# Localization and Spreading of Diseases in Complex Networks

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# Susceptible-Infected-Susceptible epidemic model

*SIS* model: a standard paradigm for disease spreading in networked systems

Individuals (vertices) can be in one of two states:

1. Susceptible (or healthy) - $S$

2. Infected - $I$

An infected vertex becomes susceptible with unit rate:

$$I \xrightarrow{1} S\,,$$

and infects its susceptible neighbor at rate $\lambda$:

$$\begin{array}{l} S \xrightarrow{\lambda} I\,. \\ \Uparrow \\ I_{nn} \end{array}$$

[ arXiv:1202.4411; PRL 109, 128702 (2012) ]

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Simplest model undergoing an epidemic phase transition between an absorbing, healthy phase, and an active phase with a stationary endemic state.

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For this kind of homogeneous networks one can safely say that disease spreading is well understood and $\lambda_c \sim 1/\langle q \rangle$ .

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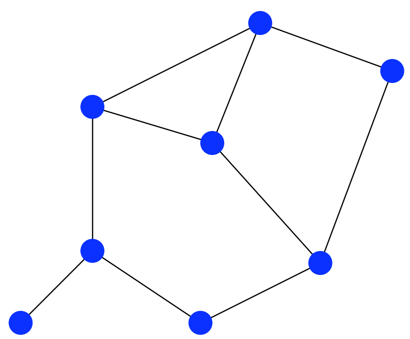

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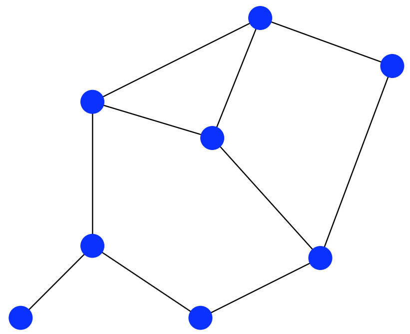

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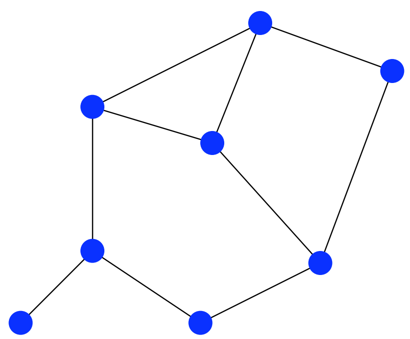

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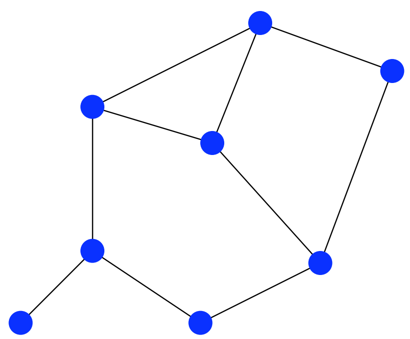

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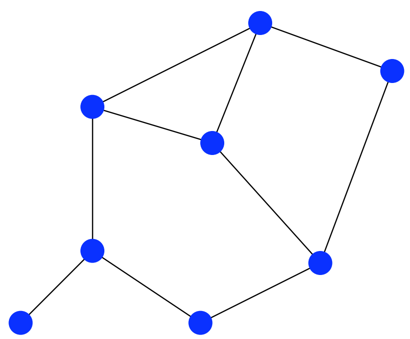

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Heterogeneous networks!

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$$\rho \equiv \sum_{i=1}^{N} \rho_i/N$$ is nonzero .

