Identity-Preserving Lax Extensions and Where to Find Them

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Abstract

Generic notions of bisimulation for various types of systems (nondeterministic, probabilistic, weighted etc.) rely on identity-preserving (*normal*) lax extensions of the functor encapsulating the system type, in the paradigm of universal coalgebra. It is known that preservation of weak pullbacks is a sufficient condition for a functor to admit a normal lax extension (the Barr extension, which in fact is then even strict); in the converse direction, nothing is currently known about necessary (weak) pullback preservation conditions for the existence of normal lax extensions. In the present work, we narrow this gap by showing on the one hand that functors admitting a normal lax extension preserve *1/4-iso* pullbacks, i.e. pullbacks in which at least one of the projections is an isomorphism. On the other hand, we give sufficient conditions, showing that a functor admits a normal lax extension if it weakly preserves either 1/4-iso pullbacks and 4/4-epi pullbacks (i.e. pullbacks in which all morphisms are epic) or inverse images. We apply these criteria to concrete examples, in particular to functors modelling neighbourhood systems and weighted systems.

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1 Introduction

Branching-time notions of behavioural equivalence of reactive systems are typically cast as notions of *bisimilarity*, which in turn are based on notions of *bisimulation*, the paradigmatic example being Park-Milner bisimilarity on labelled transition systems [29]. A key point about this setup is that while bisimilarity is an equivalence on states, individual bisimulations can be much smaller than the full bisimilarity relation, and in particular need not themselves be equivalence relations. In a perspective where one views bisimulations as certificates for bisimilarity, this feature enables smaller certificates.

The concept of bisimilarity via bisimulations can be transferred to many system types beyond basic labelled transition systems, such as monotone neighbourhood systems [20], probabilistic transition systems, or weighted transition systems. In fact, such systems can be treated uniformly within the framework of universal coalgebra [34], in which the system type is encapsulated in the choice of a set functor (the powerset functor for non-deterministic branching, the distribution functor for probabilistic branching etc.). Coalgebraic notions of bisimulation were originally limited to functors that preserve weak pullbacks [34], equivalently admit a strictly functorial extension to the category of relations [5, 40]. They were later extended to functors admitting an *identity-preserving* or *normal lax extension* [27, 28] to the category of relations (this is essentially equivalent to notions of bisimilarity based on modal logic [14]). While there is currently no formal definition of what a notion of bisimulation constitutes except via normal lax extensions, there is a reasonable claim that notions of bisimulation in the proper sense, i.e. with bisimulations not required to be equivalence relations, will not go beyond functors admitting a normal lax extension.

The *Barr extension* that underlies the original notion of coalgebraic bisimulation for weak-pullback-preserving functors [34] is, in particular, a normal lax extension; that is, preservation of weak pullbacks is sufficient for existence of a normal lax extension. However, this condition is far from being necessary; there are numerous functors that fail to preserve weak pullbacks but do admit a normal lax extension, such as the monotone neighbourhood functor [27, 28]. It has been shown that a finitary functor admits a normal lax extension if and only if it admits a separating set of finitary monotone modalities [27, 28] (a similar result holds for unrestricted functors if one considers class-sized collections of infinitary modalities [13]). The latter condition amounts to existence of an expressive modal logic that has monotone modalities [31, 35], and as such admits μ-calculus-style fixpoint extensions [10]. In a nutshell, a system type admits a good notion of bisimulation if and only if it admits an expressive temporal logic. The characterization via sets of predicate liftings, however, is often similarly elusive in that it demands the construction of a fairly complicated object.

In the present work, we narrow the gap between weak pullback preservation as a sufficient condition for admitting a normal lax extension, and no known necessary pullback preservation condition. On the one hand, we establish a necessary preservation condition, showing that functors admitting a normal lax extension (weakly) preserve *1/4-iso pullbacks*, i.e. pullbacks in which at least one of the projections is isomorphic. (We often put 'weakly' in brackets because for many of the pullback types we consider, notably for inverse images and 1/4-iso pullbacks, weak preservation coincides with preservation.) This is a quite natural condition: A key role in the field is played by *difunctional relations* [33], which may be thought of as relations obtained by chopping the domain of an equivalence in half; for instance, given labelled transition systems X, Y, the bisimilarity relation from X to Y is difunctional. In a nutshell, we show that a functor preserves 1/4-iso pullbacks iff it acts in a well-defined and monotone manner on difunctional relations. A first application of this necessary condition is a very quick proof of the known fact that the neighbourhood functor does not admit a normal lax extension [28].

We then go on to establish two separate sets of sufficient conditions: We show that a functor admits a normal lax extension if it (weakly) preserves either inverse images or 1/4-iso pullbacks and *4/4-epi pullbacks*, i.e. pullbacks in which all morphisms are epi (these are also known as *surjective pullbacks* [38], and weak preservation of 4/4-epi pullbacks is equivalent to weak preservation of kernel pairs [15]). These sufficient conditions are technically substantially more involved. As indicated above, they imply that finitary functors (weakly) preserving either inverse images or 1/4-iso pullbacks and 4/4-epi pullbacks admit a separating set of finitary modalities; this generalizes a previous result showing the same for functors preserving all weak pullbacks [25]. We summarize our main contributions in Figure 1.

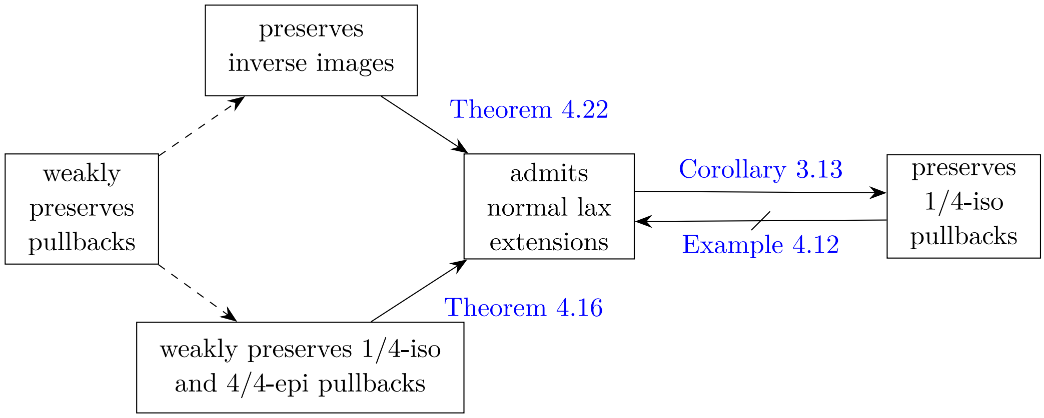


Figure 1 Summary of main results. Solid arrows are present contributions, dashed arrows are trivial. All implications indicated by arrows are non-reversible; in particular, Example 4.12 shows this for Corollary 3.13.

The criterion of weak preservation of 1/4-iso pullbacks and 4/4-epi pullbacks is satisfied by the monotone neighbourhood functor and generalizations thereof (e.g. [38]), and thus in particular reproves the above-mentioned known fact that functors admitting separating sets of monotone modalities have normal lax extensions. The criterion of (weak) preservation of inverse images, in connection with the necessary criterion, implies that a monoid-valued functor for a commutative monoid M (whose coalgebras are M-weighted transition systems) admits a normal lax extension if and only if M is positive (which in turn is equivalent to the functor preserving inverse images [17]).

Related work With variations in the axiomatics and terminology, lax extensions go back to an extended strand of work on relation liftings (e.g. [3, 39, 21, 26, 37, 36]). We have already mentioned work by Marti and Venema relating lax extensions to modal logic [27, 28]; at the same time, Marti and Venema prove that the notion of bisimulation induced by a normal lax extension captures the standard notion of behavioural equivalence. *Lax relation liftings*, constructed for functors carrying a coherent order structure [23], also serve the study of coalgebraic simulation but obey a different axiomatics than lax extensions [28, Remark 4]). Strictly functorial extensions of set functors to the category of sets and relations are known to be unique when they exist, and exist if and only if the functor preserves weak pullbacks [7, 40]; this has been extended to other base categories [3, 8]. There has been recent interest in quantitative notions of lax extensions that act on relations taking values in a quantale, such as the unit interval, in particular with a view to obtaining notions of quantitative bisimulation [22, 12, 42, 43, 13] that witness low behavioural distance (the latter having first been treated in coalgebraic generality by Baldan et al. [4]). The correspondence between normal lax extensions and separating sets of modalities generalizes to the quantitative setting [42, 43, 13].

Organization We review material on relations, in particular difunctional relations, and lax extensions in Section 2. In Section 3, we introduce our necessary pullback preservation condition and show that it characterizes well-definedness of the natural functor action on difunctional relations. We prove our main results in Section 4. In Subsection 4.1 we show that a functor that weakly preserves 1/4-iso pullbacks and 4/4-epi pullbacks admits a normal lax extension, and in Subsection 4.2 we show the same for functors that preserve 1/4-mono pullbacks.

2 Preliminaries: Relations and Lax Extensions

We work in the category Set of sets and functions throughout. We assume basic familiarity with category theory (e.g. [2]). A central role in the development is played by (weak) pullbacks: A commutative square $f \cdot p = g \cdot q$ is a pullback (of f, g) if for every competing square $f \cdot p' = g \cdot q'$, there exists a unique morphism k such that $p \cdot k = p'$ and $q \cdot k = q'$; the notion of weak pullback is defined in the same way except that k is not required to be unique. A functor F *weakly preserves* a given pullback if it maps the pullback to a weak pullback; it is known that weak preservation of pullbacks of a given type is equivalent to preservation of weak pullbacks of the same type [16, Corollary 4.4]. Our interest in functors $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ is driven mainly by their role as encapsulating types of transition systems in the paradigm of *universal coalgebra* [34]: An F-***coalgebra*** (X, α) consists of a set X of ***states*** and a ***transition map*** $\alpha\colon X \to \mathsf{F}X$ assigning to each state $x \in X$ a collection $\alpha(x)$ of successors, structured according to F. For instance, coalgebras for the ***powerset functor*** $\mathcal{P}$ assign to each state a *set* of successors, and hence are just standard relational transition systems, while coalgebras for the ***distribution functor*** $\mathcal{D}$ (which maps a set X to the set of discrete probability distributions on X) assign to each state a distribution on successor states, and are thus Markov chains.

A ***morphism*** $f\colon (X, \alpha) \to (Y, \beta)$ of F-coalgebras is a map $f\colon X \to Y$ for which $\beta \cdot f = \mathsf{F}f \cdot \alpha$. Such morphisms are thought of as preserving the behaviour of states, and correspondingly, states x and y in coalgebras (X, α) and (Y, β), respectively, are ***behaviourally equivalent*** if there exist a coalgebra (Z, γ) and morphisms $f\colon (X, \alpha) \to (Z, \gamma)$, $g\colon (Y, \beta) \to (Z, \gamma)$ such that $f(x) = g(y)$. One is then interested in notions of bisimulation relation that characterize behavioural equivalence in the sense that two states are behaviourally equivalent iff they are related by some bisimulation [34, 28]; this motivates the detailed study of relations and their liftings along F.

We write $r\colon X \nrightarrow Y$ to indicate that r is a relation from the set X to the set Y (i.e. $r \subseteq X \times Y$), and we write $x\ r\ y$ when $(x, y) \in r$. Both for functions and for relations, we use *applicative* composition, i.e. given $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$, their composite is $s \cdot r\colon X \nrightarrow Z$ (defined as $s \cdot r = \{(x, z) \mid \exists y \in Y.\, x\ r\ y\ s\ z\}$). We say that r, s of type $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$ are ***composable***, and we extend this terminology to sequences of relations in the obvious manner. Relations between the same sets are ordered by inclusion, that is $r \leq r' \iff r \subseteq r'$. We denote by $1_X\colon X \to X$ the identity map (hence relation) on X, and we say that a relation $r\colon X \nrightarrow X$ is a ***subidentity*** if $r \leq 1_X$. Given a relation $r\colon X \nrightarrow Y$,

$r^\circ\colon Y \nrightarrow X$ denotes the corresponding converse relation; in particular, if $f\colon X \to Y$ is a function, then $f^\circ\colon Y \nrightarrow X$ denotes the converse of the corresponding relation. For a relation $r\colon X \nrightarrow Y$, we denote by $\mathsf{dom}\, r \subseteq X$ and $\mathsf{cod}\, r \subseteq Y$ the respective domain and codomain (i.e. $\mathsf{dom}\, r = \{x \in X \mid \exists y \in Y.\, x\ r\ y\}$ and $\mathsf{cod}\, r = \{y \in Y \mid \exists x \in X.\, x\ r\ y\}$). A special class of relations of interest are ***difunctional relations*** [33], which are relations factorizable as $g^\circ \cdot f$ for some functions $f\colon X \to Z$ and $g\colon Y \to Z$, i.e. $x\ r\ y$ iff $f(x) = g(y)$. In the following we record some folklore facts about difunctional relations.

