Signal Reconstruction and Error Control Coding Techniques*

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Abstract: *In this article we consider the problem of reconstructing oversampled signals corrupted with impulsive noise. More precisely, we show how to find the number, positions and amplitudes of the corrupted samples* without any error. *Note that this is a nonlinear problem. The technique used presents certain similarities to BCH*[1] *coding, but in the field of complex numbers. We show that the problem can be cast as a linear system of equations with a* Toeplitz *and* Hermitian *matrix, whose structure is independent of the number of errors, and which can therefore be solved using Levinson's iteration. The solution thus found determines a polynomial with roots in the unit circle. The phase angle of the zeros of the polynomial (determined using the FFT) give the position of the incorrect samples. A study of the conditioning of this system as a function of the errors position and amplitude is made.*

Keywords: *Signal Reconstruction, Error Correction Codes, Band Limited Signals*

1. Introduction

The problem of signal reconstruction was studied by several authors and its connections to the field of error control coding (ECC) were first noticed by Blahut [1, 2] and later explored by Wolf[13] and Marshall[9]. These techniques are related to error control coding in the complex field. The basic idea beyond all the methods of discrete signal reconstruction is always to make use of some known information about the signal (redundancy) to correct a certain number of samples. In the field of signal processing several authors have studied the problem of signal reconstruction [12] with ECC techniques applied to the recovery of erasures. In [6], some connections between techniques used in the fields of signal processing, spectrum analysis, information theory and fault tolerant computing were established.

In this article we focus only on sampled finite duration signals in the field of real and complex numbers, corrupted by impulsive noise. The results obtained generalize those obtained in [11, 10, 5, 3, 4] and other works where both the number of incorrect samples and their locations are assumed to be known.

The techniques used in the present work are related to BCH coding in the complex field $\mathbb{C}$ instead of the Galois Field GF(p). The paper begins by showing that a band-limited signal can be reconstructed by solving a linear system of equations. This system can assume several forms, one of which is more appropriate to an efficient solution. The solution to this system of equations determines the coefficients of a polynomial whose roots are on the unit circle. The phase angle of the roots (which can be computed using a FFT) gives the positions of the incorrect samples.

The stability analysis of the algorithm is a particularly difficult and important problem. We study the conditioning of the system as a function of the error positions and amplitudes. These results are entirely original and decouple the nonlinear error location problem from the linear, error correction or interpolation problem. To close the paper we discuss some possible applications of the algorithm.

2. Problem definition

The problem of signal reconstruction can be formulated as follows: Consider an original signal $m(n)$ called message with K samples defined by the vector $m = [m_0, m_1, \ldots, m_{K-1}]$, with discrete Fourier transform $\widehat{m}$. The discrete Fourier transform is defined here as

$$\hat{x}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-\mathrm{j}\frac{2\pi}{N}kn}$$

with inverse transform

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}(k) e^{\mathrm{j}\frac{2\pi}{N}kn}.$$

Adding $2t = N - K$ samples with value zero to the Fourier transform $\hat{m}$ at the locations given by

$$\hat{s} = r + l, \qquad l = 0, 1, 2, \ldots, 2t-1 \tag{1}$$

(syndrome), and evaluating its inverse we obtain the oversampled signal $x(n)$ with N samples. Now sup-

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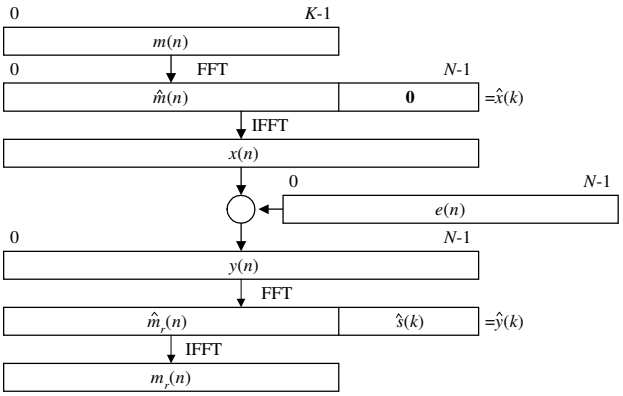