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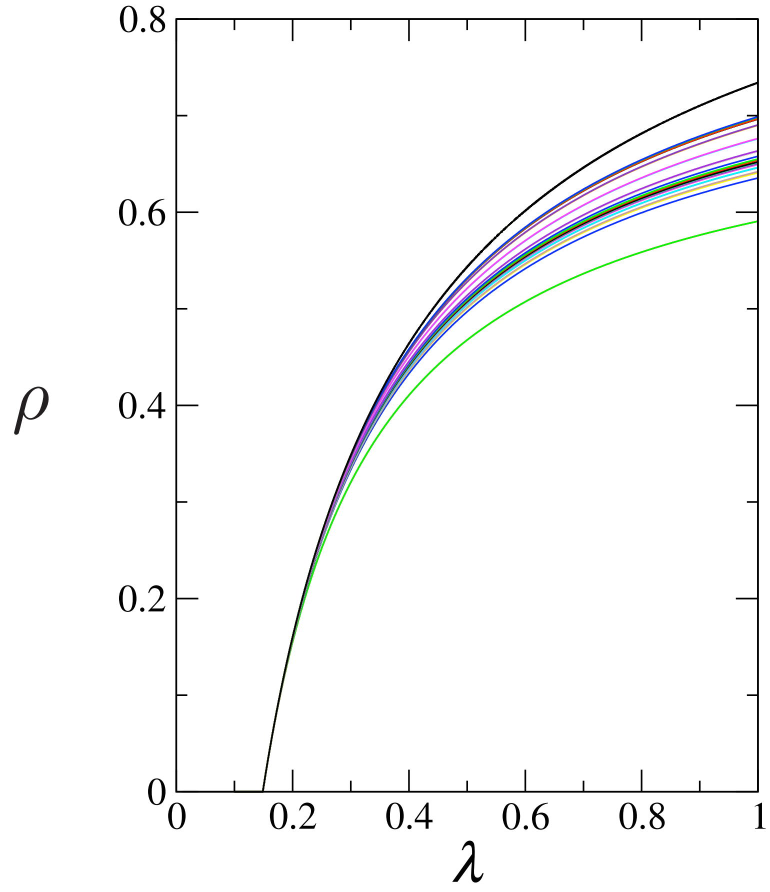
# Example of the SIS model on a real network*




*Network of social ties between people belonging to a karate club.

The prevalence $\rho$ is the most upper curve (black line).

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Since $\mathbf{A}$ is real and symmetric, its $N$ eigenvectors

$$\vec{f}(\Lambda) \;\; (\Lambda_{\max} \equiv \Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_N)$$

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Solving Eq. (2) with respect to $c(\Lambda_1)$ and setting it to zero gives:

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Considering the two largest eigenvalues $\Lambda_1$ and $\Lambda_2$ , and their eigenvectors, gives

$$\rho(\lambda) \approx \alpha_1 \tau + \alpha_2 \tau^2$$

[ arXiv:1202.4411; PRL 109, 128702 (2012) ]

# Localized and delocalized eigenvectors

The usual point of view is that a finite fraction of vertices is infected immediately above $\lambda_c$ . This corresponds to $\alpha_1$ of order $\mathcal{O}(1)$ in our analysis.

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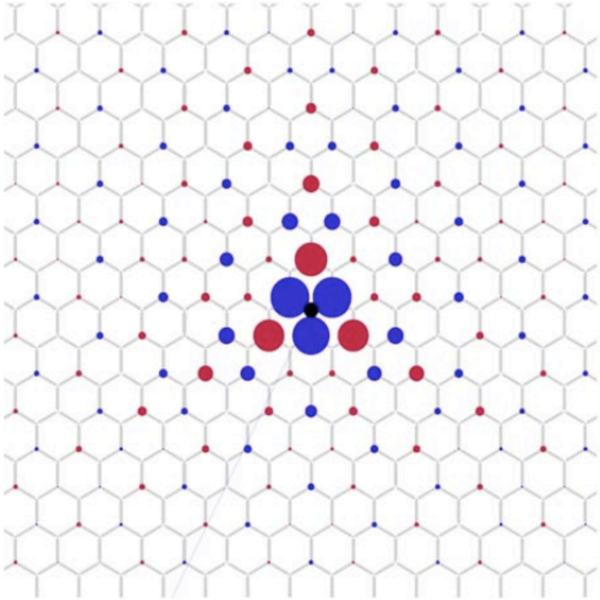

