Relators and Notions of Simulation Revisited

Sergey Goncharov
University of Birmingham
United Kingdom
s.goncharov@bham.ac.uk

Dirk Hofmann
CIDMA, University of Aveiro
Portugal
dirk@ua.pt

Pedro Nora
Radboud Universiteit
Netherlands
pedro.nora@ru.nl

Lutz Schröder
Friedrich-Alexander-Universität Erlangen-Nürnberg
Germany
lutz.schroeder@fau.de

Paul Wild
Friedrich-Alexander-Universität Erlangen-Nürnberg
Germany
paul.wild@fau.de

***Abstract*—Simulations and bisimulations are ubiquitous in the study of concurrent systems and modal logics of various types. Besides classical relational transition systems, relevant system types include, for instance, probabilistic, weighted, neighbourhood-based, and game-based systems. *Universal coalgebra* abstracts system types in this sense as set functors. Notions of (bi)simulation then arise by extending the functor to act on relations in a suitable manner, turning it into what may be termed a *relator*. We contribute to the study of relators in the broadest possible sense, in particular in relation to their induced notions of (bi)similarity. Specifically, (i) we show that every functor that preserves a very restricted type of pullbacks (termed 1/4-iso pullbacks) admits a sound and complete notion of bisimulation induced by the *coBarr relator*; (ii) we establish equivalences between properties of relators and closure properties of the induced notion of (bi)simulation, showing in particular that the full set of expected closure properties requires the relator to be a lax extension, and that soundness of (bi)simulations requires preservation of diagonals; and (iii) we show that functors preserving inverse images admit a *greatest* lax extension. In a concluding case study, we apply (iii) to obtain a novel highly permissive notion of *twisted bisimulation* on labelled transition systems.**

I. Introduction

State-based systems as used, for instance, in concurrency or in the semantics of modal logics, come in many different flavours, and in particular may vary in their branching type. While classically, attention has focused on relational systems, in particular nondeterministic ones such as labelled transition systems (LTS) or Kripke frames, there has been long-standing interest in other types where branching is probabilistic (e.g. [1]), weighted (e.g. [2], [3]), game-based [4], or neighbourhood-based as in the Montague-Scott semantics of modal logic [5], concurrent dynamic logic [6], or game logic [7].

A unifying framework for such diverse system types is available in the shape of *universal coalgebra* [8], in which the system type is abstracted as a set functor, whose coalgebras then play the role of systems. For instance, coalgebras for the powerset functor are relational transition systems, and coalgebras for the distribution functor are probabilistic transition systems. The coalgebraic framework induces a canonical abstract notion of *behavioural equivalence*, which instantiates to standard branching-time equivalences (such as Park-Milner bisimilarity on LTS) in all examples [9]. Roughly speaking, two states are behaviourally equivalent if they may be identified under suitable coalgebra morphisms. The situation is more complicated regarding the task of *certifying* behavioural equivalence by means of (ideally, small) witnessing relations, in generalization of bisimulations on Kripke frames or LTS; the existence of such relations typically depends on properties of the underlying functor. One of the first insights in universal coalgebra was that *Aczel-Mendler bisimulations* [10] are sound and complete for behavioural equivalence if the functor preserves weak pullbacks; here, we call a notion of bisimulation *sound* if bisimilar states are behaviourally equivalent, and *complete* if the converse holds, where as usual two states are bisimilar if they are related by some bisimulation.

Generally, coalgebraic notions of bisimulation rely on extending the action of the underlying functor to relations; we refer to such extensions in the most general sense, essentially only requiring well-typedness, as *relators*, following Thijs [11] (noting that the term has been used with different meanings in the field, e.g. [12], [13]). For instance, the above-mentioned notion of Aczel-Mendler bisimulation is induced by a particular form of relator, the *Barr extension* [14]. Relators have been equipped with various sets of axioms that ensure good properties of the induced class of bisimulations. Notably, notions of bisimulation induced by *normal* (or *diagonal-preserving*) *lax extensions* [11], [13], [15], [16] are sound and complete for behavioural equivalence and closed under relational composition, and moreover allow for sound up-to techniques such as bisimulation up to transitivity.

Depending on the exact set of required properties, suitable relations and corresponding notions of bisimulation may or may not exist for given functors and their associated system types. For instance, the above-mentioned Barr extension is a lax extension iff the functor weakly preserves pullbacks [14], [17]. It has recently been shown that normal lax extensions exist for functors that preserve either inverse images or weakly preserve kernel pairs and so-called 1/4-iso pullbacks, i.e. pullbacks

This work is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project numbers 434050016, 501369690 and 531706730 – and by CIDMA under the FCT (Portuguese Foundation for Science and Technology) Multi-Annual Financing Program for R&D Units.

in which at least one leg is isomorphic, and moreover that preservation of 1/4-iso pullbacks is a necessary condition for existence of a normal lax extension [18].

In the present work, we further analyse the landscape of relators and lax extensions, in particular strengthening the perspective that a given functor potentially comes with a whole range of relevant relators and associated notions of (bi)simulation. In detail, our contributions are as follows.

- We show that functors preserving 1/4-iso pullbacks admit a sound and complete notion of bisimulation induced by the *coBarr relator* (Corollary III.7), which however is not in general a lax extension.
- We show that a given relator is a lax extension iff the induced class of of (bi)simulations contains all coalgebra homomorphisms and their converses and is closed under composition (Theorem III.3). Moreover, we show that normality is essentially a necessary condition for soundness of bisimulations (Theorem III.15).
- We show that functors that preserve inverse images admit a greatest normal lax extension (Theorem IV.9), and hence a maximally permissive notion of bisimulation.
- In a concluding case study, we illustrate the latter result on functors of the form $(-)^A$, which model deterministic A-labelled transition systems. We arrive at a new notion of *twisted bisimulation*, which we extend to obtain a notion of twisted bisimulation on LTS that is more permissive than the standard notion (Section V). Both notions are illustrated for $A = \{a, b\}$ in Fig. 1. The left hand diagram recalls the standard definition: A relation r between the state sets of two LTS is a bisimulation if whenever $x \mathrel{r} y$ for states x, y, then the *forth* condition (for every labelled transition $x \xrightarrow{u} x'$, there is y' such that $x' \mathrel{r} y'$ and $y \xrightarrow{u} y'$) and the symmetric *back* condition hold. The definition of twisted bisimulation alternatively allows x, y to satisfy the right-hand clause in Fig. 1, in which actions are mismatched. Explicitly, r is a twisted bisimulation if whenever $x \; r \; y$, then either the standard forth and back conditions shown on the left hold, or the alternative clauses shown on the right hold for all $(u, v) \in \{(a, b), (b, a), (b, b)\}$. Indeed one can allow a third set of alternative clauses with the roles of a and b interchanged. Since twisted bisimulation is induced by a normal lax extension, it remains sound for standard bisimilarity while allowing for smaller bisimulation relations.

Related work: Up to variations in the axiomatics, lax extensions go back to work on coalgebraic simulation [19] (under the name *relational extensions*). The axiom for lax preservation of composition first appears explicitly in work on simulation quotients [13], where the term *relator* is used. We have already mentioned work by Marti and Venema relating lax extensions to modal logic [15], [16]; at the same time, Marti and Venema prove that the notion of bisimulation induced by a normal lax extension captures the standard notion of behavioural equivalence. *Lax relation liftings*, constructed for functors carrying a coherent order structure [20], also serve the study of coalgebraic simulation but obey a different axiomatics than lax extensions (cf. [16, Remark 4]). There has been recent interest in quantitative notions of lax extensions that act on relations taking values in a quantale, such as the unit interval, in particular with a view to obtaining notions of quantitative bisimulation [21], [22], [23], [24], [25] that witness low behavioural distance (the latter having first been treated in coalgebraic generality by Baldan et al. [26]). The correspondence between normal lax extensions and separating sets of modalities generalizes to the quantitative setting [23], [24], [25].

II. Preliminaries: Relations and Coalgebras

We assume basic familiarity with category theory (e.g. [27]).

Unless stated otherwise, we work in the category Set of sets and functions throughout. Another relevant category is the category Rel whose objects are again sets, but whose morphisms are the corresponding relations. We use the notation $r\colon X \mathrel{+\!\!\!\to} Y$ to indicate that r is a relation $r \subseteq X \times Y$.

We say that $r\colon X \mathrel{+\!\!\!\to} Y$ and $s\colon Y' \mathrel{+\!\!\!\to} Z$ are ***composable*** if $Y = Y'$, and extend this terminology to sequences of relations in the obvious manner. Both for functions and for relations, we use *applicative composition*, i.e. given $r\colon X \mathrel{+\!\!\!\to} Y$ and $s\colon Y \mathrel{+\!\!\!\to} Z$, their composite $s \cdot r\colon X \mathrel{+\!\!\!\to} Z$ is $\{(x, z) \mid \exists y \in Y.\, x \; r \; y \; s \; z\}$. Relations between the same sets are ordered by inclusion, hence we write $r \leq r'$ as a synonym to $r \subseteq r'$. We denote by $1_X\colon X \to X$ the identity map (hence relation) on X, and we say that a relation $r\colon X \mathrel{+\!\!\!\to} X$ is a ***subidentity*** if $r \leq 1_X$. The identity endofunctor will generally be denoted as Id.

Given a relation $r\colon X \mathrel{+\!\!\!\to} Y$, $r^\circ\colon Y \mathrel{+\!\!\!\to} X$ denotes the corresponding converse defined by $y \; r^\circ \; x \iff x \; r \; y$; in particular, if $f\colon X \to Y$ is a function, then $f^\circ\colon Y \mathrel{+\!\!\!\to} X$ denotes the converse of the corresponding relation. For a relation $r\colon X \mathrel{+\!\!\!\to} Y$, we denote by $\mathsf{dom}\, r \subseteq X$ and $\mathsf{img}\, r \subseteq Y$ its respective domain ($\mathsf{dom}\, r = \{x \in X \mid \exists y \in Y.\, x \; r \; y\}$ and image ($\mathsf{img}\, r = \{y \in Y \mid \exists x \in X.\, x \; r \; y\}$). A special class of relations of interest are ***difunctional relations*** [28], which are relations factorizable as $g^\circ \cdot f$ for some functions $f\colon X \to Z$ and $g\colon Y \to Z$, i.e. $x \; r \; y$ iff $f(x) = g(y)$. The ***difunctional closure*** of a relation $r\colon X \mathrel{+\!\!\!\to} Y$ is the least difunctional relation $\hat{r}\colon X \mathrel{+\!\!\!\to} Y$ greater than or equal to r. More explicitly, $\hat{r} = \bigvee_{n \in \mathbb{N}} r \cdot (r^\circ \cdot r)^n$ (e.g. [28], [29]).

Given an endofunctor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$, an F-***relator***, or simply a relator, is a monotone map $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$ that sends a relation from X to Y to a relation from $\mathsf{F}X$ to $\mathsf{F}Y$. (See [11], [12], [13] for other uses of the term "relator"). A ***relax extension*** of F [25] is a relator that satisfies $\mathsf{F}f \leq \mathsf{R}f$ and $(\mathsf{F}f)^\circ \leq \mathsf{R}(f^\circ)$ for all functions f. A ***lax extension*** is such a relax extension R that satisfies *lax preservation of composition*: $\mathsf{R}s \cdot Rr \leq \mathsf{R}(s \cdot r)$ for all $r\colon X \mathrel{+\!\!\!\to} Y$, $s\colon Y \mathrel{+\!\!\!\to} Z$. A relator $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$ is a ***relational connector*** [32] if for every set X, $1_{\mathsf{F}X} \leq \mathsf{R}1_X$ and for every relation $r\colon X \mathrel{+\!\!\!\to} Y$, and all function $f\colon A \to X$ and $g\colon B \to Y$, $R(g^\circ \cdot r \cdot f) = (\mathsf{F}g)^\circ \cdot \mathsf{R}r \cdot \mathsf{F}f$. It is well-known that every lax extension is a relational connector (e.g. [21]), and it is easy to see that every relational connector is a relax extension.

Figure 1: Standard bisimulation (left) and alternative additional clauses for twisted bisimulation (right).

In summary:

$$\begin{array}{ccccc} & \text{Relational connector} & & \text{Relator} & \\ & \subsetneq & \subsetneq & \subsetneq & \\ \text{Lax extension} & & \text{Relax extension} & & \end{array}$$

We say that a relator $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$ is ***symmetric*** if $\mathsf{R}(r^\circ) = (\mathsf{R}r)^\circ$ and ***normal*** if $\mathsf{R}1_X = 1_{\mathsf{F}X}$. We order F-relators pointwise, i.e, given F-relators R_1 and R_2, we write $\mathsf{R}_1 \leq R_2$ if for every relation r, $\mathsf{R}_1 r \leq \mathsf{R}_2 r$.

Given a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$, we will be interested in its (weak) pullback preservation properties, specifically in (weak) preservation of pullbacks of the form:

$$\begin{array}{ccc} P & \longrightarrow & B \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \longrightarrow & Y \end{array}$$

where f is a mono (which are pullbacks iff they are weak pullbacks, so weak preservation and preservation of such pullbacks coincide). We say that F ***preserves 1/4-iso pullback*** if F preserves the above pullbacks when f is an isomorphism, and that F ***preserves inverse images*** if F preserves the above pullbacks when g is a monomorphism. Both properties are significantly weaker than weak pullback preservation, i.e. preservation of weak pullbacks by F [8].

An F-***coalgebra*** (X, α) for an endofunctor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ consists of a set X of ***states*** and a ***transition map*** $\alpha\colon X \to \mathsf{F}X$. A ***morphism*** $f\colon (X, \alpha) \to (Y, \beta)$ of F-coalgebras is a map $f\colon X \to Y$ for which $\beta \cdot f = \mathsf{F}f \cdot \alpha$. Such morphisms are thought of as preserving the behaviour of states, and correspondingly, states x and y in coalgebras (X, α) and (Y, β), respectively, are ***behaviourally equivalent*** if there exist a coalgebra (Z, γ) and morphisms $f\colon (X, \alpha) \to (Z, \gamma)$, $g\colon (Y, \beta) \to (Z, \gamma)$ such that $f(x) = g(y)$.

III. Simulations induced by Relators

We fix a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ for the remainder of the technical development until the end of Section IV.

