Partial Mal'tsevness and category of quandles

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Outline

Monoids and partial pointed protomodularity

Mal'tsev and Σ-Mal'tsev category

Quandles

Naturally Mal'tsev and Σ-naturally Mal'tsev category

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Mal'tsev and Σ-Mal'tsev category

Quandles

Naturally Mal'tsev and Σ-naturally Mal'tsev category

- **Definition (B. 1990)**

  A pointed category $\mathbb{C}$ is protomodular when, for any split epimorphism (f, s), the following pullback,:

$$\begin{tikzcd} K[f] \arrow[r, tail, "k_f"] \arrow[d, shift right] & X \arrow[d, shift right, "f"'] \\ 1 \arrow[r, tail, "\alpha_Y"'] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

  is such that the pair (k_f, s) is jointly extremally epic, or in other words $1_X = sup(k_f, s)$.

  - Examples: Groups, non-unital Rings, K-algebras of any non-unitary kind, Lie_K-algebras;
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- protomodularity is the right context to deal with exact sequences and homological lemmas in **a non-abelian setting**.

- on the other hand, any protomodular category is a Mal'tsev one.

- **Definition (Carboni, Lambek, Pedicchio 1990)**
  A Mal'tsev category is such that any reflexive relation is an equivalence relation.

  - a first simple consequence:
    in a Mal'tsev category, on any reflexive graph there is at most one structure of internal category
    which is necessarily a groupoid structure.

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- More importantly, there are two major structural facts for a Mal'tsev category $\mathbb{D}$:
- 1) when, in addition, $\mathbb{D}$ is regular, the reflexive relations can be composed and do permute; i.e. $R \circ S = S \circ R$; [Carboni, Lambek, Pedicchio]
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- The idea of partial protomodularity only relative to a class Σ of split epimorphims.

- The category *Mon* of monoids.

- **Definition (Martins-Ferreira, Montoli, Sobral 2013)**
  A split monoid homomorphism is a Schreier one when the application $\mu_y : Kerf \to f^{-1}(y)$ defined by $\mu_y(k) = s(y) \cdot k$ is bijective.

  - Any Schreier split homomorphism is such that in the following diagram:

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  - The class Σ of Schreier split epimorphisms is:
    -stable under composition and pullback
    -contains the isomorphisms.
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Definition (B., Martins-Ferreira, Montoli, Sobral 2014)

A pointed category $\mathbb{C}$ is said to be Σ-protomodular provided:

- *the class Σ is point-congruous: i.e. is stable under pullback, contains the isomorphisms and is stable under finite limits inside the class of all split epimorphisms.*
- *any split epimorphism $(f, s) \in \Sigma$ is strongly split: i.e. such that in the following diagram:*

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- Examples: *Mon*, and on strictly the same model as *Mon*, the category *SRg* of semi-rings by means of $U : SRg \to CoM$ with the class $U^{-1}(\Sigma)$.

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Main tools:

- Σ-relation: a relation which is reflexive and such that (d_0, s_0) belongs to Σ:

$$\begin{tikzcd} R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X \arrow[l, "s_0"'] \end{tikzcd}$$

- Σ-special morphism $f : X \to Y$: when the kernel relation $R[f]$ is a Σ-relation

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Main results= aspects of partial pointed protomodularity:

- the Σ-exact sequences, where f is a Σ-special regular epimorphism

$$1 \to K[f] \rightarrowtail X \twoheadrightarrow Y \to 1$$

  satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

$$1 \to A \rightarrowtail X \twoheadrightarrow Y \to 1$$

- the full subcategory $\Sigma\mathbb{C}_\sharp$ of Σ-special objects (called the core of the pointed Σ-protomodular category) is protomodular
- the core of *Mon* is the category *Gp* of goups
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- From some limitations of this example to some questions:
- 1) only one kind of example; how distinguish what is important from what is incidental for the class Σ
  concerning this question of partial pointed protomodularity
- 2) only pointed case, although protomodularity is not a pointed concept
- 3) how to unknot what comes from partial Mal'tsevness
  and what comes from partial protomodularity
- 4) to produce a discriminating example: here comes the notion of quandle.

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Back to Mal'tsev categories. We have the following characterization:

- **Proposition (B. 1996)**
  A category $\mathbb{D}$ is a Mal'tsev one if and only if any fibre of the fibration of points $\P_{\mathbb{D}}$ is unital.

  - which means that in the following rightward pullback of split epimorphisms:

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- Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.
- A quandle is a set X endowed with a binary idempotent operation: $\triangleright : X \times X \to X$ such that for any object x the translation $- \triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $- \triangleright^{-1} x$.
- A homomorphism of quandles is an application $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary operation. This defines the category Qnd of quandles.
- Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G, .)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.
- Since $\emptyset$ belongs to Qnd, no hope for any kind of Σ-protomodularity, so it would be the desired discriminating example.

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- A homomorphism of quandles is an application $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary operation. This defines the category Qnd of quandles.
- Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G, .)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.
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Let us introduce the following definitions:

- A split epimorphism $(f, s) : X \rightleftarrows Y$ in *Qnd* is called: *puncturing* when, for any element $y \in Y$, the application $s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)$ is surjective (the class Σ)
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- The class $\Sigma' \subset \Sigma$ is point-congruous (-second level of left exactness: in addition, Σ' is stable under finite limits inside the class of all split epimorphims).
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- **Definition**
  A category $\mathbb{C}$ is a Σ-Mal'tsev category when Σ is a class of split epimorphims stable under pullback, containing the isomorphisms and such the previous condition on pullback is satisfied.

