# On the various definitions of cyclic operads 

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## Overview: different definitions of (cyclic) operads

Operad (a.k.a. (one object) multi-category) $=$ operations + (associative) compositions + permutation of variables (+ identities)

|  | Biased (individual compositions) |  |  | Unbiased (monad of trees) | Algebraic <br> (microcosm principle) |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Classical | Partia |  |  |  |  |
| Symmetric Operads | Boardman, Vogt, May | Markl |  | Smirnov, May, Getzler, Jones | Classical | Partial |
|  |  |  |  | $\begin{gathered} \hline \text { May, Smirnov, } \\ \text { Kelly } \end{gathered}$ | Fiore |
| Cyclic Operads | Getzler, Kapranov | Exchangeable output | Entries only |  | Getzler, Kapranov | Exchangeable output | Entries only |
|  |  | Markl | Markl | ?? |  | ?? |

+ Two flavours: skeletal and non-skeletal
Plan:
- Examine these definitions
- Introduce a $\lambda$-calculus-style syntax: the $\mu$-syntax

- Fill in the question marks


## Symmetric operads

## Classical + Skeletal

- $\mathcal{P}: \boldsymbol{\Sigma}^{o p} \rightarrow \mathbf{C}$
- $\gamma: \mathcal{P}(n) \otimes \mathcal{P}\left(k_{1}\right) \otimes \cdots \otimes \mathcal{P}\left(k_{n}\right)$

$$
\rightarrow \mathcal{P}\left(k_{1}+\cdots+k_{n}\right)
$$



- $\eta: \mathbf{1} \rightarrow \mathcal{P}(1)$


## Partial + Non-skeletal

- $\mathcal{S}: \mathrm{Bij}^{\circ p} \rightarrow \mathbf{C}$
- $\circ_{x}: \mathcal{S}(X) \times \mathcal{S}(Y) \rightarrow \mathcal{S}((X \cup Y) \backslash\{x\})$

- $i d_{x} \in \mathcal{S}(\{x\})$


## Unbiased

An operad is an algebra over the monad of rooted, decorated, labeled trees (which constitute the category $\operatorname{Tree}_{n}$ ).


## Exchangeable output: from ordinary to cyclic operads

If we extend the relabeling of the leaves of a rooted tree to an action of interchanging the labels of all its half-edges, including the label given to the root, we arrive at cyclic operads.

This is achieved by enriching the operad structure (classical or partial) with an action of the cycle $\tau_{n}=(0,1, \ldots, n)$ :


This action makes the distinction between inputs and the output of an operation no longer visible, leading us to an alternative axiomatization of cyclic operads...

## Cyclic operads: entries only

## Definition 1 (Partial + non-skeletal)

A cyclic operad is a functor $\mathcal{E}: \mathbf{B i j}^{\circ p} \rightarrow \mathbf{S e t}$, together with a distinguished element $i d_{x, y} \in \mathcal{C}(\{x, y\})$ for each two-element set $\{x, y\}$, and a partial composition operation

$$
x^{\circ} y: \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}((X \cup Y) \backslash\{x, y\})
$$

These data are required to satisfy the associativity, equivariance, unitality and commutativity equations.

Associativity.
$\left(f_{x} \circ_{y} g\right) u \circ_{z} h=f_{x} \circ_{y}\left(g_{u} \circ_{z} h\right)$
$\left(f_{x} \circ_{y} g\right) u \circ_{z} h=\left(f_{u} \circ_{z} h\right)_{x} \circ_{y} g$
Equivariance.

$$
f^{\sigma_{1}}{ }_{x}{ }_{y} g^{\sigma_{2}}=\left(f_{\sigma_{1}(x)^{\circ}{ }^{\sigma_{2}(y)}} g\right)^{\sigma}
$$

Unitality.
$f_{x}{ }^{\circ} y d_{y, z}=f^{\sigma}$
$i d_{y, z} y^{0_{x}} f=f^{\sigma}$
Commutativity.

$$
f_{x} \circ_{y} g=g_{y} \circ_{x} f
$$

This definition induces a natural combinator syntax.