▶ **Lemma 2.1.** *Let $r\colon X \nrightarrow Y$ be a relation. Then the following are equivalent:*

(i) r is difunctional;
(ii) for all x_1, x_2 in X and $y_1, y_2 \in Y$, if $x_1\ r\ y\ r^\circ\ x_2\ r\ y_2$, then $x_1\ r\ y_2$.
(iii) for every span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ such that $r = \pi_2 \cdot \pi_1^\circ$, the pushout square

$$\begin{array}{ccc} R & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & \ulcorner & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow[p_1]{} & O \end{array}$$

is a weak pullback.

As we can see in Lemma 2.1(iii) above, difunctional relations are characterized as weak pullbacks, and in this regard we recall that generally, a commutative square $f \cdot p = g \cdot q$ is a weak pullback iff $q \cdot p^\circ = g^\circ \cdot f$, equivalently $p \cdot q^\circ = f^\circ \cdot g$.

The ***difunctional closure*** of a relation $r\colon X \nrightarrow Y$ is the least difunctional relation $\hat{r}\colon X \nrightarrow Y$ greater than or equal to r. It follows from Lemma 2.1 that the difunctional closure of a relation $r\colon X \nrightarrow Y$ given by a span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ is obtained by computing its pushout $X \xrightarrow{p_1} O \xleftarrow{p_2} Y$; i.e., the difunctional closure $\hat{r}$ of r is the relation $p_2^\circ \cdot p_1$. More explicitly, $\hat{r} = \bigvee_{n \in \mathbb{N}} r \cdot (r^\circ \cdot r)^n$ (e.g. [33, 19]). A ***lax extension*** L of an endofunctor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ is a mapping that sends any relation $r\colon X \nrightarrow Y$ to a relation $\mathsf{L}r\colon \mathsf{F}X \nrightarrow \mathsf{F}Y$ in such a way that

(L1) $r \le r' \implies \mathsf{L}r \le \mathsf{L}r'$,
(L2) $\mathsf{L}s \cdot \mathsf{L}r \le \mathsf{L}(s \cdot r)$,
(L3) $\mathsf{F}f \le \mathsf{L}f$ and $(\mathsf{F}f)^\circ \le \mathsf{L}f^\circ$,

for all $r\colon X \nrightarrow Y$, $s\colon Y \nrightarrow Z$ and $f\colon X \to Y$.

We define ***relax extensions*** in the same way, without however requiring property 2. We call a (re)lax extension ***identity-preserving***, or ***normal***, if $\mathsf{L}1_X = 1_{\mathsf{F}X}$ for every set X, and we say that a (re)lax extension ***preserves converses*** if $\mathsf{L}(r^\circ) = (\mathsf{L}r)^\circ$. A tactical advantage of using the term "relax extension" is that we can thus refer to constructions that produce lax extensions most of the time, except for some cases when 2 may fail. A prototypical example of this sort is the ***Barr extension*** $\overline{\mathsf{F}}$ [6], which for weak-pullback-preserving F is even a strict extension, and is defined as follows. Given a relation $r\colon X \nrightarrow Y$, choose a factorization $\pi_2 \cdot \pi_1^\circ$ for some span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ and put $\overline{\mathsf{F}}r = \mathsf{F}\pi_2 \cdot (\mathsf{F}\pi_1)^\circ$. This assignment is independent of the factorization of r, and r admits a ***canonical factorization*** which is given by projecting into X and Y the subset of $X \times Y$ of pairs of elements related by r. It is well-known that for every Set-functor, the Barr extension is a normal relax extension, but it is a lax extension precisely when F preserves weak pullbacks [24].

In this case, the Barr extension is also the least lax extension of F, for it follows from 2–2 that $\mathsf{F}\pi_2 \cdot (\mathsf{F}\pi_1)^\circ \le \mathsf{L}r$ for every lax extension L. Lax extensions have been used extensively to treat the notion of bisimulation coalgebraically (e.g. [21, 26, 28]).

Given a lax extension $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ of a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$, an L-***simulation*** between F-coalgebras (X, α) and (Y, β) is a relation $s\colon X \nrightarrow Y$ such that $\beta \cdot s \leq \mathsf{L}s \cdot \alpha$. If L preserves converse, then L-simulations are more suitably called L-***bisimulations***. Between two given coalgebras, there is a greatest L-(bi)simulation, which is termed L-***(bi)similarity***. It has been shown [28] that if L is normal and preserves converses, then L-bisimilarity coincides with coalgebraic behavioural equivalence as recalled above.

▶ Remark 2.2. As mentioned in the introduction, a functor F admits a normal lax extension iff F admits a separating class of monotone predicate liftings [28, 13]. For readability, we discuss only the case where both the functor and the predicate liftings are finitary [28]. An *n-ary predicate lifting* λ for F is a natural transformation of type $\lambda\colon \mathcal{Q}^n \to \mathcal{Q} \cdot \mathsf{F}^{\mathrm{op}}$ where $\mathcal{Q}$ denotes the contravariant powerset functor; that is, for a set X, λ_X lifts n predicates on X to a predicate on $\mathsf{F}X$. Predicate liftings determine modalities in coalgebraic modal logic [31, 35]; a basic example is the unary predicate lifting λ for the (covariant) powerset functor $\mathcal{P}$ given by $\lambda_X(A) = \{B \in \mathcal{P}X \mid B \subseteq A\}$ for a predicate $A \subseteq X$, which determines the standard box modality on $\mathcal{P}$-coalgebras, i.e. on standard relational transition systems. A set of predicate liftings is *separating* if distinct elements of $\mathsf{F}X$ can be separated by lifted predicates; this condition ensures that the associated instance of coalgebraic modal logic is *expressive*, i.e. separates behaviourally inequivalent states [31, 35].

Monotonicity of predicate liftings allows the definition of modal fixpoint logics for temporal specification [10]. In the mentioned correspondence between lax extensions and predicate liftings, the construction of predicate liftings from a lax extension L roughly speaking involves application of L to the elementhood relation.

3 Functor Actions on Difunctional Relations

Our pullback preservation criterion for existence of normal lax extensions grows from an analysis of how functors act on difunctional relations. To start off, it is well-known that normal lax extensions of a given Set-functor are given on difunctional relations by the action of the functor (e.g. [28, 22]):

▶ **Proposition 3.1.** *Let L be an assignment of relations $\mathsf{L}r\colon \mathsf{F}X \nrightarrow \mathsf{F}Y$ to relations $r\colon X \nrightarrow Y$ that satisfies 2, 2 as well as $\mathsf{L}1_X \leq 1_{\mathsf{F}X}$ for all $X \in \mathsf{Set}$. Then L is a lax extension of F (it is then normal) iff for all functions $f\colon W \to X$, $g\colon Z \to Y$ and relations $r\colon X \nrightarrow Y$, $\mathsf{L}(g^\circ \cdot r \cdot f) = (\mathsf{F}g)^\circ \cdot \mathsf{L}r \cdot \mathsf{F}f$.*

▶ **Corollary 3.2.** *All normal lax extensions of a given Set-functor coincide on difunctional relations. Specifically, for every normal lax extension L of $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$, $\mathsf{L}(g^\circ \cdot f) = \mathsf{F}g^\circ \cdot \mathsf{F}f$ for all $f\colon X \to A$ and $g\colon Y \to A$.*

Therefore, a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ that admits at least one normal lax extension must be ***monotone on difunctional relations*** in the following sense: for all difunctional relations $g^\circ \cdot f\colon X \nrightarrow Y$ and $g'^\circ \cdot f'\colon X \nrightarrow Y$, if $g^\circ \cdot f \leq g'^\circ \cdot f'$ then $(\mathsf{F}g)^\circ \cdot \mathsf{F}f \leq (\mathsf{F}g')^\circ \cdot \mathsf{F}f'$. This property no longer mentions lax extensions, and implies that the functor is ***well-defined on difunctional relations***, i.e. that F sends cospans that determine the same difunctional relation to cospans that determine the same difunctional relation. In this section, we show that being monotone on difunctional relations is equivalent to preserving 1/4-iso (2/4-mono) pullbacks in the sense defined next; as indicated in the introduction, this allows for a quick proof of the fact that the neighbourhood functor fails to admit a normal lax extension [28].

▶ **Definition 3.3.** *We say that a functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *preserves* **1/4-iso 2/4-mono pullbacks**, **1/4-iso pullbacks**, **1/4-mono pullbacks** *and* **inverse images** *if it sends pullbacks of the following forms, respectively, to pullbacks, with arrows* $\rightarrowtail$ *and* $\xrightarrow{\simeq}$ *indicating injectivity and bijectivity correspondingly.*

$$\begin{tikzcd} P \arrow[r,"\simeq"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\ X \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r,"\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ X \arrow[r] & Y. \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\ X \arrow[r] & Y \end{tikzcd}$$

▶ Remark 3.4. 1/4-Iso 2/4-mono pullbacks are special inverse images, characterized by the property that the fibre over every element in the image of the function $B \rightarrowtail Y$ is a singleton. In particular, the inverse image of the empty subset is a 1/4-iso 2/4-mono pullback.

Due to the following proposition, for consistency, we tend to use "preservation of 1/4-mono pullbacks" instead of "preservation of inverse images".

▶ **Proposition 3.5.** *A* Set*-functor preserves 1/4-mono pullbacks iff it preserves inverse images.*

Similarly, we will see in Theorem 3.12 that preservation of 1/4-iso pullbacks is equivalent to preservation of 1/4-iso 2/4-mono pullbacks. We thus tend to use the terms "1/4-iso 2/4-mono pullback preserving" and "1/4-iso pullback preserving" interchangeably. Furthermore, in Example 3.10 we will see that preservation of 1/4-mono pullbacks is properly stronger than preservation of 1/4-iso pullbacks.

Each of the preservation properties introduced in Definition 3.3 implies preservation of monomorphisms, even if we only require that the corresponding pullbacks are weakly preserved. Hence, as at least one of the projections of the pullbacks is monic, preserving the pullbacks mentioned is equivalent to weakly preserving them, and, therefore, each of the properties is implied by weakly preserving pullbacks. Also, note that weakly preserving limits of a given shape is equivalent to preserving weak limits of that shape (e.g. [16, Corollary 4.4]). Furthermore, weakly preserving pullbacks is known to be sufficient for the existence of a normal lax extension – the Barr extension – and this condition can be decomposed as follows:

▶ **Theorem 3.6.** *[18, Theorem 2.7] A* Set*-functor weakly preserves pullbacks iff it weakly preserves inverse images and kernel pairs.*

It turns out that weakly preserving kernel pairs is equivalent to weakly preserving 4/4-epi pullbacks as defined next.

▶ **Definition 3.7.** *We say that a functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *weakly preserves* **4/4-epi pullbacks**, *if it sends pullbacks of the form*

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads] \\ X \arrow[r, two heads] & Y, \end{tikzcd}$$

with arrows $\twoheadrightarrow$ *indicating surjectivity, to weak pullbacks (necessarily of surjections).*

▶ **Theorem 3.8.** *[15, Corollary 5] A* Set*-functor weakly preserves kernel pairs iff it weakly preserves 4/4-epi pullbacks.*

Therefore, the condition of weakly preserving pullbacks can be decomposed as:

▶ **Corollary 3.9.** *A* Set*-functor weakly preserves pullbacks iff it weakly preserves 1/4-mono pullbacks and 4/4-epi pullbacks.*

In Section 4, we will show that either preserving 1/4-mono pullbacks or weakly preserving 1/4-iso pullbacks and 4/4-epi pullbacks is sufficient for the existence of a normal lax extension.

▶ **Example 3.10.** **1.** The subfunctor $(-)^3_2\colon \mathsf{Set} \to \mathsf{Set}$ of the functor $(-)^3\colon \mathsf{Set} \to \mathsf{Set}$ that sends a set X to the set of triples of elements of X consisting of at most two distinct elements does not preserve pullbacks weakly [1] but it preserves inverse images.

