Noniterative Reconstruction of Band Limited Signals and Coding Techniques

José M. N. Vieira, and Paulo Jorge S. G. Ferreira

Dep. de Electrónica e Telecomunicações da Univ. de Aveiro and INESC
3810 Aveiro, Portugal

ABSTRACT

In this paper we propose linear iterative and noniterative algorithms which can efficiently correct a subset of *t* samples of a signal, whose values might have been corrupted, possibly due to noise, clipping, or any other reasons. We discuss the applicability of techniques used in error correcting codes to this problem, and the possibility of determining the location of the erroneous samples in oversampled band limited signals. The method borrows ideas from error-correcting codes such as BCH codes, but works in the complex field rather than in a Galois field. This work complements works in reconstruction theory which usually assume that the positions of lost samples or pixels are known, and improves on previously reported nonlinear iterative algorithms.

1. INTRODUCTION

The theory of error-correcting codes is a well established branch of communication theory. See [1,2] for an introduction. There are several approaches to error-correction, but the coding procedures in common practice can be classified in two types: the convolutional codes and the block codes. Both are based on the addition of redundant data to a digital signal called message. It is this redundancy that makes error correction possible. In the signal processing field, and more precisely in sampling and reconstruction theory, there is a similar concept known as oversampling. An oversampled signal is a signal that has been sampled at a rate higher than the minimum (Nyquist) rate. An oversampled finite-dimensional signal can be artificially generated by taking the discrete Fourier transform (*DFT*) of the original signal, zero-padding, and then inverting back to the time domain. This is the well known interpolation procedure based on zero-padding.

The convolutional codes are comparable to low-pass filtering data that have been made redundant by inserting zeros [3], whereas block coding is reminiscent of interpolation through zero padding. Both techniques are implemented in a finite field (a Galois field). In fact, finite fields are as common in coding theory as the complex field is in signal processing and reconstruction. Block coding has been applied to implement the coding and decoding of BCH codes in Galois fields [1].

The connection between the two fields was first noticed by Marvasti [3], who explored it to derive algorithms capable of restoring a signal with missing samples. The interesting methods proposed by Marvasti are nonlinear and/or iterative, and at each step operate on the entire data record (that is, the algorithms manipulate both the known and unknown samples). Contrarily to most other works in reconstruction, he did not assume that the positions of the missing samples were known.

In the present work we follow this path, import part of the spectral techniques common in block coding to the complex field, and apply them to the problem of restoring random errors in a signal. To summarize the results in one sentence: given a band-limited signal with N samples, and $2t$ contiguous zero samples in its *DFT*, it is possible to evaluate the position and amplitude of t errors, noniteratively or even iteratively using a linear algorithm, in at most $O(t^2)$ flops/time, if t is small; or asymptotically in $O(t \log t)$ flops/time, if t is large.

2. NOTATION AND TERMINOLOGY

It is important to define the notation used in this work because it is a combination of notations used in signal processing and reconstruction theory and in error correcting codes. We concentrate on block coding, where we want to transmit a discrete signal, called the message, denote by $m(n)$, and with K samples. To correct a maximum of t errors in that

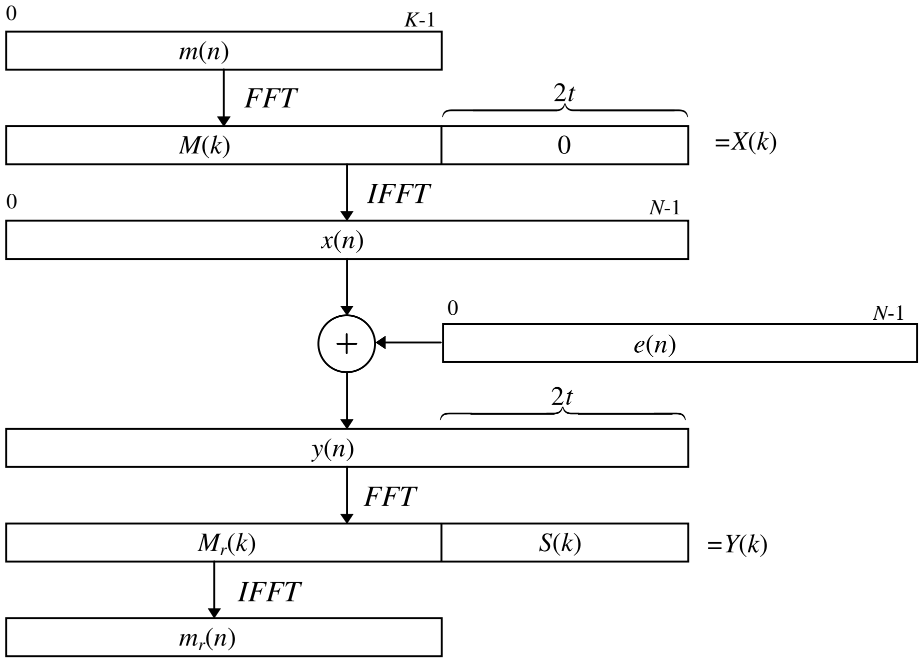

Figure 1 - Encoding and decoding of digital signal $m(n)$ based on spectral techniques.

message it is necessary to add $2t$ samples of redundancy leading to a total of $N=2t+K$ samples to transmit (assuming that the positions are not known beforehand). Such a code is called a (N,K) block code. As we will see, it enables both the error positions and their amplitudes to be found.