Relators induce notions of simulation and similarity between (possibly distinct) F-coalgebras. Given a relator $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$, a relation $r\colon X \nrightarrow Y$ is an R-***simulation*** from a coalgebra $\alpha\colon X \to \mathsf{F}X$ to a coalgebra $\beta\colon Y \to FY$ if

$$r \leq \beta^\circ \cdot \mathsf{R}r \cdot \alpha,$$

i.e, if $x\ r\ y$ entails $\alpha(x)\, \mathsf{R}r\, \beta(y)$, for all $x \in X$ and $y \in Y$. When we speak of ***notions of simulation***, we always understand them as being induced by a relator in this way. Relators are naturally composed. In particular, given an F-relator R and a relator U of the identity functor on Set, $\mathsf{R} \cdot \mathsf{U}$ yields an F-relator, and we say that an $\mathsf{R} \cdot \mathsf{U}$-simulation is an R-simulation ***up-to*** U. Whenever R_1 and R_2 are F-relators such that $\mathsf{R}_1 \leq \mathsf{R}_2$, then every R_1-simulation is an R_2-simulation; in the running discussion, we phrase this by saying that R_2 induces a ***more permissive*** notion of simulation than R_1. As we assume relators to be monotone, given coalgebras $\alpha\colon X \to \mathsf{F}X$ and $\beta\colon Y \to \mathsf{F}Y$, the map sending a relation $r\colon X \nrightarrow Y$ to the relation $\beta^\circ \cdot \mathsf{R}r \cdot \alpha\colon X \nrightarrow Y$ is monotone. Therefore, by the Knaster-Tarski fixed point theorem, this map has a greatest fixed point, the greatest R-simulation from α to β, which we call R-***similarity*** from α to β. When R is symmetric, we often speak about R-***bisimulations*** instead of R-simulations, and about R-***bisimilarity*** instead of R-similarity; this will be justified in Theorem III.3. For every relator $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$, if for all F-coalgebras α and β, R-similarity from α to β is greater or equal than behavioural equivalence from α to β, then we say that R-similarity is ***complete***. Conversely, if R-similarity from α to β is smaller or equal than behavioural equivalence from α to β, then we say that R-similarity is ***sound***.

In this section, we study general properties of notions of simulation. We begin by providing sufficient conditions for soundness and for completeness. The next result substantially generalizes [16, Theorem 1], which states that every symmetric normal lax extension induces a sound and complete notion of bisimilarity. Throughout the paper we will see examples of relators, such as the coBarr relator, that satisfy the conditions of theorem below but are not symmetric lax extensions.

Theorem III.1. *Let* R *be an* F*-relator.*

1) If for all functions $f\colon X \to A$ *and* $g\colon Y \to A$, $\mathsf{R}(g^\circ \cdot f) \geq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$, *then* R*-similarity is complete.*

2) If R *preserves difunctional relations and for every epi-cospan* $(f\colon X \to A, g\colon Y \to A) \in \mathsf{Set}$, $\mathsf{R}(g^\circ \cdot f) \leq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$, *then* R*-similarity is sound.*

Proof. 1) Let (X, α) and (Y, β) be F-coalgebras, and let $x \in X$ and $y \in Y$ be behaviourally equivalent elements. Then, there are coalgebra homomorhpisms $f\colon (X, \alpha) \to (Z, \gamma)$ and $g\colon (Y, \beta) \to (Z, \gamma)$ such that $f(x) = g(y)$. Hence, with $r = g^\circ \cdot f$, we have $x\, r\, y$ and as f, g are coalgebra homomorphisms, by hypothesis, we obtain $r \leq \beta^\circ \cdot (\mathsf{F}g)^\circ \cdot \mathsf{F}f \cdot \alpha \leq \beta^\circ \cdot \mathsf{R}r \cdot \alpha$. Therefore, x and y are R-similar.

2) Let $r\colon X \nrightarrow Y$ be an R-simulation from (X, α) to (Y, β) and $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$ be a span in Set such that

$r = \pi_2 \cdot \pi_1^\circ$. Consider the pushout $(p_1 : X \to O,\, p_2 : Y \to O)$ of (f, g). Then, $p_2^\circ \cdot p_1$ is the difunctional closure $\hat{r}$ of r and, in particular, for every $x \in X$ and $y \in Y$, if $x\ r\ y$, then $p_1(x) = p_2(y)$. Hence, the assertion follows once we show that there is a coalgebra (O, γ) s.t. $p_1 : (X, \alpha) \to (O, \gamma)$ and $p_2 : (Y, \beta) \to (O, \gamma)$ are coalgebra homomorphisms. To this end, put $\gamma(o) = \mathsf{F}p_1 \cdot \alpha(x)$, if $p_1(x) = o$, for some $x \in X$ and $\gamma(o) = \mathsf{F}p_2 \cdot \beta(y)$, if $p_2(y) = o$, for some $y \in Y$. Note that (p_1, p_2) is an epi-cocone, so we are assigning an element of $\mathsf{F}O$ to every element of O. We claim that this assignment is well-defined. To see this, we begin by observing that as R is monotone, $r \le \beta^\circ \cdot \mathsf{R}r \cdot \alpha \le \beta^\circ \cdot \mathsf{R}\hat{r} \cdot \alpha$, and, as R preserves difunctional relations, $\beta^\circ \cdot \mathsf{R}\hat{r} \cdot \alpha$ is a difunctional relation greater or equal than r, which entails, by definition of difunctional closure, $\hat{r} \le \beta^\circ \cdot \mathsf{R}\hat{r} \cdot \alpha$; i.e, $\hat{r}$ is an R-simulation. Hence, by hypothesis, we obtain $\hat{r} \le \beta^\circ \cdot (\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \cdot \alpha$, i.e, for all $x \in X$ and $y \in Y$, if $p_1(x) = p_2(y)$, then $\mathsf{F}p_1 \cdot \alpha(x) = \mathsf{F}p_2 \cdot \beta(y)$. Furthermore, by the way that pushouts are constructed in Set, we have:

a) for all $x, x' \in X$ such that $x \neq x'$, $p_1(x) = p_1(x')$ iff there is $y \in Y$ such that $p_1(x) = p_2(y) = p_1(x')$;
b) for all $y, y' \in Y$ such that $y \neq y'$, $p_2(y) = p_2(y')$ iff there is $x \in X$ such that $p_2(y) = p_1(x) = p_2(y')$.

Now, the claim follows straightforwardly by case distinction. Moreover, by definition of γ we obtain immediately that $p_1 : (X, \alpha) \to (O, \gamma)$ and $p_2 : (Y, \beta) \to (O, \gamma)$ are coalgebra homomorphisms. Therefore, R-similarity is sound. □

Corollary III.2. *Let R be an F-relator such that for all functions $f : X \to A$ and $g : Y \to A$, $\mathsf{R}(g^\circ \cdot f) = (\mathsf{F}g)^\circ \cdot \mathsf{F}f$. Then, R-similarity is sound and complete.*

The notion of relator is quite liberal and, naturally, the properties of the corresponding class of simulations can vary significantly from one relator to the other. In this work we think of simulations as witnesses for behavioural equivalence and, therefore, as coalgebra homomorphisms are thought to be "behaviour preserving maps", we are interested in relators whose classes of simulations contain all coalgebra homomorphism and their converses since behavioural equivalence is a symmetric relation. And, among such relators, we are particularly interested in those whose corresponding classes of simulations are closed under composition or, at least, under composition with coalgebra homomorphisms.

Theorem III.3. *Let R be an F-relator . The class of all R-simulations*

1) *contains all homomorphisms of F-coalgebras and their converses iff for every non-empty function, $\mathsf{F}f \le \mathsf{R}f$ and $(\mathsf{F}f)^\circ \le \mathsf{R}(f^\circ)$;*
2) *is closed under composition iff for all relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ such that $s \cdot r$ is non-empty, $\mathsf{R}s \cdot \mathsf{R}r \le \mathsf{R}(s \cdot r)$;*
3) *is closed under converses iff for every non-empty relation $r : X \nrightarrow Y$, $\mathsf{R}(r^\circ) = (\mathsf{R}r)^\circ$.*

Proof. We show (2), the other claims follow analogously. Suppose that $r : X \nrightarrow Y$ is an R-simulation from (X, α) to (Y, β) and $s : Y \nrightarrow Z$ is an R-simulation from (Y, β) to (Z, γ). If $s \cdot r$ is empty, then it is trivially an R-simulation. On the other hand, suppose that $s \cdot r$ is non-empty. Then, as r and s are R-simulations, by hypothesis, $s \cdot r \le \gamma^\circ \cdot \mathsf{R}s \cdot \beta \cdot \beta^\circ \cdot \mathsf{R}r \cdot \alpha \le \gamma^\circ \cdot \mathsf{R}(s \cdot r) \cdot \alpha$. Therefore, $s \cdot r$ is an R-simulation. To see that the converse statement holds, let $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ be relations such that $s \cdot r$ is non-empty. Suppose that there are $u \in \mathsf{F}X$, $v \in \mathsf{F}Y$ and $w \in \mathsf{F}Z$ such that $u\, \mathsf{R}r\, v$ and $v\, \mathsf{R}s\, w$. Consider the coalgebras $u : X \to \mathsf{F}X$, $v : Y \to \mathsf{F}Y$ and $w : Z \to \mathsf{F}Z$ given by constant maps into u, v, w, respectively. Then, r is an R-simulation from u to v and s is an R-simulation from v to s. Hence, by hypothesis, $s \cdot r$ is an R-simulation, which means that $s \cdot r \le w^\circ \cdot \mathsf{R}(s \cdot r) \cdot u$, and $w^\circ \cdot \mathsf{R}(s \cdot r) \cdot u$ non-empty implies $u\, \mathsf{R}(s \cdot r)\, w$. Therefore, as $s \cdot r$ is non-empty, $u\, \mathsf{R}(s \cdot r)\, w$. □

A key feature of behavioural equivalence is that it is invariant under coalgebra homomorphisms. For R-similarity, this typically follows by showing that the class of R-simulations is closed under (pre/post)composition with coalgebra homomorphisms.

Theorem III.4. *Let R be an F-relator. The class of all R-simulations*

1) *is closed under postcomposition with coalgebra homomorhpisms iff for every relation $r : X \nrightarrow Y$ and every function $f : A \to X$ such that $r \cdot f$ is non-empty, $\mathsf{R}r \cdot \mathsf{F}f \le \mathsf{R}(r \cdot f)$;*
2) *is closed under postcomposition with converses of coalgebra homomorhpisms iff for every relation $r : X \nrightarrow Y$ and every function $f : X \to A$ such that $r \cdot f^\circ$ is non-empty, $\mathsf{R}r \cdot (\mathsf{F}f)^\circ \le \mathsf{R}(r \cdot f^\circ)$;*
3) *is closed under precomposition with coalgebra homomorhpisms iff for every relation $r : X \nrightarrow Y$ and every function $g : Y \to B$ such that $g \cdot r$ is non-empty, $\mathsf{F}g \cdot \mathsf{R}r \le \mathsf{R}(g \cdot r)$;*
4) *is closed under precomposition with converses of coalgebra homomorhpisms iff for every relation $r : X \nrightarrow Y$ and every function $g : B \to Y$ such that $g^\circ \cdot r$ is non-empty, $(\mathsf{F}g)^\circ \cdot \mathsf{R}r \le \mathsf{R}(g^\circ \cdot r)$.*

Proof. Analogous to the proof of Theorem III.3.

□

The previous results motivate the study of simulations induced by (re)lax extensions and relational connectors (cf. Section II).

The prototypical example of relax extension is the Barr relator [14] which is normal, symmetric, and it is a lax extension iff it is a relational connector iff the functor preserves weak pullbacks. The ***Barr*** relator $\overline{\mathsf{F}} : \mathsf{Rel} \to \mathsf{Rel}$ of a functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$ is defined as follows. Given a relation $r : X \nrightarrow Y$, take a span $(\pi_1 : A \to X, \pi_2 : A \to Y)$ such that $r = \pi_2 \cdot \pi_1^\circ$. Then, put $\overline{\mathsf{F}}r = \mathsf{F}\pi_2 \cdot (\mathsf{F}\pi_1)^\circ$.

Example III.5. The powerset functor $\mathsf{P} : \mathsf{Set} \to \mathsf{Set}$ weakly preserves pullbacks and its Barr extension coincides with the well-known Egli-Milner extension: Given $r : X \nrightarrow Y$, $S \in \mathsf{P}X$, and $T \in \mathsf{P}Y$, we have $S\ Lr\ T$ iff for all $x \in S$, there exists $y \in T$ such that $x\ r\ y$, and symmetrically. For relational transition systems, understood as P-coalgebras, a $\overline{\mathsf{P}}$-bisimulation is then just a bisimulation in the standard sense.

The Barr relator is the standard relator used to define a notion of bisimulation coalgebraically. Regarding soundness and completeness, it is known that, independently of the functor, Barr-bisimilarity is sound [10], and that for functors that weakly preserve pullbacks, Barr-bisimilarity is complete [16].

However, Corollary III.2 suggests a different canonical construction to obtain sound and complete notions of similarity that is dual to the Barr relator on difunctional relations. Given a difunctional relation $r\colon X \nrightarrow Y$, take a cospan $(p_1\colon X \to O, p_2\colon Y \to O) \in \mathsf{Set}$ such that $r = p_2^\circ \cdot p_1$. Then, put $\underline{\mathsf{F}}r = (\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1$. Of course, for such construction to be well-defined, $\underline{\mathsf{F}}r$ must be independent of the choice of the cospan, and it has been shown recently [18] that this is equivalent to the functor preserving 1/4-iso pullbacks which is equivalent to the functor being ***monotone on difunctional relations*** in the following sense: for all difunctional relations factorized as $g^\circ \cdot f\colon X \nrightarrow Y$ and $g'^\circ \cdot f'\colon X \nrightarrow Y$, if $g^\circ \cdot f \le g'^\circ \cdot f'$ then $(\mathsf{F}g)^\circ \cdot \mathsf{F}f \le (\mathsf{F}g')^\circ \cdot \mathsf{F}f'$. This means that whenever F preserves 1/4-iso pullbacks, sending a relation $r\colon X \nrightarrow Y$ to $\underline{\mathsf{F}}\hat{r}$ as described above, with $\hat{r}$ being the difunctional closure of r, defines an F-relator, which we call the ***coBarr relator*** of F and denote by $\underline{\mathsf{F}}$. In the sequel we record some properties of this construction that follow straighforwardly from the definition.