## Cyclic operads: exchangeable output

## Definition 2 (Partial + non-skeletal)

A cyclic operad is an ordinary operad $\mathcal{S}$, augmented with actions

$$
D_{x y}: S(X) \rightarrow S((X \backslash\{x\}) \cup\{y\})
$$

indexed by variables $x \in X$ and $y \notin X \backslash\{x\}$, and subject to the following list of axioms:
Identity. $D_{x x}(f)=f$
Equivariance.
Coherence. $D_{z x}\left(D_{x y}(f)\right)=D_{z y}(f)$
$D_{\sigma(x) \sigma(y)}\left(f^{\sigma}\right)=D_{x y}(f)^{\sigma}$
$\alpha$-conversion. $D_{x a}(f)=D_{x^{\prime} a}\left(f^{\sigma}\right)$, where $\sigma(x)=x^{\prime}$, and $\sigma=i d$ elsewhere

Compatibility with compositions.
$D_{x z}\left(f \circ_{y} g\right)=D_{x u}(g) \circ_{u} D_{y z}(f)$
$D_{x z}\left(f \circ_{y} g\right)=D_{x z}(f) \circ_{y} g$
Notice that from the second axiom, by taking $y=z$, it follows that each $D_{x y}$ has the action $D_{y x}$ as an inverse.

## Cyclic operads: unbiased definition

The entries-only characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along any edge.

The pasting shemes for cyclic operads come from the category CTree $_{n}$ of unrooted (cyclic), decorated, labeled trees.


Given a functor $\mathcal{P}: \mathbf{B i j}^{o p} \rightarrow \mathbf{C}$, we build the free operad $F(\mathcal{P})$ by grafting of such trees. The free operad functor $F$ and the forgetiful functor $U$ constitute a monad $\Gamma=U F$ in $\mathbf{C}^{\mathrm{Bij}^{\circ p}}$, called the monad of unrooted trees.

## Definition 3

A cyclic operad is an algebra over this monad.

|  | Biased (individual compositions) |  | Unbiased <br> (monad of trees) |
| :---: | :---: | :---: | :---: |
|  | Classical | Partial |  |
| Symmetric Operads | Boardman, Vogt, May | Markl | Smirnov, May, Getzler, Jones |
| Cyclic Operads | Getzler, Kapranov | Exchangable <br> output Entries <br> only <br> Markl Markl | Getzler, Kapranov |
|  |  | $\mu$-syntax |  |

## $\mu$-syntax for cyclic operads

The $\mu$-syntax consists of two kinds of expressions:
commands $c: X$ (no entry selected) terms $X \mid s$ (one entry selected)

$$
c::=\langle s \mid t\rangle\left|\underline{f}\left\{t_{x} \mid x \in X\right\} \quad s, t::=x\right| \mu x . c
$$

The typing rules are as follows:

$$
\overline{\{x\} \mid x} \quad \frac{f \in \mathcal{S}(X) \ldots Y_{x} \mid t_{x} \ldots}{\underline{f}\left\{t_{X} \mid x \in X\right\}: \bigcup Y_{x}} \quad \frac{X|s \quad Y| t}{\langle s \mid t\rangle: X \cup Y} \quad \frac{c: X}{X \backslash\{x\} \mid \mu x . c}
$$

The equations are $\langle s \mid t\rangle=\langle t \mid s\rangle$ and (oriented from left to right):

$$
\langle\mu x . c \mid s\rangle=c[s / x] \quad \mu x .\langle x \mid y\rangle=y
$$

## $\mu$-syntax: intuition

$\langle\mu x . c \mid s\rangle$ and $c[s / x]$ describe two ways to build the same underlying tree!


$$
\begin{aligned}
\langle\mu y \cdot \underline{f}\{\mu a \cdot \underline{g}\{a, b, c, d\}, y, z, w\} \mid \mu p \cdot \underline{h}\{p, q\}\rangle & =\underline{f}\{\mu a \cdot \underline{g}\{a, b, c, d\}, y, z, w\}[\mu p \cdot \underline{h}\{p, q\} / y] \\
& =\underline{f}\{\mu a \cdot \underline{g}\{a, b, c, d\}, \mu p \cdot \underline{h}\{p, q\}, z, w\}
\end{aligned}
$$

## $\mu$-syntax as a rewriting system

Non-confluent - critical pairs arise from the second equation viewed as a rewriting rule:

$$
c_{2}\left[\mu x \cdot c_{1} / y\right] \longleftarrow\left\langle\mu x \cdot c_{1} \mid \mu y \cdot c_{2}\right\rangle \longrightarrow c_{1}\left[\mu y \cdot c_{2} / x\right]
$$

Terminating (modulo the commutativity of $\langle s \mid t\rangle$ ) - the set NF of normal forms consists of terms produced only with the following rules:

$$
\begin{aligned}
& x \in N F \\
& \text { if } f \in \mathcal{C}(X) \text { and } t_{x} \in N F \text { for all } x \in X \text {, then } \underline{f}\left\{t_{x} \mid x \in X\right\} \in N F \\
& \text { if } c \in N F \text {, then } \mu x . c \in N F
\end{aligned}
$$

From the viewpoint of trees, we observe that the commands in normal form correspond to different tree traversals of finite unrooted trees.