2. The neighbourhood functor $\mathcal{N}\colon \mathsf{Set} \to \mathsf{Set}$ (whose coalgebras are neighbourhood frames [9]) sends a set X to the set $\mathcal{N}X = \mathcal{P}\mathcal{P}X$ of neighbourhood systems over X, and a function $f\colon X \to Y$ to the function $\mathcal{N}f\colon \mathcal{N}X \to \mathcal{N}Y$ that assigns to every element $\mathcal{A} \in \mathcal{N}X$ the set $\{B \subseteq Y \mid f^{-1}[B] \in \mathcal{A}\}$. The monotone neighbourhood functor $\mathcal{M}\colon \mathsf{Set} \to \mathsf{Set}$ is the subfunctor of the neighbourhood functor that sends a set X to the set of upward-closed subsets of $(\mathcal{P}X, \subseteq)$. Its coalgebras are monotone neighbourhood frames, which feature, e.g., in the semantics of game logic [30] and concurrent dynamic logic [32]. A closely related functor is the clique functor $\mathcal{C}\colon \mathsf{Set} \to \mathsf{Set}$, which is the subfunctor of $\mathcal{M}$ given by $\mathcal{C}X = \{\alpha \in \mathcal{M}X \mid \forall A, B \in \alpha.\, A \cap B \neq \emptyset\}$. The functors $\mathcal{M}$ and $\mathcal{C}$ do not preserve inverse images: Consider the sets $3 = \{0, 1, 2\}$ and $2 = \{a, b\}$. Let $e\colon 3 \to 2$ be the function that sends $0, 1$ to a and 2 to b, and $B = \{a\}$. Then $\mathcal{M}e(\Uparrow\{0,1\} \cup \Uparrow\{1,2\}) = \Uparrow\{a\}$ where $\Uparrow$ denotes upwards closure, but $\Uparrow\{0,1\} \cup \Uparrow\{1,2\}$ does not belong to $\mathcal{M}(\{0,1\}) = \mathcal{M}(e^{-1}[B])$. However, routine calculations show that these functors do preserve 1/4-iso (2/4-mono) pullbacks and weakly preserve 4/4-epi pullbacks (for the first functor, see [38, Proposition 4.4]).

3. Given a commutative monoid $(M, +, 0)$ (or just M), the *monoid-valued functor* $M^{(-)}$ maps a set X to the set $M^{(X)}$ of functions $\mu\colon X \to M$ with *finite support*, i.e. $\mu(x) \neq 0$ for only finitely many x. The coalgebras of $M^{(-)}$ are M-weighted transition systems. It is known that $M^{(-)}$ preserves inverse images iff M is *positive*, i.e. does not have non-zero invertible elements. Moreover, $M^{(-)}$ preserves weak pullbacks iff M is positive and *refinable*, i.e. whenever $m_1 + m_2 = n_1 + n_2$ for $m_1, m_2, n_1, n_2 \in M$, then there exists a 2×2-matrix with entries in M whose i-th column sums up to m_i and whose j-th row sums up to n_j, for $i, j \in \{1, 2\}$ [17]. Positive but not refinable monoids are fairly common [11]; the simplest example is the additive monoid $\{0, 1, 2\}$ where $2 + 1 = 2$.

The functor $M^{(-)}$ preserves 1/4-iso (2/4-mono) pullbacks iff it preserves inverse images iff M is positive. Indeed, suppose that M is not positive. Consider the function $!_2\colon 2 \to 1$. Then, for mutually inverse non-zero elements u and v of M, the function $M^{!_2}$ sends the pair $(0,0)$ and the pair (u,v) to $0 \in M^1$ wich belongs to the image of $M^{!_\varnothing}\colon M^\varnothing \to M^1$.

4. In recent work [15], it has been shown that the functor of a monad induced by a variety of algebras preserves inverse images iff whenever a variable x is canceled from a term when identified with other variables, then the term does not actually depend on x. This provides a large reservoir of functors that preserve inverse images but do not always have easily guessable normal lax extensions (whose existence will however be guaranteed by our main results). One example is the functor that maps a set X to the free semigroup over X quotiented by the equation $xxx = xx$, as neither idempotence nor associativity cancel any variables. Notice that this functor does not preserve 4/4-epi pullbacks.

Finally, we show that being monotone on difunctional relations is equivalent to preserving 1/4-iso (2/4-mono) pullbacks. The next lemma connects the order on difunctional relations and pullbacks of such type.

▶ **Lemma 3.11.** *Let $X \xrightarrow{f} A \xleftarrow{g} Y$ and $X \xrightarrow{f'} A' \xleftarrow{g'} Y$ be cospans for which there is a map $h\colon A \to A'$ such that $f' = h \cdot f$ and $g' = h \cdot g$. Moreover, consider the commutative square*

$$\begin{array}{ccc} f[X] \cap g[Y] & \xrightarrow{h'} & f'[X] \cap g'[Y] \\ \downarrow & & \downarrow \\ A & \xrightarrow{h} & A' \end{array} \tag{1}$$

where $h'\colon f[X] \cap g[Y] \to f'[X] \cap g'[Y]$ is the restriction of h to $f[X] \cap g[Y]$ and the vertical arrows denote subset inclusions.

1. *If $g^\circ \cdot f \geq g'^\circ \cdot f'$, then h' is a bijection.*
2. *If $g^\circ \cdot f \geq g'^\circ \cdot f'$ and the cospan (f, g) is epi, then (1) is a pullback.*
3. *If h' is a bijection and (1) is a pullback, then $g^\circ \cdot f \geq g'^\circ \cdot f'$.*

▶ **Theorem 3.12.** *The following clauses are equivalent for a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$:*

(i) F *preserves 1/4-iso 2/4-mono pullbacks.*
(ii) F *is well-defined on difunctional relations.*
(iii) F *is monotone on difunctional relations.*
(iv) F *preserves 1/4-iso pullbacks.*

▶ **Corollary 3.13.** *If a* Set*-functor admits a normal lax extension, then it preserves 1/4-iso pullbacks.*

Therefore, the following functors do not admit a normal lax extension.

▶ **Example 3.14.** **1.** The neighbourhood functor $\mathcal{N}\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve 1/4-iso pullbacks: the element $\mathcal{P}1 \in \mathcal{N}1$ belongs to the image of the function $\mathcal{N}!_\varnothing$, with $!_\varnothing\colon \varnothing \rightarrowtail 1$, however, its fiber w.r.t. $\mathcal{N}!_2$, with $!_2\colon 2 \to 1$, is not a singleton.

2. For every non-positive commutative monoid, the monoid valued functor $M^{(-)}\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve 1/4-iso pullbacks (Example 3.10(3)).

3. By (the proof of) [11, Proposition 4.4], the functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ of the monad induced by a variety of algebras that admit a weak form of subtraction (for instance, groups, rings and vector spaces) does not preserve 1/4-iso pullbacks.

4. For every set A with at least two elements, consider the functor $\mathsf{Set}(A, -)/\sim$ that sends a set X to the quotient of the set $\mathsf{Set}(A, X)$ by the smallest equivalence relation $\sim$ on $\mathsf{Set}(A, X)$ that identifies all non-injective maps, and sends a function $f\colon X \to Y$ to the following one between the corresponding equivalence classes: $[g]_\sim \mapsto [f \cdot g]_\sim$. The resulting functor does not preserve 1/4-iso pullbacks. For instance, for $A = \{0, 1\}$, consider the sets $3 = \{a, b, c\}$ and $B = \{0\}$. Then, the fibre of each element of $B \subseteq A$ w.r.t. the function $f\colon 3 \to A$ that sends a to 0 and b, c to 1 is a singleton; however, the fibre of the equivalence class of the constant map into 0 w.r.t. $\mathsf{Set}(A, f)_\sim$ is not a singleton. Similar counterexamples can be constructed for arbitrary A with at least two elements.

4 Existence of Normal Lax Extensions

We proceed to present the main results of the paper: a Set-functor that weakly preserves 1/4-iso pullbacks and 4/4-epi pullbacks, or that preserves 1/4-mono pullbacks admits a normal lax extension. In view of the facts recalled in Section 2, this means that these functors admit a notion of bisimulation that captures behavioural equivalence, or equivalently, that they admit a separating class of monotone predicate liftings.

We begin by showing that the smallest lax extension of a Set-functor is obtained by "closing its Barr relax extension under composition". As a consequence, in Corollary 4.5 we obtain a criterion to determine if a Set-functor admits a normal lax extension.

Consider the partially ordered classes $\mathsf{Lax}(\mathsf{F})$ and $\mathsf{ReLax}(\mathsf{F})$ of lax and relax extensions of F, respectively, ordered pointwise. With the following result we can construct lax extensions from relax extensions in a universal way.

▶ **Proposition 4.1.** *Let* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor. The inclusion* $\mathsf{Lax}(\mathsf{F}) \rightarrowtail \mathsf{ReLax}(\mathsf{F})$ *has a left adjoint* $(-)^\bullet\colon \mathsf{ReLax}(\mathsf{F}) \to \mathsf{Lax}(\mathsf{F})$ *that sends a relax extension* $\mathsf{R}\colon \mathsf{Rel} \to \mathsf{Rel}$ *of* F *to its* **laxification** $\mathsf{R}^\bullet\colon \mathsf{Rel} \to \mathsf{Rel}$*, which is defined on* $r\colon X \nrightarrow Y$ *by*

$$\mathsf{R}^\bullet r = \bigvee\nolimits_{\substack{r_1,\dots,r_n:\\ r_n\cdot\dots\cdot r_1 \le r}} \mathsf{R}r_n \cdot \dots \cdot \mathsf{R}r_1. \tag{2}$$

Furthermore, if a relax extension $\mathsf{R}\colon \mathsf{Set} \to \mathsf{Set}$ *preserves converses, then so does its laxification.*

Since every lax extension of a functor is greater or equal than the Barr relax extension (cf. Section 2), we thus have:

▶ **Corollary 4.2.** *The smallest lax extension of a functor is given by the laxification of its Barr relax extension.*

For the Barr relax extension of a Set-functor, the supremum in the formula (2) can be restricted as follows.

▶ **Lemma 4.3.** *For every composable sequence* $r_1, \dots, r_n$ *such that* $r_n \cdot \dots \cdot r_1 \le r$*, for some relation* r*, there is a composable sequence* $r'_1, \dots, r'_n$ *such that* $r'_n \cdot \dots \cdot r'_1 = r$ *and* $\overline{\mathsf{F}}r_n \cdot \dots \cdot \overline{\mathsf{F}}r_1 \le \overline{\mathsf{F}}r'_n \cdot \dots \cdot \overline{\mathsf{F}}r'_1$.

▶ **Corollary 4.4.** *Let* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor. For every relation* $r\colon X \nrightarrow Y$*,*

$$(\overline{\mathsf{F}})^\bullet r = \bigvee\nolimits_{\substack{r_1,\dots,r_n:\\ r_n\cdot\dots\cdot r_1 = r}} \overline{\mathsf{F}}r_n \cdot \dots \cdot \overline{\mathsf{F}}r_1.$$

Therefore, as normality of a lax extension also implies normality of any lax extension below it, we have

▶ **Corollary 4.5.** *A functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *admits a normal lax extension iff the laxification of its Barr relax extension is normal. More concretely, a functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *admits a normal lax extension iff for every set* X *and every composable sequence of relations* $r_1, \dots, r_n$*, whenever* $r_n \cdot \dots \cdot r_1 = 1_X$*, then* $\overline{\mathsf{F}}r_n \cdot \dots \cdot \overline{\mathsf{F}}r_1 \le 1_{\mathsf{F}X}$.

▶ Remark 4.6. It is well-known [6] that for every functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ and all relations $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$, $\overline{\mathsf{F}}(s \cdot r) \le \overline{\mathsf{F}}s \cdot \overline{\mathsf{F}}r$. Hence, once we show the inequality of Corollary 4.5 we actually have equality.

In general terms, our main results follow by showing that in Corollary 4.5, under certain conditions on Set-functors, it suffices to consider composable sequences of relations that satisfy nice properties. In this regard, it is convenient to introduce the following notion.

▶ **Definition 4.7.** *Let $r_1, \ldots, r_n$ be a composable sequence of relations. A composable sequence $s_1, \ldots, s_k$ is said to be a* **Barr upper bound** *of the sequence $r_1, \ldots, r_n$ if $r_n \cdot \ldots \cdot r_1 = s_k \cdot \ldots \cdot s_1$ and $\overline{\mathsf{F}} r_n \cdot \ldots \cdot \overline{\mathsf{F}} r_1 \leq \overline{\mathsf{F}} s_k \cdot \ldots \cdot \overline{\mathsf{F}} s_1$.*

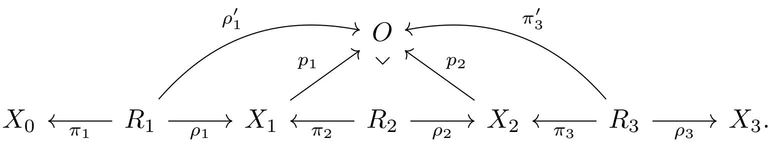
In Section 3 we have seen that every Set-functor that admits a normal lax extension preserves 1/4-iso pullbacks, or equivalently, it is monotone on difunctional relations (Theorem 3.12). As we show next, the latter condition is also equivalent to satisfying the criterion of Corollary 4.5 for pairs of composable relations.