Figure 1. Graphical decription of the coder and decoder based on the Fourier transform. The signal is oversampled by a factor of $\frac{K}{N}$ and then transmited. If the signal $x(n)$ suffers a number of errors less than t, then the spectrum of the error can be extrapolated from the Syndrome $\hat{s}\,(k)$.

pose that this vector is stored or transmitted and during this process it is corrupted in $g \leq t$ samples. Under certain conditions it is possible to correct all of those errors. Referring to the fig.1, we can see all the steps necessary to encode and decode the vector $m\,(n)$ in order to correct t errors.

Consider the received vector

$$y(n) = x(n) + e(n) \qquad n = 0, 1, \ldots, N-1, \tag{2}$$

where $e\,(n)$ is a sparse error vector: only the samples whose indexes are in a certain set, say, $U = \{i_0, i_1, \ldots, i_{g-1}\}$, are different from zero. Evaluating the $DFT\,\{y(n)\}$, we obtain a vector that will be zero in the samples belonging to $\hat{s}\,(k)$ if $e\,(n)$ is zero. If some errors occur, then $\hat{s}\,(k)$ will not be zero anymore and it is only a function of the error signal $e(n)$. We can look at the syndrome as a window over the error spectrum $\hat{e}(k)$.

Consider now the error locator polynomial

$$P\,(z) = \sum_{i=0}^{g} h_i z^i, \tag{3}$$

defined by

$$P\left(e^{-\mathrm{j}\frac{2\pi}{N} i_m}\right) = 0, \qquad m = 1, 2, \ldots, g-1, \tag{4}$$

which by construction is zero at the error locations i_m. Substituting $z = e^{-\mathrm{j}\frac{2\pi}{N} i_m}$ in (3), using (4), multiplying by $\frac{1}{\sqrt{N}} e\,(i_m)\, e^{-\mathrm{j}\frac{2\pi}{N} i_m (r-l)}$ and summing over m leads to:

$$\sum_{k=0}^{g} h_k \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\,(i_m)\, e^{-\mathrm{j}\frac{2\pi}{N} i_m (k+r-l)} = 0. \tag{5}$$

[1]Bose, Chaudhuri and Hocquenghem.

The DFT of the error signal is given by

$$\hat{e}\,(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\,(i_m)\, e^{-\mathrm{j}\frac{2\pi}{N} i_m k}. \tag{6}$$

Substituting this equation in (5), we get the equation known in the field of ECC as the *key equation*:

$$\sum_{k=0}^{g} h_k \hat{e}\,(k + r - l) = 0. \tag{7}$$

As $2t$ samples (locations r to $r + 2t - 1$) of the error spectrum are known, we can form an homogeneous system of t equations and $t + 1$ unknowns. We can impose an arbitrary value on one of the roots, corresponding to the fact that the value of the error locator polynomial $P\,(z)$ is arbitrary in the nonzero positions in the time domain. In practice the condition imposed is h_0 or h_g equal to one. Solving this system of equations we obtain $\hat{e}\,(k)$. Evaluating $e\,(n)$ with an IFFT, and using (2) to obtain $x\,(n)$, we finally get $m\,(n)$. The system of equations (7)can take several forms some of which are more easily handled than the others. We are going to classify those possible forms.

2.1. Case 1

Consider $h_g = 1$ and $l = 0, 1, \ldots, g-1$.

In this form the equation (7) becomes

$$\sum_{k=0}^{g-1} h_k \hat{e}\,(k + r - l) = -\hat{e}\,(g + r - l), \tag{8}$$

which gives the system of equations

$$\begin{bmatrix} \hat{e}\,(r) & \hat{e}\,(r+1) & \cdots & \hat{e}\,(r+g-1) \\ \hat{e}\,(r-1) & \hat{e}\,(r) & \cdots & \hat{e}\,(r+g-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}\,(r-g+1) & \hat{e}\,(r-g+2) & \cdots & \hat{e}\,(r) \end{bmatrix} \times \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{g-1} \end{bmatrix} = - \begin{bmatrix} \hat{e}\,(r+g) \\ \hat{e}\,(r+g-1) \\ \vdots \\ \hat{e}\,(r+1) \end{bmatrix} \tag{9}$$

that can be written in the compact form $\mathbf{A}_1 \mathbf{x} = \mathbf{b}_1$, where $\mathbf{A}_1$ is a *Toeplitz* matrix.