## $\mu$-syntax does the job!

## Theorem 1

The set of commands of the $\mu$-syntax generated by a cyclic operad $\mathcal{C}$ and quotiented by the equations, is in one-to-one correspondence with the set of unrooted trees with node decorations and half-edge labels induced by $\mathfrak{C}$.

The steps of the proof are as follows:

- Using Markl's formalism of trees with half-edges, we associate with every term of the syntax its underlying tree.
- We show that this assignment is well defined: the underlying trees are invariant under the equations of the syntax.
- We show that this assignment is surjective: for any tree one can build a command that represents it.
- We show the injectivity by proving that if two normal forms have the same underlying graph, then they are provably equal.


## Some interesting parts of the injectivity item

The equality relation $=$ generated by the reductions $\langle\mu x . c \mid s\rangle \longrightarrow c[s / x]$ and $\mu x .\langle x \mid y\rangle \longrightarrow y$ lives in the set of all commands.
We introduce an equality $=^{\prime}$ that relates normal forms only:

$$
\text { if } \sigma(x)=\mu y . c, \text { then } f\{\sigma\}==^{\prime} c[\mu x . f\{\sigma[x / x]\} / y]
$$



$$
\underline{f}\{\mu y \cdot \underline{g}\{y, p, q, r, s\}, a, b, c\}=\underline{g}\{\mu x \cdot \underline{f}\{x, a, b, c\}, p, q, r, s\}
$$

It has been put to direct use in the injectivity proof as follows:

$$
c_{1}=c_{2} \Rightarrow T\left(c_{1}\right)=T\left(c_{2}\right) \Rightarrow c_{1}=^{\prime} c_{2} \Rightarrow c_{1}=c_{2} .
$$

What we got as a bonus:

- Commands/= $\cong N F /={ }^{\prime}$
- a clear connection with the reversible terms syntax of Lamarche


## Algebraic



## Algebraic environment: Species of structures (Joyal)

$\mathcal{S}: \mathbf{B i j}^{\text {op }} \rightarrow$ Set is a contravariant version of Joyal's species of structures!
The operadic composition structure in this context is recognized by examining the properties of basic operations on species.

The sum of species $S$ and $T$ :

$$
(S+T)(X):=S(X)+T(X)
$$

The product of species $S$ and $T$ :

$$
(S \cdot T)(X):=\sum_{\left(X_{1}, X_{2}\right)} S\left(X_{1}\right) \times T\left(X_{2}\right)
$$

The substitution product of species $S$ and $T$ (with $S(\emptyset)=\emptyset$ ):

$$
(S \circ T)(X)=\sum_{\pi \in P(X)} S(\pi) \times \prod_{p \in \pi} T(p)
$$

The derivative of a species $S$ :

$$
\partial S(X)=S(X+\{*\})
$$

## Symmetric operads: classical, algebraic

How does an element of $(T \circ S)(X)$ look like?

$$
\underline{g}\left\{\underline{f_{y}}\left\{x \mid x \in X_{y}\right\} \mid y \in Y\right\}
$$

And what are the properties of the substitution product?

- It is associative (up to isomorphism of species)
- It has the species of singletons I as neutral element.
$\longrightarrow$ The substitution product makes the category of species a monoidal category (with unit I).


## Definition 1

An operad is a monoid $(S, \mu: S \circ S \rightarrow S)$ in the category of species.
Specifying a monoid in a monoidal category is a typical instance of what is known as the microcosm principle of higher algebra (Baez-Dolan):

Certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.

## Symmetric operads: partial, algebraic

The product on species needed to formulate partial composition algebraically is

$$
T * S:=(\partial T) \cdot S
$$

Microcosm principle: What are the properties of this product?
Comparing the species $U *(T * S)$ and $(U * T) * S$, we conclude that

- The product is not associative
- The product satisfies the pre-Lie equality, given by the isomorphism

$$
\beta:((U * T) * S)+(U *(S * T)) \rightarrow(U *(T * S))+((U * S) * T)
$$

## Definition 2

An operad is a pair $(S, \nu: S * S \rightarrow S)$, such that $\nu_{2} \circ \beta=\nu_{1}$, where $\nu_{1}$ and $\nu_{2}$ are induced by $\nu$.