Proposition III.6. *Suppose that F preserves 1/4-iso pullbacks. Then:*

1) *$\underline{\mathsf{F}}$ is a normal symmetric relax extension.*
2) *If F weakly preserves pullbacks, then for every relation r, $\underline{\mathsf{F}}r = \overline{\mathsf{F}}\hat{r}$, where $\hat{r}$ denotes the difunctional closure of r.*
3) *$\overline{\mathsf{F}} \le \underline{\mathsf{F}}$ and $\overline{\mathsf{F}} \le \mathsf{R} \le \underline{\mathsf{F}}$, for every normal relational connector R of F.*

Regarding coBarr-bisimulations, from Corollary III.2 and Proposition III.6(2) we obtain:

Corollary III.7. *Suppose that F preserves 1/4-iso pullbacks. Then:*

1) *$\underline{\mathsf{F}}$-bisimilarity is sound and complete.*
2) *If F weakly preserves pullbacks, then the $\underline{\mathsf{F}}$-bisimulations are precisely the $\overline{\mathsf{F}}$-bisimulations up-to difunctional closure.*

Remark III.8. As far as we know, the coBarr relator was first proposed by Kurz in a private communication to Hansen et al. [30]. We note that Hansen et al. do not require the functor to preserve 1/4-iso pullbacks, instead, given a relation $r\colon X \nrightarrow Y$, they define $\underline{\mathsf{F}}r$ as we have done here but w.r.t a pushout of the canonical span that determines r. This construction is independent of the pushout and defines a relator for every functor, and, in particular, they show that coBarr-bisimilarity is sound. Corollary III.7(1) complements their result by showing that coBarr-bisimilarity is complete under the assumption that the functor preserves 1/4-iso pullbacks.

Preserving 1/4-iso pullbacks is a significantly weaker condition than weakly preserving pullbacks, which means that coBarr-bisimulations provide a sound and complete proof method for behavioural equivalence for a much larger class of functors than what is currently known generically for Barr-bisimulations. However, a major drawback of the coBarr relator is that it is rarely a lax extension or even a relational connector, and, therefore, as we have seen in Theorem III.3, the class of coBarr-bisimulations is rarely closed under composition.

Example III.9. Even for the coBarr relator $\underline{\mathsf{Id}}$ of the identity functor Id, lax preservation of relational composition fails. (Of course, this functor by itself has trivial behavioural equivalence, but it serves as a building block in composite functors where triviality disappears, e.g. $2 \times \mathsf{Id}$.) Consider the endorelation $z = \{(a,a),(b,a),(b,b)\}$ on the set $2 = \{a,b\}$, the map $a\colon 1 \to 2$ that selects the element a, and the map $!_2\colon 2 \to 1$. The coBarr relator $\underline{\mathsf{Id}}$ sends every relation to its difunctional closure. Therefore, $\underline{\mathsf{Id}}(z \cdot a) = a$ but $\underline{\mathsf{Id}}z \cdot \underline{\mathsf{Id}}a = !_2^\circ$. The fact that the class of $\underline{\mathsf{Id}}$-bisimulations fails to be closed under composition often implies the same for the class of R-bisimulations up-to difunctional closure. For instance, the class of Barr-bisimulations up-to difunctional closure of the functor $2 \times \mathsf{Id}$ is not closed under composition.

Nevertheless, an important consequence of Proposition III.6 is that the class of simulations induced by a normal relational connector is contained in the class of coBarr-simulations. Hence, for a functor that preserves 1/4-iso pullbacks, the notion of coBarr-simulation is sound and complete, and it is more permissive than any notion of simulation induced by a normal relational connector. It has been shown that preserving 1/4-iso pullbacks is a necessary condition for a functor to admit a normal relational connector [25] and, therefore, coBarr-simulations help us understanding the simulations induced by normal relational connectors, such as the simulations induced by normal lax extensions which are closed under composition. Our interest in simulations induced by normal relational connectors stems from the following application of Corollary III.2.

Corollary III.10. *Every notion of similarity induced by a relational connector of a set functor is complete, and it is sound if the relational connector is normal.*

In particular, this result shows that symmetry of lax extensions is not necessary to obtain sound and complete notions of similarity induced by lax extensions as assumed by Marti and Venema [16]; we will see how to construct non-symmetric normal lax extensions for exponential functors in Section V, and a concrete example is given in Example V.7(2). This fact may be counter-intuitive at first sight, but the reason is that normal lax extensions coincide with the coBarr relator on difunctional relations which is symmetric. And, as we have seen in Theorem III.1 to show soundness and completeness we only need to inspect the action of a relator on difunctional relations.

This motivates us to understand next when normality of a relational connector is necessary for soundness. We stress that preserving monomorphisms is a very mild condition. In fact, every set functor is "naturally isomorphic up-to $\varnothing$" to a functor that preserves monomorphisms [31] and, hence, the category of coalgebras of every set functor is isomorphic to the category of coalgebras of a set functor that preserves monomorphisms.

Theorem III.11. *Suppose that* F *admits a terminal coalgebra* (Z, γ). *Let* R *be a relational connector of* F.

1) *R-similarity is sound iff* $\mathsf{R}1_Z = 1_{\mathsf{F}Z}$.
2) *If* F *preserves monomorphisms, then* R*-similarity is sound iff for every set* A *of cardinality less or equal than* $|Z|$, $\mathsf{R}1_A = 1_{\mathsf{F}A}$.

Proof. 1) It is well-known that two states in a terminal coalgebra are behaviorally equivalent iff they are equal and that behavioural equivalence is invariant under coalgebra homomorphisms. Therefore, the condition is clearly necessary, and it is also sufficient since, for every relational connector, R-similarity is invariant under coalgebra homomorphisms [32, Lemma 4.2].

2) Let (X, α) be an F-coalgebra of cardinality less or equal than the cardinality of Z. Then, there is an injective map $i \colon X \rightarrowtail Z$. Hence, we have $1_X = i^\circ \cdot 1_Z \cdot i$, and, as R is a relational connector, we obtain $\mathsf{R}1_X = (\mathsf{F}i)^\circ \cdot \mathsf{R}1_Z \cdot \mathsf{F}i$. Therefore, by the previous item, $\mathsf{R}1_X = (\mathsf{F}i)^\circ \cdot \mathsf{F}i$, and since F preserves monomorphisms, $\mathsf{R}1_X = 1_{\mathsf{F}X}$. ◻

Theorem III.11 makes it easy to see that normality is not necessary. In fact, as the next example shows, there are functors that do not admit a normal relational connector but even admit a lax extension that induces a sound and complete notion of similarity.

Example III.12. Consider the functor $(-)^2/\Delta$ that sends a set X to the quotient of the set X^2 by the equivalence relation that identifies exactly the elements of the diagonal of $X \times X$, and acts in the obvious way on functions.

This functor does not preserve 1/4-iso pullbacks and, hence, does not admit a normal relational connector [18]. However, since it preserves the terminal object, behavioural equivalence is trivial, therefore, its greatest lax extension which sends a relation $r \colon X \nrightarrow Y$ to the relation $\mathsf{F}X \times \mathsf{F}Y$ induces a sound and complete notion of similarity.

On the other hand, the next result shows that normality is typically necessary.

Definition III.13. A functor $\mathsf{G} \colon \mathsf{Set} \to \mathsf{Set}$ is ζ***-bounded*** if it admits a terminal coalgebra $Z \to \mathsf{G}Z$ and for every set X and every pair of elements $u, v \in \mathsf{G}X$ there is a set A of cardinality less or equal than $|Z|$ and an injective map $i \colon A \rightarrowtail X$ such that $u, v \in \mathsf{img}(\mathsf{G}i)$.

(Except in the corner case that Z is finite, one can rephrase this definition as saying that G has a final coalgebra G and is κ-accessible where κ is the cardinal successor of $|Z|$.)

Example III.14. The following set functors are ζ-bounded.

1) Every constant functor.
2) Every finitary functor with an infinite terminal coalgebra, e.g., the finite powerset functor and the finite multiset functor.

Failures of ζ-boundedness typically relate to triviality of behavioural equivalence: If a set functor satisfies $\mathsf{F}1 = 1$, then 1 is the terminal coalgebra, so all states are behaviourally equivalent. For instance, the identity functor, more generally all exponential functors $(-)^A$ except the constant functor $(-)^\emptyset$, and the discrete distribution functor all fail to be ζ-bounded for this reason. ζ-Boundedness is typically reinstated when such functors are combined with others to allow for actual observations; e.g. $2 \times \mathsf{Id}$ is ζ-bounded. More generally, we show in the appendix that all polynomial functors except the exponential functors $(-)^A$ are ζ-bounded.

Theorem III.15. *Suppose that* F *is* ζ*-bounded and preserves monomorphisms. Let* R *be a relational connector of* F.

If R*-similarity is sound, then* R *is normal.*

Proof. Let (Z, γ) be a terminal F-coalgebra, and let (X, α) be an F-coalgebra. If the cardinality of X is less or equal than the cardinality of Z, then, by Theorem III.11, $\mathsf{R}1_X = 1_{\mathsf{F}X}$. On the other hand, suppose that the cardinality of X is greater or equal than the cardinality of Z. Let $u, v \in \mathsf{F}X$. As F is ζ-bounded, there is set A of cardinality less or equal than Z and an injective map $i \colon A \rightarrowtail X$ u, v belong to $\mathsf{img}(\mathsf{F}i)$. Then, since i is injective, we have $1_A = i^\circ \cdot 1_X \cdot i$. Hence, as the cardinality of A is less or equal than the cardinality of Z, by Theorem III.11 and the fact that R is a relational connector, $1_{\mathsf{F}A} = \mathsf{R}1_A = (\mathsf{F}i)^\circ \cdot \mathsf{R}1_X \cdot \mathsf{F}i$. Therefore, $u \,\mathsf{R}1_X\, v$ iff $u = v$. ◻

Example III.16. Let $M = (M, +, 0)$ be a commutative monoid. Then the *monoid-valued functor* $M^{(-)}$ is defined on sets X by $M^{(X)}$ being the set of maps $\mu \colon X \to M$ with *finite support*, i.e. with $\mu(x) = 0$ for all but finitely many x. On maps $f \colon X \to Y$, $M^{(-)}$ is defined by $M^{(f)}(\mu)(y) = \sum_{f(x)=y} \mu(x)$ for $\mu \in M^{(X)}$ and $y \in Y$. Coalgebras for $M^{(-)}$ are *M-weighted transition systems*, i.e. finitely branching transition systems in which every transition is labelled with an element of M [33]. It has been shown recently that $M^{(-)}$ admits a normal lax extension iff M is *positive*, i.e. if $m+n = 0$ implies $m = n = 0$ for $m, n \in M$. For non-positive M, for instance for M being the additive group of the integers, we obtain by Theorems III.3 and III.15 that there is no notion of simulation on M-weighted transition systems that is closed under composition and contains all bounded maps (i.e. morphisms of $M^{(-)}$-coalgebras) and their converses, and is sound and complete for behavioural equivalence.

IV. The Greatest Normal Lax Extension

We proceed to capitalize on the fact that a given set functor potentially comes with a whole range of relevant relators with associated sound and complete notions of simulation. In the main result of this section, we show that set functors that preserve inverse images admit a greatest normal lax extension w.r.t. the pointwise order on relators. As we have seen in

Section III, this means essentially that for these functors, there is a maximally permissive notion of simulation such that similarity is sound and complete and whose class of simulations contains all coalgebra homomorphisms, their converses and is closed under composition. The case of normal relational connectors is less interesting since it is easy to see that non-empty pointwise suprema of relational connectors are relational connectors. We begin by describing the supremum of lax extensions with the help of relax extensions.

The pointwise infimum of a family of lax extensions of F is again a lax extension of F and, therefore, the partially ordered class of lax extensions of F is complete, with infimum given by pointwise infimum. Since the subclass of normal lax extensions is downwards closed, it follows that F admits a greatest normal lax extension iff the supremum of every family of normal lax extensions of F is normal.

However, while the non-empty pointwise supremum of relax extensions is a relax extension, as the next example shows, in general the non-empty supremum of lax extensions does not coincide with the pointwise supremum.

Example IV.1. Consider the 'upper' and 'lower' lax extensions $\mathsf{L}^{\Box}\colon \mathsf{Rel} \to \mathsf{Rel}$ and $\mathsf{L}^{\Diamond}\colon \mathsf{Rel} \to \mathsf{Rel}$, respectively, of the powerset functor $\mathsf{P}\colon \mathsf{Set} \to \mathsf{Set}$ that are defined on relations $r\colon X \nrightarrow Y$ and sets $A \subseteq X$ and $B \subseteq Y$ by

$$A\ (\mathsf{L}^{\Box} r)\ B \Leftrightarrow \forall a \in A. \exists b \in B.\ a\ r\ b;$$
$$A\ (\mathsf{L}^{\Diamond} r)\ B \Leftrightarrow \forall b \in B. \exists a \in A.\ a\ r\ b.$$

The relax extension $\mathsf{R}^{\vee}\colon \mathsf{Rel} \to \mathsf{Rel}$ given by the pointwise supremum of $\mathsf{L}^{\Box}$ and $\mathsf{L}^{\Diamond}$ does not preserve composition laxly and, hence, is not a lax extension. For instance, over $2 = \{0,1\}$, we have $\{0\}\ \mathsf{L}^{\Box} 1_2\ \{0,1\}$ and $\{0,1\}\ \mathsf{L}^{\Diamond} 1_2\ \{1\}$, which implies $\{0\}(\mathsf{R}^{\vee} 1_2 \cdot \mathsf{R}^{\vee} 1_2)\{1\}$; but $\mathsf{R}^{\vee} 1_2$ does not relate $\{0\}$ to $\{1\}$.

Indeed, the supremum of lax extensions is given by the laxification [18] of their pointwise supremum. The ***laxification*** $\mathsf{R}^{\bullet}\colon \mathsf{Rel} \to \mathsf{Rel}$ of a relax extension $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$ of F is the lax extension of F defined on $r\colon X \nrightarrow Y$ by

$$\mathsf{R}^{\bullet} r = \bigvee_{\substack{r_1,\dots,r_n:\\ r_n\cdot\ldots\cdot r_1 \le r}} \mathsf{R} r_n \cdot \ldots \cdot \mathsf{R} r_1. \tag{1}$$

Proposition IV.2. *Let $(\mathsf{L}_i)_{i\in\mathcal{I}}$ be a family of lax extensions of F, and let $\mathsf{L}^{\vee}$ be the relax extension given by pointwise supremum of the lax extensions in the family. The supremum of the lax extensions in the family is given by the laxification of $\mathsf{L}^{\vee}$.*

Proof. Immediate consequence of the fact that sending a relax extension to its laxification defines a left adjoint [18, Proposition 4.1]. □

Our proof strategy to show that the supremum of normal lax extensions of F is normal relies on F having the property that every normal lax extension of F preserves composition with subidentities. This allows restricting the collection of composable sequences of relations needed in (1) to produce laxifications. We first fix some notation that we need throughout this section.