▶ **Proposition 4.8.** *Let $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. The following clauses are equivalent:*

(i) *The functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ preserves 1/4-iso pullbacks.*
(ii) *For all relations $r_1\colon X \nrightarrow Y$, $r_2\colon Y \nrightarrow X$ such that $r_2 \cdot r_1 \leq 1_X$, $\overline{\mathsf{F}} r_2 \cdot \overline{\mathsf{F}} r_1 \leq 1_{\mathsf{F}X}$.*
(iii) *For all relations $r_1\colon X \nrightarrow Y$, $r_2\colon Y \nrightarrow X$ such that $r_2 \cdot r_1 = 1_X$, $\overline{\mathsf{F}} r_2 \cdot \overline{\mathsf{F}} r_1 \leq 1_{\mathsf{F}X}$.*

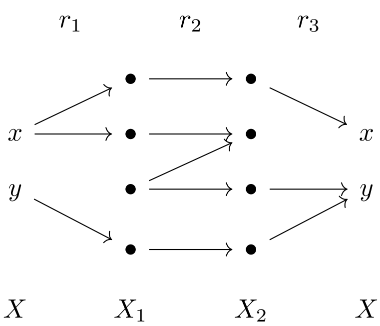
Now, suppose that we want to extend the previous result in inductive style to composable triples of relations. Due to the next lemma, a simple idea to reduce the case of composable triples to the case of composable pairs of relations is to take the difunctional closure of the second relation in the sequence.

▶ **Lemma 4.9.** *Let $r_1\colon X_0 \nrightarrow X_1$, $r_2\colon X_1 \nrightarrow X_2$ and $r_3\colon X_2 \nrightarrow X_3$ be relations given by spans that form the base of the commutative diagram*

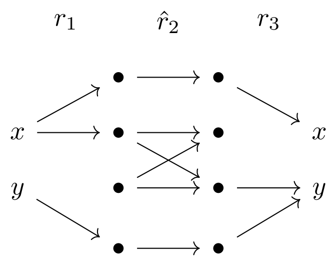


Then, with $r_1'\colon X \nrightarrow O$ and $r_3'\colon O \nrightarrow X_3$ defined by the spans $X_0 \xleftarrow{\pi_1} R_1 \xrightarrow{\rho_1'} O$ and $X_0 \xleftarrow{\pi_3'} R_3 \xrightarrow{\rho_3} X_3$, respectively, $\overline{\mathsf{F}} r_3 \cdot \overline{\mathsf{F}} r_2 \cdot \overline{\mathsf{F}} r_1 \leq \overline{\mathsf{F}} r_3 \cdot \overline{\mathsf{F}} \hat{r}_2 \cdot \overline{\mathsf{F}} r_1 \leq \overline{\mathsf{F}} r_3' \cdot \overline{\mathsf{F}} r_1'$.

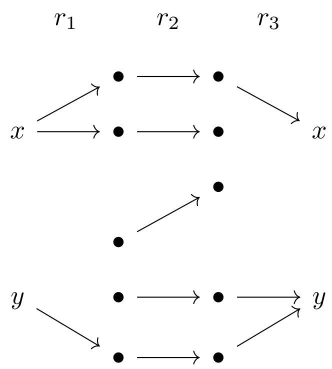
Indeed, let $r_1\colon X \nrightarrow X_1$, $r_2\colon X_1 \nrightarrow X_2$ and $r_3\colon X_2 \nrightarrow X$ be relations such that $r_3 \cdot r_2 \cdot r_1 = 1_X$. Then, by Proposition 4.8 and Lemma 4.9, we conclude that $\overline{\mathsf{F}} r_3 \cdot \overline{\mathsf{F}} r_2 \cdot \overline{\mathsf{F}} r_1 \leq 1_{\mathsf{F}X}$ once we show that $r_3' \cdot r_1' = 1_X$. Of course, in general, this does not hold. Consider the following example where the arrows depict pairs of related elements.



By taking the difunctional closure $\hat{r}_2$ of r_2 we get



So, $r_3 \cdot \hat{r}_2 \cdot r_1 = r_3' \cdot r_1'$ is not a subidentity. Now the property of preserving 1/4-iso pullbacks is helpful again. As we will see in Lemma 4.10, under this condition, the sequence below is a Barr upper bound of the first one and it is obtained from it by "splitting" where necessary the elements of X_1 that do not belong to the codomain of r_1 and the elements of X_2 that do not belong to the domain of r_3.



In this situation we can apply the difunctional closure to r_2 (which in this particular example is already difunctional) to reduce the number of relations as disussed in Lemma 4.9.

▶ **Lemma 4.10.** *Let* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor that preserves 1/4-iso pullbacks, and let* $r_1\colon X \nrightarrow Y$, $r_2\colon Y \nrightarrow Z$ *and* $r_3\colon Z \nrightarrow W$ *be relations. Then, there are relations* $s_1\colon X \nrightarrow Y'$, $s_2\colon Y' \nrightarrow Z'$ *and* $s_3\colon Z' \nrightarrow W$ *such that* s_1, s_2, s_3 *is a Barr upper bound of* r_1, r_2, r_3 *and*

1. *for all* $y, y' \in Y'$ *and all* $z \in Z'$, *if* $y \neq y'$, $y\ s_2\ z$ *and* $y'\ s_2\ z$, *then* $z \in \mathsf{dom}(s_3)$;
2. *for all* $y \in Y'$ *and* $z, z' \in Z'$, *if* $z \neq z'$, $y\ s_2\ z$ *and* $y\ s_2\ z'$, *then* $y \in \mathsf{cod}(s_1)$.

The previous lemma essentially closes the argument that we have been crafting so far.

▶ **Theorem 4.11.** *Let* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor. The following clauses are equivalent:*

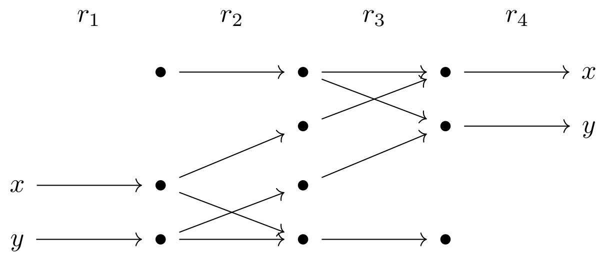
(i) *The functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *preserves 1/4-iso pullbacks.*
(ii) *For all relations* $r_1\colon X \nrightarrow Y$, $r_2\colon Y \nrightarrow Z$ *and* $r_3\colon Z \nrightarrow X$ *such that* $r_3 \cdot r_2 \cdot r_1 \leq 1_X$, $\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$.
(iii) *For all relations* $r_1\colon X \nrightarrow Y$, $r_2\colon Y \nrightarrow Z$ *and* $r_3\colon Z \nrightarrow X$ *such that* $r_3 \cdot r_2 \cdot r_1 = 1_X$, $\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$.

However, as we see next, Theorem 4.11 is as far as we can go under the assumption of 1/4-iso pullbacks preservation. In other words, the fact that a Set-functor preserves 1/4-iso pullbacks is *not* sufficient to conclude that it admits a normal lax extension.

▶ **Example 4.12.** Let us define a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ as a quotient of $\coprod_{n\in\{f,g\}}\{n\}\times X^5 \cong X^5 + X^5$ under the equivalence defined by the clauses:

$$
\begin{aligned}
f(y,x,z,x,t) &\sim f(y',x,z',x,t') & \qquad f(t,x,x,y,y) &\sim f(t',x,x,y,y)\\
g(y,x,z,x,t) &\sim g(y',x,z',x,t') & \qquad g(x,x,y,y,t) &\sim g(x,x,y,y,t')\\
f(y,x,z,x,t) &\sim g(y',x,z',x,t') & \qquad f(t,x,z,y,z) &\sim g(t,x,t,y,z)
\end{aligned}
$$

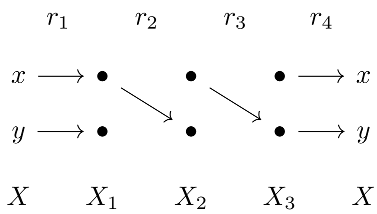
where $f(x_1,\dots,x_5)$ and $g(x_1,\dots,x_5)$ denote the corresponding elements $(f,x_1,\dots,x_5)$, $(g,x_1,\dots,x_5)\in\coprod_{n\in\{f,g\}}\{n\}\times X^5$. Let $2=\{x,y\}$ and consider the composable sequence of relations depicted below.



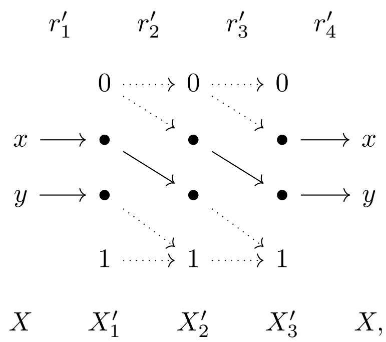
Then, F preserves 1/4-iso pullbacks and $r_4\cdot r_3\cdot r_2\cdot r_1 = 1_2$, however, $\overline{\mathsf{F}}r_4\cdot\overline{\mathsf{F}}r_3\cdot\overline{\mathsf{F}}r_2\cdot\overline{\mathsf{F}}r_1 \not\leq 1_{\mathsf{F}2}$.

4.1 The case of functors that weakly preserve 4/4-epi pullbacks

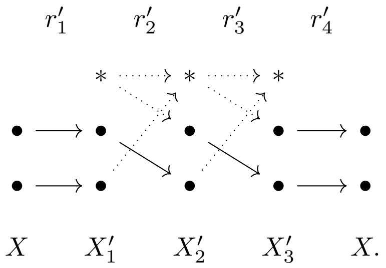
From Theorem 4.11 it basically follows that a functor that weakly preserves 1/4-iso pullbacks and 4/4-epi pullbacks admits a normal lax extension. But to see this, first we need to sharpen Corollary 4.5. The goal is to show that it suffices to consider composable sequences of relations where all relations other than the first and the last are total and surjective. To illustrate how we achieve this, let us consider the sequence of relations depicted below.



Then, by adding new elements 0 and 1 to X_1, X_2 and X_3 we can extend this sequence to the sequence



where the dotted arrows indicate pairs of elements that were added to the correponding relation as follows: for $i = 2, 3$, r'_i relates $0 \in X_{i-1}$ to every element of $X_i \cup \{0\}$ that does not belong to the codomain of r_i and relates every element of $X_{i-1} \cup \{1\}$ that does not belong to the domain of r_i to $1 \in X_i$. In this way, we guarantee that r'_2 and r'_3 are total and surjective and that $r'_4 \cdot r'_3 \cdot r'_2 \cdot r'_1 = r_4 \cdot r_3 \cdot r_2 \cdot r_1 = 1_X$. We could have extended r_2 and r_3 to total and surjective relations by adding just a single element $*$ to X_1, X_2 and X_3 that would simultaneously take the role of 0 and 1. However, composing the resulting sequence of relations would not yield a subidentity:



In other words, by splitting $*$ in two elements 0 and 1, the former to make the relations r_2 and r_3 surjective and the latter to make them total, we obtain a subidentity because we never create paths between elements of X_1 that are not part of the domain of r_2 and elements of X_3 that are not part of the codomain of r_3. In the next lemma we formalize this procedure for arbitrary composable sequences of relations and show that it yields Barr upper bounds.

▶ **Lemma 4.13.** *A functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *that preserves 1/4-iso pullbacks admits a normal lax extension iff for every composable sequence of relations* $r_1, \ldots, r_n$ *such that* $n \geq 4$ *and* $r_2, \ldots, r_{n-1}$ *are total and surjective, whenever* $r_n \cdot \ldots \cdot r_1 = 1_X$*, for some set* X*, then* $\overline{\mathsf{F}}r_n \cdot \ldots \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$*.*

▶ Remark 4.14. In a composable sequence of relations that satisfies the conditions of Lemma 4.13 the first relation is necessarily total while the last one is necessarily surjective.

Now, our first main result follows straightforwardly. Since the composite of total and surjective relations is total and surjective, due to the following fact, every composable sequence of relations where all relations other than the first and the last are total and surjective admits a Barr upper bound consisting of three relations.

▶ **Proposition 4.15.** *A functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *weakly preserves 4/4-epi pullbacks iff for all relations* $r\colon X \nrightarrow Y$ *and* $s\colon Y \nrightarrow Z$*, whenever* r *is surjective and* s *is total,* $\overline{\mathsf{F}}s \cdot \overline{\mathsf{F}}r = \overline{\mathsf{F}}(s \cdot r)$*.*

▶ **Theorem 4.16.** *A* Set*-functor that weakly preserves 1/4-iso pullbacks and 4/4-epi weak pullbacks admits a normal lax extension.*

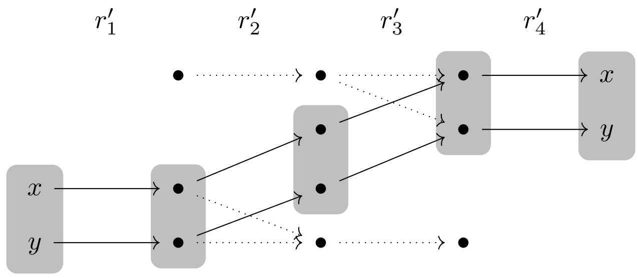
▶ Remark 4.17. Preservation of 4/4-epi pullbacks plays a role in the analysis of interpolation in coalgebraic logic [38]. In particular, this analysis implies that given a separating set Λ of monotone predicate liftings for a finite-set-preserving functor F, which induces an expressive modal logic $\mathcal{L}(\Lambda)$ for F-coalgebras, the logic $\mathcal{L}(\Lambda)$ has interpolation iff F weakly preserves 4/4-epi pullbacks [38, Theorem 37]. In connection with the fact that a functor has a normal lax extension iff it has a separating set of monotone predicate liftings [28], we obtain the following application of Theorem 4.16 and Corollary 3.13: A finite-set preserving functor F has a separating set of monotone predicate liftings such that the associated modal logic has uniform interpolation iff F weakly preserves 1/4-iso pullbacks and 4/4-epi pullbacks.

