}$ be the (unique) word in L representing $(i, bc, 1)$. Defining

$$L'_{\overline{w}} = \{(u_1, u_2)\delta_A : u_1 \in \mathrm{Pref}(L), u_2 \in L, u_1\overline{w} = u_2\}$$

and observing that for $w_1 \in \mathrm{Pref}(K), w_2 \in K$ with $w_1 \equiv b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \ldots b_{\alpha_h} c_{j_{\alpha_h} i} bc$ and $w_2 \equiv b_{\beta_1} c_{j_{\beta_1} i_{\beta_2}} b_{\beta_2} \ldots b_{\beta_k}$ we have

$$\begin{aligned}
& w_1 = w_2 \\
\Longleftrightarrow\ & (1, s_{\alpha_1}, j_{\alpha_1}) \ldots (i_{\alpha_h}, s_{\alpha_h}, j_{\alpha_h})(i, bc, 1) = (1, s_{\beta_1}, j_{\beta_1}) \ldots (j_{\beta_k}, s_{\beta_k}, 1) \\
\Longleftrightarrow\ & a_{\alpha_1} \ldots a_{\alpha_h}\overline{w} = a_{\beta_1} \ldots a_{\beta_k} \\
\Longleftrightarrow\ & (a_{\alpha_1} \ldots a_{\alpha_h}, a_{\beta_1} \ldots a_{\beta_k})\delta_A \in L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A,
\end{aligned}$$

we can write

$$\begin{aligned}
X_{(i,b,c)} = & (\textstyle\bigcup_{\alpha\in\{1,\ldots,n\}} (B^+\{b_\alpha c_{j_\alpha i} bc\} \times B^+)\delta_B) \cap (\mathrm{Pref}(K) \times K)\delta_B \cap \\
& (\textstyle\bigcup_{\alpha\in\{1,\ldots,n\}} \{((u_1 f)c_{j_\alpha i} bc, u_2 f)\delta_B : (u_1, u_2)\delta_A \in L'_{\overline{w}} \cap \\
& (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A\}).
\end{aligned}$$

The languages

$$(\mathrm{Pref}(L_{11}) \times L_{11})\delta_A,\ (B^+\{b_\alpha c_{j_\alpha i} bc\} \times B^+)\delta_B,\ (\mathrm{Pref}(K) \times K)\delta_B$$

are regular by Proposition 1.6. The language $L'_{\overline{w}}$ is regular because we can write

$$L'_{\overline{w}} = \{(u_1, u_2)\delta_A : (\exists u_3 \in A^+)((u_1, u_3)\delta_A \in L_{=}\tau \ \&\ (u_3, u_2)\delta_A \in L_{\overline{w}})\}$$

and use Proposition 1.9. The language

$$N = (L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A)\pi_A\zeta_{\mathcal{A}}\delta_B$$

is regular by Theorem 3.1 and so for a fixed $\alpha \in \{1, \ldots, n\}$ the language

$$\begin{aligned}&\{(w_1 c_{j_\alpha i} bc, w_2)\delta_B : (w_1, w_2)\delta_B \in N\}\\ &= \{((u_1 f)c_{j_\alpha i} bc, u_2 f)\delta_B : (u_1, u_2)\delta_A \in L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A\}\end{aligned}$$

is also regular. Hence $X_{(i,b,c)}$ is regular. We note that

$$Y = (L'_{=}\tau \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A)\pi_A\zeta_{\mathcal{A}}\delta_B$$

and by Theorem 3.1 it is regular. Since Z is finite it is proved that $K'_{=}$ is regular and so U is prefix-automatic. ■

5 Rees matrix semigroups with zero

In this section we show that the previous results are still valid if we consider Rees matrix semigroups with zero. The Rees matrix semigroup with zero $S = \mathcal{M}^0[U; I, J; P]$ over the semigroup U, where $P = (p_{ji})_{j\in J, i\in I}$ is a matrix with entries in U^0 (U with a zero adjoined to it), is the semigroup with the support set $(I \times U \times J) \cup \{0\}$ and multiplication defined by

$$\begin{aligned}(l_1, s_1, r_1)(l_2, s_2, r_2) &= \begin{cases}(l_1, s_1 p_{r_1 l_2} s_2, r_2) & \text{if } p_{r_1 l_2} \neq 0,\\ 0 & \text{otherwise,}\end{cases}\\ (l_1, s_1, r_1)0 &= 0(l_2, s_2, r_2) = 0 \cdot 0 = 0.\end{aligned}$$

Alternatively, S can be viewed as the Rees quotient S'/M (see Appendix A), where $S' = \mathcal{M}[U^0; I,\ J; P]$ (a Rees matrix semigroup without zero), and $M = I \times \{0\} \times J$ (an ideal). With this in mind, the following result from [23] will prove useful:

Proposition 4.8 *If S is an automatic semigroup and if I is a finite ideal of S then S/I is automatic as well.*

In general the converse does not hold; see [23]. However, in our context it does:

Proposition 4.9 *If $S = \mathcal{M}^0[U; I, J; P]$ is automatic (resp. prefix-automatic) then so is $T = \mathcal{M}[U^0; I, J; P]$.*

Proof. First we note that S has an automatic (resp. prefix-automatic) structure with uniqueness of the form $(A\cup\{\iota\}, L\cup\{\iota\})$, where ι represents 0, and no element of A or L represents 0. Indeed, let (B, K) be any automatic structure with uniqueness for S, and let w be the only element of K representing 0. Define $A = \{b \in B : b \neq 0\}$ and $L = K\backslash\{w\}$. That $(A\cup\{\iota\}, L\cup\{\iota\})$ is an automatic structure with uniqueness for S follows from Propositions 1.10, 1.11 and 1.12. Moreover, if (B, K) is a prefix-automatic structure for S, then so is $(A \cup \{\iota\}, L \cup \{\iota\})$, because

$$(L \cup \{\iota\})'_= = (K'_=\backslash\{(u, v)\delta_A \in K'_= : v \in \mathrm{Pref}(w)\}) \cup \{(\iota, \iota)\},$$

and the set $\{(u, v)\delta_A \in K'_= : v \in \mathrm{Pref}(w)\}$ is finite.

If $A = \{a_1, \dots, a_n\}$, and if a_h is mapped onto (i_h, s_h, j_h), then obviously T is generated by the set $C = A \cup \{\iota_{ij} : i \in I, j \in J\}$ under the mapping

$$a_h \mapsto (i_h, s_h, j_h),\ \ \iota_{ij} \mapsto (i, 0, j).$$

Let also

$$M = L \cup \{\iota_{ij} : i \in I, j \in J\}.$$

Clearly, M is a set of unique normal forms for T.

Denoting for a moment the multiplication in T by $*$ we see that

$$(l_1, s_1, r_1) * (l_2, s_2, r_2) = \begin{cases} (l_1, s_1, r_1)(l_2, s_2, r_2) & \text{if } (l_1, s_1, r_1)(l_2, s_2, r_2) \neq 0, \\ (l_1, 0, r_2) & \text{otherwise.} \end{cases}$$

Therefore, using the regular languages

$$L^l = L \cap \{a_h \in A : i_h = l\}A^* \ (l \in I),$$
$$L^{(a_h,0)} = \{w \in A^+ : (w, \iota)\delta_A \in (L \cup \{\iota\})_{a_h}\} \ (a_h \in A),$$

we see that

$$\begin{aligned} M_{a_h} =& ((L \cup \{\iota\})_{a_h} \cap (A^+ \times A^+)\delta_A) \cup (\textstyle\bigcup_{l \in I}((L^l \cap L^{(a_h,0)}) \times \{\iota_{lj_h}\})\delta_C) \cup \\ & \{(\iota_{ij}, \iota_{ij_h}) : i \in I, j \in J\}, \\ M_{\iota_{ij}} =& (\textstyle\bigcup_{l \in I}(L^l \times \{\iota_{lj}\})\delta_C) \cup \{(\iota_{lr}, \iota_{ij}) : l \in I, r \in J\} \end{aligned}$$

are all regular, and so (C, M) is an automatic structure for T. Moreover, if $(A \cup \{\iota\}, L \cup \{\iota\})$ is a prefix-automatic structure for S then

$$M'_= = ((L \cup \{\iota\})'_= \cap (A^+ \times A^+)\delta_A) \cup \{(\iota_{ij}, \iota_{ij}) : i \in I, j \in J\}$$

is also regular, so that (C, M) is a prefix-automatic structure for T. ■

Combining the above two propositions with Theorems 4.3, 4.6 and 4.7, we obtain the following result:

Theorem 4.10 *Let $S = \mathcal{M}^0[U; I, J; P]$ be a Rees matrix semigroup with zero.*

(i) *If U is automatic and if S is finitely generated then S is automatic as well.*

(ii) *If S is automatic and there is an entry p in the matrix P such that $pU^1 = U$ then U is automatic.*

(iii) *If S is prefix-automatic then U is prefix-automatic.* ■

We end this chapter with the following question:

Question 4.11 Let $\mathcal{M}[U; I, J; P]$ be an automatic semigroup. Is the base semigroup U necessarily automatic?

The results contained in this chapter are also contained in [15].

Chapter 5

Other semigroup constructions

We consider in this chapter further two standard semigroup constructions: the Bruck–Reilly extensions and the wreath products. We will prove that, in some particular situations, the automaticity of the base semigroups implies the automaticity of the construction.

1 Bruck–Reilly extensions

Let T be a monoid and $\theta : T \mapsto T$ be a monoid homomorphism. The set

$$\mathbb{N}_0 \times T \times \mathbb{N}_0$$

with the operation defined by

$$(m, t_1, n)(p, t_2, q) = (m - n + k, (t_1\theta^{k-n})(t_2\theta^{k-p}), q - p + k) \ (k = max\{n, p\}),$$

where θ^0 denotes the identity map on M, is called the Bruck–Reilly extension of T determined by θ and is denoted by $\mathrm{BR}(T, \theta)$. The semigroup $\mathrm{BR}(T, \theta)$ is a monoid with identity $(0, 1_T, 0)$, denoting by 1_T the identity of T. This is a generalization of the constructions from [5, 32, 37], also considered in [2].

We consider some particular situations where the automaticity of the monoid T implies the automaticity of its Bruck–Reilly extensions $\mathrm{BR}(T, \theta)$.

Theorem 5.1 *If T is a finite monoid, then any Bruck–Reilly extension of T is automatic.*

Proof. Let $T = \{t_1, \ldots, t_l\}$ and let $\overline{T} = \{\overline{t_1}, \ldots, \overline{t_l}\}$ be an alphabet in bijection with T. We define the alphabet $A = \{b, c\} \cup \overline{T}$ and the regular language

$$L = \{c^m \bar{t} b^n : m, n \geq 0, \bar{t} \in \overline{T}\}$$

on A. Defining the homomorphism

$$\psi : A^+ \to \mathrm{BR}(T, \theta);\ \bar{t} \mapsto (0, t, 0),\ c \mapsto (1, 1_T, 0),\ b \mapsto (0, 1_T, 1)$$

it is clear that A is a generating set for $\mathrm{BR}(T, \theta)$ with respect to ψ and, in fact, given an element $(m, t, n) \in \mathbb{N}_0 \times T \times \mathbb{N}_0$, the unique word in L representing it is $c^m \bar{t} b^n$.

In order to prove that (A, L) is an automatic structure with uniqueness for $BR(T, \theta)$ we only have to prove that, for each generator $a \in A$, the language L_a is regular. To prove that L_b is regular we observe that

$$(c^m \overline{t_i} b^n) b = (m, t_i, n)(0, 1_T, 1) = (m, t_i, n + 1) = c^m \overline{t_i} b^{n+1}$$

and so we can write

$$\begin{aligned} L_b &= \bigcup_{i=1}^{l} \{(c^m \overline{t_i} b^n, c^m \overline{t_i} b^{n+1}) \delta_A : n, m \in \mathbb{N}_0\} \\ &= \bigcup_{i=1}^{l} (\{(c, c)\}^* \cdot \{(\overline{t_i}, \overline{t_i})\} \cdot \{(b, b)\}^* \cdot \{(\$, b)\}) \end{aligned}$$

which is a finite union of regular languages and so is regular. With respect to L_c we have

$$\begin{aligned} &(c^m \bar{t}) c = (m, t, 0)(1, 1_T, 0) = (m + 1, t\theta, 0) = c^{m+1} \overline{t\theta}, \\ &(c^m \bar{t} b^{n+1}) c = (m, t, n + 1)(1, 1_T, 0) = (m, t, n) = c^m \bar{t} b^n \ (n, m \in \mathbb{N}_0; \bar{t} \in \overline{T}) \end{aligned}$$

and so we can write

$$\begin{aligned} L_c = & \bigcup_{i=1}^{l} \{(c^m\overline{t_i}, c^{m+1}\overline{t_i\theta})\delta_A : m \in \mathbb{N}_0\} \cup \\ & \bigcup_{i=1}^{l} \{(c^m\overline{t_i}b^{n+1}, c^m\overline{t_i}b^n)\delta_A : m, n \in \mathbb{N}_0\} \\ = & \bigcup_{i=1}^{l} (\{(c,c)\}^* \cdot \{(\overline{t_i}, c)(\$, \overline{t_i\theta})\}) \cup \\ & \bigcup_{i=1}^{l} (\{c,c)\}^* \cdot \{(\overline{t_i}, \overline{t_i})\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}) \end{aligned}$$

and we conclude that L_c is a regular language as well.

We now fix an arbitrary $\overline{t} \in \overline{T}$ and prove that $L_{\overline{t}}$ is regular. For any words $c^m\overline{t_\alpha}b^n, c^p\overline{t_\beta}b^q \in L$ we have

$$c^m\overline{t_\alpha}b^n\overline{t} = c^p\overline{t_\beta}b^q$$

if and only if $m = p, n = q$, and $t_\alpha(t\theta^n) = t_\beta$, because

$$c^m\overline{t_\alpha}b^n\overline{t} = (m, t_\alpha, n)(0, t, 0) = (m, t_\alpha(t\theta^n), n).$$

Since T is finite the set $\{t\theta^n : n \in \mathbb{N}_0\}$ is finite as well. Taking j to be minimum such that the set $C_j = \{k \geq j : t\theta^j = t\theta^{k+1}\}$ is non empty and k to be the minimum element of C_j, we will now show that

$$\{t\theta^n : n \in \mathbb{N}_0\} = \{t, t\theta, \ldots, t\theta^j, \ldots, t\theta^k\}.$$

Given $n \geq j$ we have $n = j + h$ with $h \geq 0$ and, dividing h by $k+1-j$, we obtain $n = j + q(k+1-j) + r$ with $q \geq 0$ and $0 \leq r < k+1-j$. We now prove, by induction on q, that $t\theta^{j+r+q(k+1-j)} = t\theta^{j+r}$ for $q \geq 0$. For $q = 0$ it holds trivially and for $q > 0$ we have

$$\begin{aligned} t\theta^{j+r+q(k+1-j)} &= t\theta^{j+r+k+1-j+(q-1)(k+1-j)} = (t\theta^r)(t\theta^{k+1})(t\theta^{(q-1)(k+1-j)}) \\ &= (t\theta^r)(t\theta^j)(t\theta^{(q-1)(k+1-j)}) = t\theta^{j+r+(q-1)(k+1-j)}. \end{aligned}$$

We can then write

$$\begin{aligned}
L_{\overline{t}} = & \bigcup_{n=0}^{j-1} \{(c^m \overline{t_\alpha} b^n, c^m \overline{t_\alpha(t\theta^n)} b^n)\delta_A : m \in \mathbb{N}_0, t_\alpha \in T\} \cup \\
& \bigcup_{n=j}^{k} \{(c^m \overline{t_\alpha} b^{n+q(k+1-j)}, c^m \overline{t_\alpha(t\theta^n)} b^{n+q(k+1-j)})\delta_A : m, q \in \mathbb{N}_0, t_\alpha \in T\} \\
= & \bigcup_{n=0}^{j-1} (\{(c,c)\}^* \cdot \{(\overline{t_\alpha}, \overline{t_\alpha(t\theta^n)}) : t_\alpha \in T\} \cdot \{(b,b)\}^*) \cup \\
& \bigcup_{n=j}^{k} (\{(c,c)\}^* \cdot \{(\overline{t_\alpha}, \overline{t_\alpha(t\theta^n)}) : t_\alpha \in T\} \cdot \{(b,b)^n\} \cdot \{(b,b)^{k+1-j}\}^*)
\end{aligned}$$

and since all sets in this union are regular we conclude that $L_{\overline{t}}$ is regular as well. ■

From now on we assume that T is an automatic monoid and we fix an automatic structure (X, K) with uniqueness for T, where $X = \{x_1, \ldots, x_n\}$ is a set of semigroup generators for T with respect to the homomorphism

$$\phi : X^+ \to T.$$

We define the alphabet

$$A = \{b, c\} \cup X \tag{5.1}$$

to be a set of semigroup generators for $\mathrm{BR}(T, \theta)$ with respect to the homomorphism

$$\psi : A^+ \to \mathrm{BR}(T, \theta), x_i \mapsto (0, x_i\phi, 0), c \mapsto (1, 1_T, 0), b \mapsto (0, 1_T, 1),$$

and the regular language

$$L = \{c^i w b^j : w \in K; i, j \in \mathbb{N}_0\} \tag{5.2}$$

on A^+, which is a set of unique normal forms for $\mathrm{BR}(T, \theta)$, since we have $(c^i w b^j)\psi = (i, w\phi, j)$ for $w \in K$, $i, j \in \mathbb{N}_0$. As usual, to simplify notation, we will avoid explicit use of the homomorphisms ψ and ϕ, associated with the generating sets, and it will be clear from the context whenever a word $w \in X^+$ is being identified with an element of T, with an element of $\mathrm{BR}(T,\theta)$ or considered as a word. In particular, for a word $w \in X^+$ we write $w\theta$ instead of $(w\phi)\theta$, seeing θ also as a homomorphism $\theta : X^+ \to T$, and we will often write (i,w,j) instead of $(i,w\phi,j)$ for $i,j \in \mathbb{N}_0$.

For (A,L) to be an automatic structure for $\mathrm{BR}(T,\theta)$ the languages

$$\begin{aligned} L_b = \ & \{(c^iwb^j, c^iwb^{j+1})\delta_A : w \in K; i,j \in \mathbb{N}_0\}, \\ L_c = \ & \{(c^iwb^{j+1}, c^iwb^j)\delta_A : w \in K; i,j \in \mathbb{N}_0\} \cup \\ & \{(c^iw_1, c^{i+1}w_2)\delta_A : w_1,w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\}, \\ L_{x_r} = & \{(c^iw_1b^j, c^iw_2b^j)\delta_A : (w_1,w_2)\delta_X \in K_{x_r\theta^j}; i,j \in \mathbb{N}_0\} \ (x_r \in X), \end{aligned}$$

must be regular. The language L_b is regular, since we have

$$L_b = \{(c,c)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(\$,b)\},$$

but there is no obvious reason why the languages L_c and L_{x_r} should also be regular. We will consider particular situations where (A,L) is an automatic structure for $\mathrm{BR}(T,\theta)$.

Theorem 5.2 *If T is an automatic monoid and $\theta : T \to T;\ t \mapsto 1_T$ then $\mathrm{BR}(T,\theta)$ is automatic.*

Proof. To show that the pair (A,L) defined by (5.1) and (5.2) is an automatic structure for $\mathrm{BR}(T,\theta)$ we just have to prove that the languages L_c and L_{x_r} ($x_r \in X$) are regular. But now, denoting by w_{1_T} the unique word in K representing 1_T, we have

$$\begin{aligned} L_c = & \{(c^iwb^{j+1}, c^iwb^j)\delta_A : w \in K; i,j \in \mathbb{N}_0\} \cup \{(c^iw, c^{i+1}w_{1_T})\delta_A : w \in K; i \in \mathbb{N}_0\} \\ = & (\{(c,c)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}) \cup \\ & ((\{(c,c)\}^* \cdot \{(\$,c)\}) \odot (K \times \{w_{1_t}\})\delta_X), \end{aligned}$$

where $\odot$ denotes the padded product of languages defined in Chapter 3, which is a regular language by Theorem 3.3. We have

$$\begin{aligned}
L_{x_r} =& \{(c^i w b^j, c^i w b^j)\delta_A : w \in K, i \in \mathbb{N}_0, j \in \mathbb{N}\} \cup \\
& \{(c^i w_1, c^i w_2)\delta_A : (w_1, w_2)\delta_X \in K_{x_r}; i \in \mathbb{N}_0\} \\
=& (\{(c,c)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^+) \cup \\
& (\{(c,c)\}^* \cdot K_{x_r})
\end{aligned}$$

because, for any $c^i w b^j \in L$ with $j \geq 1$, we have

$$(c^i w b^j) x_r = (i, w, j)(0, x_r, 0) = (i, w(x_r \theta^j), j) = (i, w, j) = c^i w b^j$$

and for $c^i w \in L$ we have

$$(c^i w) x_r = (i, w, 0)(0, x_r, 0) = (i, w x_r, 0).$$

Therefore L_{x_r} is also a regular language and so $\mathrm{BR}(T, \theta)$ is automatic. ■

Theorem 5.3 *If T is an automatic monoid and θ is the identity in T then $\mathrm{BR}(T, \theta)$ is automatic.*

PROOF. We use the generating set A defined by equation (5.1) but we now define $L = \{c^i b^j w : w \in K\}$ observing that, since θ is the identity, for any $x_r \in X$, we have

$$\begin{aligned}
x_r c &= (0, x_r, 0)(1, 1_T, 0) = (1, x_r \theta, 0) = (1, x_r, 0) = (1, 1_T, 0)(0, x_r, 0) = c x_r, \\
x_r b &= (0, x_r, 0)(0, 1_T, 1) = (0, x_r, 1) = (0, x_r \theta, 1) = (0, 1_T, 1)(0, x_r, 0) = b x_r.
\end{aligned}$$

The language L is regular and it is a set of unique normal forms for $\mathrm{BR}(T, \theta)$. Also the languages

$$\begin{aligned}
L_b \ =& \{(c^i b^j w, c^i b^{j+1} w)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \\
=& (\{(c,c)\}^* \cdot \{(b,b)\}^* \cdot \{(\$, b)\}) \odot \{(w,w)\delta_X : w \in K\}, \\
L_c \ =& \{(c^i b^{j+1} w, c^i b^j w)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \\
& \{(c^i w, c^{i+1} w)\delta_A : i \in \mathbb{N}_0, w \in K\} \\
=& ((\{(c,c)\}^* \cdot \{(b,b)\}^* \cdot \{(b, \$)\}) \odot \{(w,w)\delta_X : w \in K\}) \cup \\
& ((\{(c,c)\}^* \cdot \{(\$, c)\}) \odot \{(w,w)\delta_X : w \in K\}), \\
L_{x_r} =& \{(c^i b^j w_1, c^i b^j w_2)\delta_A : (w_1, w_2)\delta_X \in K_{x_r}\} \\
=& (\{(c,c)\}^* \cdot \{(b,b)\}^*) \cdot K_{x_r}
\end{aligned}$$

are regular, by Theorem 3.3, and so (A, L) is an automatic structure for $\mathrm{BR}(T, \theta)$. ■

We say that a semigroup T is of *finite geometrical type* (*fgt*) (see [49]) if for every $t_1 \in T$, there exists $k \in \mathbb{N}$ such that the equation

$$xt_1 = t_2$$

has at most k solutions for every $t_2 \in M$.

We start by presenting some examples of fgt and non-fgt semigroups. A group is fgt because the equations have always one solution. Free semigroups and free commutative semigroups are fgt because the equations have at most one solution. To see that the bicyclic monoid is fgt we will show that given an arbitrary $c^i b^j \in \mathbf{B}$ the equation

$$c^u b^v c^i b^j = c^k b^l$$

as at most $i+1$ solutions for any $c^k b^l \in \mathbf{B}$. In the case where $j \neq l$ we must have $u = k$, $v \geq i$ and $v = l + i - j$ which is only possible if $l \geq j$. Hence we have at most one solution. In the case where $j = l$ we must have $v \leq i$ and $u = k + v - i$, and so we have at most $i + 1$ solutions (precisely $i + 1$ solutions if $k \geq i$).

Any infinite semigroup with a zero, for example, is not fgt because the equation $x0 = 0$ has infinitely many solutions. In [25] the authors give an example of a commutative semigroup that is not automatic. This semigroup is defined by the presentation

$$\begin{aligned}\langle a, b, x, y \mid\ & aax = bx, bby = ay, ab = ba, ax = xa, \\ & ay = ya, bx = xb, by = yb, xy = yx\rangle\end{aligned}$$

(for an introduction and notation about semigroup presentations see Section 3 in Chapter 8). This semigroup is not fgt because, for example, the equation $Xxy = bxy$ has infinitely many solutions of the form $X = a^{2^n}$ with n odd (see [23] and [25]). In the above examples the semigroups are not fgt because there exists an equation with infinitely many solutions. We give a further example of non-fgt semigroup without such equations. The semigroup defined by the presentation

$$\langle a, b, c \mid ac = c^2, ca = c^2, bc = c^2, cb = c^2 \rangle$$

is not fgt, because any $x \in \{a, b\}^+$ with $|x| = i - 1$ is a solution of the equation $xc = c^i$, for $i > 1$. Different words in $\{a, b\}^+$ correspond to different elements in the semigroup since no relations can be applied to them. Therefore given $k \in \mathbb{N}$ we can always choose $c^i \in S$ such that the equation $xc = c^i$ has more than k solutions. Since all relations have both sides with length 2 there are no equations with infinitely many solutions.