From [ Pereira et al., PRB 77, 115109 (2008) ]

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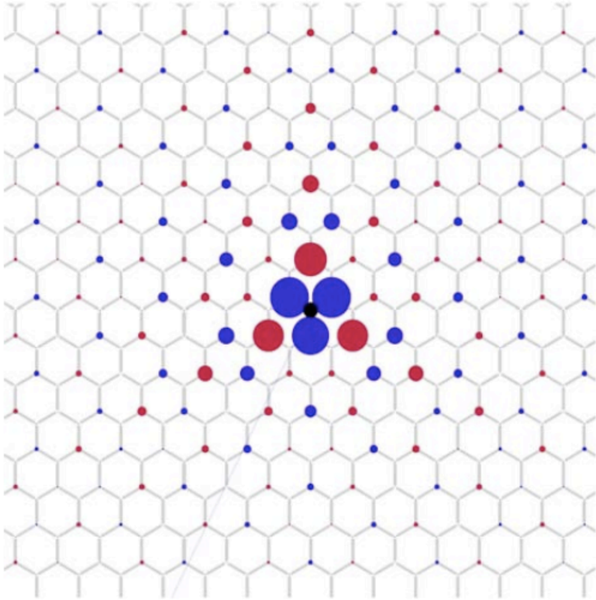
The wave function is localized on a finite number of sites around the impurity.



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How to quantify localization?

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As an illustration, consider two limiting cases:

(i) a vector with identical components $f_i = 1/\sqrt{N}$ ,

(ii) a vector with one component $f_i = 1$ and the remainders zero.

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Case (**ii**) gives $\mathrm{IPR} = 1\,.$

Thus : a delocalized state: $\mathrm{IPR}(\Lambda) \stackrel{N\to\infty}{\longrightarrow} 0$

a localized state: $\mathrm{IPR}(\Lambda) \stackrel{N\to\infty}{\longrightarrow} const. > 0$

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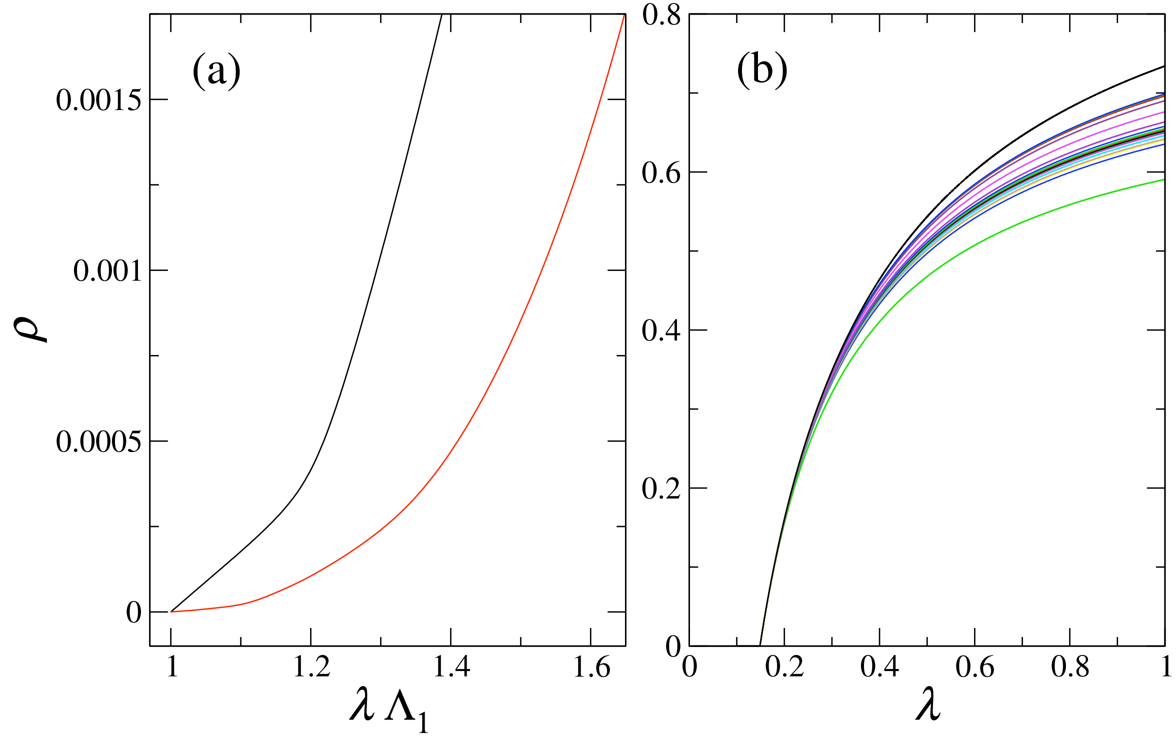
If $\vec{f}(\Lambda_1)$ is delocalized, then $\rho$ is of order $\mathcal{O}(1)$ and the disease infects a finite fraction of vertices right above $\lambda_c$ .