Given a relation $r\colon X \nrightarrow Y$, we denote the inclusions $\mathsf{dom}(r) \rightarrowtail X$ and $\mathsf{img}(r) \rightarrowtail Y$ by d_r and i_r, respectively. Furthermore, for an injective map $i\colon A \rightarrowtail X$, we denote by $\lceil i \rceil$ the induced subidentity relation $i \cdot i^{\circ}\colon X \nrightarrow X$. In particular, for a relation $r\colon X \nrightarrow Y$ we have subidentities $\lceil \mathsf{d}_r \rceil \le 1_X$ and $\lceil \mathsf{i}_r \rceil \le 1_Y$.

Lemma IV.3. *Let R be a relax extension of F that preserves composition with subidentities. Suppose that $\mathsf{R} r_n \cdot \ldots \cdot \mathsf{R} r_1 \le 1_{\mathsf{F}X}$ for every set X and every composable sequence $r_1, \dots, r_n$ of relations such that $r_n \cdot \ldots \cdot r_1 \le 1_X$ and $\mathsf{img}(r_{i-1}) = \mathsf{dom}(r_i)$ for $i = 2, \dots, n$. Then the laxification of R is normal.*

Proof. Let X be a set, and let $r_1, \dots, r_n$ be a composable sequence of relations such that $r_n \cdots r_1 \le 1_X$. By (1), to show that the laxification of R is normal we need to show that $\mathsf{R} r_n \cdot \ldots \cdot \mathsf{R} r_1 \le 1_{\mathsf{F}X}$.

By [18, Lemma A.1], we obtain composable sequences of relations

1) $s_1, \dots, s_n$ defined by $s_n = r_n$, and $s_i = \lceil \mathsf{d}_{s_{i+1}} \rceil \cdot r_i$, for $i = 1, \dots, n-1$;
2) $t_1, \dots, t_n$ defined by $t_1 = s_1$, and $t_i = s_i \cdot \lceil \mathsf{i}_{t_{i-1}} \rceil$, for $i = 2, \dots, n$,

that satisfy $t_n \cdots t_1 = s_n \cdots s_1 = r_n \cdots r_1$ and $\mathsf{img}(t_{i-1}) = \mathsf{dom}(t_i)$ for $i = 2, \dots, n$. Therefore, since R preserves composition with subidentities and $\mathsf{R} t_n \cdot \ldots \cdot \mathsf{R} t_1 \le 1_{\mathsf{F}X}$ by hypothesis,

$$\begin{aligned}
\mathsf{R} r_n \cdots \mathsf{R}_1 r_1 &= \mathsf{R} s_n \cdot \mathsf{R}\lceil \mathsf{d}_{s_n} \rceil \cdot \mathsf{R} r_{n-1} \cdots \mathsf{R} r_1 \\
&= \mathsf{R} s_n \cdot \mathsf{R}(\lceil \mathsf{d}_{s_n} \rceil \cdot r_{n-1}) \cdots \mathsf{R} r_1 \\
&= \mathsf{R} s_n \cdot \mathsf{R} s_{n-1} \cdot \mathsf{R}\lceil \mathsf{d}_{s_{n-1}} \rceil \cdot \mathsf{R} r_{n-2} \cdots \mathsf{R} r_1 \\
&\ \ \vdots \\
&= \mathsf{R} s_n \cdots \mathsf{R} s_1 \\
&= \mathsf{R} s_n \cdots \mathsf{R} s_2 \cdot \mathsf{R}\lceil \mathsf{i}_{t_1} \rceil \cdot \mathsf{R} t_1 \\
&= \mathsf{R} s_n \cdot \ldots \mathsf{R}(s_2 \cdot \lceil \mathsf{i}_{t_1} \rceil) \cdot \mathsf{R} t_1 \\
&= \mathsf{R} s_n \cdots \mathsf{R} s_3 \cdot \mathsf{R}\lceil \mathsf{i}_{t_2} \rceil \cdot \mathsf{R} t_2 \cdot \mathsf{R} t_1 \\
&\ \ \vdots \\
&= \mathsf{R} t_n \cdots \mathsf{R} t_1 \le 1_{\mathsf{F}X}.
\end{aligned}$$
□

Remark IV.4. If a relax extension satisfies the condition of Lemma IV.3 then it is necessarily normal.

To apply Lemma IV.3 to relax extensions given as pointwise suprema of normal lax extensions, we next show that preservation of composition with subidentities is stable under suprema.

Proposition IV.5. *The relax extension given by the pointwise supremum of a non-empty family of normal lax extensions L_i, $i \in I$, of F preserves composition with subidentities iff all L_i preserve composition with subidentities.*

It turns out that the functors whose normal lax extensions preserve composition with subidentities are essentially the ones that preserve inverse images. To see this, first we record some useful properties of lax extensions:

Lemma IV.6. *Let* L *be a lax extension of* F*. Then:*

1) *For every relation* $r\colon X \nrightarrow Y$, $\mathsf{img}(\mathsf{F}\,\mathsf{d}_r) \subseteq \mathsf{dom}(\mathsf{L}r)$.
2) *If* L *is normal, then for every difunctional relation* $r\colon X \nrightarrow Y$, $\mathsf{L}(r^\circ) = (\mathsf{L}r)^\circ$.
3) *If* L *is normal and* F *preserves empty intersections, then for all injective maps* $i\colon A \rightarrowtail X$ *and* $j\colon A \rightarrowtail Y$,

$$\mathsf{L}(j \cdot i^\circ) = \mathsf{F}j \cdot (\mathsf{F}i)^\circ.$$

In the next result we show that the normal lax extensions of a functor that preserve inverse images are precisely the ones that when applied to a relation r return a relation whose (co)domain is completely determined by the action of the functor on the (co)domain of r.

Lemma IV.7. *The following clauses are equivalent:*

(i) F *preserves inverse images.*

(ii) F *admits a normal lax extension and, for every normal lax extension* L *of* F *and every difunctional relation* $r\colon X \nrightarrow Y$*, the map*

$$\mathsf{F}\mathsf{d}_r\colon \mathsf{F}\,\mathsf{dom}(r) \rightarrowtail \mathsf{F}X$$

corestricts to an isomorphism $\mathsf{F}(\mathsf{dom}\, r) \cong \mathsf{dom}(\mathsf{L}r)$.

(iii) F *admits a normal lax extension and, for every normal lax extension* L *of* F *and every relation* $r\colon X \nrightarrow Y$*, the maps*

$$\mathsf{F}\mathsf{d}_r\colon \mathsf{F}\,\mathsf{dom}(r) \rightarrowtail \mathsf{F}X \text{ and } \mathsf{F}\mathsf{i}_r\colon \mathsf{F}\,\mathsf{img}(r) \rightarrowtail \mathsf{F}Y$$

corestrict to isomorphisms $\mathsf{F}(\mathsf{dom}\, r) \cong \mathsf{dom}(\mathsf{L}r)$ *and* $\mathsf{F}(\mathsf{img}\, r) \cong \mathsf{img}(\mathsf{L}r)$*, respectively.*

(iv) F *admits a normal lax extension and for every normal lax extension* L *of* F*, every relation* $r\colon X \nrightarrow Y$ *and all injective maps* $i\colon X \rightarrowtail A$ *and* $j\colon Y \rightarrowtail B$,

$$\mathsf{L}(j \cdot r \cdot i^\circ) = \mathsf{F}j \cdot \mathsf{L}r \cdot (\mathsf{F}i)^\circ.$$

(v) F *admits a normal lax extension that satisfies the condition of (iv).*

Finally, as a consequence of the previous result, we obtain the above mentioned characterization of the functors that preserve inverse images in terms of their normal lax extensions. We stress that preserving empty intersections is a very mild condition that from a coalgebraic point of view is comparable to preserving monomorphisms.

Proposition IV.8. *The following clauses are equivalent:*

(i) F *preserves inverse images.*

(ii) F *admits a normal lax extension and* each *of its normal lax extensions preserves composition with partial monomorphisms.*

(iii) F *admits a normal lax extension that preserves composition with partial monomorphisms.*

(iv) The functor F *preserves empty intersections, admits a normal lax extension and* each *of its normal lax extensions preserves composition with subidentities.*

(v) F *preserves empty intersections and admits a normal lax extension that preserves composition with subidentities.*

Proof. (i) ⇒ (ii). Every functor that preserves inverse images admits a normal lax extension [18], and the fact that each normal lax extension preserves composition with partial monomorphisms is a straightforward consequence of Lemma IV.7(v). Indeed, let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a normal lax extension of a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$. Suppose that $r\colon X \nrightarrow Y$ is a relation and $s\colon A \nrightarrow X$ is a partial monomorphism. Then, there are injective maps $i\colon S \to A$ and $j\colon S \to X$ such that $s = j \cdot i^\circ$. Hence, by Lemma IV.7(v) and the fact that L is a relational connector, $\mathsf{L}(r \cdot j \cdot i^\circ) = \mathsf{L}r \cdot \mathsf{F}j \cdot (\mathsf{F}i)^\circ$. Therefore, by Lemma IV.6(3), $\mathsf{L}(r \cdot j \cdot i^\circ) = \mathsf{L}r \cdot \mathsf{L}s$. The case of postcomposition with a partial monomorphism follows analogously.

(ii) ⇒ (iii). Trivial.

(ii) ⇒ (iv) and (iii) ⇒ (v). Every subidentity is a partial monomorphism and, hence, the claims follow immediately due to Lemma IV.6(3).

(iv) ⇒ (v). Trivial.

(v) ⇒ (i). Let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a normal lax extension of F that preserves composition with subidentities. We will see that for every relation $r\colon X \nrightarrow Y$ and all injective maps $i\colon X \rightarrowtail A$ and $j\colon Y \rightarrowtail B$, $\mathsf{L}(j \cdot r \cdot i^\circ) = \mathsf{F}j \cdot \mathsf{L}r \cdot (\mathsf{F}i)^\circ$. Then, the claim follows by Lemma IV.7. Let $r\colon X \nrightarrow Y$ be a relation and $i\colon X \rightarrowtail A$ be an injective map. Then $i \cdot i^\circ$ is a subidentity and as F preserves empty intersections, by Lemma IV.6(3), $\mathsf{L}(i \cdot i^\circ) = \mathsf{F}i(\mathsf{F}i)^\circ$. Therefore, by hypothesis, $\mathsf{L}(r \cdot i^\circ) = \mathsf{L}(r \cdot i^\circ \cdot i \cdot i^\circ) = \mathsf{L}(r \cdot i^\circ) \cdot \mathsf{L}(i \cdot i^\circ) = \mathsf{L}(r \cdot i^\circ) \cdot \mathsf{F}i \cdot (\mathsf{F}i)^\circ = \mathsf{L}r \cdot (\mathsf{F}i)^\circ$. Similarly, we obtain that for every injective map $j\colon Y \rightarrowtail B$, $\mathsf{L}(j \cdot r) = \mathsf{F}j \cdot \mathsf{L}r$, and the claim follows. □

Now we are ready to show the main result of this section:

Theorem IV.9. *Every set functor that preserves inverse images admits a greatest normal lax extension.*

Proof. Let $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ be a functor that preserves inverse images. Then, by [18], F has a normal lax extension and, therefore, the least lax extension of a functor is normal. To show that the non-empty supremum of normal lax extensions is normal, due to Proposition IV.8 and Proposition IV.5, we use the criterion of Lemma IV.3. Let $(\mathsf{L}_i)_{i \in \mathcal{I}}$ be a non-empty family of normal lax extensions of F and let $\mathsf{L}^\vee$ be the corresponding relax extension given by pointwise supremum. We have to show that for every composable sequence of relations $r_1, \ldots, r_n$ such that $r_n \cdot \ldots \cdot r_1 \le 1_X$, for some set X, and $\mathsf{img}(r_{i-1}) = \mathsf{dom}(r_i)$, for $i = 2, \ldots, n$, $\mathsf{L}^\vee r_n \cdot \ldots \cdot \mathsf{L}^\vee r_1 \le 1_{\mathsf{F}X}$. First, we note that we can assume w.l.o.g. that all relations in the sequences are total and surjective, which entails $r_n \cdot \ldots \cdot r_1 = 1_X$. Indeed, let $r_1, \ldots, r_n$ be a composable sequence of relations such that $r_n \cdot \ldots \cdot r_1 \le 1_X$, for some set X, and $\mathsf{img}(r_{i-1}) = \mathsf{dom}(r_i)$, for $i = 2, \ldots, n$. Then, by (co)restricting each relation in the sequence to its (co)domain we obtain a composable sequence $r'_1, \ldots, r'_n$ of total and surjective relations such that $r'_n \cdot \ldots \cdot r'_1 = 1_A$, where $A = \mathsf{dom}(r_n \cdot \ldots \cdot r_1)$. Furthermore, as F preserves monomorphisms, it follows from Lemma IV.7(v) that $\mathsf{L}^\vee r_n \cdot \ldots \cdot \mathsf{L}^\vee r_1 = \mathsf{F}i \cdot \mathsf{L}r'_n \cdot \ldots \cdot \mathsf{L}r'_1 \cdot (\mathsf{F}i)^\circ$, where $i\colon A \rightarrowtail X$ is the inclusion of A into X. Therefore, as F preserves monomorphisms, $\mathsf{L}^\vee r_n \cdot \ldots \cdot \mathsf{L}^\vee r_1 \leq 1_{\mathsf{F}X} \Leftrightarrow \mathsf{L}r'_n \cdot \ldots \cdot \mathsf{L}r'_1 \leq 1_{\mathsf{F}A}$.