We will need the following result:

Lemma 5.4 *Let T be a fgt monoid with an automatic structure with uniqueness (X, K). Then for every $w \in X^+$ there is a constant C such that $(w_1, w_2)\delta_X \in K_w$ implies $||w_1| - |w_2|| < C$.*

Proof. Since the language K_w is regular by Proposition 1.9, we can take C to be the number of states of an automaton recognizing K_w. Suppose we have $(w_1, w_2)\delta_X \in K_w$ with $|w_2| - |w_1| > C$. Then we have $w_2 \equiv w_3 w_4$ with $|w_3| = |w_1|$ and $|w_4| > C$. By the Pumping Lemma we have $w_4 \equiv w_5 w_6 w_7$ with $|w_6| > 0$ and $(w_1, w_3 w_5 w_7)\delta_X \in K_w$. But then we have $w_2 = w_3 w_5 w_7$ and $w_2, w_3 w_5 w_7 \in K$ are different words, which contradicts the uniqueness of K. Suppose now that $|w_1| - |w_2| > C$. Using the Pumping Lemma again we can write $w_1 \equiv w_3 w_4 w_5 w_6$ with $|w_3| = |w_2|$, $|w_5| > 1$ and we obtain infinitely many words $(w_3 w_4 w_5^i w_6, w_2)\delta_X \in K_w$ and therefore infinitely many solutions of the equation $xw = w_2$, which is not possible since T is *fgt*. ■

Theorem 5.5 *Let T be a fgt automatic monoid and let $\theta : T \to T$ be a monoid homomorphism. If $T\theta$ is finite then* $\mathrm{BR}(T, \theta)$ *is automatic.*

Proof. We will prove that the pair (A, L) defined by (5.1) and (5.2) is an automatic structure for $\mathrm{BR}(T, \theta)$. We have

$$\begin{aligned} L_c = &\{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \\ &\{(c^i w_1, c^{i+1} w_2)\delta_A : w_1, w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\} \end{aligned}$$

and, since the language

$$\begin{aligned} &\{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} = \\ &\{(c, c)^i\}^* \cdot \{(w, w)\delta_X : w \in K\} \cdot \{(b, b)\}^* \cdot \{(b, \$)\} \end{aligned}$$

is regular, we just have to prove that the language

$$M = \{(c^i w_1, c^{i+1} w_2)\delta_A : w_1, w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\}$$

is also regular. For any $t \in T\theta$ let w_t be the unique word in K representing t. Let

$$\begin{aligned} N = &\{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\theta\} = \\ &\bigcup_{t \in T\theta} \{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\theta = t\} = \\ &\bigcup_{t \in T\theta} \{(w_1, w_t)\delta_X : w_1 \in K; w_1 \in (t\theta^{-1})\phi^{-1}\} = \\ &\bigcup_{t \in T\theta} (((t\theta^{-1})\phi^{-1} \cap K) \times \{w_t\})\delta_X. \end{aligned}$$

We can define $\psi : X^+ \to T\theta;\ w \mapsto w\phi\theta$ and, since $T\theta$ is finite, for any $t \in T\theta$, we can apply Theorem 1.7 and conclude that $(t\theta^{-1})\phi^{-1} = t\psi^{-1}$ is regular. Therefore N is a regular language and, since we have

$$\begin{aligned} M = &\{(c^i w_1, c^{i+1} w_2)\delta_A : (w_1, w_2)\delta_X \in N; i \in \mathbb{N}_0\} = \\ &(\{(c, c)\}^* \cdot \{(\$, c)\}) \odot N, \end{aligned}$$

by Theorem 3.3, M is a regular language as well. We will now prove that the language

$$L_{x_r} = \{(c^i w_1 b^j, c^i w_2 b^j)\delta_A : (w_1, w_2)\delta_X \in K_{x_r\theta^j}; i, j \in \mathbb{N}_0\}$$

is regular. Since $T\theta$ is finite we can, as in the proof of Lemma 5.1, take j,k to be minimum with $x_r\theta^j = x_r\theta^{k+1}$ and $j \le k$, and we have $x_r\theta^{j+r+q(k+1-j)} = x_r\theta^{j+r}$ for $j \le j+r < k+1$ and $q \ge 0$. Therefore we can write

$$\begin{aligned} L_{x_r} =& \bigcup_{n=0}^{j-1} \{(c^i w_1 b^n, c^i w_2 b^n)\delta_A : (w_1,w_2)\delta_X \in K_{x_r\theta^n}; i \in \mathbb{N}_0\} \cup \\ & \bigcup_{n=j}^{k} \{(c^i w_1 b^{n+q(k+1-j)}, c^i w_2 b^{n+q(k+1-j)})\delta_A : (w_1,w_2)\delta_X \in K_{x_r\theta^n}; i,q \in \mathbb{N}_0\} \\ =& \bigcup_{n=0}^{j-1} (\{(c,c)\}^* \cdot (K_{x_r\theta^n} \odot \{(b,b)\}^*)) \cup \\ & \bigcup_{n=j}^{k} (\{(c,c)\}^* \cdot (K_{x_r\theta^n} \odot (\{(b,b)^n\} \cdot \{(b,b)^{k+1-j}\}^*))). \end{aligned}$$

Since T is fgt, by Lemma 5.4 there is a constant C such that

$$(w_1,w_2)\delta_X \in K_{x_r\theta^n} \implies ||w_1| - |w_2|| < C$$

for any $n = 0,\ldots,k$, and therefore we can apply Theorem 3.3 and we conclude that L_{x_r} is a regular language. ■

Since automatic groups are characterized by the fellow traveller property and Bruck–Reilly extensions of groups are somehow "almost groups" the following is a natural question:

Question 5.6 Is a Bruck–Reilly extension of a group automatic if and only if it has the fellow traveller property?

2 Wreath products

We consider the automaticity of the wreath product of semigroups, $S \operatorname{wr} T$, in the case where T is a finite semigroup. We start by giving the necessary and sufficient conditions, obtained in [39], for the wreath product in this case to be finitely generated. Finite generation of the wreath product is related to finite generation of the diagonal S-act. We use the conditions obtained for the case where the diagonal S-act is not finitely generated to prove that, in this case, the wreath product $S \operatorname{wr} T$ is automatic whenever it is finitely generated and S is an automatic semigroup.

We start by giving the definitions we require. If S is a semigroup and X is a set, then the set S^X of all mappings $X \to S$ forms a semigroup under *component-wise* multiplication of mappings: for $f, g \in S^X$, $fg : X \to S;\ x \mapsto (xf)(xg)$; this semigroup is called the *Cartesian power* of S by X. If S has a distinguished idempotent e, then the *support* of $f \in S^X$ relative to e is defined by

$$\operatorname{supp}_e(f) = \{x \in X : xf \neq e\}.$$

The set

$$S^{(X)_e} = \{f \in S^X : |\operatorname{supp}_e(f)| < \infty\}$$

is a subsemigroup of S^X; it is called the *direct power* of S relative to e. When there is no danger of confusion the subscript e is usually omitted. If X is finite of size n then S^X and $S^{(X)_e}$ coincide, and they are isomorphic to the semigroup $S^{(n)}$ consisting of n-tuples of elements of S under the component-wise multiplication. In this context, we write $S^{(X)_e}$ even if S has no idempotents; we can think of this as computing supports with respect to an identity adjoined to S.

The *unrestricted wreath product* $S \operatorname{Wr} T$ of two semigroups is the set $S^T \times T$ under multiplication

$$(f, t)(g, u) = (f\, {}^t g, tu),$$

where ${}^t g \in S^T$ is defined by

$$(x)\, {}^t g = (xt)g.$$

Let $e \in S$ be a distinguished idempotent. The *(restricted) wreath product* $S_e \operatorname{wr} T$ (with respect to e) is the subsemigroup of $S \operatorname{Wr} T$ generated by the set $\{(f, t) \in S \operatorname{Wr} T : |\operatorname{supp}_e(f)| < \infty\}$. Again the subscript e is often omitted.

The wreath product $S \operatorname{wr} T$ coincides with the unrestricted wreath product $S \operatorname{Wr} T$ in the case where T is finite, as observed in [50, Chapter 3].

An *action* of a semigroup S on a set X is a mapping $X \times S \to X$, $(x, s) \mapsto xs$, satisfying $(xs_1)s_2 = x(s_1s_2)$. The set X, together with an action, is called an *S-act*. It is said to be generated by a set $U \subseteq X$ if $US^1 = X$, and *finitely generated* if there exists a finite such U.

The *diagonal act* of a semigroup S is the set $S \times S$ with the action $(s_1, s_2)s = (s_1s, s_2s)$. For example, the diagonal acts of infinite groups, free semigroups, free commutative semigroups and completely simple semigroups are not finitely generated. The diagonal act of the full transformation monoid $T_{\mathbb{N}}$ on positive integers can be generated by a single element; see [6]. In [40] the authors give an example of an infinite, finitely presented monoid with a finitely generated diagonal act.

We will only state the conditions obtained in [39] for the case where T is finite and S is infinite.

Proposition 5.7 *Let S be an infinite semigroup and let T be a finite non-trivial semigroup. If the diagonal S-act is finitely generated then $S \operatorname{wr} T$ is finitely generated if and only if the following conditions are satisfied:*

(i) *$S^2 = S$ and $T^2 = T$;*

(ii) *S is finitely generated.*

If the diagonal S-act is not finitely generated then $S \operatorname{wr} T$ is finitely generated if and only if the following conditions are satisfied:

(i) *$S^2 = S$;*

(ii) *S is finitely generated;*

(iii) *every element of T is contained in the principal right ideal generated by a right identity.*

We will now consider the automaticity of the wreath product $S \operatorname{wr} T$ in the case where T is finite. In the case where S is also finite, $S \operatorname{wr} T$ is finite as well, and, in particular, it is automatic. We will consider the case where S is infinite and the diagonal S-act is not finitely generated.

Theorem 5.8 *If S and T are semigroups satisfying the following conditions:*

(i) *T is finite;*

(ii) *S is automatic;*

(iii) *the diagonal S-act is not finitely generated;*

(iv) *the wreath product $S \,\mathrm{wr}\, T$ is finitely generated;*

then $S \,\mathrm{wr}\, T$ is automatic.

PROOF. We assume that T is non-trivial and write $T = \{t_1, \ldots, t_m\}$ with $m > 1$. By using Proposition 5.7 we know that S is finitely generated and $S^2 = S$. So, by Proposition 1.17, we conclude that the direct product $S^{|T|}$ is automatic. Let (F, K) be an automatic structure for $S^{|T|}$ with uniqueness with $F = \{f_1, \ldots, f_k\}$. Since $S^2 = S$, we can use Theorem 1.16, and assume that K does not have words of length 1. Given $t \in T$, using again Proposition 5.7, there is a right identity $e \in T$ such that $t = eq$ for some $q \in T$. So we can define a generating set

$$Y = \{e_1, \ldots, e_m\} \cup \{q_1, \ldots, q_m\}$$

for T such that $t_i = e_i q_i$ for $i = 1, \ldots, m$ and $e_1, \ldots, e_m$ represent (not necessarily distinct) right identities in T. We define a new alphabet A by

$$A = \{(f, e_i) : f \in F, i = 1, \ldots, m\} \cup \{(f, q_i) : f \in F, i = 1, \ldots, m\}$$

and a language L on A by

$$L = \bigcup_{i=1,\ldots,m} \{(f_{\alpha_1}, e_i) \ldots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i) : f_{\alpha_1} \ldots f_{\alpha_n} \in K\}.$$

We will prove that the pair (A, L) is an automatic structure for $S \,\mathrm{wr}\, T$ (with uniqueness). To see that A generates $S \,\mathrm{wr}\, T$ and that L is a set of unique representatives for $S \,\mathrm{wr}\, T$ we observe that, given $(f, t_i) \in S \,\mathrm{wr}\, T$ there is only one word $f_{\alpha_1} \dots f_{\alpha_n}$ in K such that $f = f_{\alpha_1} \dots f_{\alpha_n}$. So there is only one word in L representing (f, t_i) which is

$$(f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i).$$

To prove that L is a regular language we now define a gsm $\mathcal{A}$ such that $K\eta_{\mathcal{A}} = L$. Let

$$\mathcal{A} = (Q, F, A, \mu, q_0, \{\chi\})$$

with $Q = \{q_0, \dots, q_m\} \cup \{\chi\}$, where q_0 is the initial state, χ is the only accept state and μ is a partial function from $Q \times F$ to finite subsets of $Q \times A^+$ defined by:

$$(q_0, f)\mu = \{(q_i, (f, e_i))\} \ (i = 1, \dots, m),$$
$$(q_i, f)\mu = \{(q_i, (f, e_i)), (\chi, (f, q_i)))\} \ (i = 1, \dots, m).$$

We will now prove that $L_{(f,e_r)}$ is a regular language, for $(f, e_r) \in A$. If we define

$$L^{(i)}_{(f,e_r)} = L_{(f,e_r)} \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+)\delta_A \ (i = 1, \dots, m)$$

then we can write

$$L_{(f,e_r)} = \bigcup_{i=1,\dots,m} L^{(i)}_{(f,e_r)}$$

and it suffices to prove that, for each $i \in \{1, \dots, m\}$, the language $L^{(i)}_{(f,e_r)}$ is regular. To achieve that, we will use Theorem 3.1, and we start by showing that

$$L^{(i)}_{(f,e_r)} = K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_i) : f \in F\})\delta_A$$

where $\bar{w}$ is the word in K that represents ${}^{q_i}f \in S^{|T|}$. Let

$$(f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j) \dots (f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j) \in L.$$

Then

$$\begin{aligned}
&((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j)\dots(f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j))\delta_A \in L^{(i)}_{(f,e_r)}\\
\iff& f_{\alpha_1}\dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1}\dots f_{\beta_s} \;\&\; e_iq_ie_r = e_jq_j\\
\iff& f_{\alpha_1}\dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1}\dots f_{\beta_s} \;\&\; e_iq_i = e_jq_j\\
\iff& f_{\alpha_1}\dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1}\dots f_{\beta_s} \;\&\; t_i = t_j\\
\iff& (f_{\alpha_1}\dots f_{\alpha_n}, f_{\beta_1}\dots f_{\beta_s})\delta_F \in K_{\bar{w}} \;\&\; i = j\\
\iff& ((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j)\dots(f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j))\delta_A \in\\
& K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_i) : f \in F\})\delta_A
\end{aligned}$$

We conclude, by Theorem 3.1, that $L^{(i)}_{(f,e_r)}$ is a regular language. For a generator $(f, q_r) \in A$ will we prove that $L_{(f,q_r)}$ is regular in a similar way. We can write

$$L_{(f,q_r)} = \bigcup_{i=1,\dots,m} L^{(i)}_{(f,q_r)}$$

where

$$L^{(i)}_{(f,q_r)} = L_{(f,q_r)} \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+)\delta_A \;(i = 1, \dots, m).$$

We let $i \in \{1, \dots, m\}$ arbitrary and we will prove that $L^{(i)}_{(f,q_r)}$ is a regular language. Let j the unique element in $\{1, \dots, m\}$ such that $e_iq_iq_r = e_jq_j$ and let $\bar{w}$ be the word in K that represents ${}^{q_i}f \in S^{|T|}$. Let

$$(f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k)\dots(f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k) \in L.$$

Then

$$\begin{aligned}
&((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k)\dots(f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k))\delta_A \in L^{(i)}_{(f,q_r)}\\
\iff& f_{\alpha_1}\dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1}\dots f_{\beta_s} \;\&\; e_iq_iq_r = e_kq_k\\
\iff& (f_{\alpha_1}\dots f_{\alpha_n}, f_{\beta_1}\dots f_{\beta_s})\delta_F \in K_{\bar{w}} \;\&\; e_iq_iq_r = e_kq_k \;\&\; k = j\\
\iff& ((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k)\dots(f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k))\delta_A \in\\
& K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_j) : f \in F\})\delta_A
\end{aligned}$$

We can use again Theorem 3.1 to conclude that, for each i, the language

$$L^{(i)}_{(f,q_r)} = K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_j) : f \in F\})\delta_A$$

is regular. ■

In the case where the semigroups S and T are monoids, necessary and sufficient conditions for the wreath product $S \operatorname{wr} T$ to be finitely generated, given in [38], are the following:

Proposition 5.9 *Let S and T be monoids, and let G be the group of units of T. Then the wreath product $S \operatorname{wr} T$ is finitely generated if and only if both S and T are finitely generated, and either S is trivial, or $T = VG$ for some finite subset V of T.*

By using this result, our theorem has the following consequence:

Corollary 5.10 *Let S be an automatic monoid and T be a finite monoid. Then the wreath product $S \operatorname{wr} T$ is automatic.*

Proof. We assume that S is not trivial. We can apply Proposition 5.9, with $V = T$, and so $S \operatorname{wr} T$ is finitely generated. Moreover, the three conditions in Proposition 5.7, for the case where the diagonal S-act is not finitely generated, hold trivially since S and T are monoids. The proof of our theorem is based on these conditions and therefore the wreath product $S \operatorname{wr} T$ is automatic. ■

It is still an open question whether or not the wreath product $S \operatorname{wr} T$ is always automatic when it is finitely generated. Of course, because of the above result, it only remains to consider the case where the diagonal S-act is finitely generated. In [38] and [50] we can find some examples of wreath products with finitely generated diagonal S-act which, as the authors observe, is in some way the less common case. Another interesting problem is that of the automaticity of the wreath product in the case where the semigroup T is also infinite. A natural starting point here is to use Proposition 5.9 and investigate the case where S and T are monoids.

Chapter 6

Subsemigroups

We will consider the automaticity of subsemigroups of free semigroups and free products of semigroups. In Section 1 we will prove that a subsemigroup of a free semigroup is automatic (a known result) and in Section 2 we will consider subsemigroups of free products, proving in particular that subsemigroups of free products, with all generators having length greater than one in the free product, are automatic.

1 Subsemigroups of a free semigroup

The following result was proved in [11] but, since we will use the idea of that proof in the following section, we include it here with some additional detail.

Theorem 6.1 *If F is a free semigroup and S is a finitely generated subsemigroup of F, then S is automatic.*

Proof. Since we are going to consider finitely generated subsemigroups of F we may assume, without loss of generality, that F is a free semigroup on a finite set X. We will show that S^1 is automatic, which is sufficient by Proposition 1.14. Suppose that S is generated by $\{\alpha_1, \ldots, \alpha_n\}$ where each α_i is an element of F. Let m_i be the length of α_i when considered as a word in X^+. Let $A = \{a_1, \ldots, a_n, 1\}$

be an alphabet, and let

$$L = \{a_1 1^{m_1 - 1}, a_2 1^{m_2 - 1}, \ldots, a_n 1^{m_n - 1}\}^+ \cup \{1\},$$

be a regular language on A^+. We have a natural homomorphism

$$\rho : (A \cup \{\$\})^+ \to X^*;\ 1, \$ \mapsto \epsilon;\ a_i \mapsto \alpha_i\ (i = 1, \ldots, n)$$

satisfying the property

$$|w\rho| = |w| \text{ for any } w \in L \backslash \{1\}.$$

We will prove that (A, L) is an automatic structure for S^1. Let $K = L\backslash\{1\}$, so that

$$\begin{aligned} L_= &= L_1 = K_= \cup \{(1,1)\}, \\ L_{a_i} &= K_{a_i} \cup \{(1, w)\delta_A : w \in L, w\rho \equiv \alpha_i\} \end{aligned}$$

for each i. The set $\{(1,1)\}$ is finite and the sets $\{(1, w)\delta_A : w \in L, w\rho \equiv \alpha_i\}$ are finite as well since there are finitely many words $w \in L$ with length $|\alpha_i|$ and $w\rho \equiv \alpha_i$ implies $|w| = |w\rho| = |\alpha_i|$. Hence it is enough to show that $K_=$ is regular and that K_{a_i} is regular for each i. We will define automata that recognize these languages but before that we need to prove the following:

Claim 1 *There is a finite set $W \subseteq X^*$ such that for any word*

$$(\beta, \gamma)\delta_A \in K_= \cup (\bigcup_{i=1,\ldots,n} K_{a_i}),$$

given an arbitrary integer t, the prefixes $\beta(t)\rho, \gamma(t)\rho$ of the words $\beta\rho, \gamma\rho \in K\rho$ are such that, one of them is a prefix of the other and the remainder suffix of the longer word belongs to the finite set W. Formally,

$$(\forall t \in \mathbb{N})(\beta(t)\rho \in (\gamma(t)\rho)W \text{ or } \gamma(t)\rho \in (\beta(t)\rho)W).$$

Proof. To prove this claim we define $N = \max\{m_i : i = 1, \ldots, n\}$ and $W = \{w \in X^* : |w| \leq N\}$. We note that for any $w \in K$ we have

$$|w| = |w\rho|, \quad |w(t)\rho| \leq |w(t+1)\rho|\, (\forall t \in \mathbb{N}).$$

For $(\beta,\gamma)\delta_A \in K_=$ we have $\beta\rho \equiv \gamma\rho$ and so $|\beta| = |\gamma|$. We can write $\beta(t) \equiv b_1 \dots b_t$ with $b_1, \dots, b_t \in A$ for any $t \leq |\beta| = |\beta\rho|$. We have

$$|b_1 \dots b_t| \leq |(b_1 \dots b_t)\rho| \leq |b_1 \dots b_t| + N$$

by definition of ρ and so $t \leq |(\beta(t))\rho| \leq t + N$. Similarly $t \leq |(\gamma(t))\rho| \leq t + N$ and we can write

$$||(\beta(t))\rho| - |(\gamma(t))\rho|| \leq N \text{ for } t \leq |\beta|. \tag{6.1}$$

For $(\beta,\gamma)\delta \in K_{a_i}$ we have $(\beta\rho)\alpha_i \equiv \gamma\rho$ and clearly, equation (6.1) still holds in this case. For $|\beta| < t \leq |\gamma|$ we have $|\beta\rho| \leq |(\gamma(t))\rho| \leq |\beta\rho| + N$ since $|\gamma| = |\beta| + |\alpha_i|$. Hence, for any $(\beta,\gamma)\delta_A \in K_= \cup (\bigcup_{i=1,\dots,n} K_{a_i})$, we have

$$||(\beta(t))\rho| - |(\gamma(t))\rho|| \leq N \text{ for any positive integer } t. \tag{6.2}$$

For $(\beta,\gamma)\delta_A \in K_=$ we have $\beta\rho \equiv \gamma\rho$ and so given a positive integer t, either $(\gamma(t))\rho$ is a prefix of $(\beta(t))\rho$ or $(\beta(t))\rho$ is a prefix of $(\gamma(t))\rho$. For $(\beta,\gamma)\delta_A \in K_{a_i}$ we have $(\beta\rho)\alpha_i \equiv \gamma\rho$ and so given t either $(\gamma(t))\rho$ is a prefix of $(\beta(t))\rho$ or $(\beta(t))\rho$ is a prefix of $(\gamma(t))\rho$ or $t > |\beta|$ and then

$$(\gamma(t))\rho \in (\beta\rho)(\mathrm{Pref}(\alpha_i)) \subseteq (\beta\rho)W = ((\beta(t))\rho)W.$$

In any case, using equation (6.2), we obtain $(\beta(t))\rho \in (\gamma(t)\rho)W$ or $(\gamma(t))\rho \in (\beta(t)\rho)W$ for any positive integer t and the claim is proved. ■

In what follows we fix a set W in the conditions of Claim 1. We now construct an automaton $\mathcal{M}$ such that $K_= \subseteq \mathcal{L}(\mathcal{M})$ and automata $\mathcal{M}_i$ such that $K_{a_i} \subseteq \mathcal{L}(\mathcal{M}_i)$. Let

$$\mathcal{M} = (Q, (A \cup \{\$\}) \times (A \cup \{\$\}), (\epsilon,\epsilon), \mu, (\epsilon,\epsilon))$$

where $Q = (W \times \{\epsilon\}) \cup (\{\epsilon\} \times W)$ and the transition μ is defined by

$$(\alpha,\beta) \xrightarrow{(x,y)}_\mu (\epsilon,\gamma) \text{ if } \alpha(x\rho)\gamma \equiv \beta(y\rho) \text{ and } \gamma \in W$$
$$(\alpha,\beta) \xrightarrow{(x,y)}_\mu (\gamma,\epsilon) \text{ if } \alpha(x\rho) \equiv \beta(y\rho)\gamma \text{ and } \gamma \in W.$$

With the same notation we let:

$$M_i = (Q, (A \cup \{\$\}) \times (A \cup \{\$\}), (\epsilon, \epsilon), \mu, (\epsilon, \alpha_i)).$$

To prove the inclusions above we will use the following:

Claim 2 *For any $u, v \in (A \cup \{\$\})^+$, with $|u| = |v|$, we have*

$$(\beta, \epsilon) \xrightarrow{(u,v)}_{\mu} (\epsilon, \gamma) \implies \beta(u\rho)\gamma \equiv v\rho, \tag{6.3}$$

$$(\beta, \epsilon) \xrightarrow{(u,v)}_{\mu} (\gamma, \epsilon) \implies \beta(u\rho) \equiv (v\rho)\gamma, \tag{6.4}$$

$$(\epsilon, \beta) \xrightarrow{(u,v)}_{\mu} (\gamma, \epsilon) \implies u\rho \equiv \beta(v\rho)\gamma, \tag{6.5}$$

$$(\epsilon, \beta) \xrightarrow{(u,v)}_{\mu} (\epsilon, \gamma) \implies (u\rho)\gamma \equiv \beta(v\rho). \tag{6.6}$$

Proof. We will prove this claim by induction in $m = |u| = |v|$. For $m = 1$ the implications follow from the definition of μ. Suppose the claim holds for words u, v of length m. Let u, v be words of length $m + 1$. Then we can write $u \equiv u'x$ and $v \equiv v'y$ where u' and v' are words of length m and $x, y \in A \cup \{\$\}$. To prove implication (6.3) suppose that $(\beta, \epsilon) \xrightarrow{(u,v)}_{\mu} (\epsilon, \gamma)$. We have either $(\beta, \epsilon) \xrightarrow{(u',v')}_{\mu} (\epsilon, \eta)$ or $(\beta, \epsilon) \xrightarrow{(u',v')}_{\mu} (\eta, \epsilon)$ for some word η. We first consider the case where $(\beta, \epsilon) \xrightarrow{(u',v')}_{\mu} (\epsilon, \eta)$. In this case $\beta(u'\rho)\eta \equiv v'\rho$ by induction hypothesis and, since $(\epsilon, \eta) \xrightarrow{(x,y)}_{\mu} (\epsilon, \gamma)$, we have $(x\rho)\gamma \equiv \eta(y\rho)$ by definition of μ. So we have

$$\beta(u\rho)\gamma \equiv \beta(u'\rho)(x\rho)\gamma \equiv \beta(u'\rho)\eta(y\rho) \equiv (v'\rho)(y\rho) \equiv v\rho.$$

In the case where $(\beta, \epsilon) \xrightarrow{(u',v')}_{\mu} (\eta, \epsilon)$ we have $\beta(u'\rho) \equiv (v'\rho)\eta$ by induction hypothesis and, since $(\eta, \epsilon) \xrightarrow{(x,y)}_{\mu} (\epsilon, \gamma)$, we have $\eta(x\rho)\gamma \equiv y\rho$, by definition of μ. Hence,

$$\beta(u\rho)\gamma \equiv \beta(u'\rho)(x\rho)\gamma \equiv (v'\rho)\eta(x\rho)\gamma \equiv (v'\rho)(y\rho) \equiv v\rho.$$

We conclude the proof of the claim by observing that implications (6.4), (6.5) and (6.6) for words u, v of length $m + 1$ can be verified similarly, using the induction hypothesis. ■

To prove that $K_= \subseteq \mathcal{L}(\mathcal{M})$ let $(u,v)\delta_A$ be an arbitrary element of $K_=$. We have $u\rho \equiv v\rho$ and so $|u| = |u\rho| = |v\rho| = |v|$. Hence we can write $u \equiv u_1 \dots u_k$ and $v \equiv v_1 \dots v_k$ with $u_1, \dots, u_k, v_1 \dots, v_k \in A$. By Claims 1 and 2 and by definition of μ there is a path

$$(\epsilon,\epsilon) \xrightarrow{(u_1,v_1)}_\mu (\gamma_1,\gamma_1') \xrightarrow{(u_2,v_2)}_\mu (\gamma_2,\gamma_2') \to \dots \xrightarrow{(u_k,v_k)}_\mu (\gamma_k,\gamma_k')$$

with $\gamma_i \equiv \epsilon$ and $\gamma_i' \in W$ or $\gamma_i \in W$ and $\gamma_i' \equiv \epsilon$ for $i = 1, \dots, k$. We now show, using Claim 2, that $\gamma_k \equiv \gamma_k' \equiv \epsilon$. If $\gamma_k \equiv \epsilon$ then, by implications (6.3) and (6.6), we have $(u\rho)\gamma_k' \equiv v\rho$ and so $\gamma_k' \equiv \epsilon$. If $\gamma_k' \equiv \epsilon$ then, by implications (6.4) and (6.5), we have $u\rho \equiv (v\rho)\gamma_k$ and so $\gamma_k \equiv \epsilon$. In any case $(\gamma_k, \gamma_k') = (\epsilon,\epsilon)$ which means that the path is successful and so $(u,v)\delta_A \in \mathcal{L}(\mathcal{M})$.

We now prove that $K_{a_i} \subseteq \mathcal{L}(\mathcal{M}_i)$. Let $(u,v)\delta_A \in K_{a_i}$ arbitrary with $u \equiv u_1 \dots u_k, v \equiv v_1 \dots v_r$ and $u_1, \dots, u_k, v_1, \dots, v_r \in A$, so that we have $(u\rho)\alpha_i \equiv v\rho$. By Claims 1 and 2 and by definition of μ, there is a path

$$\begin{aligned}&(\epsilon,\epsilon) \xrightarrow{(u_1,v_1)}_\mu (\gamma_1,\gamma_1') \xrightarrow{(u_2,v_2)}_\mu (\gamma_2,\gamma_2') \to \dots \\ &\xrightarrow{(u_k,v_k)}_\mu (\gamma_k,\gamma_k') \xrightarrow{(\$,v_{k+1})}_\mu (\gamma_{k+1},\gamma_{k+1}') \to \dots \xrightarrow{(\$,v_r)}_\mu (\gamma_r,\gamma_r')\end{aligned}$$

with $\gamma_j \equiv \epsilon$ and $\gamma_j' \in W$ or $\gamma_j \in W$ and $\gamma_j' \equiv \epsilon$ for each $j = 1, \dots, r$. Using Claim 2 we will prove that $\gamma_r \equiv \epsilon$ and $\gamma_r' \equiv \alpha_i$. In fact, if it was $\gamma_r' \equiv \epsilon$ then implications (6.4) and (6.5) would give $u\rho \equiv (v\rho)\gamma_r$ what is not possible since $(u\rho)\alpha_i \equiv v\rho$. So we have $\gamma_r \equiv \epsilon$ and, using implications (6.3) and (6.6), we obtain $(u\rho)\gamma_r' \equiv v\rho$ which implies that $\gamma_r' \equiv \alpha_i$. Hence the path is successful in $\mathcal{M}_i$ and so $(u,v)\delta_A \in \mathcal{L}(\mathcal{M}_i)$.