[ arXiv:1202.4411; PRL 109, 128702 (2012) ]

# Weighted and unweighted real-world nets




**(a)** Weighted collaboration networks of scientists posting preprints on the:

(**black line**) astrophysics archive at arXiv.org, 1995-1999

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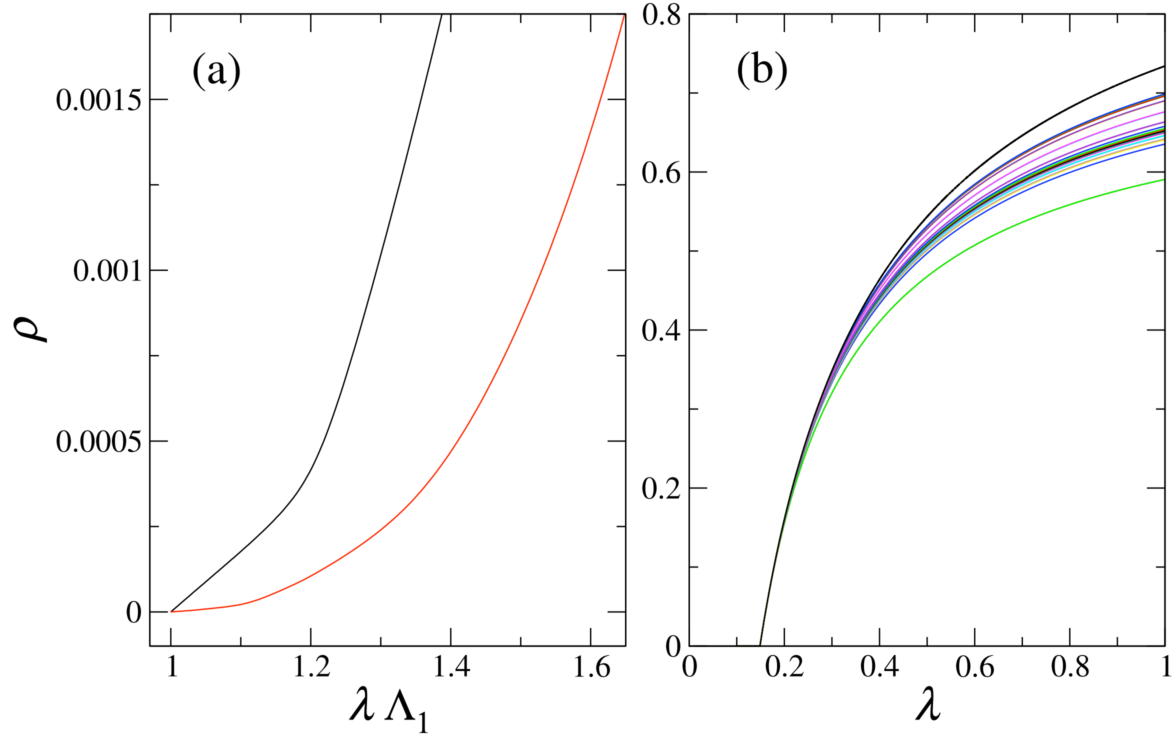