In Figure 1 we can see the classical encoding method based on spectral techniques for block codes [1,2]. First, the *DFT* of the message signal $m(n)$ is evaluated and $2t$ zeros are appended to its *DFT* $M(k)$. After evaluating the *IDFT* of this zero padded spectrum we have the signal that will be transmitted $x(n)$. This signal is an interpolated version of $m(n)$ with an added redundancy, or, in other words, oversampled by a factor of N/K. A maximum of t errors can occur during the transmission, and so the temporal error signal $e(n)$, of N samples, has only t samples different from zero. If $U=\{i_0,i_1,\ldots,i_{t-1}\}$ is the set of error indices we can write the error samples as $e(i_m)$ where $i_m \in U$ and $0 \le i_m \le N-1$. The received signal $y(n)$ has as *DFT* the signal $Y(k)$ which should have $2t$ zeros if $e(n)$=0. If some errors occur these $2t$ samples are a "window" over the spectrum $E(k)$ of $e(n)$, which is usually called the Syndrome $S(k)$. From those $2t$ samples of $E(k)$ is possible to find all the spectrum $E(k)$ and by inverse transform the error signal $e(n)$. By subtracting the error signal $e(n)$ from the received signal we obtain the transmitted signal $x(n)$.

Now that the notation and some of the basic ideas are explained we are ready to suggest the recovery algorithms.

3. RESULTS

It is possible to restore t errors if we add $2t$ samples to the spectrum of $m(n)$. We can resume this problem to the one of finding the complete *DFT* $E(k)$ of the error signal $e(n)$ knowing only $2t$ components of $E(k)$ (the syndrome) and that the signal $e(n)$ has only t components $e(i_m)$ different from zero. That way we can write $E(k)$ as

$$E(k)=\sum_{m=0}^{t-1} e(i_m)\phi^*(i_m,k) \qquad (1)$$

where $\phi^*(i_m,k)$ is the transform basis of the used transform, in our case $\phi^*(i_m,k)=e^{-j\frac{2\pi}{N}i_m k}$. In this system of equations we have $2t$ unknowns, t error positions i_m, and t error amplitudes $e(i_m)$ so we need to know $2t$ samples of $E(k)$ to solve the system. Consider now the locator polynomial:

$H(s)=s^t+\alpha_1 s^{t-1}+\ldots+\alpha_{t-1}s+\alpha_t$, where $H(s_m)=0$ with $s_m=e^{-j\frac{2\pi}{N}i_m}$. So we can write the equation: $s_m^t+\alpha_1 s_m^{t-1}+\ldots+\alpha_{t-1}s_m+\alpha_t=0$, and if we multiply it by $e(i_m)e^{-j\frac{2\pi}{N}i_m r}$ and sum over m we have:

$$\sum_{m=0}^{t-1} e(i_m) e^{-j\frac{2\pi}{N} i_m (r+t)} + \alpha_1 \sum_{m=0}^{t-1} e(i_m) e^{-j\frac{2\pi}{N} i_m (r+t-1)} + \ldots + \alpha_t \sum_{m=0}^{t-1} e(i_m) e^{-j\frac{2\pi}{N} i_m r} = 0 \quad (2)$$

The previous equation can be written in the following form:

$$E(r+t) + \alpha_1 E(r+t-1) + \ldots + \alpha_t E(r) = 0$$

or equivalently

$$\alpha_1 E(r+t-1) + \ldots + \alpha_t E(r) = -E(r+t)\,. \quad (3)$$

As we know $2t$ components of $E(k)$ we can write the following system of equations:

$$\begin{bmatrix} E(N-2) & E(N-3) & \cdots & E(N-t-1) \\ E(N-3) & E(N-4) & \cdots & E(N-t-2) \\ \vdots & \vdots & & \vdots \\ E(N-t-1) & E(N-t-2) & \cdots & E(N-2t) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_t \end{bmatrix} = - \begin{bmatrix} E(N-1) \\ E(N-2) \\ \vdots \\ E(N-t) \end{bmatrix} \quad (4).$$

The eq. (3) means that each component of $E(k)$ is a linear combination of t other components of $E(k)$. The system of equations (4) can be written in the vector form **Ax=b** where **A** is a *Hankel* matrix.