Now, we proceed by induction on n. The base case $n = 1$ is trivial as $\mathsf{L}^\vee$ is normal. To see the inductive step from n to $n+1$, let $r_1, \ldots, r_{n+1}$ be a composable sequence of total and surjective relations such that $r_{n+1} \cdot \ldots \cdot r_1 = 1_X$. Then, by [18, Lemma 4.20], $\hat{r}_{n+1} \cdot \ldots \cdot r_1 = 1_X$, where $\hat{r}$ is the difunctional closure of r_{n+1}. Furthermore, as normal lax extensions coincide on difunctional relations, relational composition preserves supremums and lax extensions preserve composition laxly, $\mathsf{L}^\vee \hat{r}_{n+1} \cdot \mathsf{L}^\vee r_n = \mathsf{L}^\vee \hat{r}_{n+1} \cdot (\bigvee_{i\in\mathcal{I}} \mathsf{L}_i r_n) = \bigvee_{i\in\mathcal{I}} (\mathsf{L}_i \hat{r}_{r+1} \cdot \mathsf{L}_i r_n) \leq \bigvee_{i\in\mathcal{I}} \mathsf{L}_i(\hat{r}_{r+1} \cdot r_n) = \mathsf{L}^\vee(\hat{r}_{n+1} \cdot r_n)$. Therefore, by induction hypothesis, $\mathsf{L}^\vee r_{n+1} \cdot \ldots \cdot \mathsf{L}^\vee r_1 \leq \mathsf{L}^\vee \hat{r}_{n+1} \cdot \ldots \cdot \mathsf{L}^\vee r_1 \leq \mathsf{L}^\vee(\hat{r}_{n+1} \cdot r_n) \cdot \ldots \cdot \mathsf{L}r_1 \leq 1_{\mathsf{F}X}$. ☐

Recall from Example III.16 that a monoid-valued functor admits a normal lax extension iff the monoid is positive [18], which in turn is known to be equivalent to preservation of inverse images by the monoid-valued functor [33]. Therefore,

Corollary IV.10. *A monoid-valued functor admits a greatest normal lax extension iff the monoid is positive.*

V. Case Study: Labelled Transitions

The first and so far only example reported in the literature of a functor that admits more than one normal lax extension, due to Paul Levy, involves a monoid-valued functor for a fairly sophisticated submonoid of the non-negative reals generated as a division semiring by a transcendental number [34, Example 4.11]. We will show that functors of the form $C + B \times \mathsf{Id}$ admit a unique normal lax extension. However, we will also show that it is not uncommon for a functor to have multiple normal lax extensions; in fact, we give a simple and widely used class of examples: Almost all exponential functors, i.e. functors of the form $(-)^A$ (which in coalgebra represent deterministic A-labelled transitions), admit multiple normal lax extensions.

For brevity, we write H_A for $(-)^A$. Building on the results of the previous sections, we describe the complete lattice of normal lax extensions of H_A, in particular obtaining a maximally permissive sound and complete notion of (bi)simulation for deterministic automata whose class of (bi)simulations is closed under composition. By combining this result with the usual notion of bisimulation on P-coalgebras (i.e. unlabelled transition systems), we obtain a new notion of *twisted bisimulation* on labelled transition systems that is more permissive than standard Park-Milner bisimulations.

Since the functor H_A preserves limits, it preserves weak pullbacks, and its Barr extension sends a relation $r\colon X \nrightarrow Y$ to the relation $\overline{\mathsf{H}_A} r\colon \mathsf{H}_A X \nrightarrow \mathsf{H}_A Y$ defined by:

$$f\ \overline{\mathsf{H}_A} r\ g \Leftrightarrow \forall a \in A, f(a)\ r\ g(a) \Leftrightarrow 1_A \leq g^\circ \cdot r \cdot f.$$

We obtain non-standard notions of simulation by additionally allowing other relations on A in place of 1_A in the last inequality: We work with a set $\mathcal{A}$ of endorelations on A and define

$$f\ \widehat{\mathsf{H}}_A^{\mathcal{A}} r\ g \Leftrightarrow (\exists \phi \in \mathcal{A}.\quad \phi \leq g^\circ \cdot r \cdot f), \tag{2}$$

or, in pointful notation, $\exists \phi \in \mathcal{A}, \forall a, b \in A, a\,\phi\,b \Rightarrow f(a)\,r\,g(b)$.

We show next that this construction yields a lax extension of H_A whenever $\mathcal{A}$ forms a submonoid of $\mathsf{Rel}(A, A)$, and that normality of this lax extension is captured as a condition on the relations in $\mathcal{A}$.

Definition V.1. An endorelation $\phi\colon A \nrightarrow A$ is ***normal*** if its difunctional closure is reflexive, i.e., if for every a in A there is a chain $a\ r\ x_1\ r^\circ\ x_2\ r\ \ldots\ r^\circ x_{n-1}\ r\ a$ of alternating r- and r°-steps of (necessarily odd) length $n \geq 1$.

Proposition V.2. *A relation $\phi\colon A \nrightarrow A$ is normal iff for every set X and every pair of functions $f, g\colon A \to X$, $\phi \leq g^\circ \cdot f$ implies $f = g$.*

Proof. Let $\phi\colon A \nrightarrow A$ be a normal relation, and $f, g\colon A \to X$ be functions such that $\phi \leq g^\circ \cdot f$. Then, by definition of difunctional closure, $\hat{\phi} \leq g^\circ \cdot f$, where $\hat{\phi}$ denotes the difunctional closure of ϕ. Hence, by normality of ϕ, $1_A \leq g^\circ \cdot f$ which is equivalent to $g \leq f$. Therefore, as f and g are functions, $f = g$. To see the converse statement, suppose that the difunctional closure $\hat{\phi}$ of ϕ is given by $g^\circ \cdot f$. Then, by definition of difunctional closure, $\phi \leq g^\circ \cdot f$. Hence, by hypothesis, $f = g$. Therefore, $\hat{\phi} = g^\circ \cdot g \geq 1_A$. ☐

Intuitively, normal relations correspond to roundabout ways of proving equality. For instance, if $A = \{a, b\}$, then two functions $f, g\colon A \to X$ are of course equal if we can show that $f(a) = g(a)$ and $f(b) = g(b)$, but also if we instead show that $f(a) = g(b)$, $g(a) = g(b)$ and $g(a) = f(b)$. This second proof corresponds to the relation $\{(a, b), (b, b), (b, a)\}$ from Figure 1.

Theorem V.3. *Let $\mathcal{A}$ be a submonoid of the monoid of endorelations on a set A. Then assigning to every relation $r\colon X \nrightarrow Y$ the relation $\widehat{\mathsf{H}}_A^{\mathcal{A}} r\colon \mathsf{H}_A X \nrightarrow \mathsf{H}_A Y$ defines a lax extension of H_A to Rel. Furthermore,*

1) *if $\mathcal{A}$ is closed under converses, then $\widehat{\mathsf{H}}_A^{\mathcal{A}}$ preserves converses, and*
2) *if every relation in $\mathcal{A}$ is normal, then $\widehat{\mathsf{H}}_A^{\mathcal{A}}$ is normal.*

Proof. Let $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$ be relations, and $f, f'\colon A \to X$, $g\colon A \to Y$, $h\colon A \to Z$ and $t\colon X \to Y$ be functions.

Monotonicity. Trivial.

Lax preservation of composition. Suppose that $f\ \widehat{\mathsf{H}}_A^{\mathcal{A}} r\ g$ and $g\ \widehat{\mathsf{H}}_A^{\mathcal{A}} s\ h$. Then there exist $\phi, \psi \in \mathcal{A}$ such that $\phi \leq g^\circ \cdot r \cdot f$ and $\psi \leq h^\circ \cdot s \cdot g$. Hence,

$$\psi \cdot \phi \leq h^\circ \cdot s \cdot g \cdot g^\circ \cdot r \cdot f \leq h^\circ \cdot s \cdot r \cdot f,$$

and the claim follows from the fact that $\mathcal{A}$ is closed under composition.

Extension of functions. We have $1_A \leq (t \cdot f)^\circ \cdot t \cdot f = f^\circ \cdot t^\circ \cdot (t \cdot f)$, and thus, since $1_A \in \mathcal{A}$, $f\ \widehat{\mathsf{H}}_A^{\mathcal{A}} t\ (t \cdot f)$ and $(t \cdot f)\ \widehat{\mathsf{H}}_A^{\mathcal{A}} t\ f$.

When $\mathcal{A}$ is closed under converses, we show 1. calculating as follows:

$$g\ \widehat{\mathsf{H}}_A r^\circ\ f \Leftrightarrow \phi \le f^\circ \cdot r^\circ \cdot g \Leftrightarrow \phi^\circ \le g^\circ \cdot r \cdot f \Leftrightarrow f\ \widehat{\mathsf{H}}_A r\ g.$$

Moreover, by definition, $f'(\widehat{\mathsf{H}}^{\mathcal{A}}_A 1_X) f$ iff there is $\phi \in \mathcal{A}$ such that $\phi \le f^\circ \cdot f'$. Hence, if every relation in $\mathcal{A}$ is normal we obtain $1_X \le f^\circ \cdot f'$. Therefore, $f = f'$, which yields 2. □

Conversely, every lax extension $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ of H_A gives rise to a set $\mathsf{S}(\mathsf{L})$ of endorelations given by

$$\mathsf{S}(\mathsf{L}) = \{\phi\colon A \nrightarrow A \mid 1_A\, \mathsf{L}\phi\, 1_A\} \tag{3}$$

Proposition V.4. *Let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a lax extension of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$. Then the set $\mathsf{S}(\mathsf{L})$ is an upwards-closed submonoid of $\mathsf{Rel}(A, A)$. Furthermore, $\mathsf{S}(\mathsf{L})$ is closed under converses if L preserves converses, and every relation in $\mathsf{S}(\mathsf{L})$ is normal if L is normal.*

Proof. All claims are straightforward except maybe the last. So suppose that L is normal. To show that $r \in \mathsf{S}(\mathsf{L})$ is normal, let $r \le d$ where d is difunctional, and hence of the form $d = g^\circ \cdot f$ where $f, g\colon A \to X$ are functions. Then, as L is a normal relational connector, $\mathsf{L}r \le \mathsf{L}d = (\mathsf{H}_A g)^\circ \cdot \mathsf{H}_A f$. . Hence, by definition of $\mathsf{S}(\mathsf{L})$, $(\mathsf{H}_A g)^\circ \cdot \mathsf{H}_A f$ relates 1_A to 1_A, which by definition of H_A entails $f = g$.

□

These two constructions are inverses of each other:

Theorem V.5. *Every lax extension $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$ is induced by the set $\mathsf{S}(\mathsf{L}) = \{r\colon A \nrightarrow A \mid 1_A\,(\mathsf{L}r)\,1_A\}$ and every upwards-closed submonoid $\mathcal{A}$ of $\mathsf{Rel}(A, A)$ is induced by the lax extension $\widehat{\mathsf{H}}^{\mathcal{A}}_A$.*

Proof. Let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a lax extension of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$, $r\colon X \nrightarrow Y$ a relation, and let $f\colon A \to X$ and $g\colon A \to Y$ be functions. First note that if there is $\phi \in \mathsf{S}(\mathsf{L})$ such that $\phi \le g^\circ \cdot r \cdot f$, then, by definition of $\mathsf{S}(\mathsf{L})$ and local monotonicity of L, $\mathsf{L}(g^\circ \cdot r \cdot f)$ relates 1_A to 1_A. Therefore, the claim follows from the fact that $\mathsf{L}r$ relates $f = \mathsf{H}_A f(1_A)$ to $g = \mathsf{H}_A g(1_A)$ iff $\mathsf{L}(g^\circ \cdot r \cdot f)$ relates 1_A to 1_A.

Conversely, let $\mathcal{A}$ be an upwards-closed submonoid of $\mathsf{Rel}(A, A)$. Then

$$\phi \in \mathsf{S}(\widehat{\mathsf{H}}^{\mathcal{A}}_A) \Leftrightarrow 1_A\, \widehat{\mathsf{H}}^{\mathcal{A}}_A \phi\, 1_A \Leftrightarrow \exists \psi \in \mathcal{A},\ \psi \le \phi \Leftrightarrow \phi \in \mathcal{A}.$$ □

Corollary V.6. *The normal lax extensions of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$ correspond precisely to the upward-closed submonoids of $\mathsf{Rel}(A, A)$ consisting only of normal relations.*

Example V.7. 1) With $A = 2 = \{a, b\}$, consider the upwards-closed submonoid $\mathcal{A}$ of $\mathsf{Rel}(2, 2)$ generated from the single relation $\Phi = \{(a, b), (b, b), (b, a)\}$. The lax extension $\widehat{\mathsf{H}}^{\mathcal{A}}_2$ of $\mathsf{H}_2\colon \mathsf{Set} \to \mathsf{Set}$ to Rel is normal, preserves converses and differs from the Barr extension of H_2 since $\widehat{\mathsf{H}}^{\mathcal{A}}_2 \Phi$ relates 1_2 to itself but $\overline{\mathsf{H}}_2 \Phi$ does not. The corresponding notion of bisimulation, combined with the standard notion of bisimulation for unlabelled transition systems, is the 'twisted' bisimulation mentioned in the introduction.

2) With $A = 3 = \{a, b, c\}$, consider the upwards-closed submonoid $\mathcal{A}$ of $\mathsf{Rel}(3, 3)$ generated from the single relation $\Phi = \mathsf{P}(3\times 3)\setminus\{(a, a), (b, c)\}$. We obtain a normal lax extension L of $\mathsf{H}_3\colon \mathsf{Set} \to \mathsf{Set}$ that does not preserve converses and, hence, differs from the Barr extension. Indeed, $\widehat{\mathsf{H}}^{\mathcal{A}}_3 \Phi$ relates 1_3 to itself but $\widehat{\mathsf{H}}^{\mathcal{A}}_3(\Phi^\circ)$ does not. As far as we know, this is the first example of a non-symmetric normal lax extension. Furthermore, since L is normal, by Corollary III.2, L-similarity is sound and complete.

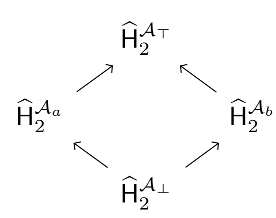
Theorem V.5 implies in particular that the class $\mathsf{Lax}(\mathsf{H}_A)$ of lax extensions of H_A is small, so we will regard it as a set. It is easy to see that the mutually inverse constructions described in Theorem V.3 and Proposition V.4 define monotone maps $\mathsf{S}\colon \mathsf{Lax}(\mathsf{H}_A) \to \mathsf{SubMon}^{\uparrow}(\mathsf{Rel}(A, A))$ and $\widehat{\mathsf{H}}^{(-)}_A\colon \mathsf{SubMon}^{\uparrow}(\mathsf{Rel}(A, A)) \to \mathsf{Lax}(\mathsf{H}_A)$ between the partially ordered set $\mathsf{Lax}(\mathsf{H}_A)$ of lax extensions of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$ and the partially ordered set $\mathsf{SubMon}^{\uparrow}(\mathsf{Rel}(A, A))$ of upwards-closed submonoids of $\mathsf{Rel}(A, A)$ ordered by inclusion.

This allows reasoning about suprema of lax extensions in terms of suprema of sets of endorelations in $\mathsf{SubMon}^{\uparrow}(\mathsf{Rel}(A, A))$ which is given by closure under composition of union of the sets; more specifically, the supremum of a family $(\mathcal{A})_{i\in\mathcal{I}}$ of elements of $\mathsf{SubMon}^{\uparrow}(\mathsf{Rel}(A, A))$, denoted as $\bigvee_{i\in\mathcal{I}} \mathcal{A}_i$, is the smallest set that contains $\bigcup_{i\in\mathcal{I}} \mathcal{A}_i$ and is closed under composition.