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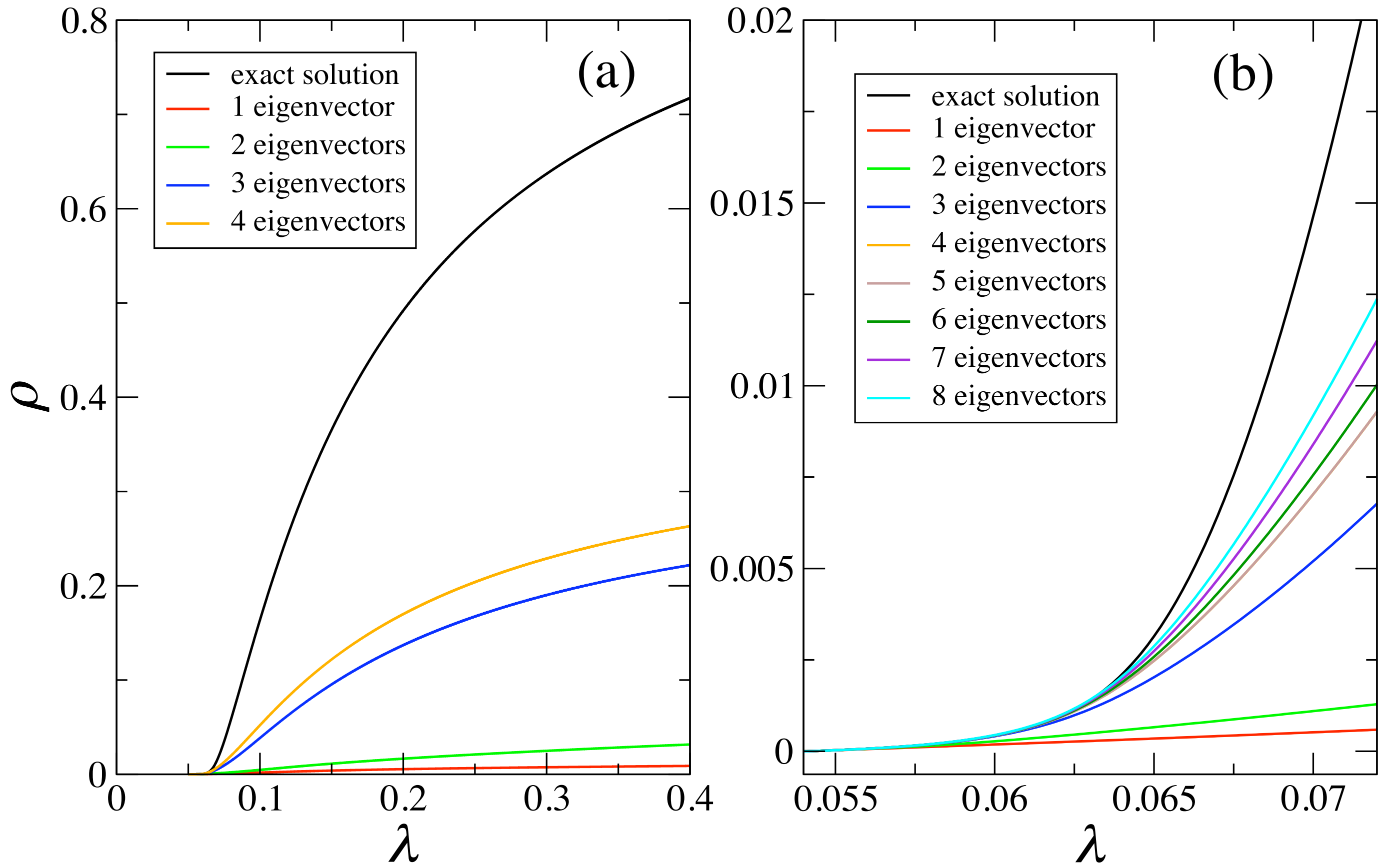
**(b)** Unweighted karate-club network: the lowest curve only accounts for the eigenstate $\Lambda_1$. The most upper curve is the exact $\rho$ .