2.2. Case 2

Consider $h_g = 1$ and $l = g-1, g-2, \ldots, 0$.

In this form the equation (8) holds, and we have the system of equations

$$\begin{bmatrix} \hat{e}\,(r-g+1) & \hat{e}\,(r-g+2) & \cdots & \hat{e}\,(r) \\ \hat{e}\,(r-g+2) & \hat{e}\,(r-g+3) & \cdots & \hat{e}\,(r+1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}\,(r) & \hat{e}\,(r+1) & \cdots & \hat{e}\,(r+g-1) \end{bmatrix} \times \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{g-1} \end{bmatrix} = - \begin{bmatrix} \hat{e}\,(r+1) \\ \hat{e}\,(r) \\ \vdots \\ \hat{e}\,(r+g) \end{bmatrix} \tag{10}$$

that can be written in the compact form $\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$, where $\mathbf{A}_2$ is a *Hankel* matrix.

It is possible to transform the system of equations (9) in the system (10) with the transformation

$$\mathbf{J}\mathbf{A}_1\mathbf{x} = \mathbf{J}\mathbf{b}_1 \Leftrightarrow \mathbf{A}_2\mathbf{x} = \mathbf{b}_2$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdot & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

2.3. Case 3

Consider $\alpha_0 = 1$ and $l = 1, 2, \ldots, g$.

In this form the equation (7) becomes

$$\sum_{k=1}^{g} \alpha_k \hat{e}\left(k + r - l\right) = -\hat{e}\left(r - l\right), \tag{11}$$

which gives the system of equations

$$\begin{bmatrix} \hat{e}\left(r\right) & \hat{e}\left(r+1\right) & \cdots & \hat{e}\left(r+g-1\right) \\ \hat{e}\left(r-1\right) & \hat{e}\left(r\right) & \cdots & \hat{e}\left(r+g-2\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}\left(r-g+1\right) & \hat{e}\left(r-g+2\right) & \cdots & \hat{e}\left(r\right) \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{g-1} \end{bmatrix} = - \begin{bmatrix} \hat{e}\left(r-1\right) \\ \hat{e}\left(r-2\right) \\ \vdots \\ \hat{e}\left(r-g\right) \end{bmatrix} \tag{12}$$

that can be written in the compact form $\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$, where $\mathbf{A}_3$ is a *Toeplitz* matrix equal to $\mathbf{A}_1$.

2.4. Case 4

Consider $\alpha_0 = 1$ and $l = g, g-1, \ldots, 1$.

In this form the equation (11) holds, and we have the system of equations

$$\begin{bmatrix} \hat{e}\left(r-g+1\right) & \hat{e}\left(r-g+2\right) & \cdots & \hat{e}\left(r\right) \\ \hat{e}\left(r-g+2\right) & \hat{e}\left(r-g+3\right) & \cdots & \hat{e}\left(r+1\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}\left(r\right) & \hat{e}\left(r+1\right) & \cdots & \hat{e}\left(r+g-1\right) \end{bmatrix} \times \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_g \end{bmatrix} = - \begin{bmatrix} \hat{e}\left(r-g\right) \\ \hat{e}\left(r-g+1\right) \\ \vdots \\ \hat{e}\left(r-1\right) \end{bmatrix} \tag{13}$$

that can be written in the compact form $\mathbf{A}_4\mathbf{x} = \mathbf{b}_4$, where $\mathbf{A}_4$ is a *Hankel* matrix equal to $\mathbf{A}_2$.