4.2 The case of functors that preserve 1/4-mono pullbacks

To obtain Theorem 4.16, we refined Corollary 4.5 to composable sequences of relations where all relations *other than* the first and the last are total and surjective. And to achieve this in Lemma 4.13, given a composable sequence of relations, we *added* pairs of related elements to the relations in the sequence. In the sequel, we will show that every functor that preserves 1/4-mono pullbacks admits a normal lax extension. We will see that for these functors it is even possible to refine Corollary 4.5 to composable sequences of relations where *all* relations are total and surjective. However, we will achieve this in Lemma 4.19 below by, given a composable sequence of relations, *removing* pairs of related elements from the relations in the sequence. Our proof strategy is justified by the next fact.

▶ **Proposition 4.18.** *A functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *preserves 1/4-mono pullbacks iff for all relations* $r\colon X \nrightarrow Y$ *and* $s\colon Y \nrightarrow Z$*, whenever* r *is the converse of a partial function or* s *is a partial function,* $\overline{\mathsf{F}}s \cdot \overline{\mathsf{F}}r = \overline{\mathsf{F}}(s \cdot r)$*.*

This result enables a "look ahead and behind" strategy for Corollary 4.5. The idea is that, given a composable sequence of relations $r_1, \ldots, r_n$ such that $r_n \cdot \ldots \cdot r_1 = 1_X$, then, with $r_i\colon X_{i-1} \nrightarrow X_i$ being a relation in the sequence, removing the elements of X_i that do not belong to the codomain of $r_i \cdot \ldots \cdot r_1$ *or* do not belong to the domain of $r_n \cdot \ldots \cdot r_{i+1}$ yields a Barr upper bound of our original sequence. For instance, consider the composable sequence of relations depicted in Example 4.12, which we used to show that there are functors that preserve 1/4-iso pullbacks but do not admit a normal lax extension. In the next lemma, in particular, we show that for functors that preserve 1/4-mono pullbacks the sequence below of total and surjective relations is a Barr upper bound of this one. The dotted arrows represent pairs of related elements that were removed, and the grey boxes represent the elements of each set that are *not* removed.



▶ **Lemma 4.19.** *A functor* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *that preserves 1/4-mono pullbacks admits a normal lax extension if for every composable sequence of total and surjective relations* $r_1, \ldots, r_n$*, whenever* $r_n \cdot \ldots \cdot r_1 = 1_X$ *for some set* X*, then* $\overline{\mathsf{F}}r_n \cdot \ldots \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$*.*

It turns out that the sufficient condition of the previous lemma is actually satisfied by every Set-functor that preserves 1/4-iso pullbacks. Indeed, due to the next result, Lemma 4.9 and the fact that surjections are stable under pushouts, every composable sequence of total and surjective relations whose composite is an identity admits a Barr upper bound consisting of three relations.

▶ **Lemma 4.20.** *Let* $r_1\colon X \nrightarrow X_1$*,* $r_2\colon X_1 \nrightarrow X_2$ *and* $r_3\colon X_2 \nrightarrow X$ *be a composable sequence of total and surjective relations, and let* $\hat{r}_2\colon X_1 \nrightarrow X_2$ *be the difunctional closure of* r_2*. If* $r_3 \cdot r_2 \cdot r_1 = 1_X$*, then* $r_3 \cdot \hat{r}_2 \cdot r_1 = 1_X$*.*

▶ **Proposition 4.21.** *Let* $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor that preserves 1/4-iso pullbacks, and let* $r_1, \ldots, r_n$ *be a composable sequence of total and surjective relations. If* $r_n \cdot \ldots \cdot r_1 = 1_X$ *for some set* X*, then* $\overline{\mathsf{F}}r_n \cdot \ldots \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$*.*

Therefore,

▶ **Theorem 4.22.** *Every* Set*-functor that preserves 1/4-mono pullbacks admits a normal lax extension.*

In particular, since in Example 3.10(3) we have seen that for (commutative) monoid-valued functors preserving 1/4-mono pullbacks is equivalent to preserving 1/4-iso pullbacks, as a consequence of Theorem 4.22 and Corollary 3.13 we obtain:

▶ **Corollary 4.23.** *A (commutative) monoid-valued functor admits a normal lax extension iff the monoid is positive.*

The class of Set-functors that admit a normal lax extension is closed under subfunctors and several natural constructions such as the sum of functors. This makes it easy to extend the reach of our sufficient conditions, but it also shows that it is easy to provide examples of functors that admit a normal lax extension and do not weakly preserve 1/4-mono pullbacks nor 4/4-epi pullbacks. A quick example is the functor given by the sum of the functor $(-)^3_2$ and the monotone neighbourhood functor. To conclude this section, we present a less obvious example that is constructed analogously to Example 4.12. Notice that, as we have seen in Example 3.14(4), the class of functors that admit a normal lax extension is not closed under quotients.

▶ **Example 4.24.** For any set X, let $\mathsf{F}X$
be the quotient of X^3 under the equivalence relation $\sim$ defined by the clauses $(x, x, y) \sim (x, x, x) \sim (y, x, x)$. This yields a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ that neither weakly preserves 1/4-mono pullbacks nor 4/4-epi pullbacks, however, F admits a normal lax extension.

5 Conclusions

Normal lax extensions of functors play a dual role in the coalgebraic modelling of reactive systems, on the one hand allowing for good notions of bisimulations on functor coalgebras and on the other hand guaranteeing the existence of expressive temporal logics. We have shown on the one hand that every functor admitting a lax extension preserves 1/4-iso pullbacks, and on the other hand that a functor admits a normal lax extension if it weakly preserves either 1/4-iso pullbacks and 4/4-epi pullbacks or inverse images. These results improve on previous results [25, 27, 28], which combine to imply that weak-pullback-preserving functors admit normal lax extensions. One application of our results implies, roughly, that a given type of monoid-weighted transition systems admits a good notion of bisimulation iff the monoid is positive.

The most obvious issue for future work is to close the remaining gap, i.e. to give a necessary and sufficient criterion for the existence of normal lax extensions in terms of limit preservation. Additionally, the structure of the lattice of normal lax extensions of a functor merits attention, in the sense that larger lax extensions induce more permissive notions of bisimulation.

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A Omitted Details and Proofs

A.1 Proof of Lemma 2.1

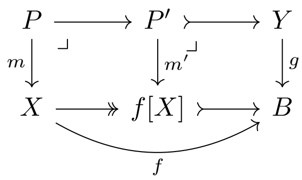
(i) $\Rightarrow$ (ii). Given $r = g^\circ \cdot f$, the hypothesis of (ii) means that $f(x_1) = g(y_1) = f(x_2) = g(y_2)$, so $x_1 \; r \; y_2$.

(ii) $\Rightarrow$ (iii). The pullback of the pushout is a relation $X \nrightarrow Y$ that relates $x \in X$ to $y \in Y$ iff x and y are equivalent under the equivalence relation on $X + Y$ generated by r. By (ii), such elements x, y are already related by r; that is, the pullback is r. If $r = \pi_2 \cdot \pi_1^\circ$ as in (iii), then R maps surjectively onto r, hence is a weak pullback.

(iii) $\Rightarrow$ (i). Immediate, since every relation r can be written in the form $r = \pi_2 \cdot \pi_1^\circ$. ◀

A.2 Proof of Proposition 3.5

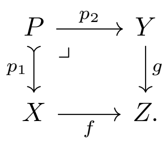
Let $\mathsf{F} \colon \mathsf{Set} \to \mathsf{Set}$ be a functor. It is clear that if F preserves 1/4-mono pullbacks, then it preserves inverse images. To see that the converse statement holds, suppose that F preserves inverse images and consider a cospan $X \xrightarrow{f} B \xleftarrow{g} Y$ in Set. Then a pullback of the cospan (f, g) can be obtained by pasting the following pullbacks where the bottom horizontal arrows are given by the image factorization of f.



Moreover, if m is injective, i.e., if we have a 1/4-mono pullback (square), then by the way pullbacks are formed in Set, m' is also injective. Therefore, in this case, the pullback of (f, g) is preserved because F preserves inverse images. ◀

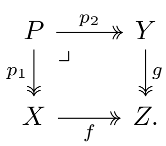
A.3 Details of Example 3.10

- To see that the functor $(-)^3_2$ preserves 1/4-mono pullbacks, consider a pullback



Note that as $p_1 \colon P \rightarrowtail X$ is injective, for every $x \in X$ such that $f(x) \in g[Y]$ there is one and only one element $y \in Y$ such that $f(x) = g(y)$. Now, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be elements of X^3_2 and Y^3_2, respectively, such that $(f(x_1), f(x_2), f(x_3)) = (g(y_1), g(y_2), g(y_3))$. Then, from the fact that x consists of at most two elements of X, we conclude that $((x_1, y_1), (x_2, y_2), (x_3, y_3))$ consists of at most two elements of P and it is clear that projecting this element to X and Y yields x and y, respectively.

- To see that the monotone neighbourhood functor and the clique functor weakly preserve 4/4-epi pullbacks, consider a pullback



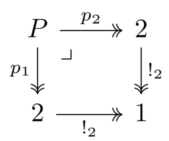
(3)

Suppose that there are $\mathcal{A} \in \mathcal{M}X$ and $\mathcal{B} \in \mathcal{M}Y$ such that $\mathcal{M}f(\mathcal{A}) = \mathcal{M}f(\mathcal{B})$. We have to show that there is $\mathcal{E} \in \mathcal{M}P$ such that $\mathcal{M}p_1(\mathcal{E}) = \mathcal{A}$ and $\mathcal{M}p_2(\mathcal{E}) = \mathcal{B}$. Put

$$\mathcal{E} = \Uparrow\{p_1^\circ[A] \mid A \in \mathcal{A}\} \cup \Uparrow\{p_2^\circ[B] \mid B \in \mathcal{B}\},$$

where, given $r\colon U \nrightarrow V$ and $C \subseteq U$, $r[C]$ denotes the relational image. Then, it is clear that $\mathcal{E}$ is monotone. Moreover, as p_1 is surjective, $\mathcal{A} \subseteq \mathcal{M}p_1(\mathcal{E})$, since for every $A \in \mathcal{A}$, $A = (p_1 \cdot p_1^\circ)[A] \in \mathsf{M}p_1(\mathcal{E})$. On the other hand, for every set $B \subseteq Y$, $(p_1 \cdot p_2^\circ)[B] = (f^\circ \cdot g)[B] = f^\circ[g[B]]$, since 3 is a (weak) pullback. In particular, for every $B \in \mathcal{B}$ we obtain $(p_1 \cdot p_2^\circ)[B] \in \mathcal{A}$, since $g[B] \in \mathcal{M}g(\mathcal{B}) = \mathcal{M}f(\mathcal{A})$. Thus, as $\mathcal{A}$ is monotone, for every set $C \subseteq P$ such that $p_2^\circ[B] \subseteq C$, for some $B \in \mathcal{B}$, $p_1[C] \in \mathcal{A}$. This means that, $\mathcal{A} \supseteq \mathcal{M}p_1(\mathcal{E})$, and, hence, $\mathcal{M}p_1(\mathcal{E}) = \mathcal{A}$. By analogous reasoning we obtain $\mathcal{M}p_2(\mathcal{E}) = \mathcal{B}$. Therefore, the monotone neighborhood functor weakly preserves 4/4-epi pullbacks. Now, suppose that $\mathcal{A}$ and $\mathcal{B}$ are cliques. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we show that $p_1^\circ[A] \cap p_2^\circ[B] \neq \varnothing$, the other cases follow from this one or from the fact that $\mathcal{A}$ and $\mathcal{B}$ are cliques. Note that, as $f[A], g[B] \in \mathcal{M}f(\mathcal{A}) = \mathcal{M}g(\mathcal{B})$ which is a clique, we have $f[A] \cap g[B] \neq \varnothing$. Hence, there is $a \in A$ and $b \in B$ such that $f(a) = g(b)$ and, therefore, $(a, b) \in p_1^\circ[A] \cap p_2^\circ[B]$, by definition of pullback.

- To see that the functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ that maps a set X to the free semigroup over X quotiened by the equation $xxx = xx$ does not preserve 4/4-epi pullbacks weakly, consider the following pullback, where $2 = \{a, b\}$.



Then, $\mathsf{F}!_2(aba) = \mathsf{F}!_2(ab)$ but there is no element p in $\mathsf{F}P$ such that $\mathsf{F}p_1(p) = aba$ and $\mathsf{F}p_2(p) = ab$ since the words have different lenght and do not contain the pattern xx.