Proposition V.8. *Let $(\mathsf{L}_i)_{i\in\mathcal{I}}$ be a family of lax extensions of the functor $\mathsf{H}_A\colon \mathsf{Set} \to \mathsf{Set}$. Then the supremum $\bigvee_{i\in\mathcal{I}} \mathsf{L}_i$ is given by the lax extension induced by set $\bigvee_{i\in\mathcal{I}} \mathsf{S}(\mathsf{L}_i)$.*

Example V.9. The squaring functor $\mathsf{H}_2\colon \mathsf{Set} \to \mathsf{Set}$ has precisely four normal lax extensions, which are induced by the upwards-closed submonoids of $\mathsf{Rel}(2, 2)$

- $\mathcal{A}_\perp$ generated by the empty set – the Barr extension;
- $\mathcal{A}_a$ generated from the single relation $\Phi_a = \{(a, b), (a, a), (b, a)\}$;
- $\mathcal{A}_b$ generated from the single relation $\Phi_b = \{(a, b), (b, b), (b, a)\}$;
- $\mathcal{A}_\top$ generated by the set $\{\Phi_a, \Phi_b\}$.



Since exponential functors preserve terminal objects, all states belonging to coalgebras for a exponential functor are behaviourally equivalent. This means that similarity w.r.t. the greatest lax extension of a exponential functor – which induces the most permissive notion of simulation but that in general fails to be normal – is sound, complete and the corresponding class of simulations is closed under composition. The situation is far more interesting for polynomial functors, whose definition we recall next.

Given a family $\mathcal{F} = (\mathsf{F}_s)_{s\in S}$ of set functors indexed by a set S, we denote by $\Sigma(\mathcal{F})\colon \mathsf{Set} \to \mathsf{Set}$ the canonical functor "sum of a family of functors" that sends a set X to the coproduct $\sum_{s\in S} \mathsf{F}_s X$, and we denote by $c^s\colon \mathsf{F}_s \to \Sigma(\mathcal{F})$ the natural transformation whose X-component is defined by the coprojection $c^s_X\colon \mathsf{F}_s X \to \sigma(\mathcal{F})$.

Definition V.10. A polynomial set functor is a functor of the form $\Sigma(\mathcal{H})$ for a family $\mathcal{H}$ of exponential functors.

Recall from Example III.14 that almost all polynomial functors (except the exponential functors) are ζ-bounded, so that Theorem III.15 applies. In particular, this means that notions of simulation on coalgebras for polynomial functors induced by a lax extension are sound iff the lax extension is normal. This further motivates the investigation of the structure of the lattice of normal lax extensions of polynomial functors. We show next that greatest normal lax extensions of polynomial functors are constructed from greatest normal lax extensions of exponential functors; we will illustrate this principle on the minimization of deterministic automata, which are coalgebras for polynomial functors of the form $2 \times \mathsf{H}_A$.

In the remainder of the paper, we fix a set S and a family $\mathcal{F} = (\mathsf{F}_s)_{s\in S}$ of set functors, and we say that a family of lax extensions $(\mathsf{L}_s)_{s\in S}$ is a family of lax extensions of $\mathcal{F}$ if for every $s \in S$, L_s is a lax extension of F_s.

It is easy to see that a family of lax extensions of $\mathcal{F}$ gives rise to a lax extension for $\Sigma(\mathcal{F})$ that sends every relation $r\colon X \nrightarrow Y$ to the relation $\Sigma^{\mathsf{L}}(\mathcal{L})r$ given by $\bigvee_{s\in S} \left(c^s_Y \cdot \mathsf{L}_s r \cdot (c^s_X)^\circ\right)$; or, in pointful notation, for all $u \in \Sigma(\mathcal{F})X$ and $v \in \Sigma(\mathcal{F})Y$, $u\ \Sigma^{\mathsf{L}}(\mathcal{L})r\ v$ iff there is $s \in S$ and $u' \in \mathsf{F}_s X$, $v' \in \mathsf{F}_s Y$ s.t. $u = c^s_X(u'), v = c^s_Y(v')$ and $u'\ \mathsf{L}_s r\ v'$. Then, assigning to every family of lax extensions of $\mathcal{F}$ the lax extension $\Sigma^{\mathsf{L}}(\mathcal{L})$ defines a monotone map $\Sigma^{\mathsf{L}}(-)\colon \mathsf{Fam}(\mathcal{F}) \to \mathsf{Lax}(\Sigma\mathcal{F})$ from the partially ordered class of families of lax extensions of $\mathcal{F}$, ordered pointwise, to the partially ordered class of lax extensions of $\Sigma(\mathcal{F})$ ordered pointwise.

Conversely, every lax extension L for the functor $\Sigma(\mathcal{F})$ gives rise to an S-indexed family $c^*(\mathsf{L})$ of lax extensions of $\mathcal{F}$ by "(co)restricting" the action of the lax extension to F_s, i.e., for every $s \in S$ and every relation $r\colon X \nrightarrow Y$, $c^*(\mathsf{L})_s(r)$ is given by $(c^s_Y)^\circ \cdot \mathsf{L}r \cdot c^s_X$; or in pointful notation, for all $u' \in \mathsf{F}_s X$ and $v' \in \mathsf{F}_s Y$, $u'\ c^*(\mathsf{L})_s(r)\ v'$ iff $c^s_X(u')\ \mathsf{L}r\ c^s_Y(v')$.

Then, assigning to every lax extension L of $\Sigma(\mathcal{F})$ the family $c^*(\mathsf{L})$ of lax extensions of $\mathcal{F}$ defines a monotone map $c^*(-)\colon \mathsf{Lax}(\Sigma\mathcal{F}) \to \mathsf{Fam}(\mathcal{F})$.

Immediately from the definions we have:

Proposition V.11. *The map $\Sigma^{\mathsf{L}}(-)$ is an order reflecting left adjoint of $c^*(-)$.*

Of course, in general, the constructions defined above are not inverse of each other.

Example V.12. Consider the pair $(\mathsf{C}_1, \mathsf{C}_1)$ where $\mathsf{C}_1\colon \mathsf{Set} \to \mathsf{Set}$ denotes the constant functor to $1 = \{*\}$. The functor $\mathsf{C}_1 + \mathsf{C}_1$ is isomorphic to the constant functor $\mathsf{C}_2\colon \mathsf{Set} \to \mathsf{Set}$ to $2 = \{0, 1\}$. The greatest lax extension $\mathsf{L}^\top$ of C_2 sends every relation to the greatest relation on 2 which is different from the identity on 2. However, $c^*(\mathsf{L}^\top) = (\overline{\mathsf{C}_1}, \overline{\mathsf{C}_1})$ and $\Sigma^{\mathsf{L}}(c^*(\mathsf{L}^\top))$ is the Barr extension of C_2 which sends every relation to the identity map on 2.

However, as we show next, they become inverse of each other when $\mathcal{F}$ consists of functors that weakly preserve pullbacks and only normal lax extensions are allowed.

Let $\mathsf{NLax}(\Sigma\mathcal{F})$ be the partially ordered subclass of $\mathsf{Lax}(\Sigma\mathcal{F})$ given by the normal lax extensions of $\Sigma\mathcal{F}$, and let $\mathsf{NFam}(\mathcal{F})$ be the partially ordered subclass of $\mathsf{Fam}(\mathcal{F})$ given by the families of normal lax extensions of $\mathcal{F}$. Clearly, the map $\Sigma^{\mathsf{L}}(-)$ (co)restricts to a map $\Sigma^{\mathsf{L}}(-)\colon \mathsf{NFam}(\mathcal{F}) \to \mathsf{NLax}(\Sigma\mathcal{F})$, and, as each natural transformation c^s is monic, it is easy to see that the map $c^*(-)$ (co)restricts to $c^*(-)\colon \mathsf{NLax}(\Sigma\mathcal{F}) \to \mathsf{NFam}(\mathcal{F})$.

Remark V.13. It has been observed that the canonical forgetful functor from the category of lax extensions to the category of Set-endofunctors is topological [35], and hence, in particular, has a left adjoint. Furthermore, this left adjoint picks, for every $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$, the smallest element of the fibre $\mathsf{Lax}(\mathsf{F})$ of F with respect to this forgetful functor. We also note that, if F weakly preserves pullbacks, this element is given by the Barr extension of F.

Suppose now that every functor in the family $\mathcal{F}$ weakly preserves pullbacks. Let $\mathcal{L} = (\overline{\mathsf{F}}_s)_{s\in S}$ be the family of the corresponding Barr extensions. It is easy to see that the functor $\Sigma(\mathcal{F})$ weakly preserves pullbacks as well, and, by adjointness, its Barr extension is given by $\Sigma^{\mathsf{L}}(\mathcal{L})$.

Theorem V.14. *If all functors in $\mathcal{F}$ weakly preserve pullbacks, then the maps $\Sigma^{\mathsf{L}}(-)\colon \mathsf{NFam}(\mathcal{F}) \to \mathsf{NLax}(\Sigma\mathcal{F})$ and $c^*(-)\colon \mathsf{NLax}(\Sigma\mathcal{F}) \to \mathsf{NFam}(\mathcal{F})$ are inverse of each other.*

Proof. Due to Proposition V.11 we are left with the task of showing $\Sigma^{\mathsf{L}}(c^*(\mathsf{L})) \geq \mathsf{L}$. Let $r\colon X \nrightarrow Y$ be a relation, and let $u \in \Sigma(\mathcal{F})X$ and $v \in \Sigma(\mathcal{F})Y$ s.t. $u\ \mathsf{L}r\ v$. Suppose that all functors in $\mathcal{F}$ weakly preserve pullbacks. Then, by Remark V.13, the functor $\Sigma(\mathcal{F})$ weakly preserves pullbacks. Hence, as L is a normal relational connector, by Proposition III.6(3), $u\ \overline{\Sigma(\mathcal{F})}(\hat{r})\ v$, where $\hat{r}$ is the difunctional closure of r. Thus, by Remark V.13, there is $s \in S$, $u' \in \mathsf{F}_s X$ and $v' \in \mathsf{F}_s Y$ s.t. $u = c^s_X(u')$ and $v = c^s_Y(v')$. Hence, by definition, $u'(c^*(\mathsf{L})r)\ v'$. Therefore, by definition of sum of a family of lax extensions, $u\ \Sigma^{\mathsf{L}}(c^*(\mathsf{L}))r\ v$. □

Thererefore:

Corollary V.15. *The greatest normal lax extension of a polynomial set functor determined by a family $\mathcal{H}$ of exponential functors is given by the sum of the family of the greatest normal lax extensions of the exponential functors in $\mathcal{H}$.*

Interestingly, as a consequence of our results, we obtain that the usual notion of (bi)simulation for stream systems with termination is the only sound notion induced by a lax extension:

Corollary V.16. *For all sets C and B, the functor $C + B \times \mathsf{Id}\colon \mathsf{Set} \to \mathsf{Set}$ admits a unique normal lax extension.*

Proof. Every functor of the form $C + B \times \mathsf{Id}$ preserves weak pullbacks and, therefore, admits a normal lax extension. On the other hand, from Corollary V.6 we conclude that the identity functor admits a unique normal lax extension. Now, the claim follows from Theorem V.14. ☐

In particular, the maximally permissive notion of simulation induced by a lax extension for deterministic automata can be obtained as follows.

Corollary V.17. *Let A, B and C be sets. The greatest normal lax extension of $C + B \times \mathsf{H}_A$ is given by postcomposing the Barr extension of $C + B \times \mathsf{Id}$ with the greatest normal lax extension of H_A.*

Example V.18 (Automata minimization)**.** Let us reinterpret Example V.9 in automata-theoretic terms.

Let $S = \{s_0, \dots, s_{n-1}\} \cup \{t_0, \dots, t_{m-1}\}$ be the state space of a deterministic automaton with all states accepting, with n and m being mutually prime, and with $\{a, b\}$ being the alphabet of actions. The transition function is as follows: the a-transitions connect every s_i with $s_{(i+1) \bmod n}$ and every t_j with $t_{(j+1) \bmod m}$; the b-transitions connect every s_i with t_0 and every t_j with s_0.

Since all states are accepting, they are all bisimilar, and thus the minimal automaton has one state. A bisimilarity relation in the usual sense – which is induced by Barr extension ($\widehat{\mathsf{H}}_2^{\mathcal{A}_\perp}$ in Example V.9) – for showing equivalence of any s_n and any t_m must include every pair (s_i, t_j), and thus is potentially quadratic in size. The following non-standard bisimulation of linear size for the greatest normal lax extension $\widehat{\mathsf{H}}_2^{\mathcal{A}_\top}$ from Example V.9 can be used instead: $R = S \times \{s_0, t_0\} \cup \{s_0, t_0\} \times S$. This is essentially because either s_0 or t_0 is reachable by a b-transition from any state in one step, and they are related to everything by R.

To conclude, we derive the notion of twisted bisimulation on labelled transition systems (LTS) mentioned in the introduction as follows. For the sake of readability, we continue to restrict to the case where the set of labels is $2 = \{a, b\}$. Then 2-labelled transition systems are coalgebras for the functor $\mathsf{F} = \mathsf{H}_2 \cdot \mathsf{P}$ where P is the covariant powerset functor. For $\mathcal{A}$ being one of the upwards closed submonoids of $\mathsf{Rel}(2, 2)$ listed in Example V.9, we obtain a normal lax extension $\widehat{\mathsf{F}}^{\mathcal{A}}$ of F by composing $\widehat{\mathsf{H}}_2^{\mathcal{A}}$ with $\overline{\mathsf{P}}$ (cf. Example III.5), i.e.

$$\widehat{\mathsf{F}}^{\mathcal{A}} r = \widehat{\mathsf{H}}_2^{\mathcal{A}}(\overline{\mathsf{P}} r) \qquad \text{for } r \colon X \nrightarrow Y.$$

Then, the most permissive notion of twisted bisimulation is the one induced by $\widehat{\mathsf{F}}^{\mathcal{A}_\top}$. In terms of the standard representation of 2-labelled LTS as pairs $(X, (\to_l)_{l \in 2})$ consisting of a set X of states and transition relations $\to_l \subseteq X \times X$ (always denoted by the same symbol if no confusion is likely), this notion is explicitly described as follows: Given LTS $(X, (\to_u)_{u \in 2})$, $(Y, (\to_u)_{u \in 2})$, a relation $r \colon X \nrightarrow Y$ is a *twisted bisimulation* (for $\mathcal{A}_\top$) if whenever $x \; r \; y$, then one of the following clauses holds (cf. Figure 1):

1) Whenever $x \to_u x'$, then there exists $y \to_u y'$ such that $x' \; r \; y'$, and whenever $y \to_u y'$, then there exists $x \to_u x'$ such that $x' \; r \; y'$, for $u \in 2$.
2) For $(u, v) \in \{(a, b), (b, a), (b, b)\}$, whenever $x \to_u x'$, then there exists $y \to_v y'$ such that $x' \; r \; y'$, and whenever $y \to_v y'$, then there exists $x \to_u x'$ such that $x' \; r \; y'$.
3) Dito, for $(u, v) \in \{(a, b), (b, a), (a, a)\}$.