[ arXiv:1202.4411; PRL 109, 128702 (2012) ]

# An uncorrelated scale-free network

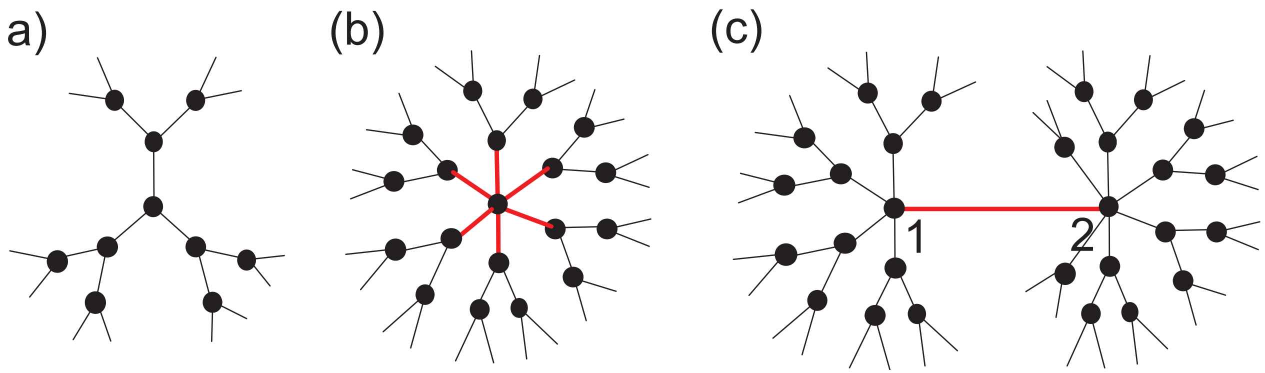



**(a)** A scale-free network of $10^5$ vertices generated by the static model with $\gamma = 4$ and $\langle q \rangle = 10$. **(b)** Zoom of the prevalence at $\lambda$ close to $\lambda_c = 1/\Lambda_1$.

Eigenvectors corresponding to $\Lambda_1$ and $\Lambda_2$ are localized. $\Lambda_3$ is delocalized.

[ arXiv:1202.4411; PRL 109, 128702 (2012) ]

# A Bethe lattice (a) with a hub (b) or two hubs (c)

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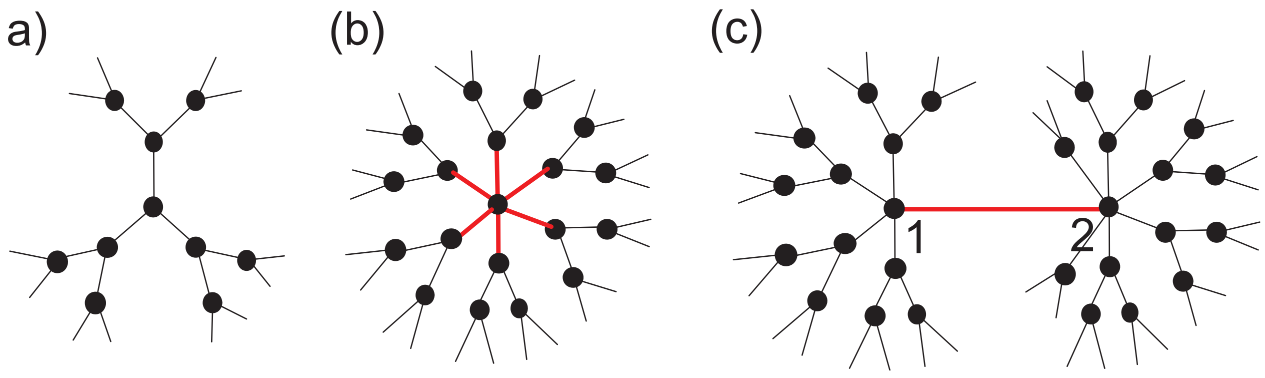