Finally, we will show that

$$\mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A \subseteq K_=, \quad \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A \subseteq K_=.$$

Let $(\beta,\gamma)\delta$ be an arbitrary element of $\mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A$. Then we have a successful path

$$(\epsilon,\epsilon) \xrightarrow{(\beta,\gamma)} (\epsilon,\epsilon)$$

in $\mathcal{M}$ and, by Claim 2, we have $\beta\rho \equiv \gamma\rho$ which means that $(\beta,\gamma)\delta \in K_=$. Analogously, given an arbitrary element $(\beta,\gamma)\delta \in \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A$, there is a successful path

$$(\epsilon,\epsilon) \xrightarrow{(\beta,\gamma)} (\epsilon,\alpha_i)$$

in $\mathcal{M}_i$ and, by Claim 2, we have $(\beta\rho)\alpha_i \equiv \gamma\rho$ which means that $(\beta,\gamma)\delta \in K_{a_i}$.

To conclude the proof of the theorem we observe that

$$K_= = \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A, \quad K_{a_i} = \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A$$

and so $K_=$ and K_{a_i} are regular languages. ∎

2 Subsemigroups of free products

If $S_1,\ldots,S_n$ are semigroups with presentations $\langle A_1 \mid R_1\rangle,\ldots,\langle A_n \mid R_n\rangle$ then their free product, $S = S_1 * \ldots * S_n$, is the semigroup defined by the presentation $\langle A_1 \cup \ldots \cup A_n \mid R_1 \cup \ldots \cup R_n\rangle$ (for an introduction and notation about semigroup presentations see Section 3 in Chapter 8). Any element $s \in S$ can be identified with a sequence

$$s_1 \ldots s_m \ (m > 1)$$

of elements of $\bigcup_{k=1}^n S_k$ such that,

$$s_i \in S_k \implies s_{i+1} \notin S_k \ (i = 1,\ldots,m-1; k = 1,\ldots,n);$$

such a sequence we call a *reduced sequence (of elements of* $\bigcup_{k=1}^n S_k$*)*. Given two elements $s = s_1 \ldots s_m, s' = s'_1 \ldots s'_p \in S$, their product ss' is the following: if the elements s_m, s'_1 do not belong to a common factor S_k then the product ss' is the concatenation of sequences and in this case we say simply that *the product ss' is the concatenation*; otherwise we have $s_m, s'_1 \in S_k$ for some k and the product ss' is the reduced sequence $s_1 \ldots s_{m-1}s'_0s'_2 \ldots s'_p$ where $s'_0 = s_ms'_1$ in S_k.

Our main result is the following:

Theorem 6.2 *Let S be a free product of finitely many semigroups. Let H be a subsemigroup of S generated by a finite set X such that no element of X belongs to a non free factor of S. Then H is automatic.*

This result has the following equivalent formulation:

Theorem 6.3 *Let S be a free product of finitely many semigroups*

$$S = S_1 * \ldots * S_n * T_1 * \ldots * T_m$$

where $T_1, \ldots, T_m$ are free semigroups on finite sets $Y_1, \ldots, Y_m$ respectively. Let $H =< t_1, \ldots, t_l >$ be a subsemigroup of S where

$$t_1, \ldots, t_l \in S \backslash (S_1 \cup \ldots \cup S_n).$$

Then H is an automatic semigroup.

Proof. Let us denote T_i by S_{n+i} for $i = 1, \ldots, m$ and let $Y = Y_1 \cup \ldots \cup Y_m$. Each generator t_i such that $t_i \notin T_1 \cup \ldots \cup T_m$ can be written as a reduced sequence of elements of $\bigcup_{k=1}^{m+n} S_k$:

$$t_i = s_{i,1} s_{i,2} \ldots s_{i,p(i)},$$

with $p(i) \geq 2$. We observe that, in these sequences, we may have an element from $\bigcup_{k=1}^{n} S_k$ appearing several times, i.e., we may have $s_{i,j} = s_{k,l}$ with $i \neq k$ or $j \neq l$, and to deal with that we are going to define alphabets that are essentially in bijection with all different elements from $\bigcup_{k=1}^{n} S_k$, that either appear in these sequences or by multiplying them. For each $k \in \{1, \ldots, n\}$ we define

$$A_k = \{ {}^k a_1, \ldots, {}^k a_{r_k} \}$$

to be an alphabet in bijection with the following finite subset of S_k:

$$F_k = \bigcup_{i=1}^{l} (\{s_{i,j} \in S_k : j = 1, \ldots, p(i)\}) \cup \{s_{i,p(i)} s_{j,1} \in S_k : i, j \in \{1, \ldots, l\}\},$$

and let $f_k : A_k \to F_k$ be that bijection (we assume that the alphabets are disjoint). We observe that, although the semigroups $S_1, \ldots, S_n$ are arbitrary and may be infinite, an arbitrary element $h \in H$ can be written as a product of the generators $t_1, \ldots, t_l$ and the essential idea for our proof is that the only elements from $\bigcup_{k=1}^{n} S_k$ that may appear when we write an element of H as a reduced sequence of elements of $\bigcup_{k=1}^{n+m} S_k$, are the elements of the finite set $\bigcup_{k=1}^{n} F_k$. Moreover, for each k, an element from F_k can now be represented by an element of the alphabet A_k.

We now define the alphabet

$$A = A_1 \cup \ldots \cup A_n \cup Y$$

and the language $L \subseteq A^+$ by

$$\begin{aligned} L = \{y_1 \ldots y_k : & y_i \in A_1 \cup \ldots \cup A_n \cup Y_1^+ \cup \ldots \cup Y_m^+, \\ & y_i \in A_j \implies y_{i+1} \notin A_j \ (i = 1, \ldots, k-1; j = 1, \ldots, n), \\ & y_i \in Y_j^+ \implies y_{i+1} \notin Y_j^+ \ (i = 1, \ldots, k-1; j = 1, \ldots, m)\}. \end{aligned}$$

The bijections f_k induce a homomorphism

$$f : A^+ \to S$$

and we will now show that any element in H has a unique representative in L. Given an element $h \in H$ it can be written as a product of the generators $t_1, \ldots, t_l$. Hence, when we write h as a reduced sequence of elements of $\bigcup_{j=1}^{n+m} S_j$: $h = u_1 \ldots u_r$, each element u_i is either some $s_{k,l}$ or a product $s_{k,p(k)} s_{l,1}$ or belongs to a free semigroup T_j. We note that here we need the fact that no generator may belong to a semigroup S_j $(j = 1, \ldots, n)$, because otherwise there could be an element $u_i \in S_j$, for some $j \in \{1, \ldots, n\}$, that could only be obtained as a product of more that two elements $s_{k,l} \in S_j$. It follows from the definition of the alphabets $A_1, \ldots, A_n$ and from the definition of L that there is a unique word $w \in L$ such that $wf = h$.

Let $\gamma_1, \ldots, \gamma_l$ be the unique words in L such that $\gamma_i f = t_i$, $i = 1, \ldots, l$. Let $X = \{x_1, \ldots, x_l, 1\}$ be a new alphabet and ρ be the homomorphism defined by

$$\rho : (X \cup \{\$\})^+ \to A^*; x_i \mapsto \gamma_i; 1, \$ \mapsto \epsilon.$$

$$\begin{tikzcd}
X^+ \arrow[r, "\rho"] & (X^+\rho)\backslash\{\varepsilon\} \arrow[r, "f"] \arrow[d, "\lambda"] & H \arrow[dl, leftrightarrow, "f"] \\
& (\overline{X^+\rho})\backslash\{\varepsilon\} &
\end{tikzcd}$$

Figure 6.1: Diagram with ρ, f, and λ.

We define the partial function

$$\lambda : A^* \to L \cup \{\epsilon\}; \epsilon \mapsto \epsilon,$$
$$w \mapsto \overline{w} \in L \text{ if there is } \overline{w} \in L \text{ such that } \overline{w} = w \text{ in } S,$$

which maps each word in A^+ to the corresponding unique "reduced word" in L if such word exists. The domain of this partial function is not A^* because there may for example exist $a, b \in A_k$ for some k, such that $(af)(bf) \notin F_k$ and in this case there is no word $w \in L$ such that $w = ab$ in S. Nevertheless, since we have

$$X^+\rho\backslash\{\epsilon\} = \{\gamma_{\alpha_1} \ldots \gamma_{\alpha_k} : k \in \mathbb{N}; \alpha_1, \ldots, \alpha_k \in \{1, \ldots, l\}\},$$

the partial function λ is defined on $X^+\rho$, and more generally, it is easy to see that it is also defined on

$$\mathrm{Subw}((X^+\rho \cup \overline{X^+\rho})^+).$$

We will now show that the set $\overline{X^+\rho}\backslash\{\epsilon\} \subseteq L \subseteq A^+$ is in bijection with H. Given an arbitrary $h \in H$ we have $h = t_{\alpha_1} \ldots t_{\alpha_k}$ if and only if $h = \overline{(x_{\alpha_1} \ldots x_{\alpha_k})\rho}$, and we have already seen that there is a unique word in L representing h. We can now identify the subsemigroup H with the set $\overline{X^+\rho}\backslash\{\epsilon\}$ which is a semigroup, defining the product of two words $w_1, w_2 \in \overline{X^+\rho}\backslash\{\epsilon\}$, representing two elements $s_1, s_2 \in H$, to be the word $\overline{w_1w_2} \in \overline{X^+\rho}\backslash\{\epsilon\}$, which represents the element $s_1s_2 \in H$. This semigroup is generated by the words $\gamma_1, \ldots, \gamma_l$. We observe that this product may be simply the concatenation or not, depending on the words w_1, w_2, but if it is not the concatenation we have $|\overline{w_1 w_2}| = |w_1 w_2| - 1$. Figure 6.1 illustrates the use of our functions by showing a diagram with the relevant subsets of their domains and ranges. The proof will now follow the lines of the proof of Theorem 6.1, keeping in mind that a product of the generators $\gamma_1, \ldots, \gamma_l$ is not precisely the concatenation but it is not far from it.

Let us consider the language $K \subseteq X^+$ defined by

$$\begin{aligned} K = \{\, & x_{\alpha_1} 1^{|\gamma_{\alpha_1}|-1} x_{\alpha_2} 1^{r(\alpha_1,\alpha_2)} x_{\alpha_3} \ldots 1^{r(\alpha_{t-2},\alpha_{t-1})} x_{\alpha_t} 1^{r(\alpha_{t-1},\alpha_t)} : \\ & t \geq 1, \alpha_i \in \{1, \ldots, l\}, i = 1, \ldots, t\} \end{aligned}$$

where

$$r(i,j) = \begin{cases} |\gamma_j| - 1 & \text{if } |\overline{\gamma_i \gamma_j}| = |\gamma_i \gamma_j| \\ |\gamma_j| - 2 & \text{if } |\overline{\gamma_i \gamma_j}| = |\gamma_i \gamma_j| - 1. \end{cases}$$

We observe that $|w| = |\overline{w\rho}|$ for any word $w \in K$. We can easily define a finite deterministic automaton that recognizes the language K and so K is a regular language. We will show that (X, K^1) is an automatic structure for H^1, where K^1 is the regular language $K \cup \{1\} \subseteq X^+$ and H^1 is the monoid obtained by adjoining an identity 1_H to H. We have

$$\begin{aligned} K^1_= &= K^1_1 = K_= \cup \{(1,1)\}, \\ K^1_{x_i} &= K_{x_i} \cup \{(1,w)\delta_X : w \in K, \overline{w\rho} \equiv \gamma_i\}. \end{aligned}$$

Since $\{(1,1)\}$ and $\{(1,w)\delta_A : w \in K, \overline{w\rho} \equiv \gamma_i\}$ are finite sets we just have to prove that $K_=$ and K_{x_i}, for each i, are regular languages.

Denoting by ${}^{i}a$, ${}^{i}b, \ldots$ generic elements in A_i, for $w_1, w_2 \in A^*$ we write $w_1 \bowtie w_2$ if one of the following situations occur:

$$\begin{aligned} &(w_1 \in \mathrm{Pref}(w_2) \ \&\ w_1 \in A^*Y) \text{ or} \\ &(w_2 \in \mathrm{Pref}(w_1) \ \&\ w_2 \in A^*Y) \text{ or} \\ &(w_1 \equiv w\,{}^{i}a \text{ and } w_2 \equiv w\,{}^{i}b w') \text{ for some } i \text{ or} \\ &(w_1 \equiv w\,{}^{i}a w' \text{ and } w_2 \equiv w\,{}^{i}b) \text{ for some } i. \end{aligned}$$

For $w_1 \bowtie w_2$ we define

$$\mathrm{Rem}(w_1, w_2) = \begin{cases} (\epsilon, w) & (w_2 \equiv w_1 w, w_1 \in A^*Y) \\ (w, \epsilon) & (w_1 \equiv w_2 w, w_2 \in A^*Y) \\ ({}^{i}a, {}^{i}b w') & (w_1 \equiv w\,{}^{i}a, w_2 \equiv w\,{}^{i}b w', i \in \{1, \ldots, k\}) \\ ({}^{i}a w', {}^{i}b) & (w_1 \equiv w\,{}^{i}a w', w_2 \equiv w\,{}^{i}b, i \in \{1, \ldots, k\}). \end{cases}$$

Intuitively, for two words $w_1, w_2 \in L$ we have $w_1 \bowtie w_2$ if one of the words is almost a prefix of the other, in the sense that it may be possible to multiply the shorter word by a word from L in order to obtain the longer word. The function Rem (which stands for remainder) gives us the remainders of the two words: the two suffixes not belonging to the common prefix.

The following result tells us that there is a finite set where we can store the remainders, if we are dealing with words from our languages.

Claim 1 *There is a finite set $W \subseteq A^*$ such that $(w_1, w_2)\delta_X \in K_= \cup (\bigcup_{i=1}^{l} K_{x_i})$ implies that, for all $t \in \mathbb{N}$, we have $\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}$ and Rem$(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) \in W \times W$.*

Proof. We take

$$N = \max\{|\gamma_i| : i = 1, \ldots, l\}$$

and we will prove that the result holds with

$$W = \{w \in \mathrm{Suff}(\overline{X^+\rho}) : |w| \leq N + 1\}.$$

Let $w_1, w_2 \in K$ and $t \leq |w_1|, |w_2|$. By the definition of K, we can write $t \leq |\overline{w_j(t)\rho}| \leq t + N$ $(j = 1, 2)$ and so we have

$$||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N.$$

If $(w_1, w_2)\delta_X \in K_=$ then $\overline{w_1\rho} \equiv \overline{w_2\rho}$ and therefore

$$\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}.$$

Let $\mathrm{Rem}(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) = (\eta_1, \eta_2)$ where $\eta_1, \eta_2 \in A^*$. Since $w_1, w_2 \in K \subseteq X^+$ we have $w_1(t)\rho, w_2(t)\rho \in X^+\rho$ and so $\overline{w_1(t)\rho}, \overline{w_2(t)\rho} \in \overline{X^+\rho}$. Therefore, by definition of Rem, $\eta_1, \eta_2 \in \mathrm{Suff}(\overline{X^+\rho})$. Since $||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N$, again by definition of Rem, we have $|\eta_1|, |\eta_2| \leq N + 1$ and we conclude that $(\eta_1, \eta_2) \in W \times W$.

Suppose now that $(w_1, w_2)\delta_A \in K_{x_i}$. Then it is $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$ and so

$$\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}$$

for any $t \in \mathbb{N}$. Since we have $|\overline{w_1\rho}| = |w_1|$ and $|\overline{w_2\rho}| = |w_2|$ it may be $|w_2| = |w_1| + |\gamma_i|$ or $|w_2| = |w_1| + |\gamma_i| - 1$ according to whether $\overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ or not. For $t \leq |w_1|$ we have as above $t \leq |\overline{w_j(t)\rho}| \leq t + N$ $(j = 1, 2)$ and so $||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N$. For $|w_1| < t \leq |w_2|$ we have

$$|\overline{w_1(t)\rho}| = |\overline{w_1\rho}| = |w_1|, \quad t \leq |\overline{w_2(t)\rho}| \leq |w_1| + |\gamma_i| \leq |w_1| + N$$

and so $||\overline{w_2(t)\rho}| - |\overline{w_1(t)\rho}|| \leq N$. Again $\overline{w_1(t)\rho}, \overline{w_2(t)\rho} \in \overline{X^+\rho}$, since $w_1, w_2 \in K \subseteq X^+$, and we have $\mathrm{Rem}(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) \in W \times W$. ■

From now on we assume that a set W satisfying the conditions of Claim 1 is fixed and we will use this set to construct automata that allow us to prove the regularity of our languages. We will prove that there is an automaton $\mathcal{M}$ such that $K_= = \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_X$ and automata $\mathcal{M}_i$ such that $K_{x_i} = \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_X$. Let

$$\mathcal{M} = (Q, (X \cup \{\$\}) \times (X \cup \{\$\}), (\epsilon, \epsilon), \mu, T)$$

where $Q = W \times W$, $T = \{(a, a) : a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}\}$ and μ is defined by

$$(\alpha, \beta) \xrightarrow{(x,y)}_\mu \mathrm{Rem}(\overline{\alpha(x\rho)}, \overline{\beta(y\rho)}) \text{ if } \overline{\alpha(x\rho)} \bowtie \overline{\beta(y\rho)} \text{ and } \mathrm{Rem}(\overline{\alpha(x\rho)}, \overline{\beta(y\rho)}) \in W \times W$$

for $\alpha, \beta \in W$ and $x, y \in X \cup \{\$\}$. For $i \in \{1, \ldots, l\}$ we define

$$M_i = (Q, (X \cup \{\$\}) \times (X \cup \{\$\}), (\epsilon, \epsilon), \mu, T_i)$$

where T_i is defined as follows. If $\gamma_i \equiv {}^j a \gamma_i'$ for some word $\gamma_i' \in A^+$ then we define

$$T_i = \{({}^j b,\, {}^j c\gamma_i') : {}^j b\, {}^j a = {}^j c \text{ in } S_j\} \cup \{({}^k b,\, {}^k b\gamma_i) : k \neq j\} \cup \{(\epsilon, \gamma_i)\}.$$

If $\gamma_i \in YA^*$ then we define

$$T_i = \{({}^k a,\, {}^k a\gamma_i)\} \cup \{(\epsilon, \gamma_i)\}.$$

The following result, relates a transition in the automata with the remainders of the pair of words involved in the transition.

Claim 2 *For any $w_1, w_2 \in (X \cup \{\$\})^+$, with $|w_1| = |w_2|$, we have*

$$(\alpha, \beta) \xrightarrow{(w_1,w_2)}_{\mu} (\theta_1, \theta_2) \implies \mathrm{Rem}(\overline{\alpha(w_1\rho)}, \overline{\beta(w_2\rho)}) = (\theta_1, \theta_2). \tag{6.7}$$

Proof. We will prove this claim by induction on $m = |w_1| = |w_2|$. For $m = 1$ the implication follows from the definition of μ. Suppose the claim holds for words of length m and let w_1, w_2 be words of length $m+1$ with $(\alpha, \beta) \xrightarrow{(w_1,w_2)}_{\mu} (\theta_1, \theta_2)$. Then we can write $w_1 \equiv w_1'x$ and $w_2 \equiv w_2'y$ where w_1' and w_2' are words of length m. We have $(\alpha, \beta) \xrightarrow{(w_1',w_2')}_{\mu} (\eta_1, \eta_2)$ and $(\eta_1, \eta_2) \xrightarrow{(x,y)}_{\mu} (\theta_1, \theta_2)$ for some words $\eta_1, \eta_2 \in W$. By the induction hypothesis and by definition of μ it is $(\eta_1, \eta_2) = \mathrm{Rem}(\overline{\alpha(w_1'\rho)}, \overline{\beta(w_2'\rho)})$ and $(\theta_1, \theta_2) = \mathrm{Rem}(\overline{\eta_1(x\rho)}, \overline{\eta_2(y\rho)})$. We can then write

$$\overline{\alpha(w_1'\rho)} \equiv w''\eta_1, \qquad \overline{\eta_1(x\rho)} \equiv w'\theta_1,$$
$$\overline{\beta(w_2'\rho)} \equiv w''\eta_2, \qquad \overline{\eta_2(y\rho)} \equiv w'\theta_2,$$

for some words $w', w'' \in A^*$.

We will now show that

$$\overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1.$$

The equation holds trivially for $\theta_1 \equiv \epsilon$. If $w' \neq \epsilon$ the equation holds as well, since $w'\theta_1 \in L$. We will now consider the case where $\theta_1 \neq \epsilon$ and $w' \equiv \epsilon$. If $w'' \in A^*Y \cup \{\epsilon\}$ then the equation clearly holds. Otherwise we have $w'' \equiv w'''\, {}^i a$ for some i, it must be $\eta_1 \neq \epsilon$ by the definition of Rem and, since $w''\eta_1 \in L$, we have either $\eta_1 \in YA^*$ or $\eta_1 \equiv {}^j b\eta_1'$ with $i \neq j$. Since $\overline{\eta_1(x\rho)} \equiv \theta_1$ we have $\theta_1 \in YA^*$

or $\theta_1 \in A_j A^*$ with $i \neq j$ as well and, in either case, $\overline{w''\theta_1} \equiv \overline{w''}\theta_1$ yielding again $\overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1$. A similar argument shows that $\overline{w''w'\theta_2} \equiv \overline{w''w'}\theta_2$.

Therefore we have

$$\begin{aligned}&\overline{\alpha(w_1\rho)} \equiv \overline{\alpha(w_1'\rho)(x\rho)} \equiv \overline{\overline{\alpha(w_1'\rho)}(x\rho)} \equiv \\ &\overline{(w''\eta_1)(x\rho)} \equiv \overline{w''\overline{\eta_1(x\rho)}} \equiv \overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1\end{aligned}$$

and

$$\begin{aligned}&\overline{\beta(w_2\rho)} \equiv \overline{\beta(w_2'\rho)(y\rho)} \equiv \overline{\overline{\beta(w_2'\rho)}(y\rho)} \equiv \\ &\overline{(w''\eta_2)(y\rho)} \equiv \overline{w''\overline{\eta_2(y\rho)}} \equiv \overline{w''w'\theta_2} \equiv \overline{w''w'}\theta_2.\end{aligned}$$

Hence $\mathrm{Rem}(\overline{\alpha(w_1\rho)}, \overline{\beta(w_2\rho)}) = (\theta_1, \theta_2)$ which concludes the proof of the claim. ■

We will now use the two claims to prove that

$$\begin{aligned}K_= &= \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A, \\ K_{x_i} &= \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A \ (i = 1, \ldots, l),\end{aligned}$$

by showing each of the four inclusions separately.

To prove that $K_= \subseteq \mathcal{L}(\mathcal{M})$ let $(w_1, w_2)\delta_X \in K_=$ arbitrary. We have $\overline{w_1\rho} \equiv \overline{w_2\rho}$, $|w_1| = |\overline{w_1\rho}| = |\overline{w_2\rho}| = |w_2|$ and we can write $w_1 \equiv y_1 \ldots y_k$ and $w_2 \equiv z_1 \ldots z_k$ with $y_1, \ldots, y_k, z_1 \ldots, z_k \in X$. Using the two claims and by definition of μ we can construct a unique path labeled by (w_1, w_2),

$$(\epsilon, \epsilon) \xrightarrow{(y_1,z_1)}_\mu (\eta_1, \eta_1') \xrightarrow{(y_2,z_2)}_\mu (\eta_2, \eta_2') \xrightarrow{(y_3,z_3)}_\mu \ldots \xrightarrow{(y_k,z_k)}_\mu (\eta_k, \eta_k'),$$

with all $\eta_i, \eta_i' \in W$. By Claim 2 it must be $(\eta_k, \eta_k') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. Since $\overline{w_1\rho} \equiv \overline{w_2\rho}$, we have $(\eta_k, \eta_k') = (a, a)$ with $a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}$, which means that $(w_1, w_2)\delta_A \in \mathcal{L}(\mathcal{M})$.

To prove that $\mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_X \subseteq K_=$ let w_1, w_2 be arbitrary words in K such that $(w_1, w_2)\delta_X \in \mathcal{L}(\mathcal{M})$. We can write $w_1 \equiv y_1 \ldots y_q$ and $w_2 \equiv z_1 \ldots z_r$ where $y_1, \ldots, y_q, z_1, \ldots, z_r \in X$. So there is a path

$$(\epsilon, \epsilon) \xrightarrow{(y_1 \ldots y_k, z_1 \ldots z_k)} (a, a)$$

in $\mathcal{M}$ where $k = max\{q,r\}$, $y_{q+1} = \ldots = y_k = z_{r+1} = \ldots = z_k = \$$ and $a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}$. By Claim 2 it is $(a,a) = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$ which implies that $w_1 = w_2$ as elements of H and so $(w_1,w_2)\delta_X \in K_=$.

To prove that $K_{x_i} \subseteq \mathcal{L}(\mathcal{M}_i)$ let $(w_1,w_2)\delta_A \in K_{x_i}$ be arbitrary. We have $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$ and we write $w_1 \equiv y_1 \ldots y_k$, $w_2 \equiv z_1 \ldots z_r$ with $y_1, \ldots, y_k, z_1, \ldots, z_r \in X$. Using the previous claims and by definition of μ we can construct a unique path in $\mathcal{M}_i$ labeled by $(w_1\$^{r-k}, w_2)$,

$$\begin{aligned}&(\epsilon,\epsilon) \xrightarrow{(y_1,z_1)}_\mu (\eta_1,\eta_1') \xrightarrow{(y_2,z_2)}_\mu (\eta_2,\eta_2') \to \ldots \\ &\xrightarrow{(y_k,z_k)}_\mu (\eta_k,\eta_k') \xrightarrow{(\$,z_{k+1})}_\mu (\eta_{k+1},\eta_{k+1}') \to \ldots \xrightarrow{(\$,z_r)}_\mu (\eta_r,\eta_r'),\end{aligned}$$

with all $\eta_j, \eta_j' \in W$. By Claim 2, $(\eta_r,\eta_r') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. If $\gamma_i \in YA^*$ then, it can be $\overline{w_1\rho} \in A^*Y$ and so $(\eta_r,\eta_r') = (\epsilon,\gamma_i) \in T_i$, or $\overline{w_1\rho} \equiv w\,{}^k\!a$ and then $(\eta_r,\eta_r') = ({}^k\!a, {}^k\!a\gamma_i) \in T_i$ as well. Otherwise we have $\gamma_i \equiv {}^j\!a\gamma_i'$ and, since $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$, there are three possibilities: it may be $\overline{w_1\rho} \equiv w'\,{}^j\!b$ and $\overline{w_2\rho} \equiv w'\,{}^j\!c\gamma_i'$ with ${}^j\!b\,{}^j\!a = {}^j\!c$ in S_j, and so $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^j\!b, {}^j\!c\gamma_i') \in T_i$; it can also be $\overline{w_1\rho} \equiv w'\,{}^k\!b$ $(k \neq j)$ and $\overline{w_2\rho} \equiv w'\,{}^k\!b\gamma_i$ and then $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^k\!b, {}^k\!b\gamma_i) \in T_i$; finally it can be $\overline{w_1\rho} \in A^*Y$ and then $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = (\epsilon,\gamma_i) \in T_i$. In any case $(\eta_r,\eta_r') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) \in T_i$ and so $(w_1,w_2)\delta_X \in \mathcal{L}(\mathcal{M}_i)$.