  - Main tools are the same as for S-protomodularity:
    Σ-relation: a relation which is reflexive and such that (d_0, s_0) belongs to Σ:

$$\begin{tikzcd} R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X \arrow[l, "s_0"] \end{tikzcd}$$

  - Σ-special morphism $f : X \to Y$: when the kernel relation $R[f]$ is a Σ-relation

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Main results= partial aspects of Mal'tsevness:

- any Σ-relation is transitive; on a Σ-graph there is at most one structure of internal category
- and similarly to the global Mal'tsev context, we have the structural facts:
- 1) in the regular context:
  given any pair of a reflexive relation R and a symmetric Σ-relation S (and so an equivalence relation) on a object X, the two relations do permute, i.e. $R \circ S = S \circ R$.
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- more important (second level of left exacness):
  when, in addition, the class Σ is point-congruous,

  the full subcategory $\Sigma\mathbb{C}_\sharp$ of Σ-special objects
  is a Mal'tsev one (called the core of the Σ-Mal'tsev category)

- the core of Qnd is the category LQd of latin quandles (when $x \triangleright -$ is bijective as well).

- even more generally
  any full subcategory $Spl_Y \subset \mathbb{C}/Y$ of the slice category
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Back to *Mon* and *SRg*, new observations:

Definition
A split monoid homomorphism is a weakly Schreier one when the application $\mu_y : Kerf \to f^{-1}(y)$ *defined by* $\mu_y(k) = s(y) \cdot k$ *is surjective.*

This class $\bar{\Sigma}$ is stable under pullback and contain the isomorphisms; it is not point-congruous.

the category *Mon* (resp. *SRg*) is a $\bar{\Sigma}$-Mal'tsev one
(resp. $U^{-1}(\bar{\Sigma})$-Mal'tsev one).

So that the previous results are already valid for the class of weakly Schreier split homomorphisms.

Outline

- Recall that the Mal'tsev context posseses an additive heart:

Definition (P.T. Johnstone 1989)

A naturally Mal'tsev category is such that any object X is endowed with a natural Mal'tsev operation $p : X \times X \times X \to X$.

- examples: a pointed category $\mathbb{A}$ is additive if and only if it is a naturally Mal'tsev one
- any slice category $\mathbb{A}/Y$ of an additive category is a non-pointed naturally Mal'tsev one

- **Proposition (B.2008)**

  When $\mathbb{C}$ is a Mal'tsev category, then any fibre $\mathrm{Grd}_Y\mathbb{C}$ of the fibration of groupoids $\mathrm{Grd}\mathbb{C} \to \mathbb{C}$ is a naturally Mal'tsev category.

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we have a list of characterization

Proposition (B. 1996)

A category $\mathbb{D}$ is a naturally Mal'tsev one if and only if any of the following conditions is satisfied:

- *1) any fibre of the fibration of points $\P_{\mathbb{D}}$ is linear*
- *1') any fibre of the fibration of points $\P_{\mathbb{D}}$ is additive*
- *2) it is a Mal'tsev category in which any pair of equivalence relations centralizes each other*
- *3) any internal reflexive graph is a groupoid (the Lawvere condition)*
- *4) any base change along any split epimorphism with respect to the fibration of points $\P_{\mathbb{D}}$ is an equivalence of categories.*

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Condition 1) means that in the following rightward pullback of split epimorphisms:

$$\begin{tikzcd}
X \times_Y Z \arrow[r, shift right, "p_X"'] \arrow[d, shift right, "p_Z"'] & X \arrow[l, shift right, "\iota_X"'] \arrow[d, shift right, "f"'] \\
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the rightward and upward square is a pushout.

- whence the idea for a notion of partial natural Mal'tsevness:

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examples:

- -the category *CoM* of commutative rings with Σ the class of Schreier split epimorphisms
- -the category *AQd* of autnomous quandles with Σ the class of acupuncturing split epimorphisms
  where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.
- expected first results:
- -any Σ-graph is endowed with a (unique) internal category structure
- -any Σ-equivalence relation centralizes every reflexive relation
- -when $\mathbb{C}$ is a Σ-Mal'tsev category, then any fibre $Grd_Y\mathbb{C}$ of the fibration of groupoids $Grd\mathbb{C} \to \mathbb{C}$ is a Σ-naturally Mal'tsev category.

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  When $\mathbb{C}$ is a point congruous Σ-naturally Mal'tsev category, then the full subcategory Spl_Y of the slices category $\mathbb{C}/Y$ are naturally Mal'tsev ones.

  In particular its Mal'tsev core $\Sigma\mathbb{C}_\sharp$ is a naturally Mal'tsev category.

  - -the Mal'tsev core of *CoM* is the category *Ab* of abelian groups.
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- Finally let us emphazise that there appears some subttle phenomenons:

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- Similarly the category AQd is a Σ'-Mal'tsev category with respect to the class Σ' of puncturing split epimorphisms and Σ-naturally Mal'tsev for the subclass Σ of acupuncturing split epimorphisms.

- When $\mathbb{C}$ is a Σ-Mal'tsev category, any fibre $Grd_Y\mathbb{C}$ is a Mal'tsev category (since it is protomodular in any category $\mathbb{C}$), and Σ-naturally Mal'tsev.

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