(In particular, every bisimulation in the standard sense is a twisted bisimulation.) Since $\widehat{\mathsf{F}}^{\mathcal{A}_\top}$ is a normal lax extension, twisted bisimulation is sound (and complete) for standard bisimilarity, i.e. two states are bisimilar if (and only if) they are connected by a twisted bisimulation [16, Theorem 11]. Our statement from the introduction to the effect that twisted bisimulations on LTS can be smaller than standard bisimulations is illustrated by Example V.18, as a deterministic automaton with all states accepting is in particular an LTS. Since the functor $\mathsf{H}_A \cdot \mathsf{P}$ preserves weak pullbacks, due to Proposition III.6(2) and Proposition III.6(3), another way of thinking about twisted bisimulations is that they form a subclass of bisimulations (in the usual sense) up-to difunctionality that, unlike the full class of bisimulations up-to difunctionality, is closed under relational composition. This phenomenon disappears when H_A is combined with functors that fail to preserve weak pullbacks:

Example V.19. Combining the greatest normal lax extension of H_A with the standard lax extension of the monotone neighbourhood functor [16] (which fails to preserve weak pullbacks) yields a notion of *twisted neighbourhood bisimulation* on labelled monotone neighbourhood frames, which are exactly the models underlying the semantics of concurrent PDL [6] that is more permissive than the standard notion of (labelled) monotone bisimulation [36].

VI. Conclusions

We have analysed aspects of notions of (bi)simulation induced by relators and lax extensions, reinforcing the view that a given functor (i.e. a given system type) can be associated with multiple relevant notions of (bi)simulation. By establishing key results on the existence and properties of lax extensions, we have clarified their role in certifying behavioural equivalence across diverse system types. Notably, we have demonstrated that functors preserving 1/4-iso pullbacks admit a sound and complete notion of bisimulation induced by the coBarr relator, and that normality is essentially a necessary condition for soundness. Furthermore, we have shown that functors preserving inverse images possess a greatest normal lax extension, providing a maximally permissive notion of bisimulation.

In a case study on functors of the form $(-)^A$, which model A-labelled transitions, we have introduced the notion of twisted bisimulation and demonstrated its greater permissiveness compared to standard bisimilarity, while retaining soundness. This result can potentially offer new tools for reasoning about state-based systems of various branching types, and allows for smaller bisimulations certifying behavioural equivalence.

Our work contributes to the general theory of bisimulations by refining the structural conditions under which sound and complete bisimulation notions exist. One direction for future investigation is to identify further sufficient conditions for a functor to admit a greatest normal lax extension; one candidate condition is weak preservation of 1/4-iso and 4/4-epi pullbacks, which has recently been shown to guarantee existence of a normal lax extension [18]. Such endeavour will require a completly new proof strategy because, as we have shown, preservation of inverse images is crucial for the techniques used in this paper. A further important open question is the uniqueness of normal lax extensions and a characterization of functors whose Barr-bisimilarity is complete.

References

[1] K. G. Larsen and A. Skou, "Bisimulation through probabilistic testing," *Inf. Comput.*, vol. 94, no. 1, pp. 1–28, 1991.

[2] O. Kupferman, U. Sattler, and M. Y. Vardi, "The complexity of the graded μ-calculus," in *Automated Deduction, CADE 02*, ser. LNCS, A. Voronkov, Ed., vol. 2392. Springer, 2002, pp. 423–437. [Online]. Available: http://link.springer.de/link/service/series/0558/bibs/2392/23920423.html

[3] P. Buchholz and P. Kemper, "Model checking for a class of weighted automata," *Discret. Event Dyn. Syst.*, vol. 20, no. 1, pp. 103–137, 2010.

[4] R. Alur, T. A. Henzinger, and O. Kupferman, "Alternating-time temporal logic," *J. ACM*, vol. 49, pp. 672–713, 2002. [Online]. Available: http://doi.acm.org/10.1145/585265.585270

[5] B. F. Chellas, *Modal Logic - An Introduction*. Cambridge University Press, 1980.

[6] D. Peleg, "Concurrent dynamic logic," *J. ACM*, vol. 34, no. 2, pp. 450–479, 1987.

[7] R. Parikh, "Propositional game logic," in *Foundations of Computer Science, FOCS 1983*. IEEE Computer Society, 1983, pp. 195–200.

[8] J. J. M. M. Rutten, "Universal coalgebra: a theory of systems," *Theor. Comput. Sci.*, vol. 249, no. 1, pp. 3–80, 2000.

[9] B. Klin, "Structural operational semantics for weighted transition systems," in *Semantics and Algebraic Specification, Essays Dedicated to Peter D. Mosses on the Occasion of His 60th Birthday*, ser. LNCS, J. Palsberg, Ed., vol. 5700. Springer, 2009, pp. 121–139.

[10] P. Aczel and N. P. Mendler, "A final coalgebra theorem," in *Category Theory and Computer Science, Manchester, UK, September 5-8, 1989, Proceedings*, ser. Lecture Notes in Computer Science, D. H. Pitt, D. E. Rydeheard, P. Dybjer, A. M. Pitts, and A. Poigné, Eds., vol. 389. Springer, 1989, pp. 357–365.

[11] A. Thijs, "Simulation and fixpoint semantics," Ph.D. dissertation, University of Groningen, 1996.

[12] R. C. Backhouse, P. J. de Bruin, P. F. Hoogendijk, G. Malcolm, E. Voermans, and J. van der Woude, "Polynomial relators (extended abstract)," in *Algebraic Methodology and Software Technology, AMAST 1991*, ser. Workshops in Computing, M. Nivat, C. Rattray, T. Rus, and G. Scollo, Eds. Springer, 1991, pp. 303–326.

[13] P. B. Levy, "Similarity quotients as final coalgebras," in *Foundations of Software Science and Computational Structures - 14th International Conference, FOSSACS 2011, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2011, Saarbrücken, Germany, March 26-April 3, 2011. Proceedings*, ser. Lecture Notes in Computer Science, M. Hofmann, Ed., vol. 6604. Springer, 2011, pp. 27–41. [Online]. Available: https://doi.org/10.1007/978-3-642-19805-2_3

[14] M. Barr, "Relational algebras," in *Reports of the Midwest Category Seminar IV*, ser. Lect. Notes Math., no. 137. Springer, 1970, pp. 39–55.

[15] J. Marti and Y. Venema, "Lax extensions of coalgebra functors," in *Coalgebraic Methods in Computer Science, CMCS 2021*, ser. LNCS, D. Pattinson and L. Schröder, Eds., vol. 7399. Springer, 2012, pp. 150–169.

[16] ——, "Lax extensions of coalgebra functors and their logic," *Journal of Computer and System Sciences*, vol. 81, no. 5, pp. 880–900, 2015. [Online]. Available: https://doi.org/10.1016/j.jcss.2014.12.006

[17] V. Trnková, "General theory of relational automata," *Fund. Inform.*, vol. 3, no. 2, pp. 189–233, 1980.

[18] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, "Identity-preserving lax extensions and where to find them," in *Theoretical Aspects of Computer Science, STACS 2025*, ser. LIPIcs, O. Beyersdorff, M. Pilipczuk, E. Pimentel, and N. K. Thang, Eds., vol. 327. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025.

[19] W. H. Hesselink and A. Thijs, "Fixpoint semantics and simulation," *Theor. Comput. Sci.*, vol. 238, no. 1-2, pp. 275–311, 2000. [Online]. Available: https://doi.org/10.1016/S0304-3975(98)00176-5

[20] J. Hughes and B. Jacobs, "Simulations in coalgebra," *Theor. Comput. Sci.*, vol. 327, no. 1-2, pp. 71–108, 2004.

[21] D. Hofmann, G. J. Seal, and W. Tholen, Eds., *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, ser. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, Jul. 2014, vol. 153, authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.

[22] F. Gavazzo, "Quantitative behavioural reasoning for higher-order effectful programs: Applicative distances," in *Logic in Computer Science, LICS 2018*, A. Dawar and E. Grädel, Eds. ACM, 2018, pp. 452–461.

[23] P. Wild and L. Schröder, "Characteristic logics for behavioural metrics via fuzzy lax extensions," in *Concurrency Theory, CONCUR 2020*, ser. LIPIcs, I. Konnov and L. Kovács, Eds., vol. 171. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020, pp. 27:1–27:23.

[24] ——, "Characteristic logics for behavioural hemimetrics via fuzzy lax extensions," *Log. Methods Comput. Sci.*, vol. 18, no. 2, 2022.

[25] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, "A point-free perspective on lax extensions and predicate liftings," *Mathematical Structures in Computer Science*, p. 1–30, 2023.

[26] P. Baldan, F. Bonchi, H. Kerstan, and B. König, "Coalgebraic behavioral metrics," *Log. Methods Comput. Sci.*, vol. 14, no. 3, 2018.

[27] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories: The joy of cats*. John Wiley & Sons Inc., 1990, republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507. [Online]. Available: http://tac.mta.ca/tac/reprints/articles/17/tr17abs.html

[28] J. Riguet, "Relations binaires, fermetures, correspondances de Galois," *Bulletin de la Société Mathématique de France*, vol. 76, pp. 114–155, 1948.

[29] H. P. Gumm and M. Zarrad, "Coalgebraic simulations and congruences," in *Coalgebraic Methods in Computer Science - 12th IFIP WG 1.3 International Workshop, CMCS 2014, Colocated with ETAPS 2014, Grenoble, France, April 5-6, 2014, Revised Selected Papers*, ser. Lecture Notes in Computer Science, M. M. Bonsangue, Ed., vol. 8446. Springer, 2014, pp. 118–134. [Online]. Available: https://doi.org/10.1007/978-3-662-44124-4_7

[30] H. H. Hansen, C. Kupke, and E. Pacuit, "Bisimulation for neighbourhood structures," in *CALCO*, ser. Lecture Notes in Computer Science, vol. 4624. Springer, 2007, pp. 279–293.

[31] M. Barr, "Terminal coalgebras in well-founded set theory," *Theor. Comput. Sci.*, vol. 114, no. 2, pp. 299–315, 1993.

[32] P. Nora, J. Rot, L. Schröder, and P. Wild, "Relational connectors and heterogeneous simulations," in *Foundations of Software Science and Computation Structures - 28th International Conference, FoSSaCS 2025*, D. Kesner and P. A. Abdulla, Eds., 2025, to appear. Preprint available on arXiv under https://doi.org/10.48550/arXiv.2410.14460.

[33] H. P. Gumm and T. Schröder, "Monoid-labeled transition systems," in *Coalgebraic Methods in Computer Science, CMCS 2001*, ser. ENTCS, A. Corradini, M. Lenisa, and U. Montanari, Eds., vol. 44(1). Elsevier, 2001, pp. 185–204.

[34] U. Dorsch, S. Milius, L. Schröder, and T. Wißmann, "Predicate Liftings and Functor Presentations in Coalgebraic Expression Languages," in *Coalgebraic Methods in Computer Science - 14th IFIP WG 1.3 International Workshop, CMCS 2018, Colocated with ETAPS 2018, Thessaloniki, Greece, April 14-15, 2018, Revised Selected Papers*, ser. Lecture Notes in Computer Science, C. Cîrstea, Ed., vol. 11202. Springer, 2018, pp. 56–77. [Online]. Available: https://doi.org/10.1007/978-3-030-00389-0_5

[35] C. Schubert and G. J. Seal, "Extensions in the theory of lax algebras," *Theory and Applications of Categories*, vol. 21, no. 7, pp. 118–151, 2008.

[36] H. H. Hansen and C. Kupke, "A coalgebraic perspective on monotone modal logic," in *Coalgebraic Methods in Computer Science, CMCS 2004*, ser. ENTCS, J. Adámek and S. Milius, Eds., vol. 106. Elsevier, 2004, pp. 121–143.

[37] V. Trnková, “Some properties of set functors,” *Commentationes Mathematicae Universitatis Carolinae*, vol. 010, no. 2, pp. 323–352, 1969. [Online]. Available: http://eudml.org/doc/16330

Appendix: Omitted Proof Details

A. Useful pullbacks

Lemma A.1. *Let $f\colon X \to A$ and $g\colon Y \to A$. Then, the following diagram is a pullback square*

$$\begin{array}{ccc} \mathsf{dom}(r) & \xrightarrow{f|_{\mathsf{dom}(r)}} & g[Y] \\ \downarrow & \lrcorner & \downarrow j \\ X & \xrightarrow[f]{} & A. \end{array}$$

where $r = g^\circ \cdot f$ and $j\colon g[Y] \rightarrowtail A$ is the obvious inclusion.

Proof. Noting the image factorization $Y \to g[Y] \xrightarrow{j} A$ of g, we form the pullback

$$\begin{array}{ccc} f^{-1}[g[Y]] & \xrightarrow{f|_{f^{-1}[g[Y]]}} & g[Y] \\ \downarrow & \lrcorner & \downarrow j \\ X & \xrightarrow[f]{} & A \end{array}$$

We are left to show that $f^{-1}[g[Y]] = \mathsf{dom}(r)$. Indeed, $\mathsf{dom}(r) = \mathsf{img}(r^\circ) = \mathsf{img}((g^\circ \cdot f)^\circ) = \mathsf{img}(f^\circ \cdot g) = f^{-1}[g[Y]]$. □

Lemma A.2 (e.g. [18][Lemma 2.2)**.** *] Let $r\colon X \nrightarrow Y$ be a relation. Then the following are equivalent:*

(i) r is difunctional;

(ii) for every span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ such that $r = \pi_2 \cdot \pi_1^\circ$, the pushout square

$$\begin{array}{ccc} R & \xrightarrow{\pi_2} & Y \\ \pi_1 \downarrow & \ulcorner & \downarrow p_2 \\ X & \xrightarrow[p_1]{} & O \end{array}$$

is a weak pullback.