To prove that $\mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_X \subseteq K_{x_i}$ let $w_1, w_2 \in K$ arbitrary such that $(w_1,w_2)\delta_X \in \mathcal{L}(\mathcal{M}_i)$. We can write $w_1 \equiv y_1 \ldots y_q$ and $w_2 \equiv z_1 \ldots z_r$ where $y_1, \ldots, y_q, z_1, \ldots, z_r \in X$. There is a successful path

$$(\epsilon,\epsilon) \xrightarrow{(y_1 \ldots y_k, z_1 \ldots z_k)} (\eta,\eta')$$

in $\mathcal{M}_i$ where $k = max\{q,r\}$, $y_{q+1} = \ldots = y_k = z_{r+1} = \ldots = z_k = \$$ and $(\eta,\eta') \in T_i$. By Claim 2 we have $(\eta,\eta') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. If $\gamma_i \equiv {}^j\!a\gamma_i'$ then, by definition of T_i, we have either $(\eta,\eta') = ({}^j\!b, {}^j\!c\gamma_i')$ with ${}^j\!b\,{}^j\!a = {}^j\!c$ in S_j, or $(\eta,\eta') = ({}^k\!b, {}^k\!b\gamma_i)$ with $k \neq j$, or $(\eta,\eta') = (\epsilon,\gamma_i)$. In the first case we have $\overline{w_1\rho} \equiv w\,{}^j\!b$ and $\overline{w_2\rho} \equiv w\,{}^j\!c\gamma_i'$ for some word $w \in A^*$ and so we can write $\overline{(w_1\rho)\gamma_i} \equiv \overline{w\,{}^j\!b\,{}^j\!a\gamma_i'} \equiv \overline{w\,{}^j\!c\gamma_i'} \equiv \overline{w_2\rho}$ which means that $w_1x_i = w_2$ in H. In the second case we have $\overline{w_1\rho} \equiv w\,{}^k\!b$ and $\overline{w_2\rho} \equiv w\,{}^k\!b\gamma_i$ for some word $w \in A^*$ and so we can write

$\overline{(w_1\rho)\gamma_i} \equiv \overline{w\,{}^k b\gamma_i} \equiv \overline{w_2\rho}$ and again $w_1x_i = w_2$ in H. In the third case we have $\overline{w_1\rho} \in A^*Y$ and so $\overline{w_2\rho} \equiv \overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ which implies $w_2 = w_1x_i$ in H. If we have $\gamma_i \in A^*Y$ then, by definition of T_i, it may be $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^k a, {}^k a\gamma_i)$ or $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = (\epsilon, \gamma_i)$. In the first case we have $\overline{w_1\rho} \equiv w\,{}^k a$ and so $\overline{w_2\rho} \equiv w\,{}^k a\gamma_i$ which implies that $\overline{(w_1\rho)\gamma_i} \equiv \overline{w\,{}^k a\gamma_i} \equiv w\,{}^k a\gamma_i \equiv \overline{w_2\rho}$ and therefore $w_1x_i = w_2$ in H. In the second case we have $\overline{w_1\rho} \in A^*Y$ and $\overline{w_2\rho} \equiv \overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ which implies again $w_2 = w_1x_i$ in H. So in any case $(w_1, w_2)\delta_X \in K_{x_i}$ and the inclusion is proved.

To conclude the proof of the theorem we observe that, since $K_= = \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A$ and $K_{x_i} = \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A$, $K_=$ and K_{x_i} are regular languages and so H^1 is automatic which implies that H is automatic. ■

Corollary 6.4 *If S is a free product of semigroups that are either finite or free then any finitely generated subsemigroup of S is automatic.*

Proof. Let $S = S_1 * \ldots * S_n * T_1 * \ldots * T_m$ where $S_1, \ldots, S_n$ are finite semigroups and $T_1, \ldots, T_m$ are free semigroups. Let H be an (infinite) subsemigroup of S. Suppose that H is generated by $A = \{t_1, \ldots, t_l\} \subseteq S$ and, without loss of generality, that $A \cap S_1 = \{t_1, \ldots, t_k\}$ $(0 < k < l)$. Since the semigroup $U =< t_1, \ldots, t_k >$ is a subsemigroup of S_1 it is finite. Let H' be the semigroup generated by the finite set

$$A' = \{U^1t_{k+1}U^1, U^1t_{k+2}U^1, \ldots, U^1t_lU^1\}.$$

We observe that

$$A \cap (S_1 \cup \ldots \cup S_n) \supsetneq A' \cap (S_1 \cup \ldots \cup S_n), \quad A' \cap S_1 = \emptyset$$

and $H \backslash H' = U$ is finite. If A' contains elements from S_2 we can remove them the same way obtaining a semigroup H'' generated by a set A'' that does not contain elements from $S_1 \cup S_2$ and such that $H \backslash H''$ is finite. Repeating this process for every S_i that contains generators we will obtain a semigroup V generated by a set B such that $B \cap (S_1 \cup \ldots \cup S_n) = \emptyset$ and $H \backslash V$ is finite. Since V is in the conditions of the previous theorem it is automatic. Since $H \backslash V$ is finite we can use Proposition 1.15 and conclude the H is automatic. ∎

Corollary 6.5 *Any finitely generated subsemigroup of a free product of finite semigroups is automatic.*

Proof. This is a particular case of the previous corollary, worth stating separately. ∎

We say that a semigroup is *monogenic* if it is generated by a single element and we have the following result:

Corollary 6.6 *Any finitely generated subsemigroup of a free product of monogenic semigroups is automatic.*

Proof. A monogenic semigroup is either free or finite and so we can use Corollary 6.4. ∎

Defining a semigroup to be *strongly automatic* if all its finitely generated subsemigroups are automatic we may ask the following question:

Question 6.7 Is the free product of strongly automatic semigroups always strongly automatic?

The answer to the same question for groups is "yes" because we can use the Kurosh Subgroup Theorem: If H is a subgroup of $G_1 * G_2$ then H is isomorphic to $F * H_1 * H_2$ where F is a free group, H_1 is isomorphic to a subgroup of G_1 and H_2 is isomorphic to a subgroup of G_2. For semigroups it is still an open question. As we will see, the bicyclic monoid is strongly automatic. So we may also consider the following question:

Question 6.8 Does Theorem 6.2 still hold if we allow generators to belong to factors isomorphic to the bicyclic monoid?

Chapter 7

Subsemigroups of the bicyclic monoid B

The bicyclic monoid is one of the most fundamental semigroups. It is one of the main ingredients in the Bruck–Reilly extensions (see [29]), and also the basis of several generalizations; see [1, 8, 21, 26]. In [30, Sec 3.4] references are given to a number of applications of the bicyclic monoid to topics outside semigroup theory. The bicyclic monoid is known to have several remarkable properties, one of which is that it is completely determined by its lattice of subsemigroups; see [45] and [46]. Also, in [31] the authors study properties of a specific subsemigroup of **B**. Slightly surprisingly, there seems to be little other work in literature regarding the subsemigroups of **B**.

In this chapter we give a description of all subsemigroups of **B**. We show that there are essentially five different types of subsemigroups. One of them is the degenerate case of subsets of $\{c^i b^i : i \geq 0\}$, and the remaining four split in two groups of two, linked by the obvious anti-isomorphism $\widehat{\ } : c^i b^j \mapsto c^j b^i$ of **B**. Each subsemigroup is characterized by a certain collection of parameters. We describe algorithms for obtaining these parameters from the generating set.

In Section 1 we define a series of distinguished subsets of **B**, which are then used as a kind of building blocks, and then we state our main theorem in Section

	0	1	2	3	
0	1	b	b^2	b^3	$\cdots$
1	c	cb	cb^2	cb^3	$\cdots$
2	c^2	c^2b	c^2b^2	c^2b^3	$\cdots$
3	c^3	c^3b	c^3b^2	c^3b^3	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 7.1: The bicyclic monoid

2. Section 3 contains the auxiliary results needed to prove the main theorem. In Sections 4 and 5 we respectively consider the two non-degenerate types of subsemigroups. Finally, Section 6 contains the algorithms for the computation of parameters.

1 Distinguished subsets

In this section we introduce the notation we will need. In order to define subsets of the bicyclic monoid we find it convenient to represent **B** as an infinite square grid, as shown in Figure 7.1. We start by defining the functions $\Phi, \Psi, \lambda : \mathbf{B} \to \mathbb{N}_0$ by $\Phi(c^i b^j) = i$, $\Psi(c^i b^j) = j$ and $\lambda(c^i b^j) = |j - i|$ and by introducing some basic subsets of **B**:

$D = \{c^i b^i : i \geq 0\} -$ the *diagonal* ,
$U = \{c^i b^j : j > i \geq 0\} -$ the *upper half*,
$R_p = \{c^i b^j : j \geq p, i \geq 0\} -$ the *right half plane (determined by p)*,
$L_p = \{c^i b^j : 0 \leq j < p, i \geq 0\} -$ the *left strip (determined by p)*,
$M_d = \{c^i b^j : d \mid j - i; i, j \geq 0\} -$ the λ-*multiples of* d,

for $p \geq 0$ and $d > 0$.

We now define the function $\widehat{\ }: \mathbf{B} \to \mathbf{B}$ by $c^i b^j \mapsto \widehat{c^i b^j} = c^j b^i$. Geometrically $\widehat{\ }$ is the reflection with respect to the main diagonal. So, for example, $\widehat{U}$ is the lower half. Algebraically this function is an anti-isomorphism ($\widehat{xy} = \widehat{y}\widehat{x}$), as is

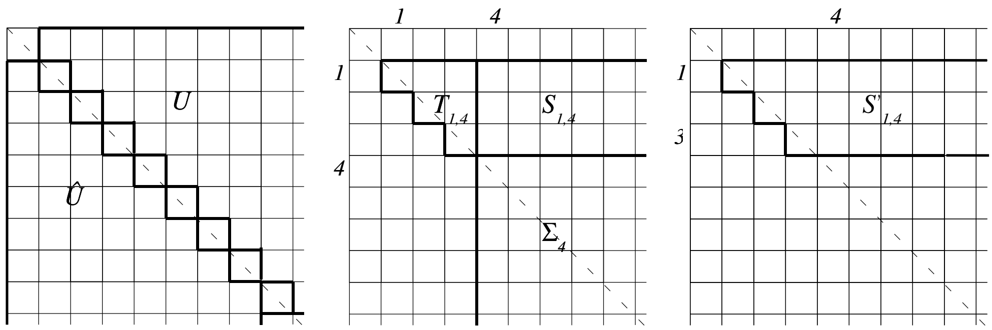

Figure 7.2: The upper and lower halves $U, \widehat{U}$, the triangle $T_{1,4}$ and the strips $S_{1,4}, S'_{1,4}$

easy to check.

By using the above basic sets and functions we now define some further subsets of **B** that will be used in our description. For $0 \le q \le p \le m$ we define the *triangle*

$$T_{q,p} = L_p \cap \widehat{R_q} \cap (U \cup D) = \{c^i b^j : q \le i \le j < p\},$$

and the *strips*

$$\begin{aligned} S_{q,p} &= R_p \cap \widehat{R_q} \cap \widehat{L_p} = \{c^i b^j : q \le i < p, j \ge p\}, \\ S'_{q,p} &= S_{q,p} \cup T_{q,p} = \{c^i b^j : q \le i < p, j \ge i\}, \\ S_{q,p,m} &= S_{q,p} \cap R_m = \{c^i b^j : q \le i < p, j \ge m\}. \end{aligned}$$

Note that for $q = p$ the above sets are empty. For $i, m \ge 0$ and $d > 0$ we define the *lines*

$$\begin{aligned} \Lambda_i &= \widehat{R_i} \cap \widehat{L_{i+1}} = \{c^i b^j : j \ge 0\}, \\ \Lambda_{i,m,d} &= \Lambda_i \cap R_m \cap M_d = \{c^i b^j : d \mid j - i, j \ge m\} \end{aligned}$$

and in general for $I \subseteq \{0, \dots, m-1\}$,

$$\Lambda_{I,m,d} = \textstyle\bigcup_{i \in I} \Lambda_{i,m,d} = \{c^i b^j : i \in I,\ d \mid j - i,\ j \ge m\}.$$

For $p \ge 0$, $d > 0$, $r \in [d] = \{0, \dots, d-1\}$ and $P \subseteq [d]$ we define the *squares*

$$\begin{aligned} \Sigma_p &= R_p \cap \widehat{R_p} = \{c^i b^j : i, j \ge p\}, \\ \Sigma_{p,d,r} &= \Sigma_p \cap \Big(\bigcup_{u=0}^{\infty} \Lambda_{p+r+ud}\Big) \cap \Big(\bigcup_{u=0}^{\infty} \widehat{\Lambda_{p+r+ud}}\Big) = \{c^{p+r+ud} b^{p+r+vd} : u, v \ge 0\}, \\ \Sigma_{p,d,P} &= \textstyle\bigcup_{r \in P} \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \ge 0\}. \end{aligned}$$

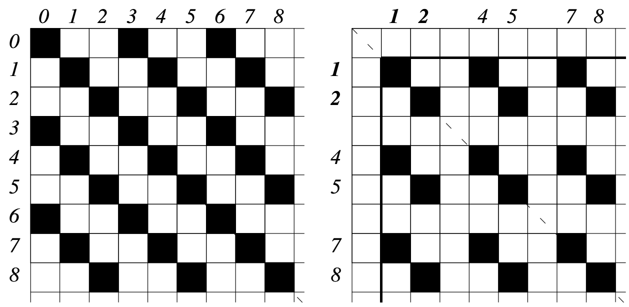

Figure 7.3: The λ-multiples of 3, M_3, and the square $\Sigma_{1,3,\{0,1\}}$

Some of our subsetes are illustrated in Figures 7.2 and 7.3.

Finally, for $X \subseteq \mathbf{B}$, we define $\iota(X) = \min(\Phi(X \cap U))$ (if $X \cap U \neq \emptyset$) and $\kappa(X) = \min(\Psi(X \cap \widehat{U}))$ (if $X \cap \widehat{U} \neq \emptyset$). Geometrically , $\iota(X)$ is the topmost line having an element in X above the diagonal and $\kappa(X)$ is the leftmost column having an element in X below the diagonal.

2 The main theorem

We now state our main theorem, that will be proved in the following sections.

Theorem 7.1 *Let S be a subsemigroup of the bicyclic monoid. Then one of the following conditions holds:*

1. *The subsemigroup is a subset of the diagonal; $S \subseteq D$.*

2. *The subsemigroup is a union of a subset of a triangle, a subset of the diagonal above the triangle, a square below the triangle and some lines belonging to a strip determined by the square and the triangle, or it is the reflection of such a union with respect to the diagonal. Formally, there exist $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \ldots, p-1\}$ with $q \in I$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$ such that S is of one of the following forms:*

   (i) $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$; or

(ii) $S = F_D \cup \widehat{F} \cup \widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P}$.

3. *There exist* $d \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$ *and sets* $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ *such that* S *is of one of the following forms:*

   (i) $S = F_D \cup \bigcup_{i \in I} S_i$;

   (ii) $S = F_D \cup \bigcup_{i \in I} \widehat{S_i}$;

   where each S_i *has the form*

$$S_i = F_i \cup \Lambda_{i,m_i,d}$$

   for some $m_i \in \mathbb{N}_0$ *and some finite set* F_i, *and*

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

   for some (possibly empty) $R \subseteq \{0, \ldots, d-1\}$, *some* $N \in \mathbb{N}_0$ *and some finite set* $I_0 \subseteq \{0, \ldots, N-1\}$.

We start by observing that if $S \subseteq D$ then there is nothing to describe because any idempotent $c^i b^i$ is an identity for the square Σ_i below it.

Condition *2.* corresponds to subsemigroups having elements both above and below the diagonal; we call them *two-sided subsemigroups*. We observe that a subsemigroup defined by condition *2.(ii)* is symmetric to the corresponding subsemigroup given by condition *2.(i)* with respect to the diagonal, and so we can use the anti-isomorphism $\widehat{\ }$ to obtain one from the other. Therefore we only need to consider subsemigroups that fall in one of these two categories. The description of two-sided subsemigroups is obtained in Section 4.

We call *upper subsemigroups* those having all elements above the diagonal and *lower subsemigroups* those having all elements below the diagonal. Condition *3.* corresponds to upper and lower subsemigroups. Again conditions *3.(i)* and *3.(ii)* give subsemigroups symmetric with respect to the diagonal and so only one of them will have to be considered. Upper subsemigroups are dealt with in Section 5.

3 Auxiliary results

In this section we will prove some useful properties of the subsets defined in Section 1. In particular, we prove that a number of the distinguished subsets are in fact subsemigroups. We start with three basic ones.

Lemma 7.2 *For any $d \in \mathbb{N}$ the λ-multiples of d, M_d, is a subsemigroup.*

PROOF. Let $c^i b^j, c^k b^l \in M_d$. Then $d \mid i - j$ and $d \mid k - l$. If $j > k$ then $c^i b^j c^k b^l = c^i b^{j-k+l}$ otherwise $c^i b^j c^k b^l = c^{i-j+k} b^l$. In any case $c^i b^j c^k b^l \in M_d$ because $d \mid i - j + k - l$. ■

Lemma 7.3 *For any $p \in \mathbb{N}$ the right half plane R_p and the strip $S'_{0,p}$ are subsemigroups.*

PROOF. Let $x = c^i b^j, y = c^k b^l \in R_p$ $(j, l \geq p)$. If $j \geq k$ then $xy = c^i b^{j-k+l} \in R_p$ since $j - k + l \geq l \geq p$. If $j < k$ then $xy = c^{i-j+k} b^l \in R_p$ since $l \geq p$. Therefore R_p is a subsemigroup. Let $x = c^i b^j, y = c^k b^l \in S'_{0,p}$ $(i, k < p, j \geq i, l \geq k)$. If $j \geq k$ then $xy = c^i b^{j-k+l} \in S'_{0,p}$ since $i < p$ and $j - k + l \geq j \geq i$. If $j < k$ then $xy = c^{i-j+k} b^l \in S'_{0,p}$ since $i - j + k \leq k < p$ and $l \geq k \geq i - j + k$. Therefore $S'_{0,p}$ is also a subsemigroup. ■

Now we use these basic subsemigroups and the fact that the image of a subsemigroup under an anti-isomorphism is also a subsemigroup, to establish some further subsemigroups:

Lemma 7.4 *For any $q, p, m \in \mathbb{N}_0$ with $q < p \leq m$ the following sets are subsemigroups:*

(i) $S_{q,p}$;	*(ii)* $S'_{q,p}$;	*(iii)* Σ_p;
(iv) $S_{q,p} \cup \Sigma_p$;	*(v)* $S_{q,p,m}$;	*(vi)* $S'_{q,p} \cup \Sigma_p$.

PROOF. To prove *(i)* – *(v)* we will just write the sets as intersections of subsemigroups given by the previous lemma and their images by the anti-isomorphism$\widehat{\ }$. We have

$$S_{q,p} = S'_{0,p} \cap \widehat{R_q} \cap R_p,\ S'_{q,p} = S'_{0,p} \cap \widehat{R_q},\ \Sigma_p = R_p \cap \widehat{R_p},$$
$$S_{q,p} \cup \Sigma_p = R_p \cap \widehat{R_q},\ S_{q,p,m} = S'_{0,p} \cap R_m \cap \widehat{R_q}.$$

To prove that $S = S'_{q,p} \cup \Sigma_p$ is a subsemigroup, it is sufficient to show that, for $x = c^i b^j \in S'_{q,p}$ ($q \le i < p, j \ge i$) and $y = c^k b^l \in \Sigma_p$ ($k, l \ge p$), we have $xy, yx \in S$. If $j \ge k$ then $xy = c^i b^{j-k+l} \in S$, because $i \ge q$ and $j - k + l \ge l \ge p$. If $j < k$ then $xy = c^{i-j+k} b^l \in S$, because $i - j + k > i \ge q$ and $l \ge p$. Since $l \ge p > i$ we have $yx = c^k b^{l-i+j} \in \Sigma_p$, because $k \ge p$ and $l - i + j \ge l \ge p$. ■

The following lemma establishes some inclusions that will also be useful.

Lemma 7.5 *For any $p, q \in \mathbb{N}_0$ with $q < p$ the following inclusions hold:*

(i) $T_{q,p} S_{q,p} \subseteq S_{q,p}$; *(ii)* $S_{q,p} T_{q,p} \subseteq S_{q,p}$;
(iii) $T_{q,p} \Sigma_p \subseteq S_{q,p} \cup \Sigma_p$; *(iv)* $\Sigma_p T_{q,p} \subseteq \Sigma_p$.

PROOF. Let

$$\alpha = c^i b^j \in T_{q,p}\ (q \le i \le j < p),$$
$$\beta = c^k b^l \in S_{q,p}\ (q \le k < p, l \ge p),$$
$$\gamma = c^u b^v \in \Sigma_p\ (u, v \ge p).$$

If $j \ge k$ then $\alpha\beta = c^i b^{j-k+l}$ and, since $j - k + l \ge l \ge p$, $\alpha\beta \in S_{q,p}$. If $j < k$ then $\alpha\beta = c^{i-j+k} b^l$ and, since $l \ge p$ and $S'_{q,p}$ is a subsemigroup, $\alpha\beta \in S_{q,p}$. So *(i)* is proved. We have $\beta\alpha = c^k b^{l-i+j}$ because $i < p \le l$. Since $l - i + j \ge l \ge p$ we have $\beta\alpha \in S_{q,p}$ and so *(ii)* is proved as well. We have $\alpha\gamma = c^{i-j+u} b^v$ because $j < p \le u$ and, since $v \ge p$ and $S'_{q,p} \cup \Sigma_p$ is a subsemigroup, $\alpha\gamma \in S_{q,p} \cup \Sigma_p$ and *(iii)* is proved. Finally, $\gamma\alpha = c^u b^{v+j-i}$ because $i < p \le v$. We have $\gamma\alpha \in \Sigma_p$, because $v + j - i \ge v \ge p$, and *(iv)* is proved as well. ■

Next we prove that every square is a subsemigroup:

Lemma 7.6 *For any $p \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $P \subseteq \{0, \dots, d-1\}$, the square $\Sigma_{p,d,P}$ is a subsemigroup.*

Proof. Let

$$\alpha = c^{p+r_1+u_1 d} b^{p+r_1+v_1 d},\ \beta = c^{p+r_2+u_2 d} b^{p+r_2+v_2 d} \in \Sigma_{p,d,P}$$

where $r_1, r_2 \in P$; $u_1, v_1, u_2, v_2 \in \mathbb{N}_0$. If $p + r_1 + v_1 d \geq p + r_2 + u_2 d$ then

$$\alpha\beta = c^{p+r_1+u_1 d} b^{p+r_1+(v_1-u_2+v_2)d}.$$

Since we have $p + r_1 + v_1 d \geq p + r_2 + u_2 d$, it follows that $r_1 + v_1 d - u_2 d \geq r_2 \geq 0$, which implies $r_1 + (v_1 - u_2 + v_2)d \geq 0$. So we have $(v_1 - u_2 + v_2)d \geq -r_1 > -d$ and hence $v_1 - u_2 + v_2 \geq 0$. Therefore $\alpha\beta \in \Sigma_{p,d,P}$. If $p + r_1 + v_1 d < p + r_2 + u_2 d$ then

$$\alpha\beta = c^{p+r_2+(u_1-v_1+u_2)d} b^{p+r_2+v_2 d}.$$

Analogously $p+r_2+u_2 d > p+r_1+v_1 d$ implies $u_1-v_1+u_2 \geq 0$ and so $\alpha\beta \in \Sigma_{p,d,P}$. ■

Also, a square 'extended' by adjoining all the λ-multiples of d in a strip above it is a subsemigroup:

Lemma 7.7 *For any $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$ and $P \subseteq \{0, \dots, d-1\}$, the set*

$$\Sigma_{p,d,P} \cup (M_d \cap S'_{q,p})$$

is a subsemigroup.

Proof. Let

$$H = \Sigma_{p,d,P} \cup (M_d \cap S'_{q,p}).$$

We know from the previous lemma that $\Sigma_{p,d,P}$ is a subsemigroup. From Lemmas 7.2 and 7.4 we know that $M_d \cap S'_{q,p}$ is a subsemigroup as well. Let

$$\alpha = c^{p+r+ud} b^{p+r+vd} \in \Sigma_{p,d,P},\ \beta = c^i b^{i+sd} \in M_d \cap S'_{q,p}.$$

We just have to prove that $\alpha\beta, \beta\alpha \in H$. Since $p + r + vd \geq p > i$,

$$\alpha\beta = c^{p+r+ud} b^{p+r+(v+s)d} \in \Sigma_{p,d,P}.$$

We have

$$\beta\alpha = c^i b^{i+sd} c^{p+r+ud} b^{p+r+vd}.$$

We note that $H \subseteq U = (\Sigma_p \cup S'_{q,p}) \cap M_d$ and, using the same two lemmas, U is a subsemigroup. Therefore, if $i + sd \geq p + r + ud$ then $\beta\alpha \notin \Sigma_p$ and, since U is a subsemigroup,

$$\beta\alpha \in S'_{q,p} \cap M_d \subseteq H.$$

If $i + sd < p + r + ud$ and $u - s < 0$ we have again

$$\beta\alpha \in S'_{q,p} \cap M_d \subseteq H.$$

Finally, if $i + sd < p + r + ud$ and $u - s \geq 0$ then

$$\beta\alpha = c^{p+r+(u-s)d} b^{p+r+vd} \in \Sigma_{p,d,P}$$

which concludes the proof. ■

Another important type of subsemigroups are the lines:

Lemma 7.8 *For any $p \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $I \subseteq \{0, \ldots, p-1\}$, the set $\Lambda_{I,p,d}$ is a subsemigroup.*

Proof. Let $\alpha = c^i b^{i+ud}$, $\beta = c^j b^{j+vd} \in \Lambda_{I,p,d}$ $(i, j < p; i + ud, j + vd \geq p)$. Then $\alpha\beta = c^i b^{i+(u+v)d}$ because $i + ud \geq p > j$. Since $i + (u+v)d \geq i + ud \geq p$ we have $\alpha\beta \in \Lambda_{I,p,d}$. ■

The following lemma describes the subsemigroups that are obtained 'extending' a square by adjoining lines above it:

Lemma 7.9 *Let $p \in \mathbb{N}_0$, $d \in \mathbb{N}$, $\emptyset \neq I \subseteq \{0, \ldots, p-1\}$, $\emptyset \neq P \subseteq \{0, \ldots, d-1\}$, and $q = \min(I)$. The set $H = \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ is a subsemigroup if and only if*

$$I' = \{p + r - ud : r \in P, u \in \mathbb{N}_0, p + r - ud \geq q\} \subseteq I.$$

Proof. We will first assume that H is a subsemigroup and prove that $I' \subseteq I$. Let $c^q b^{q+d_1}, c^{p+r+d} b^{p+r} \in H$ where $r \in P$ and $d_1 > 0$ is a multiple of d. For any $n, m \in \mathbb{N}$ such that $p + r + md - nd_1 \geq q$ we have

$$(c^q b^{q+d_1})^n (c^{p+r+d} b^{p+r})^m = c^{p+r+md-nd_1} b^{p+r} \in H$$

and so $p+r-ud \in I$ for any $r \in P$ and $u \in \mathbb{N}$ such that $p+r-ud \geq q$. Therefore $I' \subseteq I$. Let us assume now that $I' \subseteq I$ and prove that H is a subsemigroup. We know that $\Sigma_{p,d,P}$ is a subsemigroup. Let

$$\alpha = c^{p+r+ud} b^{p+r+vd} \in \Sigma_{p,d,P} \ (r \in P; u, v \in \mathbb{N}_0)$$
$$\beta = c^i b^{i+d_1} \in \Lambda_{I,p,d} (i \in I, d_1 \in \mathbb{N}, d \mid d_1).$$

We have

$$\alpha\beta = c^{p+r+ud} b^{p+r+vd+d_1} \in \Sigma_{p,d,P}.$$

If $i + d_1 \geq p + r + ud$ then

$$\beta\alpha = c^i b^{i+d_1+(v-u)d} \in \Lambda_{I,p,d},$$

because $i + d_1 + (v-u)d \geq p + r + vd \geq p$. If $i + d_1 < p + r + ud$ then

$$\beta\alpha = c^{p+r+ud-d_1} b^{p+r+vd}.$$

In this case, if $ud - d_1 \geq 0$ then $\beta\alpha \in \Sigma_{p,d,P}$ and if $ud - d_1 < 0$ then $p + r + ud - d_1 \geq q$ because $H \subseteq S_{q,p} \cup \Sigma_p$ and $S_{q,p} \cup \Sigma_p$ is a subsemigroup. Therefore $p + r + ud - d_1 \in I' \subseteq I$, implying $\beta\alpha \in \Lambda_{I,p,d}$. ■

4 Two-sided subsemigroups

In this section we describe subsemigroups that have elements both above and below the diagonal. Let S be a subsemigroup of $\mathbf{B}$ with $S \cap U \neq \emptyset$ and $S \cap \widehat{U} \neq \emptyset$. Without loss of generality we can assume that $q = \iota(S) \leq \kappa(S) = p$ observing that the other case is dual to this by using the anti-isomorphism $\widehat{\ }$.