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(**a**) $\Lambda_1 = k$ and $f_i(\Lambda_1) = 1/\sqrt{N}$ (delocalized) .

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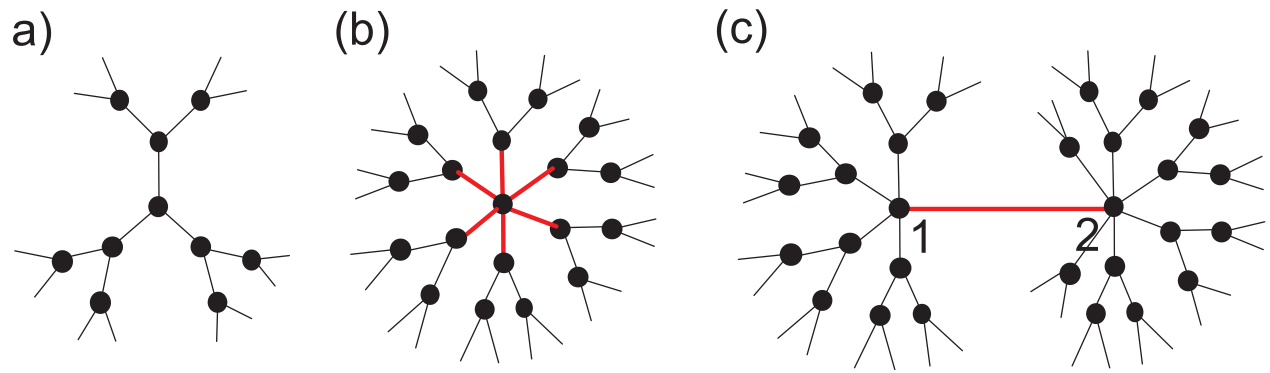


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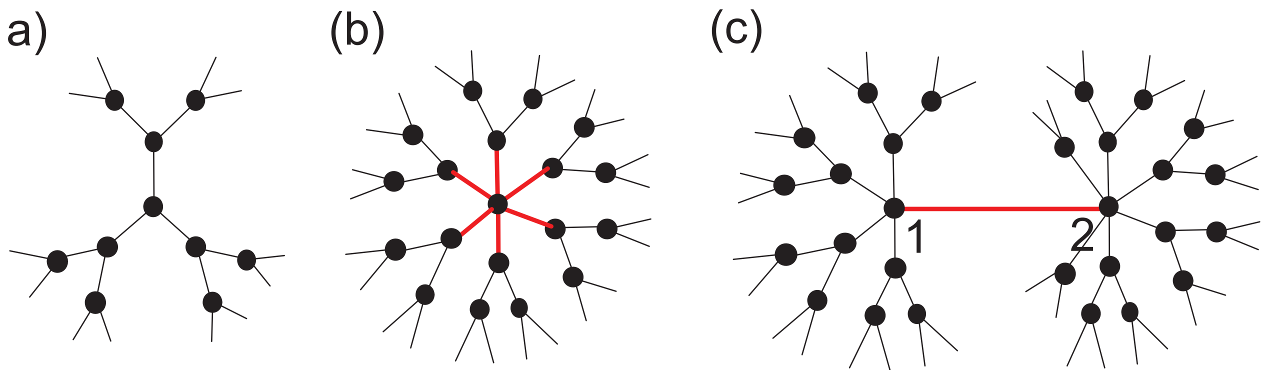