Equation (4) can be found in [3,4], but the method suggested is not based on its solution. Instead, the authors consider iterative and-or nonlinear iterative methods. So, at this point we depart from [3,4] and note that eq. (4) can be transformed using the change of variables $\alpha_k = h_{t-k+1}$ to the equivalent system of eq. (5), whose matrix is a *Toeplitz* matrix of dimension $t \times t$. The rank of this matrix is equal to the number of errors that really occurred, and for a given number N_e of occurred errors it has only N_e linear independent rows.

$$\begin{bmatrix} E(N-t-1) & E(N-t) & E(N-t+1) & \cdots & E(N-2) \\ E(N-t-2) & E(N-t-1) & E(N-t) & \cdots & E(N-3) \\ E(N-t-3) & E(N-t-2) & E(N-t-1) & \cdots & E(N-4) \\ \vdots & \vdots & \vdots & & \vdots \\ E(N-2t) & E(N-2t+1) & E(N-2t+2) & \cdots & E(N-t-1) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_t \end{bmatrix} = - \begin{bmatrix} E(N-1) \\ E(N-2) \\ E(N-3) \\ \vdots \\ E(N-t) \end{bmatrix} \quad (5).$$

An algebraic set of Toeplitz equations can be efficiently inverted with an asymptotic complexity $O(t \log t)$. But if there are fewer than t errors a more rational method is to solve (5) using Levinson´s recursive algorithm, stopping as soon as a singular submatrix is detected. This avoids having to compute the rank of **A** before evaluating (5), and yields a method that requires an amount of computation proportional to the square of the number of effective errors. This stands in contrast to the algorithms proposed in [3,4], which operate on the entire data record at each iteration. Obviously, t is much smaller than N.

Once (5) is solved and the values of h_i are known then from (3) we can write the following recursion formula to evaluate the other $K=N-2t$ elements of $E(k)$

$$E(r) = -\sum_{i=1}^{t} h_i E(r-t-1+i)\,. \quad (6)$$

To obtain the transmitted signal $x(n)$ we only need to evaluate the error signal $e(n)$ by *IFFT* of $E(k)$ and subtract it from the received signal.

$$e(n) = IFFT\{E(k)\} \qquad x(n) = y(n) - e(n)$$

The message $m(n)$ can be obtained easily by evaluating the *DFT* of $x(n)$, truncate to eliminate the zeros, and then evaluating the *IDFT* of this signal.

4. SIMULATION RESULTS

The algorithm presented here is able to perform error location in a band limited discrete signal. To illustrate this a simulation result is presented in fig.2 for a random discrete signal $m(n)$ with 32 samples with 32 zeros added to its spectrum. The reconstructed signal and the original are almost a perfect match, and the maximum reconstruction error is 5e-10.

As the number of erroneous consecutive samples increases, the algorithm rapidly misbehaves. In fig.3 we have the same vector sizes and message $m(n)$ as in fig.2 but with 7 consecutive errors, and as depicted the reconstructed signal is very different from the original. This is not really a problem with the algorithm, but rather a consequence of the very ill-posed nature of the problem, consistent with the results in [5-7].

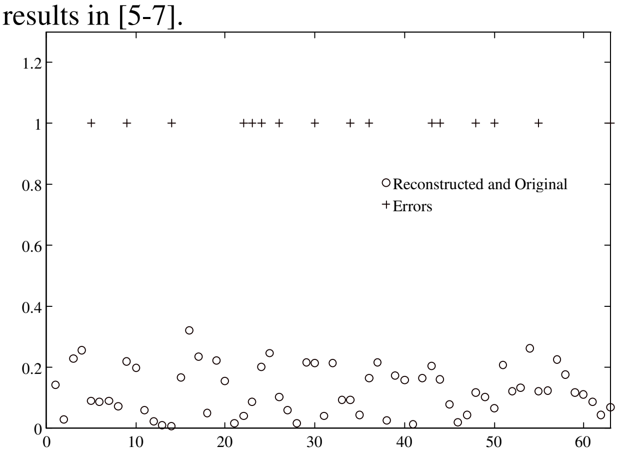

Figure 2. Example of signal reconstruction for N= 64, K= 32 and t= 16, with randomly distributed errors. The positions of the erroneous samples are signalled with a cross. The original $m(n)$ and reconstructed $m_r(n)$ signals are indistinguishable.

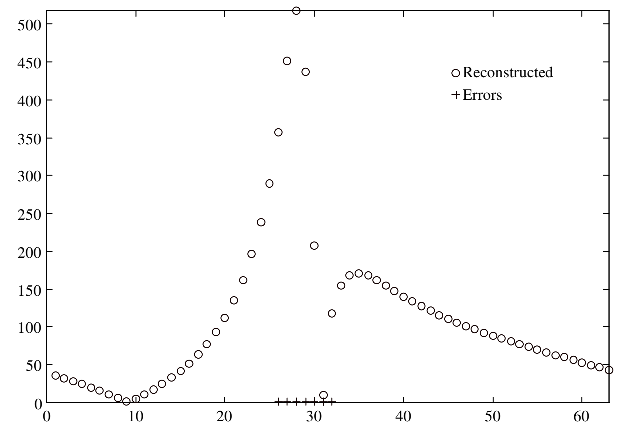

Figure 3. Example of signal reconstruction for N= 64, K= 32 and t= 16, in which there are 7 consecutive errors. The deviation between the reconstructed $m_r(n)$ and original $m(n)$ signals is so great that the original cannot be seen at the scale of the figure.

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