We now state our main result of this section:

Theorem 7.10 *Let S be a subsemigroup of $\mathbf{B}$ such that $S \cap U \neq \emptyset$, $S \cap \widehat{U} \neq \emptyset$ and $q = \iota(S) \leq \kappa(S) = p$. There exist $d \in \mathbb{N}$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$, $I \subseteq \{q, \ldots, p-1\}$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$ such that*

$$S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}.$$

To prove this theorem we need the following elementary result from number theory, the proof of which we include for completeness:

Lemma 7.11 *Let $a_1, \ldots, a_k, b_1, \ldots, b_l, r_0 \in \mathbb{N}_0$ be arbitrary with $a_1 > 0$, $b_1 > 0$ and let*

$$d = \gcd(a_1, \ldots, a_k, b_1, \ldots, b_l).$$

Then there exist numbers $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l \in \mathbb{N}_0$ such that:

(i) $\alpha_1 a_1 + \ldots + \alpha_k a_k + \beta_1 b_1 + \ldots + \beta_l b_l = d$;

(ii) $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l \geq r_0$.

Proof. We start by assuming, without loss of generality, that

$$a_1, \ldots, a_k, b_1, \ldots, b_l > 0.$$

Since $d = \gcd(a_1, \ldots, a_k, b_1, \ldots, b_l)$, we can write

$$d = \sum_{i=1}^{k} \alpha_i' a_i + \sum_{j=1}^{l} \beta_j' b_j$$

for some integers $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$. Let H be any positive integer and let

$$P = Hkla_1 \ldots a_k b_1 \ldots b_l, \quad Q = P/k, \quad R = P/l.$$

We can then write

$$\begin{aligned} d &= \sum_{i=1}^{k} \alpha'_i a_i + \sum_{j=1}^{l} \beta'_j b_j = \sum_{i=1}^{k} \alpha'_i a_i + P - P + \sum_{j=1}^{l} \beta'_j b_j \\ &= \sum_{i=1}^{k} (\alpha'_i a_i + Q) + \sum_{j=1}^{l} (\beta'_j b_j - R) = \sum_{i=1}^{k} (\alpha'_i + Q/a_i) a_i + \sum_{j=1}^{l} (\beta'_j - R/b_j) b_j \\ &= \sum_{i=1}^{k} \alpha_i a_i + \sum_{j=1}^{l} \beta_j b_j \end{aligned}$$

It is clear that when H increases all numbers $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l$ increase as well and so the result holds. ■

Proof of Theorem 7.10. Let $F_D = S \cap D \cap L_q$ and $S' = S \backslash F_D$. We have $S' = S \cap (M_d \cap (S'_{q,p} \cup \Sigma_p))$ where $d = \gcd(\lambda(S'))$ and so S' is a subsemigroup. We observe that the elements $c^i b^i \in F_D$ act as identities in S'. Let $x \in S' \cap U$ and $y \in S' \cap \widehat{U}$ such that $\Phi(x) = \iota(S) = q$ and $\Psi(y) = \kappa(S) = p$. Let $Y \subseteq S'$ be a finite set such that:

(i) $x, y \in Y$;

(ii) $\Lambda_i \cap S' \cap S'_{q,p} \neq \emptyset \implies \Lambda_i \cap Y \neq \emptyset$ for $i \in \{q \ldots, p-1\}$ (Y contains at least one representative for each line in the strip with elements in S');

(iii) $\{(i-p) \mod d : \Lambda_i \cap Y \cap \Sigma_p \neq \emptyset\} = \{(i-p) \mod d : \Lambda_i \cap S' \cap \Sigma_p \neq \emptyset\}$ (Y contains at least one representative for each class of lines in the square having a representative in S');

(iv) $\gcd(\lambda(Y)) = d$.

Such Y can be obtained by choosing a finite set Y_1 (with at most $p-q+d$ elements) satisfying $(i)-(iii)$, and a finite set Y_2 such that $\gcd(\lambda(Y_2)) = \gcd(\lambda(S'))$, and

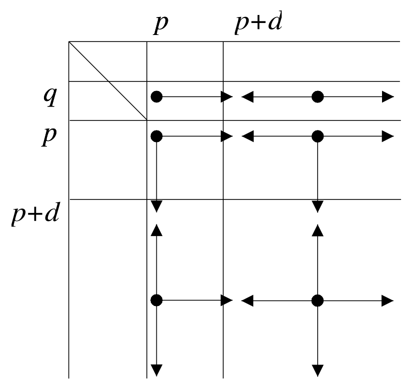

Figure 7.4: Moving using $c^p b^{p+d}$ and $c^{p+d} b^p$

letting $Y = Y_1 \cup Y_2$. Let

$$Y \cap (D \cup U) = \{c^{i_1} b^{j_1}, \ldots, c^{i_r} b^{j_r}\}$$

where $x = c^{i_1} b^{j_1}$, $q = i_1 \leq i_2 \leq \ldots \leq i_r, j_1 > i_1, j_2 \geq i_2, \ldots, j_r \geq i_r$ and let

$$Y \cap \widehat{U} = \{c^{k_1} b^{l_1}, \ldots, c^{k_s} b^{l_s}\}$$

where $y = c^{k_1} b^{l_1}$, $p = l_1 \leq l_2 \leq \ldots \leq l_s$ and $k_1 > l_1, \ldots, k_s > l_s$.

We are going to show that

$$c^{p+d} b^p, c^p b^{p+d} \in S'.$$

Before proving this we will make an observation showing the importance of these two elements. This observation is illustrated in Figure 7.4.

Let $c^i b^j$ be an element in $M_d \cap (S_{q,p} \cup \Sigma_p)$. We have $c^i b^j c^p b^{p+d} = c^i b^{j+d}$ which means intuitively that we can move d positions to the right in the grid using the element $c^p b^{p+d}$. If $i \geq p$ then we also have $c^{p+d} b^p c^i b^j = c^{i+d} c^j$ which means that we can move d positions down. If $j \geq p+d$ then we have $c^i b^j c^{p+d} c^p = c^i b^{j-d}$ which means that we can move left. Finally, if $i \geq p+d$ then we have $c^p b^{p+d} c^i b^j = c^{i-d} b^j$ and so we can move up.

In order to prove that $c^{p+d} b^p, c^p b^{p+d} \in S'$ we first note that

$$d = \gcd\{j_1 - i_1, \ldots, j_r - i_r, k_1 - l_1, \ldots, k_s - l_s\},$$

by *(iv)*. Since $i_1 - j_1 < 0$ and $k_1 - l_1 > 0$, Lemma 7.11 can be applied and we can write

$$d = \alpha_1(i_1 - j_1) + \ldots + \alpha_r(i_r - j_r) + \beta_1(k_1 - l_1) + \ldots + \beta_s(k_s - l_s) \tag{7.1}$$

with the constants

$$\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \geq \max\{i_1, \ldots, i_r, l_1, \ldots, l_s\}.$$

We can now consider the product $(c^{i_1}b^{j_1})^{\alpha_1} \ldots (c^{i_r}b^{j_r})^{\alpha_r}$ which is equal to

$$(c^{i_1}b^{i_1+\alpha_1(j_1-i_1)})(c^{i_2}b^{i_2+\alpha_2(j_2-i_2)}) \ldots (c^{i_r}b^{i_r+\alpha_r(j_r-i_r)}).$$

Since $\alpha_1 \geq \max\{i_1, \ldots, i_r\}$ and $j_1 - i_1 \geq 1$ we have

$$i_1 + \alpha_1(j_1 - i_1) > i_t, \; t = 1, \ldots, r$$

and therefore, we can compute the above product working from the left hand side to obtain

$$c^{i_1}b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)}. \tag{7.2}$$

We now consider the product $(c^{k_s}b^{l_s})^{\beta_s} \ldots (c^{k_2}b^{l_2})^{\beta_2}(c^{k_1}b^{l_1})^{\beta_1}$ which is equal to

$$(c^{l_s+\beta_s(k_s-l_s)}b^{l_s}) \ldots (c^{l_2+\beta_2(k_2-l_2)}b^{l_2})(c^{l_1+\beta_1(k_1-l_1)}b^{l_1}).$$

Since $\beta_1 \geq \max\{l_1, \ldots, l_s\}$ and $k_1 - l_1 \geq 1$ we have

$$l_1 + \beta_1(k_1 - l_1) > l_t, \; t = 1, \ldots, s$$

and we can compute this product from the right to obtain

$$c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)}b^{l_1}. \tag{7.3}$$

Multiplying the elements (7.2) and (7.3) of S' we obtain

$$\begin{aligned}&c^{i_1}b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)}c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)}b^{l_1}\\ &= c^{l_1+d}b^{l_1} = c^{p+d}b^{p}\end{aligned}$$

since $q = i_1 \leq l_1 = p$ and using equation (7.1). So $c^{p+d}b^p \in S'$.

Since $d \mid (j_1 - i_1)$ we can write $j_1 - i_1 = td$ for some $t \in \mathbb{N}$. Since $p \geq i_1$ we have $p + td \geq j_1$ and therefore

$$c^{i_1}b^{j_1}(c^{p+d}b^p)^t = c^{i_1-j_1+p+td}b^p = c^p b^p.$$

We conclude that $c^p b^p \in S'$ as well. We now take the constants

$$\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \geq \max\{i_1, \ldots, i_r, l_1, \ldots, l_s\}$$

to be such that

$$d = \alpha_1(j_1 - i_1) + \ldots + \alpha_r(j_r - i_r) + \beta_1(l_1 - k_1) + \ldots + \beta_s(l_s - k_s) \tag{7.4}$$

and we consider the following element of S':

$$c^p b^p c^{i_1} b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)} c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)} b^{l_1}.$$

Since $i_1 = q \leq p = l_1$ this element can be written as

$$c^p b^{p+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)} c^{p+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)} b^p$$

and it is equal to $c^p b^{p+d}$ by equation (7.4). Therefore we have $c^p b^{p+d}, c^{p+d}b^p \in S'$ as we wanted to show.

We are now going to prove that

$$S' \cap \Sigma_p = \Sigma_{p,d,P}$$

where

$$P = \{(i - p) \mod d : L_i \cap Y \cap \Sigma_p \neq \emptyset\}.$$

We will first show that $\Sigma_{p,d,P} \subseteq S'$. Let $c^{p+r+ud}b^{p+r+vd} \in \Sigma_{p,d,P}$. By definition of Y there is $c^i b^j \in Y \cap \Sigma_p$ such that $(i - p) \mod d = r$. Therefore, since $Y \subseteq S' \subseteq M_d$, we have $c^i b^j = c^{p+r+u'd}b^{p+r+v'd}$. As we have seen we can move from $c^i b^j$ to $c^{p+r+ud}b^{p+r+vd}$ using the elements $c^p b^{p+d}$ and $c^{p+d}b^p$ which means that $c^{p+r+ud}b^{p+r+vd}$ belongs to S', because it can be written as a product of the elements $c^p b^{p+d}$, $c^{p+d}b^p, c^i b^j \in S'$. We will now show that $S' \cap \Sigma_p \subseteq \Sigma_{p,d,P}$. Let $c^i b^j \in S' \cap \Sigma_p$. By definition of P and by *(iii)* in the definition of Y we have $(i-p) \mod d = r \in P$. Since $S' \subseteq M_d$ we have $c^i b^j = c^{p+r+ud} b^{p+r+vd}$ for some $u, v \geq 0$ and so $c^i b^j \in \Sigma_{p,d,P}$. We conclude that $S' \cap \Sigma_p = \Sigma_{p,d,P}$.

We now prove that

$$S' \cap S_{q,p} = \Lambda_{I,p,d}$$

where

$$I = \{i : q \leq i \leq p-1; c^i b^j \in S' \text{ for some } j\}.$$

In fact, from any element $c^i b^j \in S' \cap S_{q,p}$ we can move left and right using the elements $c^p b^{p+d}$ and $c^{p+d} b^p$ in order to obtain the whole line $\Lambda_{i,p,d}$. Since $S' \subseteq M_d$ it follows that $S' \cap S_{q,p} = \Lambda_{I,p,d}$. We conclude that

$$S' = F \cup \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$$

where $F = S \cap T_{q,p}$ is a finite set, and this implies

$$S = F_D \cup F \cup \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$$

as required. ■

5 Upper subsemigroups

In this section we consider subsemigroups whose elements are above (and on) the diagonal. The case where all elements are below the diagonal is again obtained by using the anti-isomorphism $\widehat{\ }$.

The following lemma shows that given a finite subset X of a strip, for each line that intersects X, the subsemigroup generated by X contains all λ-multiples of d in that line after some column.

Lemma 7.12 *Let $q, p, d \in \mathbb{N}_0$ with $q < p$ and $d > 0$, and let $X \subseteq S'_{q,p}$ be a finite set with $\iota(X) = q$ and $\gcd(\lambda(X)) = d$. For any $x \in X$ there exists $m \in \mathbb{N}_0$ such that*

$$\Lambda_{\Phi(x),m,d} \subseteq \langle X \rangle.$$

Proof. Let

$$S = \langle X \rangle, \ Y = X \cap U = \{c^{i_1}b^{i_1+d_1}, \ldots, c^{i_n}b^{i_n+d_n}\}$$

with

$$q = i_1 \leq i_2 \leq \ldots \leq i_n; d_1, \ldots, d_n \in \mathbb{N}.$$

For each $j \in \{1, \ldots, n\}$ choose $\alpha_j \in \mathbb{N}$ such that

$$\begin{aligned} & i_j + \alpha_j d_j \geq p, \\ & d = \gcd(d_1, \ldots, d_n) = \gcd(\alpha_1 d_1, \ldots, \alpha_n d_n). \end{aligned}$$

We can take $\alpha_1, \ldots, \alpha_n$ to be large enough distinct primes not appearing in the decomposition of d in prime factors. It is well known that given numbers $x_1, \ldots, x_n \in \mathbb{N}$, such that $\gcd\{x_1, \ldots, x_n\} = d$, there is a constant k such that all multiples of d greater than k can be obtained as combinations of $x_1, \ldots, x_n$ with coefficients in $\mathbb{N}$. Let $k \in \mathbb{N}$ be such that

$$\{td : td \geq k, t \in \mathbb{N}\} \subseteq \{\gamma_1(\alpha_1 d_1) + \ldots + \gamma_n(\alpha_n d_n) : \gamma_1, \ldots, \gamma_n \in \mathbb{N}\}.$$

Let $m = p + k$. We are going to prove that $\Lambda_{\Phi(x),m,d} \subseteq S$ for any $x \in X$. Let $x \in X$, $i = \Phi(x) \in \{q, \ldots, p-1\}$ and $t \in \mathbb{N}$ with $i + td \geq m$. Then $td \geq m - i = p + k - i \geq k$. Therefore we can write

$$td = \gamma_1(\alpha_1 d_1) + \ldots + \gamma_n(\alpha_n d_n)$$

with $\gamma_1, \ldots, \gamma_n \in \mathbb{N}$. If $x = c^{i_j}b^{i_j+d_j} \in Y$ then we have

$$c^i b^{i+td} = c^{i_j}b^{i_j+td} = (c^{i_j}b^{i_j+\alpha_j d_j})^{\gamma_j} \cdot \prod_{\substack{1 \leq l \leq n \\ l \neq j}} (c^{i_l}b^{i_l+\alpha_l d_l})^{\gamma_l}.$$

If $x \notin Y$ then $x = c^i b^i$ and so we have

$$c^i b^{i+td} = c^i b^i (c^{i_1} b^{i_1+\alpha_1 d_1})^{\gamma_1} \dots (c^{i_n} b^{i_n+\alpha_n d_n})^{\gamma_n} \in S,$$

which concludes the proof. ∎

Theorem 7.13 *Let S be a subsemigroup of* $\mathbf{B}$ *such that $S \cap \widehat{U} = \emptyset$ and $S \cap U \neq \emptyset$. There exist $d \in \mathbb{N}$, $I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$, and sets $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ such that*

$$S = F_D \cup \bigcup_{i \in I} S_i$$

where each S_i has the form

$$S_i = F_i \cup \Lambda_{i,m_i,d}$$

for some $m_i \in \mathbb{N}_0$ and some finite set F_i, and

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

for some (possibly empty) $R \subseteq \{0, \dots, d-1\}$, some $N \in \mathbb{N}_0$ and some finite set $I_0 \subseteq \{0, \dots, N-1\}$.

Proof. Let $q = \iota(S)$, $F_D = S \cap D \cap L_q$, $S' = S \backslash F_D$, so that we have $S = F_D \cup S'$, and let $d = \gcd(\lambda(S'))$. Since $S' \subseteq (U \cup D) \cap M_d$, letting $I = \Phi(S')$, we have

$$S = F_D \cup \bigcup_{i \in I} S_i$$

where $S_i = S' \cap \Lambda_{i,i,d}$ for $i \in I$. For any $i \in I$ we can consider a finite set $X_i \subseteq S'$ with $i \in \Phi(X_i)$ and $\gcd(X_i) = d$ and conclude, by using Lemma 7.12, that $\Lambda_{i,m_i,d} \subseteq S$ for some $m_i \in \mathbb{N}_0$. If I is finite then we can take $R = \emptyset$, $I_0 = I$ and $N = \max(I) + 1$. We will now consider the case where I is infinite. Let $X = \{c^{i_1} b^{i_1+d_1}, \dots, c^{i_k} b^{i_k+d_k}\} \subseteq S'$ such that $d = \gcd(\lambda(X))$, $i_1 \geq i_2 \geq \dots \geq i_k$. By Lemma 7.12, there is a constant M such that $td \geq M$ implies $c^{i_1} b^{i_1+td} \in S'$. Define a set $R \subseteq \{0, \dots, d-1\}$ by

$$r \in R \iff |\{i \in \mathbb{N} : \Lambda_i \cap S' \neq \emptyset \;\&\; i \bmod d = r\}| = \infty.$$

Then there exists a constant K such that

$$c^i b^j \in S' \ \&\ i \geq K \implies (i \bmod d) \in R.$$

Let $N = \max\{i_1, K\}$ and

$$I_0 = \{i : q \leq i \leq N-1, \Lambda_i \cap S' \neq \emptyset\}.$$

We claim that

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}.$$

The direct inclusion is obvious, as is $I_0 \subseteq I$. Now consider an arbitrary $r+ud \geq N$, $r \in R$. Choose an arbitrary

$$c^{r+vd} b^{r+vd+wd} \in S'$$

such that $t = v - u \geq M/d$. From $td \geq M$ it follows that $c^{i_1} b^{i_1+td} \in S'$ and so

$$c^{i_1} b^{i_1+td} c^{r+vd} b^{r+vd+wd} = c^{r+ud} b^{r+vd+wd} \in S'$$

because $r + vd = r + ud + td \geq N + td \geq i_1 + td$. We conclude that $r + ud \in I$. ∎

Observation 7.14 In the case where I is finite ($R = \emptyset$), the subsemigroup can be written as a union of two finite sets and finitely many lines all starting from the same column. Formally there exist $q, p, m \in \mathbb{N}_0$ with $q < p \leq m$, finite sets $F_D \subseteq D \cap L_q$, $F \subseteq S'_{q,p} \backslash S_{q,p,m}$ and a set $I \subseteq \{q, \ldots, p-1\}$ such that

$$S = F_D \cup F \cup \Lambda_{I,m,d}.$$

6 Computation of parameters

In this section we will show how to compute the parameters that appear in Theorem 7.1, given a finite generating set for the subsemigroup. We will first consider two-sided subsemigroups defined by condition *2.(i)* in Theorem 7.1 and then we will consider finitely generated upper subsemigroups defined by condition *3.(i)*, observing again that subsemigroups defined by *2.(ii)* and *3.(ii)* can be obtained from these two by using the anti-isomorphism $\widehat{\ }$. We observe that, given a finite set X, we can determine which kind of subsemigroup it generates:

1) $\langle X\rangle \subseteq D$ if and only if $X \subseteq D$;

2) $\langle X\rangle$ is a two-sided subsemigroup if and only if $X \cap U \neq \emptyset$ and $X \cap \widehat{U} \neq \emptyset$;

3) $\langle X\rangle$ is an upper (respectively lower) subsemigroup if and only if $X \cap U \neq \emptyset$ and $X \cap \widehat{U} = \emptyset$ (respectively $X \cap U = \emptyset$ and $X \cap \widehat{U} \neq \emptyset$).

Theorem 7.15 *Let $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ be a two-sided subsemigroup of* **B** *defined by condition 2.(i) in Theorem 7.1. Let X be a finite generating set for S. Then we have:*

(i) $q = \iota(X)$, $p = \kappa(X)$, $d = \gcd(X)$;

(ii) $F_D = X \cap D \cap L_q$;

(iii) $P = \{(i-p) \mod d : i \in \mathbb{N}_0, \Lambda_i \cap X \cap \Sigma_p \neq \emptyset\}$;

(iv) $F = \bigcup_{i=1}^{M} (X \cap T_{q,p})^i \cap T_{q,p}$ *where* $M = (p-q+1)(p-q)/2$;

(v) Defining

$$I_0 = \{p+r-ud : r \in P, u \in \mathbb{N}_0, p+r-ud \geq q\} \cup \{i : \Lambda_i \cap (F \cup (X \cap S_{q,p})) \neq \emptyset\}$$

and the left action $. : \mathbf{B} \times \mathbb{N}_0 \to \mathbb{N}_0$ *by*

$$c^i b^j.k = \begin{cases} i \text{ if } j \geq k \\ i-j+k \text{ otherwise} \end{cases}$$

we have

$$I = \bigcup_{n=0}^{p-q} F^n.I_0.$$

PROOF. Let $q' = \iota(X)$, $p' = \kappa(X)$, $d' = \gcd(\lambda(X))$, $F'_D = X \cap D \cap L_{q'}$ and $X' = X \backslash F'_D$. Then we have $S = F'_D \cup \langle X' \rangle$ and the elements of F'_D act as identities in $\langle X' \rangle$. If $q' \leq p'$ then $X' \subseteq M_d \cap (S'_{q',p'} \cup \Sigma_{p'})$ and, by Lemmas 7.2 and 7.4, this last set is a subsemigroup and so

$$\langle X' \rangle \subseteq M_d \cap (S'_{q',p'} \cup \Sigma_{p'}),$$

implying $q = q'$, $p = p'$. If $q' > p'$ then analogous reasoning gives $\langle X' \rangle \subseteq M_d \cap (\widehat{S'_{q',p'}} \cup \Sigma_{p'})$ from which it follows that $p = q' < p' = q$ which contradicts our assumption on the shape of S. So we have $q = q'$, $p = p'$ and then it immediately follows that

$$F_D = F'_D = X \cap D \cap L_{q'} = X \cap D \cap L_q.$$

Finally, from $S = \langle X \rangle \subseteq M_{d'}$ (since $M_{d'}$ is a subsemigroup) it follows that $d = d'$. This proves *(i)* and *(ii)*.

We know that $P = \{(i - p) \mod d : i \in \mathbb{N}_0, \Lambda_i \cap S \cap \Sigma_p \neq \emptyset\}$. Let

$$P' = \{(i - p) \mod d : i \in \mathbb{N}_0, \Lambda_i \cap X' \cap \Sigma_p \neq \emptyset\}.$$

It is clear that $P' \subseteq P$. We also have

$$X' \subseteq \Sigma_{p,d,P} \cup (M_d \cap S'_{q,p}) = T.$$

But T is a subsemigroup by Lemma 7.7, and so $\langle X' \rangle = S \backslash F_D \subseteq T$. Therefore $S \cap \Sigma_p \subseteq T \cap \Sigma_p$ which is equivalent to $\Sigma_{p,d,P} \subseteq \Sigma_{p,d,P'}$ and so in fact $P = P'$, proving *(iii)*.

To prove *(iv)* we observe that the inclusions in Lemma 7.5 imply that $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is an ideal of S. It then follows that the elements of F can be obtained by forming the appropriate products of the generators of X that belong to $T_{q,p}$. Since $T_{q,p}$ has $(p-q+1)(p-q)/2$ elements the desired formula follows. In practice we do not need to form all these products. Again using the fact that $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is an ideal we see that F can be computed by the following simple orbit algorithm:

$X_0 := X \cap T_{q,p}$
$F := X_0$
while not $(FX_0 \cap T_{q,p} \subseteq F)$ do
  $F := F \cup (FX_0 \cap T_{q,p})$
od.

To prove *(v)* we will first show that $I_0 \subseteq I$. Since

$$S \cap (S_{q,p} \cup \Sigma_p) = \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$$

is a subsemigroup, it follows from Lemma 7.9 that

$$\{p + r - ud : r \in P, u \in \mathbb{N}_0, p + r - ud \geq q\} \subseteq I.$$

Given $c^i b^j \in F \cup (X \cap S_{q,p})$ we can multiply this element on the right by a power of an element of the form $c^q b^{q+d_1}$ with $d_1 > 0$ (such an element must exist by definition of q) in order to obtain an element in $S \cap \Lambda_i \cap S_{q,p}$. From this element we can obtain the whole line $\Lambda_{i,p,d}$ by using the elements $c^p b^{p+d}, c^{p+d} b^p \in T$ and so $I_0 \subseteq I$.

We will now show that

$$T = \Lambda_{I_0,p,d} \cup \Sigma_{p,d,P}$$

is a right ideal ($TS \subseteq T$). We know that T is a subsemigroup, by Lemma 7.9. By the way we have defined I_0 we have

$$X \cap S_{q,p} \subseteq T.$$

We also have

$$X \cap \Sigma_p \subseteq T$$

because $S \cap \Sigma_p = \Sigma_{p,d,P} = T \cap \Sigma_p$. It remains to show that

$$T((X \cap T_{q,p}) \cup F_D) \subseteq T.$$

Let $c^k b^l \in T, c^i b^{i+d_1} \in (X \cap T_{q,p}) \cup F_D$. Since $l \geq i$, we have $c^k b^l c^i b^{i+d_1} = c^k b^{l+d_1} \in T$. Therefore T is a right ideal. Clearly if $I_0 \subseteq I' \subseteq I$ then $T' = \Lambda_{I',p,d} \cup \Sigma_{p,d,P}$ is a right ideal as well.

Finally we observe that, although multiplying two elements in F we can obtain an element in a line belonging to $I \backslash I_0$, we do not have to consider these products in order to obtain I. If $c^i b^j, c^k b^l \in F$ and $c^i b^j c^k b^l = c^{i-j+k} b^l$ where $i-j+k \in I \backslash I_0$, then I_0 contains line k and so line $i-j+k$ can also be obtained from $F.I_0$. We conclude that I can be obtained by running the orbit algorithm, starting from I_0:

$I := I_0$
while not $(F.I \subseteq I)$ do
  $I := I \cup F.I$
od.

This algorithm must stop in no more then $p-q$ iterations because it generates a strictly ascending chain of sets contained in $\{q, \ldots, p-1\}$ (normally far fewer iterations are necessary), concluding the proof of *(v)*. ■

We will now consider finitely generated upper subsemigroups. Let $X \subseteq U \cup D$ be a finite set such that $X \cap U \neq \emptyset$ and let $S = \langle X \rangle$. As already remarked after Theorem 7.13, we are in the case where I is finite ($R = \emptyset$) in condition *3.(i)* of Theorem 7.1, and our subsemigroup has the form

$$S = F_D \cup F \cup \Lambda_{I,m,d}.$$

Similarly as in the proof of Theorem 7.15 we can see that

$$q = \iota(X),\ p = \max(\Phi(X)) + 1,\ I \subseteq \{q, \ldots, p-1\},$$

$$F_D = X \cap D \cap L_q,\ d = \gcd(\lambda(X)).$$

We need to obtain the parameters F, I, and m from the generating set. Since the elements in F_D act as identities in $\langle X' \rangle$, where $X' = X \backslash F_D$, we will assume, without loss of generality, that $F_D = \emptyset$ and so $X = X' \subseteq S'_{q,p}$. We will define an algorithm to obtain these parameters; it consists in forming a sequence of unions of powers of the generating set, $X, X \cup X^2, X \cup X^2 \cup X^3, \ldots$, until we have a subsemigroup of the form $F \cup \Lambda_{I,m,d}$. For that we need a sufficient condition, that can be checked by an algorithm, for a finite subset of a strip $S'_{q,p}$ to give us a subsemigroup of this form.