## From ordinary to cyclic operads

Our cornerstone is an ordinary operad $(S, \nu: S * S \rightarrow S)$.
How to enrich this structure, so that it encompasses the actions $D_{x y}$ ?
In particular, we should translate the compatibility of $D_{x y}$ with the two possible partial compositions subject to it:


$$
\nu_{3}: \partial S \cdot \partial S \rightarrow \partial S
$$


$\nu_{4}: \partial(\partial S) \cdot S \rightarrow \partial S$

These two are induced by $\nu:(\partial S) \cdot S \rightarrow S$ and the isomorphism

$$
\varphi: \partial(\partial T \cdot S) \rightarrow(\partial(\partial T) \cdot S)+(\partial T \cdot \partial S)
$$

This suggests to mimick the actions of $D_{x y}$ by a natural transformation $D: \partial S \rightarrow \partial S$, which will commute with $\nu_{3}$ and $\nu_{4}$ in the appropriate way.

## Cyclic operads: exchangeable output

## Definition 3

A cyclic operad is a triple $(S, \nu: S * S \rightarrow S, D: \partial S \rightarrow \partial S)$, such that:

- $(S, \nu: S * S \rightarrow S)$ is an operad, and
- the morphism $D$ is satisfies the laws

$$
D^{2}=i d \text { and }(\partial D \circ e x)^{3}=i d
$$

where ex: $\partial(\partial S) \rightarrow \partial(\partial S)$ exchanges the two distinguished elements in $(X+\{*\})+\{\diamond\}$,

- as well as the laws given by the following commuting diagrams:



## Cyclic operads: entries only

The relevant product on species is now

$$
S \mathbf{\Delta} T=(\partial S) \cdot(\partial T)
$$

Following the microcosm principle, we examine its properties:

- It is commutative
- It satisfies the identity given by the isomorphism
$\gamma:(S \mathbf{\Delta} T) \mathbf{\Delta} U+T \mathbf{\Delta}(S \mathbf{\Delta} U)+(T \mathbf{\Delta} U) \mathbf{\Delta} S \rightarrow S \mathbf{\Delta}(T \mathbf{\Delta} U)+(S \mathbf{\Delta} U) \mathbf{\Delta} T+U \mathbf{\Delta}(S \mathbf{\Delta} T)$


## Definition 4

A cyclic operad is a pair $(S, \rho: S \boldsymbol{\Delta} S \rightarrow S)$, such that $\rho_{2} \circ \gamma=\rho_{1}$, where $\rho_{1}$ and $\rho_{2}$ are induced by $\rho$.

## Descent theory for species (Lamarche)

## Question:

$$
\text { Can we reconstruct a species } T \text {, given } \partial T \text { ? }
$$

The answer is no, but it becomes yes if additional data is provided.
Lamarche (private comunication) defines a descent data as a pair

$$
(S, D: \partial S \rightarrow \partial S)
$$

such that $D^{2}=i d$ and $(\partial D \circ e x)^{3}=i d$.
Lamarche has also shown that the functor

$$
T \mapsto(\partial T, e x)
$$

is an equivalence of categories when restricted to species which are empty on the empty set.

## "exchangeable output" $\Leftrightarrow$ "entries only"

## Theorem 2

This equivalence carries over to an equivalence between the previous two definitions of cyclic operads.

The steps of the proof are as follows:

- Given $\rho: T \mathbf{\Delta} T \rightarrow T$, we define $\nu: \partial T * \partial T \rightarrow \partial T$ as follows

$$
\nu: \partial \partial T \cdot \partial T \xrightarrow{\text { ex•id }} \partial \partial T \cdot \partial T \xrightarrow{i n j} \partial(\partial T \cdot \partial T) \xrightarrow{\partial \rho} \partial T
$$

- Given $\nu: \partial T * \partial T \rightarrow \partial T$, to define $\rho: T \mathbf{\Delta} T \rightarrow T$ amounts to define $\rho^{\prime}: \partial(\partial T \cdot \partial T) \rightarrow \partial T\left(\rho\right.$ is then uniquely defined via $\left.\partial \rho=\rho^{\prime}\right)$ :

$$
\begin{aligned}
& \rho_{1}^{\prime}: \partial \partial T \cdot \partial T \xrightarrow{\text { ex.id }} \partial \partial T \cdot \partial T \xrightarrow{\nu} \partial T \\
& \rho_{2}^{\prime}: \partial T \cdot \partial \partial T \xrightarrow{\text { Swap }} \partial \partial T \cdot \partial T \xrightarrow{\rho_{1}^{\prime}} \partial T
\end{aligned}
$$

$$
\rho^{\prime}=\left[\rho_{1}^{\prime}, \rho_{2}^{\prime}\right]
$$

- And then we have to check that everything commutes in the appropriate way...


## ...for example:


$23 / 24$

## Some directions of investigations:

- Figure out the microcosm principle for modular and wheeled operads
- Study the "up to homotopy" versions of all these (infinity operads, properads, wheeled properads,...)


## Hope that you enjoyed the promenade !