With the values of α_i or h_i known and with the syndrome we can evaluate (extrapolate) the missing spectrum using the recursive equation (7) in the form

$$\hat{e}(i) = \sum_{k=0}^{g-1} h_k \hat{e}(k+i)$$

concerning the cases 1 and 2 and in the form

$$\hat{e}(i) = \sum_{k=1}^{g} \alpha_k \hat{e}(k+i)$$

concerning the cases 3 and 4. So the problem of error correction can be see as an error spectrum extrapolation problem [14].

The parameters α_i and h_i are related as follow, consider the equation (11)

$$\begin{aligned} & \sum_{k=1}^{g} \alpha_k \hat{e}\left(k + r - l\right) = -\hat{e}\left(r - l\right) \Leftrightarrow \\ \Leftrightarrow \quad & \sum_{k=0}^{g-1} \alpha_k \hat{e}\left(k + r - l\right) = -\alpha_g \hat{e}\left(g + r - l\right). \end{aligned}$$

Combining this equation with (8) we get

$$\sum_{k=0}^{g-1} \alpha_k \hat{e}\left(k + r - l\right) = \alpha_g \sum_{k=0}^{g-1} h_k \hat{e}\left(k + r - l\right),$$

and

$$h_i = \frac{\alpha_i}{\alpha_g} : i = \{0, 1, 2, \ldots, g-1\},$$

with $\alpha_0 = 1$ and $h_g = 1$.

3. Solving the system

The cases 1 and 3 which leads to *Toeplitz* matrices are more useful because we can apply directly several well known methods to solve the system of equations. As we will see in the section 4, when the errors are contiguous, the matrix A becomes ill conditioned and the solution is very sensitive to round off noise introduced during the matrix computations. Due to this we can identify two main characteristics to compare algorithms: the complexity and noise sensibility.

Concerning the complexity analysis, *Levinson's* algorithm gives the best performance because it avoids having to pre evaluate the rank of $\mathbf{A}$, which equals the number of actual errors. This is only possible when we use it with $\mathbf{A}_3$, because we can start the algorithm without knowing the number of true errors. For example, consider that only one error has occurred, in the second iteration of the *Levinson* algorithm we get a singular matrix and the algorithm stop. To use this algorithm A_3 should be *Hermitian* and this can be easily obtained if we choose $r = \frac{N}{2}$. If N is even and $e\left(i_m\right)$ is real we can write

$$\begin{aligned} \hat{e}\left(\frac{N}{2} + i\right) &= \hat{e}^*\left(-\frac{N}{2} - i\right) = \\ &= \hat{e}^*\left(N - \frac{N}{2} - i\right) = \hat{e}^*\left(\frac{N}{2} - i\right). \end{aligned}$$

This symmetry of $\hat{e}\left(k\right)$ can be tested to determine if more than t errors have occurred, because the error signal $e\left(i_m\right)$ is supposed to be real. Experimental results confirmed this.

Even when no errors happen, the syndrome is not exactly zero and some $\varepsilon > 0$ should be properly chosen. Also the test for $\mathbf{A}$ singular must be done with some threshold $\varepsilon' > 0$. The amplitude of those thresholds depends on the coder and decoder implementation and on the distribution of the errors.

4. Stability analysis

In some situations the impact of the round off error upon the signal reconstruction can be considerable. To study the stability of the method we analyzed the conditioning of matrix $\mathbf{A}_3$, and used the quadratic form

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \sum_{p=0}^{g-1} \sum_{q=0}^{g-1} x_p^H x_q a_{pq} \tag{14}$$

with $a_{pq} = \hat{e}(r+q-p)$. Substituting this in (14) and using the expression (6) we get

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \sum_{p=0}^{g-1} \sum_{q=0}^{g-1} x_p^H x_q \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) e^{-\mathrm{j}\frac{2\pi}{N} i_m (r+q-p)},$$

changing the summation order and rearranging the exponents we can write

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) e^{-\mathrm{j}\frac{4\pi}{N} i_m r} \left|\phi\left(i_m\right)\right|^2 \tag{15}$$

where

$$\phi\left(i_m\right) = \sum_{p=0}^{g-1} x_p e^{-\mathrm{j}\frac{2\pi}{N} i_m p}.$$