Lemma 7.16 *Let $Y \subseteq S'_{q,p}$ be a finite set with $\gcd(Y) = d$ and $c^q b^{q+d_1} \in Y$ for some $d_1 \in \mathbb{N}$. Suppose that for any $i \in I = \Phi(Y)$ there is $m_i \in \mathbb{N}_0$ such that*

$$c^i b^{m_i}, c^i b^{m_i+d}, \ldots, c^i b^{2m_i-i-d} \in Y, \;\; c^i b^{m_i-d} \notin Y.$$

Let $m = \max\{m_i : i \in I\}$ and $F = Y \cap (S'_{q,p} \backslash S_{q,p,m})$. If $FF \cap (S'_{q,p} \backslash S_{q,p,m}) \subseteq F$ and $F.I \subseteq I$ then $\langle Y \rangle = F \cup \Lambda_{I,m,d}$. Moreover m is minimum such that $\Lambda_{I,m,d} \subseteq \langle Y \rangle$.

Proof. We start by showing that $F \cup \Lambda_{I,m,d} \subseteq \langle Y \rangle = S$. For any $i \in I$, we have

$$\Lambda_{i,m_i,d} \subseteq \langle c^i b^{m_i}, \ldots, c^i b^{2m_i-i-d} \rangle,$$

because any element in $\Lambda_{i,m_i,d}$ can be written in the form

$$c^i b^u (c^i b^{m_i})^k$$

for some $k \in \mathbb{N}_0$, and $u \in \mathbb{N}_0$ such that

$$i + (m_i - i) = m_i \leq u \leq 2m_i - i - d = i + 2(m_i - i) - d.$$

We conclude that $\Lambda_{i,m_i,d} \subseteq S$ for any $i \in I$ and therefore $F \cup \Lambda_{I,m,d} \subseteq S$ with $m = \max\{m_i : i \in I\}$. It is clear that $Y \subseteq F \cup \Lambda_{I,m,d}$, because $Y \subseteq M_d$ and $I = \Phi(Y)$, and so to prove the other inclusion we just have to show that $F \cup \Lambda_{I,m,d}$ is a subsemigroup. We have

$$FF \cap (S'_{q,p} \backslash S_{q,p,m}) \subseteq F, \;\; F.I \subseteq I$$

by hypothesis and, since $\Phi(F) \subseteq I$, we also have

$$\Phi(FF) \subseteq F.I \subseteq I$$

and we conclude that $FF \subseteq F \cup \Lambda_{I,m,d}$. It is also clear that

$$\Lambda_{I,m,d}(\Lambda_{I,m,d} \cup F) \subseteq \Lambda_{I,m,d}.$$

Finally, we have

$$F\Lambda_{I,m,d} \subseteq \Lambda_{I,m,d},$$

because $F.I \subseteq I$. ■

Clearly, it can be checked by an algorithm whether a finite set $Y \subseteq S'_{q,p}$ satisfies the conditions of Lemma 7.16; let us call such a procedure *iscomplete(Y)*. Also, provided that Y does satisfy these conditions, there is a straightforward procedure *parameters(Y)* returning the triple (F, I, m). Given these two procedures, an algorithm to compute the parameters F, I, m given any finite generating set X is:

$Y := X$
while not $\mathit{iscomplete}(Y)$ do
  $Y := Y \cup YX$
od
$(F, I, m) := \mathit{parameters}(Y)$.

Note that if we are simply interested in the index set I of lines occurring in S, we can use a much more efficient orbit algorithm:

$I := \Phi(X)$
while not $X.I \subseteq I$ do
  $I := I \cup X.I$
od.

We conclude this chapter by presenting some examples.

Example 7.17 Let S be the subsemigroup of $\mathbf{B}$ generated by the set

$$X = \{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}.$$

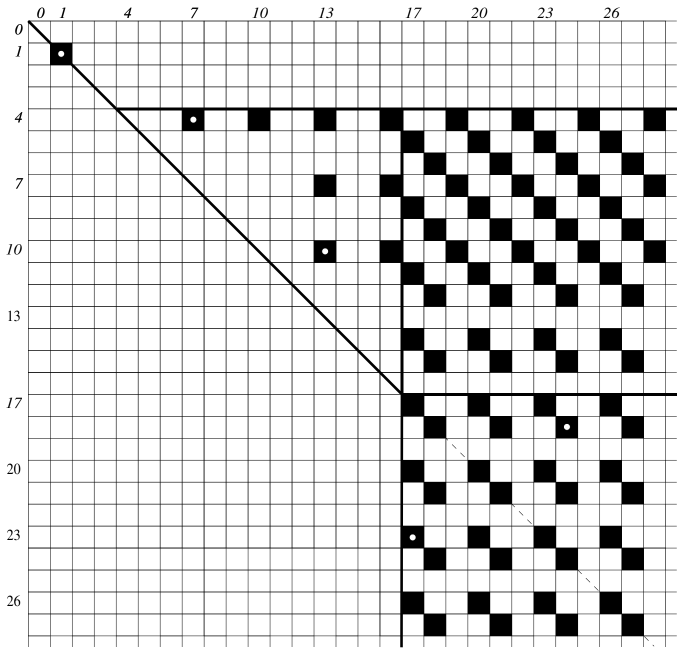

Figure 7.5: Two-sided subsemigroup generated by $\{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}$.

Then S is clearly a two-sided subsemigroup of the form $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. From the generating set we see that $F_D = \{cb\}$, $q = 4$, $p = 17$, $d = 3$ and $P = \{0, 1\}$. The remaining parameters have been obtained using our implementation of the above algorithms in the system GAP (see [22]), and they are $I = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15\}$ and

$$F = \{c^4b^7, c^4b^{10}, c^4b^{13}, c^4b^{16}, c^7b^{13}, c^7b^{16}, c^{10}b^{13}, c^{10}b^{16}\}.$$

This subsemigroup is shown in Figure 7.5.

Example 7.18 Let S to be the subsemigroup of **B** generated by the set

$$X = \{cb, c^3b^{13}, c^5b^9, c^{10}b^{16}\}.$$

Then S is clearly an upper subsemigroup of the form $S = F_D \cup F \cup \Lambda_{I,m,d}$ and from the generating set we see that $F_D = \{cb\}$ and $d = 2$. Using again our implementation in GAP we have obtained $m = 20$, $I = \{3, 5, 6, 10\}$ and

$$F = \{c^3b^{13}, c^3b^{17}, c^3b^{19}, c^5b^9, c^5b^{13}, c^5b^{17}, c^5b^{19}, c^6b^{16}, c^{10}b^{16}\}.$$

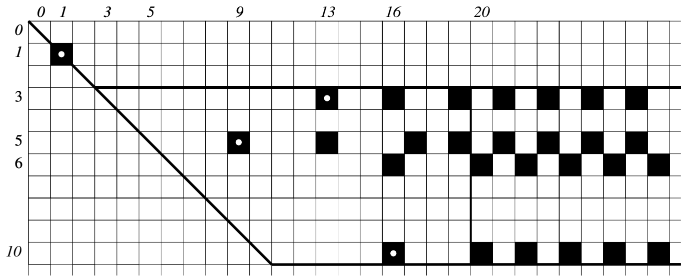

Figure 7.6: Upper subsemigroup generated by $\{cb, c^3b^{13}, c^5b^9, c^{10}b^{16}\}$.

This subsemigroup is shown in Figure 7.6.

We observe that, from these two examples we can obtain other two examples just by replacing the generating set X by $\widehat{X}$. The corresponding pictures for these two subsemigroups are obtained reflecting the given pictures with respect to the diagonal. The four examples together illustrate the all four non-degenerated cases of subsemigroups of the bicyclic monoid that, as we will see in the following chapter, are finitely generated.

The GAP program (and the session in GAP) used to obtain these examples and several others is included in Appendix B.

We will end with an example of an upper semigroup with elements in infinitely many lines.

Example 7.19 Let

$$S_1 = \Lambda_{I,12,3} \ (I = \{0, 1, 2, 3, 5, 6, 7\})$$
$$S_2 = \{c^{6+3i}b^{6+3j} : i \geq 1, j \geq 2i\}.$$

We will show that the set

$$S = S_1 \cup S_2,$$

shown in Figure 7.7, is a subsemigroup. We know that S_1 is a subsemigroup, by Lemma 7.8. To show that S_2 is a subsemigroup let

$$\alpha = c^{6+3i}b^{6+3j}, \beta = c^{6+3k}b^{6+3l} \in S_2.$$

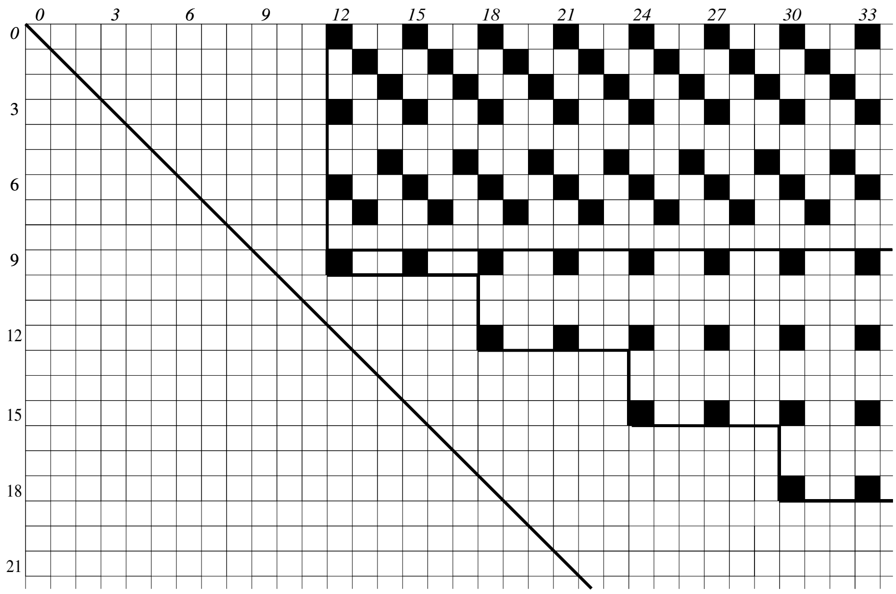

Figure 7.7: An upper semigroup with elements in infinitely many lines.

If $6+3k \geq 6+3j$ then we have

$$\alpha\beta = c^{6+3(k+i-j)}b^{6+3l} \in S_2$$

because $i-j \leq 0$ which implies $l \geq 2k \geq 2(k+i-j)$. Otherwise we still have

$$\alpha\beta = c^{6+3i}b^{6+3(j+l-k)} \in S_2$$

since $l-k \geq 0$ implies $j+l-k \geq j \geq 2i$. We will show that $S_1S_2 \subseteq S$ and $S_2S_1 \subseteq S$. Let

$$\alpha = c^i b^{i+3j} \in S_1 \ (i \in I, i+3j \geq 12)$$
$$\beta = c^{6+3k}b^{6+3l} \in S_2 \ (k \geq 1, l \geq 2k).$$

We will first consider the product $\alpha\beta$. If $i+3j \geq 6+3k$ then we have

$$\alpha\beta = c^i b^{i+3j-3k+3l} \in S_1$$

since $i+3j-3k+3l \geq i+3j \geq 12$. Otherwise we have

$$\alpha\beta = c^{6+3(k-j)}b^{6+3l}.$$

In this case there are two possibilities: if $6 + 3(k - j) < 9$ then $\alpha\beta \in S_1$ because $6 + 3l \geq 12$ and $6 + 3(k - j)$ is a multiply of 3; otherwise $k - j \geq 1$ and, since $l \geq 2k \geq 2(k - j)$, we have $\alpha\beta \in S_2$.

Again we can apply the anti-isomorphism $\widehat{\ }$ to obtain an example of lower semigroup with infinitely many columns and these two examples together illustrate, as we will see in the following chapter, the only two different kinds of non-degenerated subsemigroups of **B** that are not finitely generated.

The results contained in this chapter are also contained in [14].

Chapter 8

Properties of the subsemigroups of the bicyclic monoid

In this chapter, we use the description of the subsemigroups of the bicyclic monoid obtained in the last chapter, and we consider the finite generation, automaticity and finite presentability of the subsemigroups.

1 Finite generation

In this section we will establish necessary and sufficient conditions for a subsemigroup of the bicyclic monoid to be finitely generated proving the following:

Theorem 8.1 *Let S be a subsemigroup of the bicyclic monoid. Then S is finitely generated if and only if one of the following conditions holds:*

(i) S is a finite diagonal subsemigroup;

(ii) S is a two-sided subsemigroup;

(iii) S is an upper subsemigroup and the set $\{i \in \mathbb{N}_0 : \Lambda_i \cap S \neq \emptyset\}$ is finite;

(iv) S is a lower subsemigroup and the set $\{i \in \mathbb{N}_0 : \widehat{\Lambda}_i \cap S \neq \emptyset\}$ is finite.

Proof. *(i)* A subsemigroup of the bicyclic monoid contained in the diagonal only admits itself as a generating set and so it is finitely generated if and only if it is finite.

(ii) Let $\iota(S) = q$ and $\kappa(S) = p$ and let $d = \gcd(\lambda(X))$. We can assume that $q \leq p$, and the other case can be obtained from this by using the anti-isomorphism $\widehat{\ }$. By Theorem 7.1 we have

$$S = F_D \cup F \cup \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$$

where F and F_D are finite sets and $I \subseteq \{q, q+1, \ldots, p-1\}$ for some $q, p \in \mathbb{N}_0$. For every $i \in I$ let $i + u_i d = \min\{i + ud : i + ud \geq p\}$. We will prove that the finite set

$$Y = \{c^i b^{i+u_i d} : i \in I\} \cup \{c^p b^{p+d}, c^{p+d} b^p\} \cup \{c^{p+r} b^{p+r} : r \in P\}$$

generates the set $\Sigma_{p,d,P} \cup \Lambda_{I,p,d}$, which is a semigroup by Lemma 7.4 since

$$\Sigma_{p,d,P} \cup \Lambda_{I,p,d} = S \cap (S_{q,p} \cup \Sigma_p).$$

In fact, if $c^i b^{i+ud} \in \Lambda_{I,p,d}$ then

$$c^i b^{i+ud} = c^i b^{i+u_i d} (c^p b^{p+d})^{u-u_i}$$

and if $c^{p+r+ud} b^{p+r+vd} \in \Sigma_{p,d,P}$ then

$$c^{p+r+ud} b^{p+r+vd} = (c^{p+d} b^p)^u (c^{p+r} b^{p+r})(c^p b^{p+d})^v.$$

Therefore S is generated by the finite set $F_D \cup F \cup Y$.

(iii) We will prove that an upper semigroup S is finitely generated if and only if the set

$$K = \{i \in \mathbb{N}_0 : L_i \cap S \neq \emptyset\}$$

is finite. We first assume that K is infinite and prove that S is not finitely generated. Suppose that there exists a finite set X such that $S = \langle X \rangle$. Since $X \subseteq S \subseteq U \cup D$ and X is finite, this implies $X \subseteq S'_{0,p}$ for some $p \in \mathbb{N}_0$. Hence

$S = \langle X \rangle \subseteq S'_{0,p}$ because, by Lemma 7.4, $S'_{0,p}$ is a subsemigroup, and therefore $K \subseteq \{0, \ldots, p\}$ is finite, which contradicts our assumption. We conclude that S is not finitely generated.

We now assume that K is finite and prove that S is finitely generated. By Theorem 7.1 and by Observation 7.14 we have

$$S = F_D \cup F \cup \Lambda_{I,m,d}$$

for some finite sets $F_D \subseteq D \cap L_q$, $F \subseteq S'_{q,p} \backslash S_{q,p,m}$ and $I \subseteq \{q, q+1, \ldots, p-1\}$ with $q, p, m, d \in \mathbb{N}_0$. Let $X \subseteq S$ be a finite set such that $\Phi(X) = I$ and $\gcd(X) = d$. Using again Theorem 7.1 we have

$$< X >= F' \cup \Lambda_{I,m',d}$$

where $F' \subseteq S'_{q,p} \backslash S_{q,p,m'}$ and $m' \in \mathbb{N}_0$. This means that the set X generates all our complete lines from some column m', and so we can define

$$F'' = S \cap (S'_{q,p} \backslash S_{q,p,m'})$$

in order to write

$$S =< F_D \cup F'' \cup X >$$

and conclude that S is finitely generated. Another way to prove this implication is to see S as a finite union of subsemigroups of $\mathbb{N}$ (one in each line), which is clearly finitely generated.

(iv) Straightforward consequence of *(iii)* by using the anti-isomorphism $\widehat{\ }$. ■

2 Automaticity

In this section we will consider automaticity of the subsemigroups of the bicyclic monoid and our main result is the following:

Theorem 8.2 *All finitely generated subsemigroups of the bicyclic monoid are automatic.*

To prove this theorem we will use Proposition 1.15 and the following

Lemma 8.3 *For any numbers $p, m \in \mathbb{N}_0$ with $p \leq m$, $d \in \mathbb{N}$ and sets $I \subseteq \{0, \ldots, p-1\}$, $P \subseteq \{0, \ldots, d-1\}$ such that $0 \in P$, each of the following subsets of the bicyclic monoid is automatic whenever it is a subsemigroup:*

$$(i)\ \ \Lambda_{I,m,d}; \qquad (ii)\ \ \widehat{\Lambda_{I,m,d}};$$
$$(iii)\ \ \Sigma_{p,d,P} \cup \Lambda_{I,p,d}; \qquad (iv)\ \ \Sigma_{p,d,P} \cup \widehat{\Lambda_{I,p,d}}.$$

Proof. We observe that although the semigroups *(ii)* and *(iv)* are obtained from *(i)* and *(iii)* respectively by using the anti-isomorphism $\widehat{\ }$, our notion of automatic structure involves multiplication on the right and so we cannot just apply $\widehat{\ }$ to obtain the latter automatic structures and we need to prove each of the four cases separately.

(i) Let $i + u_i d = \min\{i + ud : i + ud \geq m\}$ for $i \in I$. Fixing $i_0 \in I$ and $u = u_{i_0}$ we define the alphabet

$$\Lambda = \bigcup_{i \in I} \{\lambda(i, 0), \ldots, \lambda(i, u-1)\}$$

and the homomorphism

$$f : \Lambda^* \longrightarrow \Lambda_{I,m,d}; \lambda(i, j) \mapsto c^i b^{i+(u_i+j)d}.$$

Defining

$$L = \bigcup_{i \in I} (\bigcup_{j=0}^{u-1} \{\lambda(i, j)\lambda(i_0, 0)^n : n \geq 0\}$$

it is clear that L is a regular language and we will show that it is a set of unique normal forms for $S = \Lambda_{I,m,d}$. Given $s \in S$ we can write $s = c^i b^{i+(u_i+k)d}$ for some $i \in I$ and $k \geq 0$. Dividing k by u we obtain $k = nu + j$ with $n \geq 0$ and $0 \leq j < u$. It is now clear that the unique word in L representing s is the word $\lambda(i, j)\lambda(i_0, 0)^n$. To prove that the pair (Λ, L) is an automatic structure for S we only have to show that the languages

$$L_{\lambda(k,l)} = \{(w_1, w_2)\delta : w_1, w_2 \in L, w_1\lambda(k, r) = w_2\}$$

are regular for every $\lambda(k,l) \in \Lambda$. We can write

$$\lambda(i,j)\lambda(i_0,0)^n\lambda(k,l) = c^ib^{i+(u_i+j)d+nud}c^kb^{k+(u_k+l)d} = c^ib^{i+(u_i+j+u_k+l)d+nud}$$

and dividing $j+u_k+l$ by u we obtain $j+u_k+l = qu+r$ with $q \geq 0$ and $0 \leq r < u$ and so we have

$$\lambda(i,j)\lambda(i_0,0)^n\lambda(k,l) = c^ib^{i+(u_i+r)d+(n+q)ud} = \lambda(i,r)\lambda(i_0,0)^{n+q}.$$

Therefore we have

$$L_{\lambda(k,l)} = \bigcup_{i\in I}(\bigcup_{j=0}^{u-1}\{\,(\lambda(i,j)\lambda(i_0,0)^n, \lambda(i,r)\lambda(i_0,0)^{n+q})\delta : \\ u_k + j + l = qu + r, 0 \leq r < u, n \geq 0\}).$$

Each inner set in the union,

$$Y_{k,l,i,j} = \{(\lambda(i,j)\lambda(i_0,0)^n, \lambda(i,r)\lambda(i_0,0)^{n+q})\delta : u_k+j+l = qu+r, 0 \leq r < u, n \geq 0\}$$

is a regular language because the numbers q and r are uniquely determined by the fixed numbers k, l, i and j, and we have in fact

$$Y_{k,l,i,j} = \{(\lambda(i,j), \lambda(i,r))\} \cdot \{(\lambda(i_0,0), \lambda(i_0,0)\}^* \cdot \{(\epsilon, \lambda(i_0,0)^q)\delta\}.$$

Hence $L_{\lambda(k,l)}$ is a finite union of regular languages and so is regular.

(ii) We define u_i $(i \in I)$, i_0, u and the alphabet Λ as in the proof of *(i)* but now our homomorphism is

$$f : \Lambda \to S; \lambda(i,j) \mapsto c^{i+(u_i+j)d}b^i$$

and our regular language is

$$L = \bigcup_{i\in I}(\bigcup_{j=0}^{u-1}\{\lambda(i_0,0)^n\lambda(i,j) : n \geq 0\},$$

where $S = \widehat{\Lambda_{I,m,d}}$. It is clear that L is a set of unique normal forms for S, since we have

$$\lambda(i_0,0)^n\lambda(i,j) = c^{i+(u_i+j)d+nud}b^i,$$

and we will prove that the languages

$$L_{\lambda(k,l)} = \{(w_1, w_2)\delta : w_1, w_2 \in L, w_1\lambda(k, r) = w_2\}$$

are regular for every $\lambda(k, l) \in \Lambda$. We can write

$$\lambda(i_0, 0)^n\lambda(i, j)\lambda(k, l) = c^{i+(u_i+j)d+nud}b^i c^{k+(u_k+l)d}b^k = c^{k+(u_k+j+u_i+l)d+nud}b^k$$

and dividing $j+u_i+l$ by u we obtain $j+u_i+l = qu+r$ with $q \geq 0$ and $0 \leq r < u$ and so we have

$$\lambda(i_0, 0)^n\lambda(i, j)\lambda(k, l) = c^{k+(u_k+r)d+(q+n)ud}b^k = \lambda(i_0, 0)^{q+n}\lambda(k, r).$$

Therefore we have

$$L_{\lambda(k,l)} = \bigcup_{i\in I}(\bigcup_{j=0}^{u-1}\{ (\lambda(i_0, 0)^n\lambda(i, j), \lambda(i_0, 0)^{n+q}\lambda(k, r))\delta : \\ u_i + j + l = qu + r, 0 \leq r < u, n \geq 0\})$$

which is a finite union of regular languages and so is regular.

(iii) Let $Y = \Lambda \cup \{x, y\} \cup \Gamma$, where $\Lambda = \{\lambda_i : i \in I\}$ and $\Gamma = \{\gamma_r : r \in P\}$, be an alphabet and

$$L = \bigcup_{i\in I}(\{\lambda_i x^u : u \geq 0\}) \cup \bigcup_{r\in P}(\{y^v\gamma_r x^u : u, v \geq 0\}$$

a regular subset of Y^+. We are going to prove that (Y, L) is an automatic structure (with uniqueness) for the semigroup $S = \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ with respect to

$$f : Y^+ \to S;\ \lambda_i \mapsto c^i b^{i+u_i d}, \gamma_r \mapsto c^{p+r}b^{p+r}, x \mapsto c^p b^{p+d}, y \mapsto c^{p+d}b^p$$

where $i + u_i d = \max\{i + ud : i + ud \geq p\}$ for $i \in I$.

To show that each element in S has a unique representative in L it suffices to observe that

$$\begin{aligned} \lambda_i x^u &= c^i b^{i+(u_i+u)d} && (i \in I; u \geq 0), \\ y^v\gamma_r x^u &= c^{p+r+vd}b^{p+r+ud} && (r \in P; u, v \geq 0). \end{aligned}$$

Therefore we only have to show that that languages $L_y = \{(w_1, w_2)\delta : w_1, w_2 \in L, w_1 y = w_2\}$ are regular for every $y \in Y$. We will first consider the case where $y = \lambda_t \in \Lambda$. Since $\Psi((\lambda_i x^u)f) \geq p > t = \Phi(\lambda_t f)$ and $\Psi((y^v\gamma_r x^u)f) \geq p > t = \Phi(\lambda_t f)$ we have

$$L_{\lambda_t} = \bigcup_{i\in I}\{(\lambda_i x^u, \lambda_i x^{u+u_t})\delta : u \geq 0\} \cup \bigcup_{r\in P}\{(y^u\gamma_r x^u, y^v\gamma_r x^{u+u_t})\delta : u, v \geq 0\}$$

which is a regular language. We will now consider $y = \gamma_t \in \Gamma$. Since for $u > 0$ we have $\Psi((\lambda_i x^u)f), \Psi((y^v\gamma_r x^u)f) \geq p + d > \Phi(\gamma_t f)$ we have

$$\begin{aligned} L_{\gamma_t} = &\bigcup_{i\in I}(\{(\lambda_i x^u, \lambda_i x^u)\delta : u > 0\} \cup \{(\lambda_i, w)\delta : w \in L, \lambda_i\gamma_t = w\})\cup \\ &\bigcup_{r\in P}(\{(y^v\gamma_r x^u, y^v\gamma_r x^u)\delta : v \geq 0, u > 0\} \cup L_{(\gamma_t,r)}) \end{aligned}$$

where

$$L_{(\gamma_t,r)} = \begin{cases} \{(y^u\gamma_r, y^u\gamma_r)\delta : u \geq 0\} \text{ if } r \geq t \\ \{(y^u\gamma_r, y^u\gamma_t)\delta : u \geq 0\} \text{ otherwise .} \end{cases}$$

We note that, for each $i \in I$, the set $\{(\lambda_i, w)\delta : w \in L, \lambda_i\gamma_t = w\})$ has only one element because L is a set of unique normal forms for S, and so the language L_{γ_t} is a finite union of regular languages and therefore it is regular. The language L_x is clearly regular since we have

$$L_x = \{(w, wx)\delta : w \in L\}.$$

Finally, we have

$$\begin{aligned} L_y = &\bigcup_{i\in I}(\{(\lambda_i x^u, \lambda_i x^{u-1})\delta : u > 0\} \cup \{(\lambda_i, w)\delta : w \in L, \lambda_i y = w\})\cup \\ &\bigcup_{r\in P}(\{(y^v\gamma_r x^u, y^v\gamma_r x^{u-1})\delta : v \geq 0, u > 0\} \cup \{(y^v\gamma_r, y^{v+1}\gamma_0)\delta : v \geq 0\}) \end{aligned}$$

because, for $v \geq 0$, we have

$$(y^v\gamma_r)y = (c^{p+r+vd}b^{p+r})(c^{p+d}b^p) = c^{p+(v+1)d}b^p = y^{v+1}\gamma_0.$$

Again, for each $i \in I$, the set $\{(\lambda_i, w)\delta : w \in L, \lambda_i y = w\})$ is regular because it has only one element and so L_y is a finite union of regular languages and hence is regular. We conclude that S is automatic.