A.4 Proof of Lemma 3.11

Note that $g^\circ \cdot f \geq g'^\circ \cdot f'$ means precisely that for all $x \in X$ and $y \in Y$, if $h \cdot f(x) = f'(x) = g'(y) = h \cdot g(y)$, then $f(x) = g(y)$.

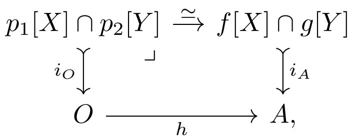
1. Suppose that $g^\circ \cdot f \geq g'^\circ \cdot f'$. Let $a' \in f'[X] \cap g'[Y]$, that is, we have $x \in X$ and $y \in Y$ such that $h \cdot f(x) = f'(x) = g'(y) = h \cdot g'(y)$. Then $a := f(x) = g(y) \in f[X] \cap g[Y]$ and $h(a) = a'$, so h' is surjective. On the other hand, let $a_1, a_2 \in f[X] \cap g[Y]$ such that $h'(a_1) = h'(a_2)$. Then there are $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $a_1 = f(x_1) = g(y_1)$, $a_2 = f(x_2) = g(y_2)$ and, hence, in particular we obtain $h \cdot f(x_1) = h \cdot g(y_2)$. Therefore, $a_1 = f(x_1) = g(y_2) = a_2$.

2. Let $a \in A$ with $h(a) \in f'[X] \cap g'[Y]$. Then there are $x \in X$ and $y \in Y$ such that $h(a) = h \cdot f(x) = h \cdot g(y)$. Moreover, since the cospan $X \xrightarrow{f} A \xleftarrow{g} Y$
is epi, w.l.o.g, there is $x' \in X$ such that $f(x') = a$. Hence, $h \cdot f(x') = h(a) = h \cdot g(y)$. Therefore, as $g^\circ \cdot f \geq g'^\circ \cdot f'$, $a = f(x') = g(y) \in f[X] \cap g[Y]$.

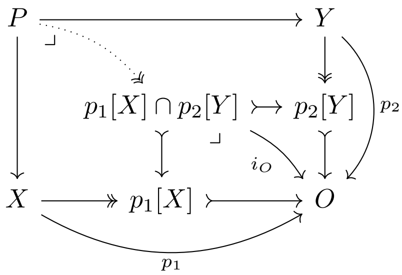
3. Let $x \in X$ and $y \in Y$ such that $f'(x) = h \cdot f(x) = h \cdot g(y) = g'(y)$. Since (1) is a pullback,
it follows that $f(x), g(y) \in f[X] \cap g[Y]$, and since h' is injective, we obtain $f(x) = g(y)$. ◀

A.5 Proof of Theorem 3.12

▬ (i) ⇒ (ii). Let $g^\circ \cdot f\colon X \nrightarrow Y$ be a difunctional relation determined by a cospan $X \xrightarrow{f} A \xleftarrow{g} Y$. Consider the pushout $X \xrightarrow{p_1} O \xleftarrow{p_2} Y$ of the pullback of the cospan $X \xrightarrow{f} A \xleftarrow{g} Y$. Note that every cospan that determines the relation $g^\circ \cdot f$ gives rise to the same pushout. Furthermore, by Lemma 2.1, $p_2^\circ \cdot p_1 = g^\circ \cdot f$, and, hence, to show that the claim holds it suffices to show $(\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 = (\mathsf{F}g)^\circ \cdot \mathsf{F}f$. By the universal property of (p_1, p_2) as a pushout, we have h such that $h \cdot p_1 = f$ and $h \cdot p_2 = g$. The inequality $(\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \leq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$ is then immediate from $\mathsf{F}h \cdot \mathsf{F}p_1 = \mathsf{F}f$ and $\mathsf{F}h \cdot \mathsf{F}p_2 = \mathsf{F}g$. To see the inequality $(\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \geq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$ we consider first the case where $g^\circ \cdot f$ is non-empty. Note that, as (p_1, p_2) is an epicocone, by Lemma 3.11 the inequality $p_2^\circ \cdot p_1 \geq g^\circ \cdot f$ entails that we have the following pullback square

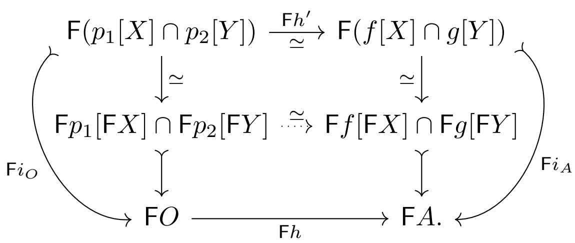


where i_O and i_A are the corresponding inclusions into O and A, respectively. Hence, since F is 1/4-iso preserving, its image under F is also a pullback. Moreover, as Set-functors preserve epimorphisms, by applying F to the commutative diagram



we conclude that $\mathsf{F}i_O\colon \mathsf{F}(p_1[X] \cap p_2[Y]) \to \mathsf{F}O$ corestricts to $\mathsf{F}p_1[\mathsf{F}X] \cap \mathsf{F}p_2[\mathsf{F}Y]$. And, as $p_1[X] \cap p_2[Y]$ is non-empty since the relation $p_2 \cdot p_1^\circ$ is non-empty, $\mathsf{F}i_O$ is a mononormphism because every Set-functor preserves monomorphisms with non-empty domain.

On the other hand, as $p_1[X] \cap p_2[Y]$ in non-empty and every Set-functor preserves non-empty intersections [41], we have $\mathsf{F}(p_1[X] \cap p_2[Y]) \simeq \mathsf{F}(p_1[X]) \cap \mathsf{F}(p_2[Y]) \simeq \mathsf{F}p_1[\mathsf{F}X] \cap \mathsf{F}p_2[\mathsf{F}Y]$, with the second isomorphism holding due to the fact that for every function $q\colon X \to Y$ with non-empty domain, the sets $\mathsf{F}(q[X])$ and $\mathsf{F}q[\mathsf{F}X]$ are isomorphic because each of them is the codomain of an epimorphism and the domain of a monomorphism of an epi-mono factorizations of $\mathsf{F}q$. Hence, $\mathsf{F}i_O\colon \mathsf{F}(p_1[X] \cap p_2[Y]) \to \mathsf{F}O$ corestricts to an isomorphism $\mathsf{F}i_O\colon \mathsf{F}(p_1[X] \cap p_2[Y]) \to \mathsf{F}p_1[\mathsf{F}X] \cap \mathsf{F}p_2[\mathsf{F}Y]$. And, by analogous reasoning for the morphism $\mathsf{F}i_A\colon \mathsf{F}(f[X] \cap g[Y]) \to \mathsf{F}A$, we obtain the commutative diagram



Thus, as the outer square is a pullback, the square

$$\begin{array}{ccc} \mathsf{F}p_1[\mathsf{F}X] \cap \mathsf{F}p_2[\mathsf{F}Y] & \xrightarrow{\simeq} & \mathsf{F}f[\mathsf{F}X] \cap \mathsf{F}g[\mathsf{F}Y] \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{F}O & \xrightarrow[\mathsf{F}h]{} & \mathsf{F}A \end{array}$$

is a pullback, where the top morphism is given by restricting $\mathsf{F}h$ to $\mathsf{F}p_1[\mathsf{F}X] \cap \mathsf{F}p_2[\mathsf{F}Y]$. Therefore, from Lemma 3.11(3), $(\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \geq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$. Now suppose that $g^\circ \cdot f$ is empty. Consider the functions $f_{+1} \colon X + 1 \to A + 1$ and $g_{+1} \colon Y + 1 \to A + 1$ that are defined as f and g on X and Y, respectively, and send the element added to X and Y, respectively, to the element added to A, and the functions $p_{1,+1} \colon X+1 \to O+1$ and $p_{2,+1} \colon Y+1 \to O+1$ defined analogously. Then, $p_{2,+1}^\circ \cdot p_{1,+1} = g_{+1}^\circ \cdot f_{+1}$ is non-empty. Hence, by the previous argument, $(\mathsf{F}p_{2,+1})^\circ \cdot \mathsf{F}p_{1,+1} = (\mathsf{F}g_{+1})^\circ \cdot \mathsf{F}f_{+1}$. Now, let $\mathfrak{x} \in \mathsf{F}X$, $\mathfrak{y} \in \mathsf{F}Y$ such that $\mathsf{F}f(\mathfrak{x}) = \mathsf{F}g(\mathfrak{y})$. Then, as the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X+1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{+1}} \\ A & \longrightarrow & A+1 \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g_{+1}} \\ Y & \xrightarrow[i_Y]{} & Y+1 \end{array}$$

commutes (with the horizontal arrows denoting coprojections), $\mathsf{F}p_{1,+1}(\mathsf{F}i_X(\mathfrak{x})) = \mathsf{F}p_{2,+1}(\mathsf{F}i_Y(\mathfrak{y}))$. Hence, as the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X+1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_{1,+1}} \\ O & \xrightarrow{i_O} & O+1 \\ {\scriptstyle p_2}\uparrow & & \uparrow{\scriptstyle p_{2,+1}} \\ Y & \xrightarrow[i_Y]{} & Y+1 \end{array}$$

commutes (with the horizontal arrows denoting coprojections), $\mathsf{F}i_O(\mathsf{F}p_1(\mathfrak{x})) = \mathsf{F}i_O(\mathsf{F}p_2(\mathfrak{y}))$. Therefore, as F preserves monomorphisms $\mathsf{F}p_1(\mathfrak{x}) = \mathsf{F}p_2(\mathfrak{y})$ which entails $(\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \geq (\mathsf{F}g)^\circ \cdot \mathsf{F}f$.

- (ii) $\Leftrightarrow$ (iii) The implication (iii) $\Rightarrow$ (ii) is trivial. To show (ii) $\Rightarrow$ (iii), let $g^\circ \cdot f \colon X \nrightarrow Y$ and $g'^\circ \cdot f' \colon X \nrightarrow Y$ be difunctional relations, given by cospans $X \xrightarrow{f} O \xleftarrow{g} Y$ and $X \xrightarrow{f'} A \xleftarrow{g'} Y$ respectively, such that $g^\circ \cdot f \leq g'^\circ \cdot f'$. Since F is well-defined on difunctional relations, by Lemma 2.1 we can assume that the cospan $X \xrightarrow{f} O \xleftarrow{g} Y$ is the pushout of its pullback $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$. Then, the condition $g^\circ \cdot f \leq g'^\circ \cdot f'$ entails $f' \cdot \pi_1 = g' \cdot \pi_2$. Hence, by the pushout property, there is a map $h \colon O \to A$ such that $f' = h \cdot f$ and $g' = h \cdot g$. Therefore, $(\mathsf{F}g)^\circ \cdot \mathsf{F}f \leq (\mathsf{F}g)^\circ \cdot (\mathsf{F}h)^\circ \cdot \mathsf{F}h \cdot \mathsf{F}f = (\mathsf{F}g')^\circ \cdot \mathsf{F}g'$.
- (ii) $\Rightarrow$ (iv). Consider a pullback square of the form

$$\begin{array}{ccc} P & \xrightarrow[j]{\simeq} & Y \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ X & \xrightarrow[f]{} & Z. \end{array}$$

Then, $i \cdot j^{-1} = i \cdot j^\circ = f^\circ \cdot l$, so $1_X^\circ \cdot (i \cdot j^{-1}) = f^\circ \cdot l$. Hence, as F is well-defined on difunctional relations, $(\mathsf{F}1_X)^\circ \cdot \mathsf{F}(i \cdot j^{-1}) = (\mathsf{F}f)^\circ \cdot \mathsf{F}l$. Thus, $\mathsf{F}i \cdot (\mathsf{F}j)^\circ = \mathsf{F}i \cdot (\mathsf{F}j)^{-1} = \mathsf{F}(i \cdot j^{-1}) = (\mathsf{F}f)^\circ \cdot \mathsf{F}l$, i.e, $(\mathsf{F}i, \mathsf{F}j)$ is a weak-pullback of $(\mathsf{F}f, \mathsf{F}l)$.

- (iv) $\Rightarrow$ (i) trivial. ◀

A.6 Proof of Proposition 4.1

We begin by showing that, given a relax extension R of F, $\mathsf{R}^\bullet \colon \mathsf{Rel} \to \mathsf{Rel}$ is a lax extension of F.