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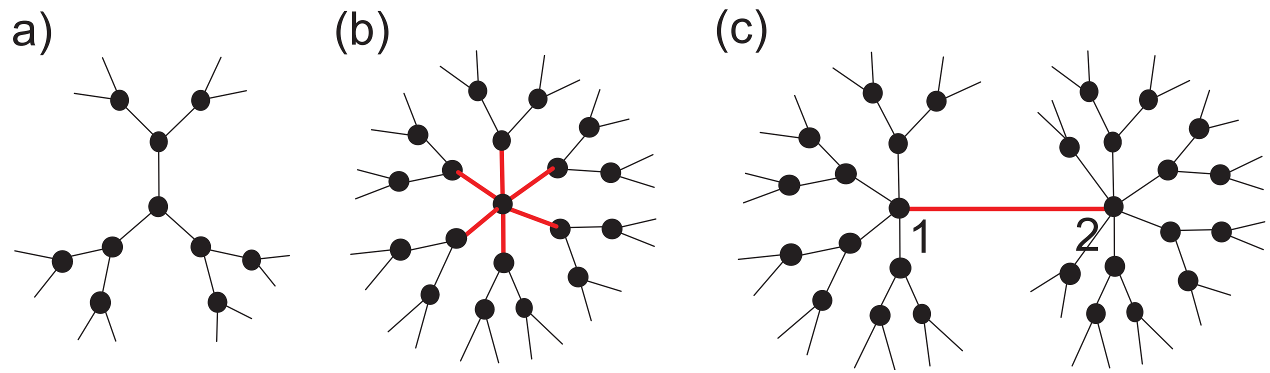


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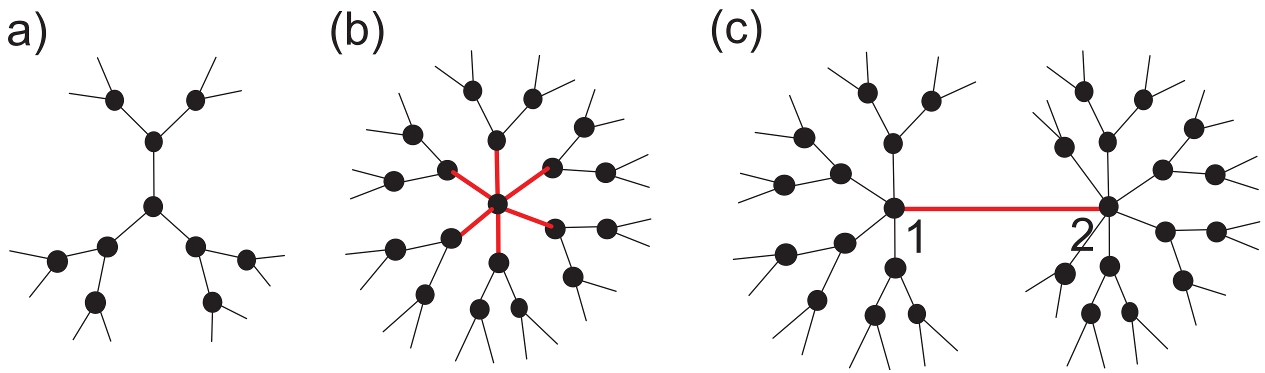


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# Conclusion

If the principal eigenvector of the adjacency matrix is localized,

then, immediately above the threshold $1/\Lambda_1$,

the disease is localized on a finite number of vertices.

In this case, a real epidemic affecting a finite fraction of

vertices occurs after a smooth crossover, and

the notion of the epidemic threshold is meaningless.

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