When $r = \frac{N}{2}$, $\mathbf{A}_3$ is *Hermitian* and (15) becomes

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) \left(-1\right)^{i_m} \left|\phi\left(i_m\right)\right|^2. \tag{16}$$

In practical applications the signals are sampled and have finite amplitude and resolution we only considered the case where

$$e_{\min} \leq |e(i_m)| \leq 1. \tag{17}$$

To study the conditioning of $\mathbf{A}$ we can work with the equivalent problem[8] of finding

$$\lambda_{\max} = \max_{\mathbf{x}^H \mathbf{x}=1} \mathbf{x}^H \mathbf{A} \mathbf{x}$$

and

$$\lambda_{\min} = \min_{\mathbf{x}^H \mathbf{x}=1} \mathbf{x}^H \mathbf{A} \mathbf{x},$$

where $\lambda_{\max}$ and $\lambda_{\min}$, are the largest and the smallest eigenvalue of $\mathbf{A}$. The summation in (16) is a function of the error positions and their amplitude. The term $\left|\phi\left(i_m\right)\right|^2$ is always positive and is only a function of the errors position. Due to those facts and the error bound in (17), we can easily identify the worst cases for the amplitudes as

$$e\left(i_m\right) = \left(-1\right)^{i_m} \tag{18}$$

and

$$e(i_m) = e_{\min} (-1)^{i_m}, \tag{19}$$

and we can write

$$\frac{e_{\min}}{\sqrt{N}} \sum_{m=0}^{g-1} \left|\phi\left(i_m\right)\right|^2 \leq \mathbf{x}^H \mathbf{A} \mathbf{x} \leq \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} \left|\phi\left(i_m\right)\right|^2.$$

Experimental work confirms that the worst case for the positions of the erroneous samples is the contiguous case $i_m = m$. Considering this case for the error positions, when (19) is satisfied the $\lambda_{\max}$ has its maximum value $\tilde{\lambda}_{\max}$, and when (18) is satisfied $\lambda_{\min}$ has its minimum value $\tilde{\lambda}_{\min}$. So we can write that for a given set U of error positions, the following inequality holds:

$$\tilde{\lambda}_{\min} \leq \lambda_{\min} \leq \lambda_{\max} \leq \tilde{\lambda}_{\max}. \tag{20}$$

The equation (20) gives a very conservative bound for the condition number of $\mathbf{A}$ as

$$\operatorname{cond}\left(\mathbf{A}\right) \leq \frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}}.$$

On the other hand we can establish some simple bounds on the eigenvalues of $\mathbf{A}$ using the fact that for every *Hermitian* matrix $\mathbf{A}$ the vector of diagonal entries majorizes the vector of eigenvalues of $\mathbf{A}$ (see [8], theorem 4.3.26 and [7]). Applying this to $\mathbf{A}_3$ we have

$$A_{pq} = \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) \left(-1\right)^{i_m} e^{-\mathrm{j}\frac{2\pi}{N} i_m (q-p)} \tag{21}$$

with diagonal elements

$$A_{pp} = \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) \left(-1\right)^{i_m}$$

and

$$\sum_{j=0}^{g-1} \lambda_j = \sum_{p=0}^{g-1} A_{pp} = \frac{g}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) \left(-1\right)^{i_m}. \tag{22}$$

It is not very difficult to see that

$$g\lambda_{\min} \leq \sum_{j=0}^{g-1} \lambda_j \leq g\lambda_{\max}$$

and using (22) we have

$$\lambda_{\min} \leq \frac{1}{\sqrt{N}} \sum_{m=0}^{g-1} e\left(i_m\right) \left(-1\right)^{i_m} \leq \lambda_{\max}.$$