(iv) We define the alphabet Y as in the proof of *(iii)* and our regular language over Y^+ is now

$$L = \bigcup_{i\in I}(\{y^v\lambda_i : v \geq 0\}) \cup \bigcup_{r\in P}(\{y^v\gamma_r x^u : u, v \geq 0\}.$$

We are going to prove that (Y, L) is an automatic structure (with uniqueness) for the semigroup $S = \Sigma_{p,d,P} \cup \widehat{\Lambda_{I,p,d}}$ with respect to

$$f : Y^+ \to S;\ \lambda_i \mapsto c^{i+u_i d}b^i, \gamma_r \mapsto c^{p+r}b^{p+r}, x \mapsto c^p b^{p+d}, y \mapsto c^{p+d}b^p$$

again with $i + u_i d = \max\{i + ud : i + ud \geq p\}$ for $i \in I$.

It is again clear that L is a set of unique normal forms for S and we will show that the languages $L_y = \{(w_1, w_2)\delta : w_1, w_2 \in L, w_1 y = w_2\}$ are regular for every $y \in Y$. We start by showing that, for any $\lambda_t \in \Lambda$, we have

$$\begin{aligned} L_{\lambda_t} = &\bigcup_{i\in I}\{(y^v\lambda_i, y^{v+u_i}\lambda_t)\delta : v \geq 0\}\cup \\ &\bigcup_{r\in P}(\{(y^v\gamma_r x^u, y^v\gamma_r x^{u-u_t})\delta : v \geq 0, u \geq u_t\} \cup L_{(\lambda_t,r)} \\ &\bigcup_{u=1}^{u_t-1}\{(y^v\gamma_r x^u, y^{v+u_t-u-u_k}\lambda_k)\delta : v \geq 0, k = p + r + (u - u_t)d\}) \end{aligned}$$

where

$$L_{(\lambda_t,r)} = \begin{cases} \{(y^v\gamma_r, y^v\lambda_t)\delta : v \geq 0\} \text{ if } p + r \leq t + u_t d \\ \{(y^v\gamma_r, y^{v+u_t-u_k}\lambda_k)\delta : k = p + r - u_t d\} \text{ otherwise.} \end{cases}$$

We have

$$y^v\lambda_i\lambda_t = c^{i+u_i d+vd}b^i c^{t+u_t d}b^t = c^{t+u_t d+(v+u_i)d}b^t = y^{v+u_i}\lambda_t.$$

If $u \geq u_t$ then

$$y^v\gamma_r x^u\lambda_t = c^{p+r+vd}b^{p+r+ud}c^{t+u_t d}b^t = c^{p+r+vd}b^{p+r+(u-u_t)d} = y^v\gamma_r x^{u-u_t}.$$

For $u \in \{1, \ldots, u_t - 1\}$ we define $k = p + r + (u - u_t)d$ and we have

$$\begin{aligned} z &= y^v \gamma_r x^u \lambda_t = c^{p+r+vd} b^{p+r+ud} c^{t+u_t d} b^t = c^{p+r+vd} b^{p+r+(u-u_t)d} \\ &= c^{k+(v+u_t-u)d} b^k = c^{k+u_k d+(v+u_t-u-u_k)d} b^k. \end{aligned}$$

Since S is a semigroup and $k < p$ we have $z \in \widehat{\Lambda_{I,p,d}}$ and therefore, observing the definition of u_k, it must be $v + u_t - u - u_k \geq 0$ and we can write $z = y^{v+u_t-u-u_k} \lambda_k$. We will now consider the multiplication of a word of the form $y^v \gamma_r$ by λ_t and so we define

$$z = y^v \gamma_r \lambda_t = c^{p+r+vd} b^{p+r} c^{t+u_t d} b^t.$$

If $p + r \leq t + u_t d$ then $z = c^{t+u_t d+vd} b^t = y^v \lambda_t$. If $p + r > t + u_t d$ we have $z = c^{p+r+vd} b^{p+r-u_t d}$. We observe that $u_t > 0$ because $t < p$ and $t + u_t d \geq p$ and therefore $z \in \widehat{\Lambda_{I,p,d}}$. Hence, defining $k = p + r - u_t d$ we can write

$$z = c^{k+(v+u_t)d} b^k = c^{k+u_k d+(v+u_t-u_k)d} b^k$$

and, from the definition of u_k, it follows that $v + u_t - u_k \geq 0$ and so we have

$$z = y^{v+u_t-u_k} \lambda_k.$$

We conclude that L_{λ_t} can be defined as a finite union of regular languages and so it is a regular language.

It is easy to see that

$$\begin{aligned} L_{\gamma_t} = & \bigcup_{i \in I} \{(y^v \lambda_i, y^{v+u_i} \gamma_t)\delta : v \geq 0\} \cup L_{(\gamma_t, r)} \\ & \bigcup_{r \in P} \{(y^v \gamma_r x^u, y^v \gamma_r x^u)\delta : u > 0, v \geq 0\} \end{aligned}$$

where

$$L_{(\gamma_t, r)} = \begin{cases} \{(y^v \gamma_r, y^v \gamma_r)\delta : v \leq 0\} \text{ if } r \geq t \\ \{(y^v \gamma_r, y^v \gamma_t)\delta : v \geq 0\} \text{ otherwise} \end{cases}$$

and so it is a regular language. The language L_x is regular because we have

$$L_x = \bigcup_{i \in I} \{(y^v \lambda_i, y^{u_i+v} \gamma_0 x)\delta : v \geq 0\} \cup \bigcup_{r \in P} \{(y^v \gamma_r x^u, y^v \gamma_r x^{u+1})\delta : u, v \geq 0\}$$

and since

$$L_y = \bigcup_{i \in I} \{(y^v \lambda_i, y^{v+u_i+1}\gamma_0)\delta : v \geq 0\} \cup \bigcup_{r \in P} (\{(y^v \gamma_r x^u, y^v \gamma_r x^{u-1})\delta : v \geq 0, u > 0\} \cup \{(y^v \gamma_r, y^{v+1}\gamma_0)\delta : v \geq 0\})$$

L_y is a regular language as well. We conclude that (Y, L) is an automatic structure for S. ■

Proof of Theorem 8.2 We know from the previous section that any finitely generated subsemigroup is either a finite subset of the diagonal, and so it is automatic, or it has one of the forms:

$$F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}, \ F_D \cup F \cup \widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P},$$
$$F_D \cup F \cup \Lambda_{I,p,d}, \ F_D \cup F \cup \widehat{\Lambda_{I,p,d}}$$

where $I \subseteq \{q, q+1, \ldots, p-1\}$ for some numbers $q, p \in \mathbb{N}_0$ and the sets F and F_D are finite. In each case we can remove the finite set $F_D \cup F$ from our subsemigroup and we still have a subsemigroup, because we are in fact intersecting it with the set $S_{q,p} \cup \Sigma_p$ (or the set $\widehat{S_{q,p}} \cup \Sigma_p$), which by Lemma 7.4 is itself a subsemigroup. Hence every finitely generated subsemigroup S of $\mathbf{B}$ has a subsemigroup U such that $S \backslash U$ is finite and that, by the previous lemma, is automatic. It follows from Proposition 1.15 that S is automatic as well. ■

3 Finite presentability

Let A be an alphabet and R be a relation on A^+. We say that the semigroup S is defined by the presentation $\langle A \mid R \rangle$ if $S \cong A^+/\rho$ where ρ is the smallest congruence on A^+ that contains R (see Appendix A).

Given a semigroup S with a presentation $\langle A \mid R \rangle$, for two words $w, v \in A^+$ we write $w \rightarrow^* v$, and we say that $w = v$ *is a consequence of* R (or that the word w *can be reduced to* v by applying relations from R), to mean that either $w \equiv v$ or there is a sequence a words $w \equiv w_1, w_2, \ldots, w_n \equiv v$ where $w_i \equiv w_i' v_i w_i''$ with $w_i', w_i'' \in A^*$ and v_i in A^+ $(i = 1, \ldots, n)$ such that either $(v_i, v_{i+1}) \in R$ or $(v_{i+1}, v_i) \in R$ for each $i = 1, \ldots, n-1$. We will need the following result:

Proposition 8.4 *Let S be a semigroup generated by a set A and let $R \subseteq A^+ \times A^+$. Then $\langle A \mid R \rangle$ is a presentation for S if and only if the following conditions hold:*

(i) *S satisfies all the relations from R;*

(ii) *If $u, v \in A^+$ are two words such that $u = v$ in S, then $u = v$ is a consequence of R.*

Proof. See [42]. ■

The following straightforward consequence of this proposition will be used, whenever we have a set of unique normal forms $L \subseteq A^+$ for the semigroup S, to prove that a given pair $\langle A \mid R \rangle$ is a presentation for S.

Proposition 8.5 *Let S be a semigroup generated by a set A, let $R \subseteq A^+ \times A^+$ and let $L \subseteq A^+$ be a set of unique normal forms for S. If the following conditions hold then $\langle A \mid R \rangle$ is a presentation for S.*

(i) *S satisfies all the relations from R;*

(ii) *Any word $w \in A^+$ can be reduced to the corresponding unique normal form in L by using relations from R.*

For further details about semigroup presentations we refer the reader to [42].

Our main result of this section is the following:

Theorem 8.6 *All finitely generated subsemigroups of the bicyclic monoid are finitely presented.*

From [43] we have the following:

Proposition 8.7 *Let S be a semigroup and T be a subsemigroup of S such that $S \backslash T$ is finite. Then S is finitely presented if and only if T is finitely presented.*

Our main result will be proved using this proposition and the following result:

Lemma 8.8 *For any numbers $p, m \in \mathbb{N}_0$ with $p \leq m$, $d \in \mathbb{N}$ and sets*

$$I \subseteq \{0, \ldots, p-1\}, \ P \subseteq \{0, \ldots, d-1\}$$

such that $0 \in P$, each of the following subsets of the bicyclic monoid is finitely presented whenever it is a subsemigroup:

(i) $\Lambda_{I,m,d}$;

(ii) $\Lambda_{I,m,d} \cup \Sigma_{p,d,P}$.

PROOF. *(i)* We consider the automatic structure (Λ, L) obtained in the proof of Lemma 8.3 *(i)*, which gives us a finite generating set and a set of unique normal forms for $\Lambda_{I,m,d}$. We are going to prove that $< \Lambda \mid R >$ is a finite presentation for T, defining R to be a set of equations that allow us to re-write each product of two generators into a word in L. More precisely, R consists of the following relations:

$$\lambda(i,j)\lambda(k,l) = \lambda(i,r)\lambda(i_0,0)^q \text{ where } j + u_k + l = qu + r, \ 0 \leq r < u$$
$$(i, k \in I, j, l \in \{0, \ldots, u-1\}).$$

That the relations hold follows from the definition of $L_{\lambda(k,l)}$ in the proof of Lemma 8.3 *(i)*. We are going to show that any word $w \in \Lambda^+$ can be reduced to a word in L by applying relations from R, using induction in the length $|w|$ of the word w. If $|w| = 1$ then $w \in L$ by definition of L. If $|w| = 2$ then $w = \lambda(i,j)\lambda(k,l)$ and therefore

$$w \to^* \lambda(i,r)\lambda(i_0,0)^q \in L$$

with

$$j + u_k + l = qu + r \ (0 \leq r < u),$$

which is a relation in R. Let $n \geq 2$ and suppose that any word w such that $|w| \leq n$ can be reduced to a word in L by using relations from R. Let $w \in \Lambda^+$ with $|w| = n+1$. We have $w = \lambda(i_1, j_1) \ldots \lambda(i_n, j_n)\lambda(i_{n+1}, j_{n+1})$. Therefore

$$w \rightarrow^* \lambda(i_1, j_1) \ldots \lambda(i_{n-1}, j_{n-1})\lambda(i_n, r)\lambda(i_0, 0)^q$$

where

$$j_n + u_{i_{n+1}} + j_{n+1} = qu + r \ (0 \leq r < u).$$

Letting $w' = \lambda(i_1, j_1) \ldots \lambda(i_{n-1}, j_{n-1})\lambda(i_n, r)$ we have $|w'| = n$ and, using the induction hypothesis, we have

$$w' \rightarrow^* \lambda(i, j)\lambda(i_0, 0)^m \in L$$

for some $i \in I, \ j \in \{0, \ldots, u-1\}, \ m \in \mathbb{N}_0$, implying

$$w \rightarrow^* \lambda(i, j)\lambda(i_0, 0)^{m+q} \in L.$$

(ii) We will use the automatic structure (Y, L) obtained in the proof of Lemma 8.3 *(iii)* to prove that $T = \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ is finitely presented. We will show that $< Y \mid R >$ is a finite presentation for T, defining R to be a set of relations that

allows us to re-write words of length smaller than three into words in L:

$$x = \gamma_0 x \tag{8.1}$$

$$y = y\gamma_0 \tag{8.2}$$

$$\lambda_i\lambda_j = \lambda_i x^{u_j} \quad (i,j \in I) \tag{8.3}$$

$$x\lambda_i = x^{1+u_i} \quad (i \in I) \tag{8.4}$$

$$y\lambda_i = yx^{u_i} \quad (i \in I) \tag{8.5}$$

$$\gamma_r\lambda_i = \gamma_r x^{u_i} \quad (r \in P, i \in I) \tag{8.6}$$

$$xy = \gamma_0 \tag{8.7}$$

$$\lambda_i y = \lambda_j \quad (i \in I, u_i > 1, j = p + d - u_i d) \tag{8.8}$$

$$\lambda_i y = \gamma_0 \quad (i \in I, u_i = 1) \tag{8.9}$$

$$\gamma_r y = y \quad (r \in P) \tag{8.10}$$

$$x\gamma_r = x \quad (r \in P) \tag{8.11}$$

$$\lambda_i\gamma_r = \lambda_i \quad (i \in I, r \in P, i + u_i d \geq p + r) \tag{8.12}$$

$$\lambda_i\gamma_r = \lambda_j \quad (i \in I, r \in P, i + u_i d < p + r, j = p + r - u_i d) \tag{8.13}$$

$$\gamma_r\gamma_t = \gamma_r \quad (r \geq t) \tag{8.14}$$

$$\gamma_r\gamma_t = \gamma_t \quad (r < t) \tag{8.15}$$

To see that a relation holds we just have to prove that both sides of it correspond to the same word in $\{c^i b^j : i,j \geq 0, (i,j) \neq (0,0)\}$. We will only prove that equations (8.8), (8.9), (8.12) and (8.13) hold since for the others it is straightforward.

To prove that relations (8.8) and (8.9) hold we observe that, by definition of u_i, we have

$$\lambda_i y = c^i b^{i+u_i d} c^{p+d} b^p = c^{p+d-u_i d} b^p.$$

If $u_i = 1$ then $\lambda_i y = c^p b^p = \gamma_0$ and relation (8.9) holds. If $u_i > 1$ then $p+d-u_i d < p$ and so, defining $j = p + d - u_i d$, we have $\lambda_i y = c^j b^{j+(u_i-1)d} \in \Lambda_{I,p,d}$. But we have $j + (u_i - 1)d = p$ which implies, by definition of u_j, that $u_i - 1 = u_j$ which means that $\lambda_i y = \lambda_j$ and relation (8.9) holds as well.

To prove that relations (8.12) and (8.13) hold we start by writing

$$\lambda_i\gamma_r = c^i b^{i+u_i d} c^{p+r} b^{p+r}.$$

If $i + u_i d \geq p + r$ then $\lambda_i\gamma_r = c^i b^{i+u_i d} = \lambda_i$ and relation (8.12) holds. Otherwise we have $\lambda_i\gamma_r = c^{p+r-u_i d} b^{p+r} \in \Lambda_{I,p,d}$ because $u_i > 0$. Defining $j = p + r - u_i d$ we have $\lambda_i\gamma_r = c^j b^{j+u_i d}$ and, since $j + u_i d = p+r < p+d$ and using the definition of u_j, it must be $u_i = u_j$ what implies $\lambda_i\gamma_r = \lambda_j$ and relation (8.13) holds as well.

We are now going to prove that any word in $w \in Y^+$ can be reduced to a word in L, using our relations, by induction on the length of w. If $|w| = 1$ then either $w \in L$ or it can be reduced to a word in L by using one of the relations (8.1) and (8.2). We now consider words of length 2. The word $\lambda_i\lambda_t$ reduces to $\lambda_i x^{u_t} \in L$ using relation (8.3); $\lambda_i x \in L$; $\lambda_i y$ either reduces to $\gamma_0 \in L$ using relation (8.9) or to $\lambda_j \in L$ for some j using relation (8.8); $\lambda_i\gamma_r$ reduces to $\lambda_j \in L$ for some j using relations (8.12) or (8.13); xx reduces to $\gamma_0 x^2 \in L$ using (8.1); xy reduces to $\gamma_0 \in L$ using relation (8.7); $x\lambda_i$ reduces to $\gamma_0 x^{1+u_i} \in L$ using relations (8.4) and (8.1); $x\gamma_t$ reduces to $\gamma_0 x \in L$ using relations (8.11) and (8.1); yx reduces to $y\gamma_0 x \in L$ using (8.1); yy reduces to $y^2\gamma_0 \in L$ using (8.2); $y\lambda_i$ reduces to $y\gamma_0 x^{u_i} \in L$ using (8.5) and (8.2); $y\gamma_t \in L$; $\gamma_i x \in L$; $\gamma_i y$ reduces to $y\gamma_0 \in L$ using (8.10) and (8.2); $\gamma_i\lambda_t$ reduces to $\gamma_i x^{u_t} \in L$ using (8.6); finally $\gamma_i\gamma_r$ reduces to $\gamma_j \in L$ for some j using (8.14) or (8.15).

In the following induction step we use that fact that if a word w belongs to L then wx^n belongs to L as well for any $n \in \mathbb{N}_0$, which follows immediately from the definition of L. Let $n \geq 2$ and suppose that all words $w \in Y^+$ with $|w| \leq n$ can be reduced to a word in L. Let $w \in Y^+$ be a word of length $n+1$. Then we have $w = w_1 g_1 g_2$ with $w_1 \in Y^+$ and $g_1, g_2 \in Y$. We will consider all possible pairs of generators $g_1, g_2 \in Y$ and prove that in every case w reduces to a word in L using the relations.

Case 1: $g_1 g_2 \in \{\lambda_i y, \lambda_i\gamma_t, xy, x\gamma_t, \gamma_t y, \gamma_t\gamma_i\}$. In these case we can apply one of the relations to reduce $g_1 g_2$ to a generator g. We can then apply the induction hypothesis to reduce $w_1 g$ to a word in L.

Case 2: $g_1g_2 \equiv g_1x$. In these cases we can reduce w_1g_1 to a word $w_2 \in L$ using the induction hypothesis and so we can reduce w to $w_2x \in L$.

Case 3: $g_1g_2 \equiv \lambda_i\lambda_t$. Using relation (8.3) we have $w \to^* w_1\lambda_i x^{u_t}$ and, since $|w_1\lambda_i| = n$, using the induction hypothesis we have $w_1\lambda_i \to^* w_2 \in L$ and therefore $w \to^* w_2x^{u_t} \in L$.

Case 4: $g_1g_2 \equiv x\lambda_t$. Using relation (8.4) we have $w \to^* w_1x^{1+u_t}$. Since $|w_1| \leq n$, using the hypothesis we can write $w_1 \to^* w_2 \in L$ and so $w \to^* w_1x^{1+u_t} \to^* w_2x^{1+u_t} \in L$.

Case 5: $g_1g_2 \equiv y\lambda_t$. Using relation (8.5) we reduce $y\lambda_t$ to yx^{u_t}. We can then apply the induction hypothesis to w_1y to obtain $w_1y \to^* w_2 \in L$ implying $w \to^* w_2x^{u_t} \in L$.

Case 6: $g_1g_2 = yy$. We start by reducing w_1y to a word $w_2 \in L$ using the induction hypothesis. It can be $w_2 = \lambda_i x^u$ or $w_2 = y^v\gamma_r x^u$. If $w_2 = \lambda_i$ then $w \to^* \lambda_i y$ and applying relations (8.8) or (8.9) it reduces to a word in L. If $w_2 = \lambda_i x$ then $w \to^* \lambda_i xy \to^* \lambda_i\gamma_0$ by applying relation (8.7). Therefore by applying now relations (8.12) or (8.13), w reduces to word in L. If $w_2 = \lambda_i x^u$ with $u > 1$ then

$$w \to^* \lambda_i x^{u-1}xy \to^* \lambda_i x^{u-2}x\gamma_0 \to^* \lambda_i x^{u-1} \in L,$$

by applying relations (8.7) and (8.11). If $w_2 = y^v\gamma_r$ then

$$w \to^* y^v\gamma_r y \to^* y^v y \to^* y^{v+1}\gamma_0 \in L,$$

using relations (8.10) and (8.2). If $w_2 = y^v\gamma_r x$ then $w \to^* y^v\gamma_r xy$ and we can apply relation (8.7) to reduce xy to γ_0. Then we can reduce $\gamma_r\gamma_0$ to γ_r by applying relation (8.14) and so $w \to^* y^v\gamma_r \in L$. If $w_2 = y^v\gamma_r x^u$ with $u > 1$ then

$$w \to^* y^v\gamma_r x^{u-1}xy \to^* y^v\gamma_r x^{u-2}x\gamma_0 \to^* y^v\gamma_r x^{u-1} \in L$$

by applying relations (8.7) and (8.11).

Case 7: $g_1g_2 \equiv y\gamma_t$. We start again by reducing w_1y to a word $w_2 \in L$. It can be $w_2 = \lambda_i x^u$ or $w_2 = y^v\gamma_r x^u$. If $w_2 = \lambda_i$ then $w \to^* \lambda_i y$ and applying relation (8.8) or relation (8.9) we can reduce w to a generator that belongs to L. If $w_2 = \lambda_i x^u$ with $u > 0$ then we can apply relation (8.11) giving

$$w \to^* \lambda_i x^u \gamma_t \to^* \lambda_i x^u \in L.$$

If $w_2 = y^v \gamma_r$ then $w \to^* y^v \gamma_r \gamma_t$ and so applying relations (8.14) or (8.15) we have $w \to^* y^v g \in L$ with $g \in \{\gamma_r, \gamma_t\}$. Finally, if $w_2 = y^v \gamma_r x^u$ with $u > 0$ then we have

$$w \to^* y^v \gamma_r x^u \gamma_t \to^* y^v \gamma_r x^u \in L$$

by applying relation (8.11).

Case 8: $g_1 g_2 \equiv \gamma_t \lambda_i$. Applying relation (8.6) we get $\gamma_t \lambda_i \to^* \gamma_t x^{u_i}$. Since $|w_1 \gamma_t| \leq n$, using the hypothesis, we have $w_1 \gamma_t \to^* w_2 \in L$ and so $w \to^* w_2 x^{u_i} \in L$. ■

Lemma 8.9 *If S is a finitely presented subsemigroup of the bicyclic monoid* $\mathrm{B} =< b, c \mid bc = 1 >$ *then the semigroup* $T = \{c^i b^j : c^j b^i \in S\}$ *is finitely presented as well.*

Proof. If $< A \mid R >$ is a finite presentation for S then $< A \mid R' >$ is a finite presentation for T where $x_1 \dots x_i = y_1 \dots y_j$ belongs to R if and only if $x_i \dots x_1 = y_j \dots y_1$ belongs to R'. In fact, T is the opposite of S and so it is finitely presented if and only if S is finitely presented. ■

Proof of Theorem 8.6 We know from Section 1 that any finitely generated subsemigroup is either a finite subset of the diagonal, and so it is finitely presented, or it has one of the forms:

$$F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P},\ F_D \cup F \cup \widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P},$$
$$F_D \cup F \cup \Lambda_{I,p,d},\ F_D \cup F \cup \widehat{\Lambda_{I,p,d}}$$

where $I \subseteq \{q, q+1, \dots, p-1\}$ for some numbers $q, p \in \mathbb{N}_0$ and the sets F and F_D are finite. The previous lemma allows us to consider only subsemigroups of the form

$$F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}, \;\; F_D \cup F \cup \Lambda_{I,p,d}.$$

In both cases we can remove the finite set $F_D \cup F$ from our subsemigroup and we still have a subsemigroup. Hence, in both cases, our subsemigroup S has a subsemigroup U such that $S \backslash U$ is finite and that, by Lemma 8.8, is finitely presented. It follows from Proposition 8.7 that S is finitely presented as well. ■

4 Residual finiteness

In this section we give necessary and sufficient conditions for a subsemigroup of the bicyclic monoid to be residually finite.

We say that a semigroup S is *residually finite* if, for any two elements $s_1, s_2 \in S$, there is a finite semigroup F and a homomorphism $\phi : S \to F$ that separates s_1 and s_2 (such that $s_1\phi \neq s_2\phi$). We start by showing that the bicyclic monoid $B = \langle b, c \mid bc = 1\rangle$ is not residually finite.

We need the following facts:

Lemma 8.10 *Let $C_j = \{x, x^2, \ldots, x^{j-1}, 1\}$ be a cyclic group of order j, for some $j \in \mathbb{N}$. Then the function $\phi : \mathbf{B} \to C_j;\; c^m b^n \mapsto x^{m-n}$ is a homomorphism.*

Proof. Let $c^m b^n, c^p b^q \in \mathbf{B}$ arbitrary. We have

$$(c^m b^n)\phi(c^p b^q)\phi = x^{n-m}x^{q-p} = x^{n-m+q-p}.$$

If $n \geq p$ then

$$(c^m b^n c^p b^q)\phi = (c^m b^{n-p+q})\phi = x^{n-p+q-m} = (c^m b^n)\phi(c^p b^q)\phi$$

and otherwise we still have

$$(c^m b^n c^p b^q)\phi = (c^{m-n+p} b^q)\phi = x^{q-m+n-p} = (c^m b^n)\phi(c^p b^q)\phi$$

which concludes the proof. ■

Lemma 8.11 *Let F be a finite semigroup and let $\phi : \mathbf{B} \to F$ be an onto homomorphism. Then F is a cyclic group $C_j = \{x, x^2, \ldots, x^{j-1}, 1\}$ of order j, for some $j \in \mathbb{N}$, and $(c^m b^n)\phi = x^{m-n}$ for any $m, n \in \mathbb{N}_0$.*

Proof. Let $c\phi = x$ and $b\phi = y$. Then F is a monoid with identity $1\phi = (bc)\phi = yx = 1$. The subsemigroup of F generated by x is finite and so we can take minimum i, j with $i \leq j$ such that $x^{j+1} = x^i$, and the elements $x, x^2, \ldots, x^j$ are all distinct. But $x^i = x^{j+1}$ implies $y^i x^i = y^i x^{j+1}$ and so

$$x^{j+1-i} = c^{j+1-i}\phi = (b^i c^{j+1})\phi = y^i x^{j+1} = y^i x^i = (b^i c^i)\phi = 1\phi = 1.$$

Supposing that $i > 1$ we have $j + 1 - i < j$ and so $\gamma = j + 1 - i + 1 \leq j$. Hence

$$x^\gamma = x^{j+1-i+1} = x^{j+1-i}x = x$$

what contradicts the fact that the elements $x, x^2, \ldots, x^j$ are all distinct. Therefore it must be $i = 1$ and the semigroup $\langle x \rangle$ is in fact the cyclic group, of order j, $C_j = \{x, x^2, \ldots, x^{j-1}, 1\}$. Moreover, the associativity in F implies that $y = y(xx^{j-1}) = (yx)x^{j-1} = x^{j-1} = x^{-1}$ and so for any $n, m \in \mathbb{N}_0$ we have $(c^n b^m)\phi = x^n (x^{-1})^m = x^{n-m}$. ■

Theorem 8.12 *The bicyclic monoid is not residually finite.*

Proof. It follows from the previous lemmas that, for example, two different idempotents $c^n b^n, c^m b^m \in \mathbf{B}$ cannot be separated by any homomorphism $\phi : \mathbf{B} \to F$ with F being a finite semigroup. ■

The following was shown in [31] and we include our proof for completeness.

Lemma 8.13 *A subset of the form $I_p = \{c^i b^j : 0 \leq i \leq j, j \geq p\}$ ($p \in \mathbb{N}_0$) is an ideal of U.*

Proof. Let $\alpha = c^i b^j \in I_p$ and $\beta = c^k b^l \in U$. We first consider the product $\alpha\beta$. If $j \geq k$ then $\alpha\beta = c^i b^{j-k+l} \in I_p$ since $j - k + l \geq j \geq p$ and otherwise we have $\alpha\beta = c^{i-j+k} b^l \in I_p$, since $l \geq k \geq j \geq p$. We now consider the product $\beta\alpha$. If $l \geq i$ then $\beta\alpha = c^k b^{l-i+j} \in I_p$ since $l - i + j \geq j \geq p$, and otherwise we have $\beta\alpha = c^{k-l+i} b^j \in I_p$ since $j \geq p$. ■

Our main result follows.