2 Trivial.

2 Let $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow X$ be relations. Moreover, suppose that $r_1, \dots, r_m$ and $s_1, \dots, s_n$ are finite sequences of relations such that $r_1 \cdot \ldots \cdot r_m \leq r$ and $s_1 \cdot \ldots \cdot s_n \leq s$. Then, $s_1, \dots, s_n, r_1, \dots, r_m$ is a finite sequence of relations such that $s_n \cdot \ldots \cdot s_1 \cdot r_m \cdot \ldots \cdot r_1 \leq s \cdot r$. Therefore, as relational composition preserves suprema in each variable,

$$\begin{aligned}\mathsf{R}^\bullet s \cdot \mathsf{R}^\bullet r &= \bigvee_{\substack{s_1,\dots,s_n:\\ s_n\cdot\ldots\cdot s_1 \leq s}} \bigvee_{\substack{r_1,\dots,r_m:\\ r_m\cdot\ldots\cdot r_1 \leq r}} \mathsf{R}s_n \cdot \ldots \cdot \mathsf{R}s_1 \cdot \mathsf{R}r_m \cdot \ldots \cdot \mathsf{R}r_1 \\ &\leq \bigvee_{\substack{t_1,\dots,t_k:\\ t_k\cdot\ldots\cdot t_1 \leq s\cdot r}} \mathsf{R}t_k \cdot \ldots \cdot \mathsf{R}t_1 = \mathsf{R}^\bullet(s \cdot r).\end{aligned}$$

2 Trivial.

Now, it is clear that $(-)^\bullet \colon \mathsf{ReLax}(\mathsf{F}) \to \mathsf{Lax}(\mathsf{F})$ is a monotone map and that the laxification of a relax extension produces a relax extension that is greater or equal than the starting one, and that, since lax extensions preserve composition laxly, equality is attained precisely when the starting relax extension is a lax extension. Furthermore, suppose that R preserves converses. Let $r\colon X \nrightarrow Y$ be a relation. Then, since $r_1, \dots, r_n$ is a composable sequence of relations such that $r_n \cdots r_1 \leq r$ iff $s_1 = r_n^\circ, \dots, s_n = r_1^\circ$ is a composable sequence of relations such that $s_n \cdots s_1 \leq r^\circ$ and R preserves converses, we obtain:

$$\begin{aligned}(\mathsf{R}^\bullet r)^\circ &= \bigvee_{\substack{r_1,\dots,r_n:\\ r_n\cdot\ldots\cdot r_1 \leq r}} (\mathsf{R}r_n \cdot \ldots \cdot \mathsf{R}r_1)^\circ \\ &= \bigvee_{\substack{r_1,\dots,r_n:\\ r_n\cdot\ldots\cdot r_1 \leq r}} \mathsf{R}r_1^\circ \cdot \ldots \cdot \mathsf{R}r_n^\circ \\ &= \bigvee_{\substack{s_1,\dots,s_n:\\ s_n\cdot\cdots\cdot s_1 \leq r^\circ}} \mathsf{R}s_n \cdot \ldots \cdot \mathsf{R}s_1 \\ &= \mathsf{R}^\bullet(r^\circ)\end{aligned}$$

◀

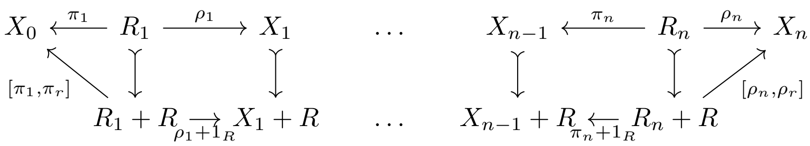
A.7 Proof of Lemma 4.3

Let $r\colon X_0 \nrightarrow X_n$ be a relation, and let $r_1, \dots, r_n$ be a composable sequence of relations such that $r_n \cdot \ldots \cdot r_1 \leq r$. For $i = 1, \dots, n$, let $X_{i-1} \xleftarrow{\pi_i} R_i \xrightarrow{\rho_i} X_i$ be a span in Set such that $r_i = \rho_i \cdot \pi_i^\circ$ and let $X_0 \xleftarrow{\pi_r} R \xrightarrow{\rho_r} X_n$ be a span in Set such that $r = \rho_r \cdot \pi_r^\circ$. We construct a sequence $r'_1, \dots, r'_n$ with the desired properties as follows.

If $n = 1$, we just take $r'_1 = r$, otherwise, with $[f, g]\colon X + Y \to Z$ denoting the copairing of $f\colon X \to Z$ and $g\colon Y \to Z$,

- r'_1 is given by the span $X_0 \xleftarrow{[\pi_1,\pi_r]} R_1 + R \xrightarrow{\rho_1 + 1_R} X_1 + R$;
- for $i = 2, \dots, n-1$, r'_i is given by the span $X_{i-1} + R \xleftarrow{\pi_i + 1_R} R_i + R \xrightarrow{\rho_i + 1_R} X_i + R$;
- $r_n\colon (X_{n-1} + R) \nrightarrow X_n$ is given by the span $X_{n-1} \xleftarrow{\pi_n + 1_R} R_n + R \xrightarrow{[\rho_n,\rho_r]} X_n$.

Then it is clear that by construction we have $r'_n \cdot \ldots \cdot r'_1 = r$ and, hence, the claim follows from the fact that the diagram below commutes



where the vertical arrows denote the corresponding coprojections. ◀

A.8 Proof of Proposition 4.8

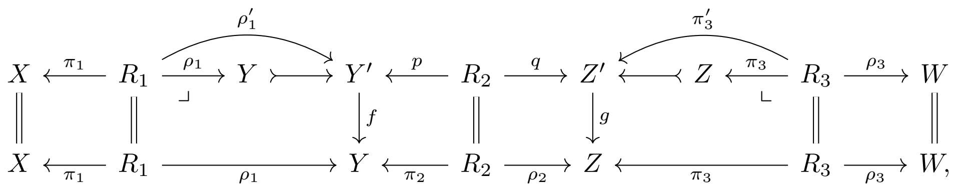
Note that for all relations $r_1 \colon X \nrightarrow Y$ and $r_2 \colon Y \nrightarrow X$ such that $r_1 = \rho_1 \cdot \pi_1^\circ$ and $r_2 = \rho_2 \cdot \pi_2^\circ$, for spans $X \xleftarrow{\pi_1} R_1 \xrightarrow{\rho_1} Y$ and $Y \xleftarrow{\pi_2} R_2 \xrightarrow{\rho_2} X$ in Set, $r_2 \cdot r_1 \leq 1_X \iff \pi_2^\circ \cdot \rho_1 \leq \rho_2^\circ \cdot \pi_1$. Therefore, the equivalence between (i) and (ii) follows from the fact that preserving 1/4-iso pullbacks is equivalent to being monotone on difunctional relations (Theorem 3.12). The equivalence between (ii) and (iii) is an immediate consenquence of Lemma 4.3. ◀

A.9 Proof of Lemma 4.9

Let $X_1 \xrightarrow{\pi'_2} P \xleftarrow{\rho'_2}$ be the pushout of the cospan (p_1, p_2). So, $\hat{r}_2 = \rho'_2 \cdot \pi_2^\circ$ and, as F is monotone and $p_1 \cdot \pi'_2 = p_2 \cdot \rho'_2$, $\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 \leq \overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}\hat{r}_2 \cdot \overline{\mathsf{F}}r_1 \leq \overline{\mathsf{F}}r_3 \cdot (\mathsf{F}p_2)^\circ \cdot \mathsf{F}p_1 \cdot \overline{\mathsf{F}}r_1 = \overline{\mathsf{F}}r'_3 \cdot \overline{\mathsf{F}}r'_1$.

A.10 Proof of Lemma 4.10

For $i = 1, 2, 3$, let $\rho_i \cdot \pi_i{}^\circ$ be the canonical factorization of r_i. We will show that "splitting the elements" of the domain of r_2 that do not belong to the codomain of r_1 and the elements of the codomain of r_2 that do not belong to the domain of r_3 yields a sequence of relations with the desired properties. W.l.o.g. assume that $Y \cap R_2 = \emptyset$ and $Z \cap R_2 = \emptyset$, and consider $Y' = X_1 \cup (R_2 \setminus \pi_2^{-1}[\rho_1[R_1]])$, $Z' = X_2 \cup (R_2 \setminus \rho_2^{-1}[\pi_3[R_3]])$, which are then disjoint unions. Consider the functions $f \colon Y' \to Y$ and $g \colon Z' \to Z$ that act identically on Y and Z and as π_2 and ρ_2 on $R_2 \setminus \pi_2^{-1}[\rho_1[R_1]]$ and $R_2 \setminus \rho_2^{-1}[\pi_3[R_3]]$ respectively. Moreover, let $p \colon R_2 \to Y'$ be the function that sends $(y, z) \in R_2$ to $y \in Y$ if $y \in \mathsf{cod}(r_1)$ and acts identically otherwise, and let $q \colon R_2 \to Z'$ be the function that sends $(y, z) \in R_2$ to $z \in Z$ if $z \in \mathsf{dom}(r_3)$ and acts identically otherwise. Then, we have the following commutative diagram



where the arrows $Y \rightarrowtail Y'$ and $Z \rightarrowtail Z'$ denote inclusions. Since the second and the fourth squares are pullbacks, we obtain equations $\rho'_1 = f^\circ \cdot \rho_1$ and $\pi'^\circ_3 = \pi_3^\circ \cdot g$. Hence,

$$\begin{aligned} \rho_3 \cdot \pi_3^\circ \cdot \rho_2 \cdot \pi_2^\circ \cdot \rho_1 \cdot \pi_1^\circ &= \rho_3 \cdot \pi_3^\circ \cdot g \cdot q \cdot p^\circ \cdot f^\circ \cdot \rho_1 \cdot \pi_1^\circ \\ &= \rho_3 \cdot \pi'^\circ_3 \cdot q \cdot p^\circ \cdot \rho'_1 \cdot \pi_1^\circ. \end{aligned}$$

Moreover, as F is 1/4-iso pullback preserving, by applying F to the commutative diagram above and reasoning analogously, we have

$$\begin{aligned}\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 &= \mathsf{F}\rho_3 \cdot (\mathsf{F}\pi_3)^\circ \cdot \mathsf{F}\rho_2 \cdot (\mathsf{F}\pi_2)^\circ \cdot \mathsf{F}\rho_1 \cdot (\mathsf{F}\rho_1)^\circ \\ &= \mathsf{F}\rho_3 \cdot (\mathsf{F}\pi_3')^\circ \cdot \mathsf{F}q \cdot (\mathsf{F}p)^\circ \cdot \mathsf{F}\rho_1 \cdot (\mathsf{F}\pi_1)^\circ.\end{aligned}$$

Therefore, with $r_1' = \rho_1' \cdot \pi_1^\circ$, $r_2' = q \cdot p^\circ$ and $r_3' = \rho_3 \cdot \pi_3'$,

$$\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 = \overline{\mathsf{F}}r_3' \cdot \overline{\mathsf{F}}r_2' \cdot \overline{\mathsf{F}}r_1.$$

Note that as $\mathsf{cod}(r_1') = \mathsf{cod}(r_1)$ and $\mathsf{dom}(r_3') = \mathsf{dom}(r_3)$, by construction, for all $y, y' \in X_1$ and $z, z' \in X_2$:

1. if $y \neq y'$, $y\ r_2'\ z$ and $y'\ r_2'\ z$, then $z \in \mathsf{dom}(r_3')$, and
2. if $z \neq z'$, $y\ r_2'\ z$ and $y\ r_2'\ z'$, then $y \in \mathsf{cod}(r_1')$.

◀

A.11 Proof of Theorem 4.11

▬ (i) ⇒ (ii) Let $r_1 \colon X \nrightarrow Y$, $r_2 \colon Y \nrightarrow Z$ and $r_3 \colon Z \nrightarrow X$ be relations such that $r_3 \cdot r_2 \cdot r_1 \leq 1_X$. By Lemma 4.10 we can assume w.l.o.g that for all y, y' in X_1 and $z, z' \in X_2$:

1. if $y \neq y'$, $y\ r_2\ z$ and $y'\ r_2\ z$, then $z \in \mathsf{dom}(r_3)$, and
2. if $z \neq z'$, $y\ r_2\ z$ and $y\ r_2\ z'$, then $y \in \mathsf{cod}(r_1)$.

Let $\hat{r}_2 \colon X_1 \nrightarrow X_2$ denote the difunctional closure of r_2. We claim that $r_3 \cdot \hat{r}_2 \cdot r_1 \leq 1_X$. Let $x\ r_1\ y_0\ r_2\ z_1\ r_2^\circ\ y_1\ r_2\ z_2 \ldots r_2^\circ\ y_{n-1}\ r_2\ z_n\ r_3\ x'$, with $n \geq 1$; we have to show that $x = x'$. We assume w.l.og that for $i = 0, \ldots, n-2$, $y_i \neq y_{i+1}$, and for $j = 1, \ldots, n-1$, $z_j \neq z_{j+1}$. The reason is that whenever $y_i = y_{i+1}$, correspondingly $z_j = z_{j+1}$, we can remove $y_i\ r_2\ z_{i+1}$ and $z_{i+1}\ r_2^\circ\ y_{i+1}$ from the chain of related elements and still obtain a chain of related elements from x to x'.