For the particular case of $e\left(i_m\right) = \epsilon \left(-1\right)^{i_m}$ we get the relations

$$\lambda_{\min} \leq \frac{g\epsilon}{\sqrt{N}} \leq \lambda_{\max},$$

and for this special case the equation (21) is a submatrix of

$$B_{pq} = \frac{\epsilon}{\sqrt{N}} \sum_{m=0}^{N-1} e^{-\mathrm{j}\frac{2\pi}{N} m(q-p)}$$

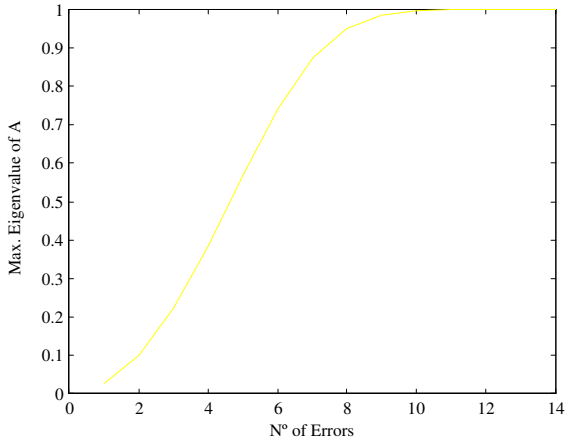

Figure 2. Evolution of the greatest eigenvalue of **A** with the number of actual errors, in the situation $N = 40$, $i_m = m$ and $t = 1 \ldots 14$, and the amplitude scale was normalized by $\sqrt{N}$.

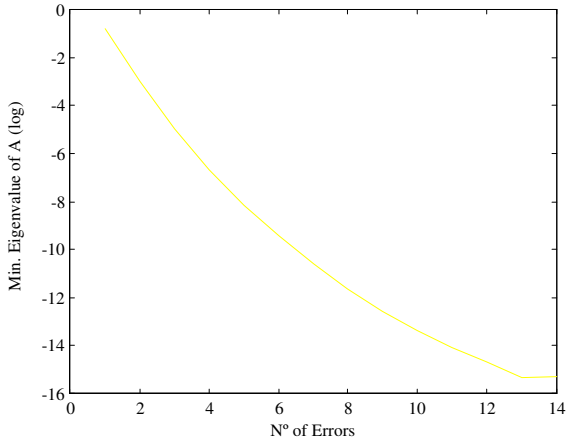

Figure 3. Evolution of the min eigenvalue of A with the number of actual errors, for the situation of the previous figure. Note that the amplitude scale is logarithmic base 10.

with diagonal elements

$$B_{pp} = \epsilon\sqrt{N},$$

giving finally the bounds

$$\boxed{0 \leq \lambda_{\min} \leq \frac{g\epsilon}{\sqrt{N}} \leq \lambda_{\max} \leq \epsilon\sqrt{N}}. \qquad (23)$$

The bounds in (23) only gives a small contribution to understand the behavior of the condition of A as a function of the error positions. To get some more insight, we have run some simulations and in the figures 2 and 3, we can see the value of the greatest eigenvalue and the smallest of matrix **A**, as a function of the number of errors when the equation (18) holds. As one can see the smallest eigenvalue of **A** rapidly decreases with the number of consecutive errors.

5. Conclusions

In this article, we have used BCH coding techniques to reconstruct time limited signals in the field of real or complex numbers. We have identified four formulations of this problem and selected one which makes possible to solve the system in $O\left(t^2\right)$ operations[2] with the recursive method due to *Levinson*. We have also studied the conditioning of **A** and gave bounds for its eigenvalues as a function of the error positions.

This technique can be used as part of a coding system to allow DSP-based error correction in a general purpose processor. As an alternative, the methods could also be used to reconstruct signals corrupted with impulsive noise, in which case long signal segments might be partitioned in smaller blocks. Yet another application is in spectrum analysis: detecting the frequencies and amplitudes of a unknown number of sinusoids (the Fourier transform of these signals is similar to our "impulsive noise", and the role of time / frequency is reversed).

[2] Algorithms $O\left(t \log t\right)$ also exist

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