Theorem 8.14 *A subsemigroup of the bicyclic monoid is residually finite if and only if it does not contain elements both above and below the diagonal.*

Proof. We first show that a two-sided semigroup is not residually finite. In fact, a two-sided semigroup S contains a subset of the form

$$X = \{c^{p+ud} b^{p+vd}; u, v \geq 0\},$$

which is a subsemigroup isomorphic to the bicyclic monoid; the function

$$\psi : \mathbf{B} \to X; c^u b^v \mapsto c^{p+ud} b^{p+vd}$$

is clearly an isomorphism. If S was residually finite then, for any two elements $x_1, x_2 \in X$ there would be an homomorphism $\phi : S \to F$, with F finite, separating x_1, x_2 and so there would be an homomorphism $\psi\phi : \mathbf{B} \to F$ separating $x\psi^{-1}, y\psi^{-1}$, which would imply, since ψ is a bijection, that the bicyclic monoid is residually finite.

We will now show that a subsemigroup S contained in U (an upper semigroup or a subset of the diagonal) is residually finite. Let $\alpha = c^i b^j$ and $\beta = c^k b^l$ be two arbitrary elements of S. Taking $p \geq \max(j, l)$ the set $S_p = S \cap I_p$ is an ideal of S. Hence the Rees homomorphism $\phi : S \to (S \backslash S_p) \cup \{0\}$ separates α and β, and $S \backslash S_p \cup \{0\}$ is finite, since $S \backslash S_p \subseteq T_{0,p}$. Analogously, any subsemigroup contained in $\widehat{U}$ is residually finite. ■

We have seen that the bicyclic monoid is strongly automatic (all its finitely generated subsemigroups are automatic). Combining this chapter with Chapter 6 we have the following natural question:

Question 8.15 Let $S = \mathbf{B} * \mathbf{B}$ be the free product of two copies of the bicyclic monoid. Is S strongly automatic?

We can consider generalizations of the bicyclic monoid, as for example in [1], and a natural problem is the following:

Question 8.16 Consider generalizations of the bicyclic monoid that are still strongly automatic.

In Chapter 5 we saw that Bruck–Reilly extensions of a finite monoid are automatic. We can ask:

Question 8.17 Are the Bruck–Reilly extensions of a finite monoid strongly automatic?

Appendix A

Semigroups

For completeness and clarity we list here the basic definitions and results used in the thesis which are not included in the introduction. This material can be found with more detail in [29].

Basic definitions

Let S be a semigroup. An element $e \in S$ is a *left identity* if for all $s \in S$ we have $es = s$ and a *right identity* if for all $s \in S$ we have $se = s$. An *identity* is an element $1 \in S$ such that for all $s \in S$ we have $1s = s1 = s$. We denote by S^1 the monoid obtained from S by adjoining an identity ($S^1 = S \cup \{1\}$ and the operation is extended by $s1 = 1s = s, s \in S$, $11 = 1$). An *idempotent* is an element $e \in S$ such that $ee = e$ and, finally, a *zero* is an element $0 \in S$ such that for all $s \in S$ we have $s0 = 0s = 0$.

Given a set X, the *full transformation semigroup on* X is the semigroup $(\mathcal{T}_X, \circ)$ where $\mathcal{T}_X$ is the set of all functions from X to X and the operation $\circ$ is the composition of functions. A *transformation semigroup* is any subsemigroup of $(\mathcal{T}_X, \circ)$.

Let X be a set. A *(binary) relation* on X is a subset ρ of the cartesian product $X \times X$; we normally write $x_1 \ \rho \ x_2$ instead of $(x_1, x_2) \in \rho$. Given two relations ρ, τ on X we can define their composition $\rho \circ \tau$ by the rule that $s \ (\rho \circ \tau) \ u$ if there is $t \in X$ such that $s \ \rho \ t$ and $t \ \tau \ u$. An *equivalence relation* on X is a relation ρ that is

reflexive: $(\forall x \in X) \ x \ \rho \ x$;

symmetric: $(\forall x_1, x_2 \in X) \ x_1 \ \rho \ x_2 \implies x_2 \ \rho \ x_1$;

and *transitive*: $(\forall x_1, x_2, x_3 \in X) \ x_1 \ \rho \ x_2 \ \& \ x_2 \ \rho \ x_3 \implies x_1 \ \rho \ x_3$.

An equivalence relation ρ defines a partition on X and each element of the partition is called an *equivalence class*; we denote by $a\rho$ the equivalence class of the element $a \in X$ and by X/ρ the set of all classes. A *(partial) order* on X is a relation, normally denoted by $\leq$, that is reflexive, transitive and

antisymmetric: $(\forall x_1, x_2 \in X) \ x_1 \leq x_2 \ \& \ x_2 \leq x_1 \implies x_1 = x_2$.

Finally, an *operation* on X is any function from $X \times X$ to X.

Ideals

Let S be a semigroup and let X be a non-empty subset of S. The set X is called a *left ideal* if $SX \subseteq X$, a *right ideal* if $XS \subseteq X$ and a (two-sided) *ideal* if it is both a left and a right ideal. Evidently every ideal (whether right, left or two-sided) is a subsemigroup, but the converse is not true. Given $a \in S$, the smallest left ideal of S containing a is $S^1 a = Sa \cup \{a\}$ and we call it the *principal left ideal generated by a*. Analogously, the *principal right ideal generated by a* is aS^1.

A semigroup without zero is called *simple* if it has no proper ideals. A semigroup S with zero is called *0-simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$.

Green's relations

The equivalence $\mathcal{L}$ on S is defined by the rule that $a \ \mathcal{L} \ b$ if and only if a and b generate the same principal left ideal, that is, if and only if $S^1 a = S^1 b$. Similarly, we define the equivalence $\mathcal{R}$ by the rule that $a \ \mathcal{R} \ b$ if and only if $aS^1 = bS^1$. The relation $\mathcal{H}$ is the intersection of $\mathcal{L}$ and $\mathcal{R}$. Since $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ we define the relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Finally we define the relation $\mathcal{J}$ by $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$. The following notation is used for equivalence classes of the Green's relations: the $\mathcal{L}$-class ($\mathcal{R}$-class; $\mathcal{H}$-class; $\mathcal{D}$-class; $\mathcal{J}$-class) containing element a is denoted by L_a (R_a; H_a; D_a; J_a). A $\mathcal{D}$-class can be represented by a table, called an *eggbox*, where each row represents an $\mathcal{R}$-class, each column represents a $\mathcal{L}$-class, and each cell represents an $\mathcal{H}$-class.

Completely simple semigroups

There is a natural (partial) order defined on the set of the idempotents of a semigroup S by the rule that

$$e \leq f \text{ if and only if } ef = fe = e.$$

If S is a semigroup with zero then the zero is the unique minimum (e is *minimum* if $e \leq f$ for every idempotent f) idempotent. The idempotents that are minimal (e is *minimal* if $f \leq e$ implies $f = e$ for every idempotent f) within the set of non-zero idempotents are called *primitive*. We say that a semigroup is *completely 0-simple* if it is 0-simple and has a primitive idempotent, and we say that a semigroup is *completely simple* if it is simple and has a primitive idempotent. Completely simple semigroups are known to be Rees matrix semigroups over groups. Completely 0-simple semigroups are Rees matrix semigroups with zero over groups where the matrix is regular (no row or column consists entirely of zeros).

Congruences

Let S be a semigroup. A relation ρ on the set S is called *compatible* (with the operation in S) if

$$(\forall s, t, s', t' \in S)\ s \;\rho\; t \;\&\; s' \;\rho\; t' \implies ss' \;\rho\; tt'.$$

A compatible equivalence relation is called a *congruence*. Given a congruence ρ on the semigroup S the set S/ρ is a semigroup with operation $(a\rho)(b\rho) = (ab)\rho$.

Rees quotient

Given a proper ideal I of a semigroup S the relation $\rho_I = (I \times I) \cup 1_S$ is a congruence (here 1_S stands for the relation $\{(s, s) : s \in S\}$), called the *Rees congruence*, and we can consider the semigroup S/ρ_I, called the *Rees quotient*. This semigroup can be seen as the set $(S\backslash I) \cup \{0\}$ where the product of elements of $S\backslash I$ is the same as their product in S if this lies in S and it is 0 otherwise. We have the natural homomorphism

$$\phi : S \to S/\rho_I;\ s \mapsto s\ (s \in S\backslash I);\ s \mapsto 0\ (s \in I),$$

called the *Rees homomorphism*.

Presentations

Let A be an alphabet. A *semigroup presentation* is a pair $\langle A \mid R\rangle$ where R is a relation in A^+. In this context, we normally write $u = v$ instead of $u\ R\ v$ and we say that $u = v$ is a *(defining) relation*. If $A = \{a_1, \ldots, a_n\}$ and $R = \{u_1 = v_1, \ldots, u_m = v_m\}$, we write $\langle a_1, \ldots, a_n \mid u_1 = v_1, \ldots, u_m = v_m\rangle$ for $\langle A \mid R\rangle$. We say that the semigroup S is defined by the presentation $\langle A \mid R\rangle$ if $S \cong A^+/\rho$ where ρ is the smallest congruence on A^+ that contains R. Replacing A^+ by A^* in the previous definition we obtain the definition of a *monoid presentation*; in this context we normally write 1 instead of ϵ (for example, the bicyclic monoid is defined by the monoid presentation $\langle b, c \mid bc = 1\rangle$).

Appendix B

GAP program

```
*** GAP PROGRAM ***

## SUBSEMIGROUPS OF THE BICYCLIC MONOID

Print("------------------------------------------------------------\n");
Print("Subsemigroups of the bicyclic monoid <b,c | bc = 1>.\n");
Print("Function \"subsembimon\" displays the subsemigroup given the\n");
Print("generating set.\n\n");
Print("An element c^i b^j is represented by [i,j] and so\n");
Print("the generating set must be a list of pairs of numbers (>=0).\n\n");
Print("Example:\n");
Print("x := [[1,2],[5,4]];\n");
Print("subsembimon(x);\n");
Print("------------------------------------------------------------\n");

## MIDDLE SEMIGROUPS

## The sets f,x must be contained in F_{q,p}
## Returns a list with the elements in f \cup f.x
## that are on the left of column p
actright := function(f,x,p)
   local k,j,r;
   r := ShallowCopy(f);
   for k in f do
      for j in x do
         if k[2] <= j[1] then
            Add(r,[k[1]-k[2]+j[1], j[2] ]);
         else
```

```
         if (k[2]-j[1]+j[2] < p) then
            Add(r,[k[1],k[2]-j[1]+j[2]]);
         fi;
      fi;
    od;
  od;
  return Set(r);
end;

## u is a subset of B, l is a list of line numbers smaller then p
## returns the list of line numbers in l \cup u.l
## that are smaller then p
actleft := function(u,l,p)
  local r,k,j,n;
  r := ShallowCopy(l);
  for k in u do
    for j in l do
      if j > k[2] then
         n := k[1]-k[2]+j;
         if n < p then
            Add(r,n);
         fi;
      fi;
    od;
  od;
  return Set(r);
end;

## MAIN FUNCTION - middle semigroups
## Assumes $X \cap D_1 \neq \emptyset$,
## $X \cap D_2 \neq \emptyset$,
## $\iota(X) \le \kappa(X)$ and
## $S \cap F_{0,\iota(X)} = \emptyset$
computemid :=function(x)
  local q,p,d,I0,I1,k,u,r,F0,F1,i,setP,n,x0;
  ## Compute d,q,p and setP
  d := AbsInt(Gcd(List(x, k -> k[1]-k[2])));
  q := Minimum(ListX(x,k -> k[1] < k[2],k -> k[1]));
  p := Minimum(ListX(x,k -> k[1] > k[2], k-> k[2]));
  setP := Set(ListX(x, k -> (k[1] >= p) and (k[2] >= p), k -> ((k[1]-p) mod d)));
  ## Creates initial F
  F1 := ListX(x, k -> k[1] >= q and k[1] < p and k[2] < p, k -> k);
  x0 := ShallowCopy(F1);
  F0 := [] ;
  ## Iteration to construct final F
```

```
   while (F1 <> F0) do
      F0:=F1;
      F1 := actright(F0,x0,p);
   od;
   ## Construct initial I
   I1 := [];
   for r in setP do
      u := 1;
      k := p+r-d;
      while (k >= q) do
         if (k < p) then
            Add(I1,k);
         fi;
         u:=u+1;
         k := p+r-u*d;
      od;
   od;
   for k in x do
      if k[1] >= q and k[1] < p then
          Add(I1,k[1]);
      fi;
   od;
   I1 := Set(I1);
   I0 := [];
   ## Iteration to produce final I
   while (I1 <> I0) do
      I0 := I1;
      I1 := actleft(F1,I0,p);
   od;
   return [q,p,d,F1,I1,setP];
end;


## UPPER SEMIGROUPS

## returns a list of the lines that form the received set
splitinlines := function(y)
   local n,i,k,l;
   i := y[1][1]; l := []; n := 1; l[n] := [];
   for k in y do
      if k[1] = i then
         Add(l[n],k);
      else
         n := n + 1;
         l[n] := [];
```

```
         Add(l[n],k);
         i:=k[1];
      fi;
   od;
   return l;
end;

## Returns the line numbers in a list of lines
linesin := function(l)
   return List(l, k -> k[1][1]);
end;

## Receives a list of lines and a list columns numbers from which
## regularity starts.
## returns the a set with the elements of S on the left of column
## max(m)
formf := function(l,m,d)
   local s,setF,i,t,j,k,mmax;
   s := Size(l); setF :=[]; mmax := Maximum(m);
   for i in [1..s] do
      k := l[i];
      t:=k[1][1];
      setF := Set(Concatenation(setF,k));
      j := m[i]+d;
      while (j < mmax) do
         Add(setF,[t,j]);
         j := j+d;
      od;
   od;
   setF := Filtered(setF, k -> k[2] < mmax);
   return Set(setF);
end;

## Checks if the subset of the line i determines a number m
## such that it generates F \cup \Lambda_{i,p,d}
linegen := function(l,d)
   local h,i,j,k,s,sucesso,jmax,t;
   h:=[]; s := Size(l);
   for i in [1..s] do
      h[i] := l[i][2];
   od;
   t:=l[1][1];
   i:=1;
   k:=h[i];
   sucesso := false;
```

```
   while i+(k-t)/d-1 <=s and not sucesso do
      sucesso:=true;
      j:=0;
      jmax:=(k-t)/d-1;
      while sucesso and j < jmax do
         j:=j+1;
         if h[i+j] <> k+j*d
         then sucesso := false;
         fi;
      od;
      i := i+1;
      if i <=s then
         k := h[i];
      fi;
   od;
   if sucesso then
      return h[i-1];
   else return -1;
   fi;
end;

## Check if a set (already split in lines) generates the final
## subsemigroup. If not returns []. If it generates
## returns the vector with the columns numbers from
## where regularity starts for each line in S
linesselfgen := function(l,d)
   local s,m,i,k,flag;
   s := Size(l); m := []; i := 1; flag := true;
   while (i <= s) and flag do
      k := linegen(l[i],d);
      if k <> -1 then
         m[i] := k;
         i := i +1;
      else
         flag := false;
      fi;
   od;
   if flag then
      return m;
   else
      return [];
   fi;
end;

## returns setI \cup setF.setI
```

```
addlines := function(setF,setI)
   local r,k,j;
   r := ShallowCopy(setI);
   for k in setF do
      for j in setI do
         if j > k[2] then
            Add(r,k[1]-k[2]+j);
         fi;
      od;
   od;
   return(Set(r));
end;

## Multiplies subsets of the bicyclic monoid
multiplica := function(a,b)
   local c,i,j;
   c := [];
   for i in a do
      for j in b do
         if (i[2] >= j[1]) then
             Add(c, [i[1], i[2]-j[1]+j[2] ]);
         else
             Add(c, [i[1]-i[2]+j[1],j[2]]);
         fi;
      od;
   od;
   return Set(c);
end;

## Checks if the operation is already closed
test_issemigroup := function(setF,setI,m)
   local mmax,flag,prod,setI2;
   mmax := Maximum(m); flag:=false;
   prod := multiplica(setF,setF);
   prod := Filtered(prod,k -> k[2] < mmax);
   prod := Set(Concatenation(setF,prod));
   if setF = prod then
      flag :=true;
   fi;
   if flag then
      setI2 := addlines(setF,setI);
      if setI <> setI2 then
         flag := false;
      fi;
   fi;
```

```
   return flag;
end;

## Checks if we have all we need
issemigroup := function(y,d)
   local l,setI,setF,m,done;
   done := false;
   l := splitinlines(y);
   m := linesselfgen(l,d);
   if m <> [] then
      setI := linesin(l);
      setF := formf(l,m,d);
      done := test_issemigroup(setF,setI,m);
   fi;
   return done;
end;

## Returns the paremeters
semigroup := function(y,d)
   local l,setI,setF,m;
   l := splitinlines(y);
   m := linesselfgen(l,d);
   setI := linesin(l);
   setF := formf(l,m,d);
   return [d,setF,setI,m];
end;

## MAIN FUNCTION - upper semigroups
## Assumes X \cap \hat{U} = \emptyset, X \cap U \neq \emptyset,
## X \cap F_{0,\iota(X)} = \emptyset
computeabove := function(x)
   local done,d,y,p,k,m;
   d := AbsInt(Gcd(List(x, k -> k[1]-k[2])));
   done := false; y :=ShallowCopy(x); p :=ShallowCopy(x);
   done := issemigroup(y,d);
   ## main cycle - in iteration n, y is equal to X^1 \cup ... X^n
   ## and it is checked if y gives us already the semigroup
   while not done do
      p := multiplica(p,x);
      y := Set(Concatenation(y,p));
      done := issemigroup(y,d);
   od;
   return semigroup(y,d);
end;
```

```
## Find the kind of semigroup
isdiagonal := function (x)
   local i,s;
   s := Size(x); i := 1;
   while x[i][1] = x[i][2] and i < s do
      i := i +1;
   od;
   if i = s and x[i][1] = x[i][2] then
      return true;
   else
      return false;
   fi;
end;

# Are all elements above (or in) the diagonal?
isabove := function (x)
   local i,s;
   s := Size(x); i := 1;
   while x[i][1] <= x[i][2] and i < s do
      i := i +1;
   od;
   if i = s and x[i][1] <= x[i][2] then
      return true;
   else
      return false;
   fi;
end;

# Are all elements below (or in) the diagonal?
isbelow := function (x)
   local i,s;
   s := Size(x); i := 1;
   while x[i][1] >= x[i][2] and i < s do
      i := i +1;
   od;
   if i = s and x[i][1] >= x[i][2] then
      return true;
   else
      return false;
   fi;
end;

# Assumes it is two-sided checks if it is two-sided upper
istwosidup := function(x)
   local q,p;
```

```
   q := Minimum(ListX(x,k -> k[1] < k[2],k -> k[1]));
   p := Minimum(ListX(x,k -> k[1] > k[2], k-> k[2]));
   if q <= p then
       return true;
   else
       return false;
   fi;
end;

## Apllies the anti-isomorphism to the generating set
invert := function(x)
   local k;
   return List(x, k -> [k[2],k[1]]);
end;

## MAIN FUNCTION
## receives the generating set
## returns the subsemigroup
subsembimon := function(x)
    local d,setFD,setF,setI,setP,m,l,q,p,x1;
    if isdiagonal(x)
    then
        Print("Contained in the diagonal\n");
    elif isabove(x)
    then
        Print("------------------------------------------------\n");
        Print("Upper semigroup: F_D \\cup F \\cup \\Lambda_{I,m,d}\n");
        Print("------------------------------------------------\n");
        q := Minimum(ListX(x, k -> k[1] < k[2],k -> k[1]));
        x1 := ListX(x, k -> k[1] >= q, k->k);
        l := computeabove(x1);
        d:=l[1]; setF:=l[2]; setI:=l[3]; m:=l[4];
        setFD := ListX(x, k -> k[1] < q, k -> k);
        Print("d=",d,"\nm=", Maximum(m),"\n");
        Print("F_D=\n",setFD,"\n");
        Print("F=\n",setF,"\n");
        Print("I=\n",setI,"\n");
    elif isbelow(x)
    then
        Print("-----------------------------------------------------\n");
        Print("Lower semigroup: F_D \\cup F \\cup \\wh{\\Lambda_{I,m,d}} \n");
        Print("-----------------------------------------------------\n");
        q := Minimum(ListX(x, k -> k[1] > k[2],k -> k[2]));
        x1 := ListX(x, k -> k[2] >= q, k->k);
        x1 := invert(x1);
```

```
        l := computeabove(x1);
        d:=l[1]; setF:=l[2]; setI:=l[3]; m:=l[4];
        setF := invert(setF);
        setFD := ListX(x, k -> k[2] < q, k -> k);
        Print("d=",d,"\nm=", Maximum(m),"\n");
        Print("F_D=\n",setFD,"\n");
        Print("F=\n",setF,"\n");
        Print("I=\n",setI,"\n");
    elif istwosidup(x)
    then
        Print("----------------------------------------------------\n");
        Print("Two sided upper semigroup:\n");
        Print("F_D \\cup F \\cup \\Lambda_{I,p,d} \\cup \\Sigma_{p,d,P} \n");
        Print("----------------------------------------------------\n");
        q := Minimum(ListX(x, k -> k[1] < k[2],k -> k[1]));
        x1 := ListX(x, k -> k[1] >= q, k->k);
        l := computemid(x1); p := l[2]; d := l[3]; setF := l[4];
        setI := l[5]; setP := l[6];
        setFD := ListX(x, k -> k[1] < q, k -> k);
        Print("d=",d,"\np=",p,"\n");
        Print("F_D=\n",setFD,"\n");
        Print("F=\n",setF,"\n");
        Print("I=\n",setI,"\n");
        Print("P=\n",setP,"\n");
    else
        Print("----------------------------------------------------------\n");
        Print("Two sided lower semigroup:\n");
        Print("F_D \\cup F \\cup \\hat{\\Lambda_{I,p,d}} \\cup \\Sigma_{p,d,P}\n");
        Print("----------------------------------------------------------\n");
        q := Minimum(ListX(x, k -> k[1] > k[2],k -> k[2]));
        x1 := ListX(x, k -> k[2] >= q, k->k);
        x1 := invert(x1);
        l := computemid(x1);
        p := l[2]; d := l[3]; setF := l[4]; setI := l[5]; setP := l[6];
        setFD := ListX(x, k -> k[1] < q, k -> k);
        setF := invert(setF);
        Print("d=",d,"\np=",p,"\n");
        Print("F_D=\n",setFD,"\n");
        Print("F=\n",setF,"\n");
        Print("I=\n",setI,"\n");
        Print("P=\n",setP,"\n");
    fi;
end;

*** GAP session ***
```

```
gap> Read("bimon.g");;
------------------------------------------------------------
Subsemigroups of the bicyclic monoid <b,c | bc = 1>.
Function "subsembimon" displays the subsemigroup given the
generating set.

An element c^i b^j is represented by [i,j] and so
the generating set must be a list of pairs of numbers (>=0).

Example:
x := [[1,2],[5,4]];
subsembimon(x);
------------------------------------------------------------
gap> x:=[[1,2],[5,4]];;
gap> subsembimon(x);;
---------------------------------------------------
Two sided upper semigroup:
F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}
---------------------------------------------------
d=1
p=4
F_D=
[  ]
F=
[ [ 1, 2 ], [ 1, 3 ] ]
I=
[ 1, 2, 3 ]
P=
[ 0 ]
gap> x:=[[1,1],[4,7],[10,13],[18,24],[23,17]];;
gap> subsembimon(x);;
---------------------------------------------------
Two sided upper semigroup:
F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}
---------------------------------------------------
d=3
p=17
F_D=
[ [ 1, 1 ] ]
F=
[ [ 4, 7 ], [ 4, 10 ], [ 4, 13 ], [ 4, 16 ], [ 7, 13 ], [ 7, 16 ],
  [ 10, 13 ], [ 10, 16 ] ]
I=
[ 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15 ]
```

```
P=
[ 0, 1 ]
gap> x:=[[1,1],[7,4],[13,10],[24,18],[17,23]];;
gap> subsembimon(x);;
---------------------------------------------------------
Two sided lower semigroup:
F_D \cup F \cup \hat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P}
---------------------------------------------------------
d=3
p=17
F_D=
[ [ 1, 1 ] ]
F=
[ [ 7, 4 ], [ 10, 4 ], [ 13, 4 ], [ 16, 4 ], [ 13, 7 ], [ 16, 7 ],
  [ 13, 10 ], [ 16, 10 ] ]
I=
[ 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15 ]
P=
[ 0, 1 ]
gap> x:=[[1,1],[3,13],[5,9],[10,16]];;
gap> subsembimon(x);;
------------------------------------------------
Upper semigroup: F_D \cup F \cup \Lambda_{I,m,d}
------------------------------------------------
d=2
m=20
F_D=
[ [ 1, 1 ] ]
F=
[ [ 3, 13 ], [ 3, 17 ], [ 3, 19 ], [ 5, 9 ], [ 5, 13 ], [ 5, 17 ], [ 5, 19 ],
  [ 6, 16 ], [ 10, 16 ] ]
I=
[ 3, 5, 6, 10 ]
gap> x:=[[1,1],[13,3],[9,5],[16,10]];;
gap> subsembimon(x);;
-----------------------------------------------------
Lower semigroup: F_D \cup F \cup \wh{\Lambda_{I,m,d}}
-----------------------------------------------------
d=2
m=20
F_D=
[ [ 1, 1 ] ]
F=
[ [ 13, 3 ], [ 17, 3 ], [ 19, 3 ], [ 9, 5 ], [ 13, 5 ], [ 17, 5 ], [ 19, 5 ],
  [ 16, 6 ], [ 16, 10 ] ]
```

```
I=
[ 3, 5, 6, 10 ]
gap> x:=[[1,1],[2,2],[3,3],[5,5],[8,8]];;
gap> subsembimon(x);;
Contained in the diagonal
gap> x:=[[9,12],[12,18],[15,24],[18,30],[21,36]];;
gap> subsembimon(x);;
------------------------------------------------
Upper semigroup: F_D \cup F \cup \Lambda_{I,m,d}
------------------------------------------------
d=3
m=36
F_D=
[  ]
F=
[ [ 9, 12 ], [ 9, 15 ], [ 9, 18 ], [ 9, 21 ], [ 9, 24 ], [ 9, 27 ],
  [ 9, 30 ], [ 9, 33 ], [ 12, 18 ], [ 12, 21 ], [ 12, 24 ], [ 12, 27 ],
  [ 12, 30 ], [ 12, 33 ], [ 15, 24 ], [ 15, 27 ], [ 15, 30 ], [ 15, 33 ],
  [ 18, 30 ], [ 18, 33 ] ]
I=
[ 9, 12, 15, 18, 21 ]
gap> x:=[[3,6],[15,12]];;
gap> subsembimon(x);;
---------------------------------------------------
Two sided upper semigroup:
F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}
---------------------------------------------------
d=3
p=12
F_D=
[  ]
F=
[ [ 3, 6 ], [ 3, 9 ] ]
I=
[ 3, 6, 9 ]
P=
[ 0 ]
gap> x:=[[3,12],[15,12]];;
gap> subsembimon(x);;
---------------------------------------------------
Two sided upper semigroup:
F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}
---------------------------------------------------
d=3
p=12
```

```
F_D=
[  ]
F=
[  ]
I=
[ 3, 6, 9 ]
P=
[ 0 ]
gap> x:=[[3,21],[15,12]];;
gap> subsembimon(x);;
---------------------------------------------------
Two sided upper semigroup:
F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}
---------------------------------------------------
d=3
p=12
F_D=
[  ]
F=
[  ]
I=
[ 3, 6, 9 ]
P=
[ 0 ]
gap> LogTo();;
```

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