B. Proof of Proposition III.6

The clauses 1, 2 and the first inequality of 3 follow straightforwardly from the definition of coBarr relator. To see the second inequality of 3, let R be a normal relational connector and let $r\colon X \nrightarrow Y$ be a relation. Furthermore let $\hat{r}$ be the difunctional closure of r and $f\colon X \to O$ and $g\colon Y \to O$ $\hat{r}\colon X \nrightarrow Y$ be maps s.t. $\hat{r} = g^\circ \cdot f$. Then, as R is a normal relational connector, $\mathsf{R}(r) \leq (\mathsf{F}g)^\circ \cdot \mathsf{F}f = \underline{\mathsf{F}}r$. On the other hand, let $p\colon A \to X$ and $q\colon A \to Y$ be maps s.t. $r = q \cdot p^\circ$. Then, $1_A \leq q^\circ \cdot r \cdot p$. Hence, as R is a normal relational connector $1_{\mathsf{F}A} \leq (\mathsf{F}q)^\circ \cdot \mathsf{L}(r) \cdot \mathsf{F}p$. Therefore, by adjointness, $\mathsf{F}q \cdot (\mathsf{F}p)^\circ \leq \mathsf{R}r$. □

C. Details for Example III.14

We prove the claim that a polynomial functor is ζ-bounded if it is not an exponential functor.

Let $\Sigma(\mathcal{H})\colon \mathsf{Set} \to \mathsf{Set}$ be a polynomial functor, with $\mathcal{H} = (\mathsf{H}_{A_s})_{s \in S}$, for some set S. If $S = \emptyset$, then $\Sigma(\mathcal{H})$ is a constant functor, hence ζ-bounded. If $|S| = 1$, then $\Sigma(\mathcal{H})$ is an exponential functor. Thus, suppose that $|S| \geq 2$. Moreover, if all A_s are empty, then $\Sigma(\mathcal{H})$ is a constant functor, hence ζ-bounded; so suppose that there is $s \in S$ such that $A_s \neq \varnothing$. Let Z be the carrier of a terminal coalgebra for $\Sigma(\mathcal{H})$, which is well-known to exist (e.g. [8]). Now for all $s \in S$ and $X \in \mathsf{Set}$, $|1 + \mathsf{H}_{A_s}(X)| \leq |\Sigma(\mathcal{H})(X)|$. Thus, for every $s \in S$, the cardinality of Z is greater or equal than the cardinality of the terminal coalgebra for $1 + \mathsf{H}_{A_s}$, which is greater or equal than the cardinality of A_s and it is infinite for A_s non-empty (cf. [8, Example 10.2(6)]). In particular, this means that the cardinality of Z is infinite.

Now, let $u, v \in \Sigma(\mathcal{H})(X)$ for some set X. We claim that there is a set Y and an injective map $i\colon Y \rightarrowtail X$ s.t. $u, v \in \mathsf{img}(\Sigma(\mathcal{H})i)$ and $|Y| \leq |Z|$. By definition of $\Sigma(\mathcal{H})$ there are $s, t \in S$ and $u' \in \mathsf{H}_{A_s}(X)$, $v' \in \mathsf{H}_{A_s}(X)$ s.t $u = c^s_X(u')$ and $v = c^t_X(v')$. Let Y be the set $\mathsf{img}(u') \cup \mathsf{img}(v')$, and let $i\colon Y \rightarrowtail X$ be the inclusion of Y into X. Then, as $|Z|$ is infinite and $|\mathsf{img}(u')| \leq |A_s| \leq |Z|$ and $|\mathsf{img}(v')| \leq |A_t| \leq |Z|$, $|Y| \leq |Z|$. Furthermore, as u' and v' corestrict to Y and $c^s\colon \mathsf{H}_{A_s} \to \Sigma(\mathcal{H})$ and $c^t\colon \mathsf{H}_{A_j} \to \Sigma(\mathcal{H})$ are natural transformations, it follows that $u, v \in \mathsf{img}(\Sigma(\mathcal{H})i)$. □

D. *Proof of Proposition IV.5*

Let L be the pointwise supremum of the L_i. Let $r\colon X \nrightarrow Y$ be a relation, and let $s\colon X \nrightarrow X$ be a subidentity. Since normal lax extensions coincide on difunctional relations (e.g. [16], [21]), in particular on subidentities, all L_i and, hence L map s to the same relation $\bar{s}$, which by normality of the L_i is a subidentity.

'If:' Since relational composition preserves suprema, $\mathsf{L}(r\cdot s) = \bigvee_{i\in\mathcal{I}} \mathsf{L}_i(r\cdot s) = \bigvee_{i\in\mathcal{I}}(\mathsf{L}_i r\cdot\bar{s}) = (\bigvee_{i\in\mathcal{I}} \mathsf{L}_i r)\cdot\bar{s} = \mathsf{L}r\cdot\bar{s} = \mathsf{L}r\cdot\mathsf{L}s$. The case of postcomposition with subidentities runs analogously.

'Only if:' We show that $\mathsf{L}_i(r\cdot s) \leq \mathsf{L}_i r\cdot \mathsf{L}_i s = \mathsf{L}_i r\cdot\bar{s}$; the other inequality holds by the definition of lax extension. So let $a\in\mathsf{F}X$ and $b\in\mathsf{F}Y$ such that $a\ \mathsf{L}_i(r\cdot s)\ b$. We claim that $a\ \mathsf{L}_i r\ b$ and $a\ \bar{s}\ a$. Indeed, as s is a subidentity, $r\cdot s\leq r$, whence $a\ \mathsf{L}_i r\ b$.

Moreover, $a\ \mathsf{L}(r\cdot s)\ b$ because L_i is below L. Since L preserves composition with subidentities, we thus have c such that $a\ \bar{s}\ c$ and $c\ Lr\ b$. But then $a = c$ because $\bar{s}$ is a subidentity. The case of postcomposition with a subidentity runs analogously. $\square$

E. *Proof of Lemma IV.6*

1) Let $r\colon X\nrightarrow Y$ be a relation. Consider a span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ such that $r = \pi_2\cdot\pi_1^\circ$. Then, π_1 factors as $e\cdot\mathsf{d}_r$, with $e\colon R\twoheadrightarrow \mathsf{dom}(r)$ and $\mathsf{d}_r\colon \mathsf{dom}(r)\rightarrowtail X$. Hence, $\mathsf{F}\pi_2\cdot(\mathsf{F}e)^\circ\cdot(\mathsf{F}\,\mathsf{d}_r)^\circ\leq \mathsf{L}r$. Therefore, as every set functor preserves epimorphisms, $\mathsf{img}(\mathsf{F}\,\mathsf{d}_r)\subseteq\mathsf{dom}(\mathsf{L}r)$.

2) Let $X\xrightarrow{f} Y \xleftarrow{i} B$ be a cospan in Set. Then, as L is a normal relational connector, $\mathsf{L}((g^\circ\cdot f)^\circ) = \mathsf{L}(f^\circ\cdot g) = (\mathsf{F}f)^\circ\cdot\mathsf{F}g = (\mathsf{L}(g^\circ\cdot f))^\circ$.

3) Let $i\colon A\rightarrowtail X$ and $j\colon A\rightarrowtail Y$ be injective maps. Then, $j\cdot i^\circ\colon X\nrightarrow Y$ is difunctional. Hence, by Lemma A.2, the pushout square of i along j

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle p_2} \\ X & \underset{p_1}{\rightarrowtail} & O \end{array}$$

is a pullback square, in fact it is the square of an intersection. Since F preserves empty intersections by hypothesis, then it preserves all intersections [37, Proposition 2.1]. Thus, as L is a normal relational connector, we obtain $\mathsf{L}(j\cdot i^\circ) = \mathsf{L}(p_2^\circ\cdot p_1) = (\mathsf{F}p_2)^\circ\cdot\mathsf{F}p_1 = \mathsf{F}j\cdot(\mathsf{F}i)^\circ$.

F. *Proof of Lemma IV.7*

(i) $\Rightarrow$ (ii). Suppose that F preserves inverse images and let L be a normal lax extension of F. Then, by [18], F has a normal lax extension. Now, suppose that $r = g^\circ\cdot f$ for some cospan $X\xrightarrow{f} A \xleftarrow{g} Y$. Then, by Lemma A.1 we have a commutative diagram

$$\begin{array}{ccccc} \mathsf{dom}(r) & \xrightarrow{f|_{\mathsf{dom}(r)}} & g[Y] & \overset{e}{\twoheadleftarrow} & Y \\ {\scriptstyle \mathsf{d}_r}\downarrow & \lrcorner & \downarrow{\scriptstyle j} & \swarrow{\scriptstyle g} & \\ X & \underset{f}{\longrightarrow} & A, & & \end{array}$$

in which the square is a pullback and $g = j\cdot e$ is the image factorization of g. Since F preserves inverse images it preserves monomorphisms, and every set functor preserves epimorphisms, so $\mathsf{F}j\cdot\mathsf{F}e$ is an epi-mono factorization of $\mathsf{F}g$. This means that $\mathsf{F}j\colon\mathsf{F}(g[Y])\rightarrowtail\mathsf{F}A$ corestricts to an isomorphism $h\colon\mathsf{F}(g[Y])\cong\mathsf{F}g[FY]$. Furthermore, as L is a normal relational connector, $\mathsf{L}r = (\mathsf{F}g)^\circ\cdot\mathsf{F}f$. Thus, by Lemma A.1 and Lemma IV.6(1) we obtain the following diagram where the outer square is a pullback since F preserves inverse images by hypothesis.

$$\begin{array}{ccc} \mathsf{F}\,\mathsf{dom}(r) & \xrightarrow{\mathsf{F}(f_{|\,\mathsf{dom}(r)})} & \mathsf{F}g[Y] \\ \vdots\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ \mathsf{dom}(\mathsf{L}r) & \xrightarrow{\mathsf{F}f_{(\mathsf{dom}(\mathsf{L}r)}} & \mathsf{F}g[\mathsf{F}Y] \\ \downarrow{\scriptstyle d_{\mathsf{L}r}} & \lrcorner & \downarrow{\scriptstyle k} \\ \mathsf{F}X & \underset{\mathsf{F}f}{\longrightarrow} & \mathsf{F}A. \end{array} \qquad (\mathsf{F}\,\mathsf{d}_r\colon \mathsf{F}\,\mathsf{dom}(r)\to\mathsf{F}X,\ \ \mathsf{F}j\colon \mathsf{F}g[Y]\to\mathsf{F}A)$$

Furthermore, as k is mono and the bottom square and the triangles commute, the top square also commutes. Hence, as h is an isomorphism, we conclude that $(\mathsf{F}\,\mathsf{d}_r, h\cdot\mathsf{F}(f_{|\,\mathsf{dom}(r)}))$ is a pullback of $(\mathsf{F}f, k)$. Therefore, as pullbacks are unique, $\mathsf{F}\,\mathsf{d}_r$ corestricts to an isomorphism between $\mathsf{F}\,\mathsf{dom}(r)$ and $\mathsf{dom}(\mathsf{L}r)$.

(ii) $\Rightarrow$ (iii). Note that the codomain of a relation is the domain of its converse and by Lemma IV.6(2) every normal lax extension preserves converses of difunctional relations. Therefore, the claim follows from (ii) by duality.

(iii) $\Rightarrow$ (iv). We show the case of precomposition with the converse of an injective map, the other case follows analogously. Let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a normal lax extension of F, and let $r\colon X \nrightarrow Y$ be a relation and $j\colon X \to A$ be an injective map. It suffices to show that $\mathsf{L}(r \cdot i^\circ) \leq \mathsf{L}r \cdot (\mathsf{F}i)^\circ$ since the other inequality holds by definition of lax extension. We begin by observing that as $i \cdot i^\circ = \mathsf{d}_{i^\circ} \cdot {\mathsf{d}_{i^\circ}}^\circ$, $\mathsf{F}i \cdot (\mathsf{F}i)^\circ = \mathsf{F}\,\mathsf{d}_{i^\circ} \cdot (\mathsf{F}\,\mathsf{d}_{i^\circ})^\circ$.

Furthermore, as i is injective, $\mathsf{L}r = \mathsf{L}(r \cdot i^\circ \cdot i) = \mathsf{L}(r \cdot i^\circ) \cdot \mathsf{F}i$. Hence, the claim follows once we show $\mathsf{L}(r \cdot i^\circ) \leq \mathsf{L}(r \cdot i^\circ) \cdot \mathsf{F}i \cdot (\mathsf{F}i)^\circ = \mathsf{L}(r \cdot i^\circ) \cdot \mathsf{F}\,\mathsf{d}_{i^\circ} \cdot (\mathsf{F}\,\mathsf{d}_{i^\circ)})^\circ$ which is equivalent to showing $\mathsf{dom}(\mathsf{L}(r \cdot i^\circ)) \subseteq \mathsf{img}(\mathsf{F}\,\mathsf{d}_{i^\circ})$. To see this note that by hypothesis $\mathsf{F}d_{r \cdot i^\circ}\colon \mathsf{F}(\mathsf{dom}(r \cdot i^\circ)) \to \mathsf{dom}(\mathsf{L}(r \cdot i^\circ))$ corestricts to an isomorphism $h\colon \mathsf{F}(\mathsf{dom}(r \cdot i^\circ)) \cong \mathsf{dom}(\mathsf{L}(r \cdot i^\circ))$. Furthermore, as $\mathsf{dom}(r \cdot i^\circ) \subseteq \mathsf{dom}(i^\circ)$, we have $\mathsf{d}_{r \cdot i^\circ} = \mathsf{d}_{i^\circ} \cdot k$, where k denotes the inclusion $\mathsf{dom}(r \cdot i^\circ) \rightarrowtail \mathsf{dom}(i^\circ)$. Hence, the inclusion $\mathsf{dom}(\mathsf{L}(r \cdot i^\circ)) \rightarrowtail \mathsf{F}X$ factors as $\mathsf{F}\,\mathsf{d}_{i^\circ} \cdot \mathsf{F}k \cdot h^{-1}$. Therefore, $\mathsf{dom}(\mathsf{L}(r \cdot i^\circ)) \subseteq \mathsf{img}(\mathsf{F}\,\mathsf{d}_{i^\circ})$.

(iv) $\Rightarrow$ (v). Trivial.

(v) $\Rightarrow$ (i). Let $X \xrightarrow{f} Y \xleftarrow{i} B$ be a cospan in Set with i injective and $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a normal lax extension of F that satisfies the condition required in (v). Consider a pullback (p_1, p_2) of (f, i). Then, since pullbacks reflect monomorphisms, p_1 is injective. Hence, as L is normal, by hypothesis we obtain $(\mathsf{F}i)^\circ \cdot \mathsf{F}f = \mathsf{L}(i^\circ \cdot f) = \mathsf{L}(p_2 \cdot p_1^\circ) = \mathsf{F}p_2 \cdot (\mathsf{F}p_1)^\circ$. Therefore, $(\mathsf{F}p_1, \mathsf{F}p_2)$ is a weak pullback of $(\mathsf{F}j, \mathsf{F}i)$ and it is in fact a pullback since $\mathsf{F}p_1$ is injective because F preserve injective maps given that it admits a normal lax extension [18]. $\square$