We proceed by induction on n, with the base case $n = 1$ holding because $r_3 \cdot r_2 \cdot r_1 \leq 1_X$, by hypothesis. Let $x\ r_1\ y_0\ r_2\ z_1\ r_2^\circ\ y_1\ r_2\ z_2 \ldots r_2^\circ\ y_{n-1}\ r_2\ z_n\ r_3\ x'$. Since $n \geq 2$, we have $y_0 \neq y_1$ by assumption, and $y_0\ r_2\ z_1$ and $y_1\ z_1$. Hence, by item 1 and the fact that $r_3 \cdot r_2 \cdot r_1 \leq 1_X$, we obtain $z_1\ r_3\ x$. This entails, by analogous reasoning using item 2, that $x\ r_1\ y_1$. Thus, we obtain a chain of related elements $x\ r_1\ y_1\ r_2\ z_2 \ldots r_2^\circ\ y_{n-1}\ r_2\ z_n\ r_3\ x'$; so $x = x'$ by the inductive hypothesis. Therefore, by Lemma 4.9 and Proposition 4.8 we obtain $\overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 \leq 1_{\mathsf{F}X}$.

▬ (ii) ⇒ (iii) Immediate consequence of Lemma 4.3.

▬ (iii) ⇒ (i) Immediate consequence of Proposition 4.8 since $\overline{\mathsf{F}}$ is normal. ◀

A.12 Details of Example 4.12

The symmetric-reflexive closure of $\sim$ is already transitive. The property that F preserves 1/4-iso pullbacks can be equivalently formulated as follows: for every rank-1 term t, $t[x/x_1, \ldots, x/x_n] \sim t[x/x_1, \ldots, x/x_n, z_1/y_1, \ldots, z_m/y_m]$ implies $t \sim t[z_1/y_1, \ldots, z_m/y_m]$ where $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ are all the variables of t, and t being rank-1 means that it contains precisely one occurrence either of f or of g. Preservation of 1/4-iso pullbacks then follows by case-by-case analysis.

That $\overline{\mathsf{F}}r_4 \cdot \overline{\mathsf{F}}r_3 \cdot \overline{\mathsf{F}}r_2 \cdot \overline{\mathsf{F}}r_1 \not\leq 1_X$ follows from the fact that we can define $h(x, y) = f(x, x, x, y, y)$, $u(x, y) = g(x, x, y, y, y)$, and then $h(x, y) \nsim u(x, y)$, which easily follows by inspecting the above clauses that define $\sim$. ◀

A.13 Proof of Lemma 4.13

Clearly, the condition is necessary due to Corollary 4.5 and to see that is also sufficient first we note that by Theorem 4.11 it suffices to consider composable sequences of four or more relations. Now, let $r_1, \ldots, r_n$ be a composable sequence of relations such that $n \geq 4$, $r_n \cdot \ldots \cdot r_1 = 1_X$, and $r_i = X_{i-1} \nrightarrow X_i$ for $i = 1, \ldots, n$ and $X_0 = X_n = X$. We show that this sequence admits a Barr upper bound such that all relations other than the first and the last are total and surjective. Then, the claim follows by Corollary 4.5.

For simplicity of notation let us assume that $0, 1 \notin X_i$, for $i = 2, \ldots, n-1$, and let $X'_i = X_i \cup \{0, 1\}$ for $i = 2, \ldots, n-1$. Consider the sequence of relations $r'_1, \ldots, r'_n$ defined as follows:

- The elements related by $r'_1 \colon X \nrightarrow X'_1$ and $r'_n \colon X'_{n-1} \nrightarrow X$ are precisely the ones related by r_1 and r_n, respectively.
- For $i = 2, \ldots, n-1$, $r'_i \colon X'_{i-1} \nrightarrow X'_i$ consist of the following pairs
  - $(0, x)$ if $x = 0$ or $x \in X_i \setminus \mathsf{cod}(r_i)$;
  - $(x, 1)$ if $x = 1$ or $x \in X_{i-1} \setminus \mathsf{dom}(r_i)$;
  - (x, y) if $x \; r_i \; y$.

Then, by construction, for $i = 2, \ldots, n-1$, r'_i is total and surjective and, with $\rho_i \cdot \pi_i^\circ$ and $\rho'_i \cdot \pi_i'^\circ$ denoting the canonical factorizations of r_i and r'_i, respectively, the following diagram where the vertical arrows denote inclusions commutes

$$\begin{array}{ccccccccccccccccc}
X & \overset{\pi'_1}{\twoheadleftarrow} & R'_1 & \overset{\rho'_1}{\rightarrow} & X'_1 & \overset{\pi'_2}{\twoheadleftarrow} & R'_2 & \overset{\rho'_2}{\twoheadrightarrow} & X'_2 & \cdots & X'_{n-2} & \overset{\pi'_{n-1}}{\twoheadleftarrow} & R'_{n-1} & \overset{\rho'_{n-1}}{\twoheadrightarrow} & X'_{n-1} & \overset{\pi'_n}{\leftarrow} & R'_n & \overset{\rho'_n}{\twoheadrightarrow} & X \\
\| & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \| \\
X & \underset{\pi_1}{\twoheadleftarrow} & R_1 & \underset{\rho_1}{\rightarrow} & X_1 & \underset{\pi_2}{\leftarrow} & R_2 & \underset{\rho_2}{\rightarrow} & X_2 & \cdots & X_{n-2} & \underset{\pi_{n-1}}{\leftarrow} & R_{n-1} & \underset{\rho_{n-1}}{\rightarrow} & X_{n-1} & \underset{\pi_n}{\leftarrow} & R_n & \underset{\rho_n}{\twoheadrightarrow} & X
\end{array}$$

This entails that $r_n \cdot \ldots \cdot r_1 \leq r'_n \cdot \ldots \cdot r'_1$ and by applying F to the diagram we conclude that $\overline{\mathsf{F}}r_n \cdot \ldots \cdot \overline{\mathsf{F}}r_1 \leq \overline{\mathsf{F}}r'_n \cdot \ldots \cdot \overline{\mathsf{F}}r'_1$. To see that $r_n \cdot \ldots \cdot r_1 \geq r'_n \cdot \ldots \cdot r'_1$ first note that by construction for $i = 2, \ldots, n-1$, $x \; (r'_i \cdot \ldots \cdot r'_2) \; 0$ iff $x = 0$ and $1 \; (r'_{n-1} \cdot \ldots \cdot r'_i) \; x$ iff $x = 1$.

Now, suppose that $x \; r'_1 \; x_1 \; r'_2 \; x_2 \ldots r'_{n-1} \; x_{n-1} \; r'_n \; y$, then, as $0 \notin \mathsf{cod}(r'_1)$ and $1 \notin \mathsf{dom}(r'_n)$, for $i = 1, \ldots, n-1$, $x_i \in X_i$. Therefore, $x \; (r_n \cdot \ldots \cdot r_1) \; y$. ◀

A.14 Proof of Proposition 4.15

Note that a relation $s \colon Y \nrightarrow Z$ is total iff it can be factorized as $f \cdot e^\circ$, for some function $f \colon A \to Z$ and some surjective map $e \colon A \to Y$. And, dually, a relation $r \colon X \nrightarrow Y$ is surjective iff it can be factorized as $h \cdot g^\circ$, for some function $g \colon A \to X$ and some surjective map $h \colon A \to Z$.

A.15 Proof of Proposition 4.18

Note that a relation $s \colon Y \nrightarrow Z$ is a partial function iff it can be factorized as $f \cdot i^\circ$, for some function $f \colon A \to Z$ and some injective map $i \colon A \to Y$. And, dually, a relation $r \colon X \nrightarrow Y$ is the converse of partial function iff it can be factorized as $j \cdot g^\circ$, for some function $A \to X$ and some injective map $j \colon A \to Z$. ◀

A.16 Proof of Lemma 4.19

▶ **Lemma A.1.** *Let $r_1, \dots, r_n$ be a composable sequence of relations. Consider the following composable sequences of relations:*

1. *$s_1, \dots, s_n$ defined by $s_n = r_n$, and $s_i = \lceil \mathsf{dom}\, s_{i+1} \rceil \cdot r_i$, for $i = 1, \dots, n-1$;*
2. *$t_1, \dots, t_n$ defined by $t_1 = s_1$, and $t_i = s_i \cdot \lceil \mathsf{cod}\, t_{i-1} \rceil$, for $i = 2, \dots, n$.*

Then, $t_n \cdots t_1 = s_n \cdots s_1 = r_n \cdots r_1$ and for every $i = 2, \dots, n$, $\mathsf{cod}(t_{i-1}) = \mathsf{dom}(t_i)$.

Proof. Clearly, $t_n \cdots t_1 = s_n \cdots s_1 = r_n \cdots r_1$. Moreover, note that $\mathsf{dom}(t_i) = \mathsf{dom}(s_i) \cap \mathsf{cod}(t_{i-1})$, and $\mathsf{cod}(t_{i-1}) = s_{i-1}[\mathsf{cod}(t_{i-2})]$ if $i > 2$ and $\mathsf{cod}(t_{i-1}) = \mathsf{cod}(s_1)$ if $i = 2$. Thus, we have $\mathsf{cod}(t_{i-1}) \leq \mathsf{cod}(s_{i-1})$ for $i = 2, \dots, n$, and since $\mathsf{cod}(s_{i-1}) \leq \mathsf{dom}(s_i)$ for $i = 2, \dots, n$, it follows that $\mathsf{dom}(t_i) = \mathsf{dom}(s_i) \cap \mathsf{cod}(t_{i-1}) = \mathsf{cod}(t_{i-1})$. ◀

Let $r_1, \dots, r_n$ be a composable sequence of relations such that $r_n \cdot \ldots \cdot r_1 = 1_X$, for some set X. We will show that there is a Barr upper bound $t'_1, \dots, t'_n$ of $r_1, \dots, r_n$ consisting only of total and surjective relations. Then, the claim follows immediately by Corollary 4.5. By Lemma A.1, we obtain sequences $s_1, \dots, s_n$ and $t_1, \dots, t_n$ such that $t_n \cdots t_1 = 1_X$ and for $i = 2, \dots, n$, $\mathsf{cod}(t_{i-1}) = \mathsf{dom}(t_i)$. Therefore, since F preserves 1/4-mono pullbacks, by Proposition 4.18,

$$
\begin{aligned}
&\overline{\mathsf{F}}r_n \cdots \overline{\mathsf{F}}r_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \overline{\mathsf{F}}\lceil \mathsf{dom}\, s_n \rceil \cdot \overline{\mathsf{F}}r_{n-1} \cdots \overline{\mathsf{F}}r_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \overline{\mathsf{F}}(\lceil \mathsf{dom}\, s_n \rceil \cdot r_{n-1}) \cdots \overline{\mathsf{F}}r_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \overline{\mathsf{F}}s_{n-1} \cdots \overline{\mathsf{F}}r_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \overline{\mathsf{F}}s_{n-1} \cdot \overline{\mathsf{F}}\lceil \mathsf{dom}\, s_{n-1} \rceil \cdot \overline{\mathsf{F}}r_{n-2} \cdots \overline{\mathsf{F}}r_1 \\
&\quad\ \ \vdots \\
&\quad = \overline{\mathsf{F}}s_n \cdots \overline{\mathsf{F}}s_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdots \overline{\mathsf{F}}s_2 \cdot \overline{\mathsf{F}}\lceil \mathsf{cod}\, t_1 \rceil \cdot \overline{\mathsf{F}}t_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \ldots \overline{\mathsf{F}}(s_2 \cdot \lceil \mathsf{cod}\, t_1 \rceil) \cdot \overline{\mathsf{F}}t_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdot \ldots \overline{\mathsf{F}}t_2 \cdot \overline{\mathsf{F}}t_1 \\
&\quad = \overline{\mathsf{F}}s_n \cdots \overline{\mathsf{F}}s_3 \cdot \overline{\mathsf{F}}\lceil \mathsf{cod}\, t_2 \rceil \cdot \overline{\mathsf{F}}t_2 \cdot \overline{\mathsf{F}}t_1 \\
&\quad\ \ \vdots \\
&\quad = \overline{\mathsf{F}}t_n \cdots \overline{\mathsf{F}}t_1.
\end{aligned}
$$

Furthermore, given spans $X \xleftarrow{f} R \xrightarrow{g} Y$, $Y \xleftarrow{f'} R \xrightarrow{g'} Z$ and a monomorphism $i \colon Y \rightarrowtail A$, we have $g' \cdot f'^{\circ} \cdot i^{\circ} \cdot i \cdot g \cdot f^{\circ} = g' \cdot f'^{\circ} \cdot g \cdot f^{\circ}$. Therefore, as F preserves monomorphisms, we obtain a Barr upper bound $t'_1, \dots, t'_n$ of $r_1, \dots, r_n$ consisting only of total and surjective relations by (co)restricting t_i to its (co)domain (Note that as $t_n \cdot \ldots \cdot t_1 = 1_X$, t_1 is total and t_n is surjective). ◀

A.17 Proof of Lemma 4.20

First note that $r_3 \cdot r_2 \leq r_1^{\circ}$ and $r_2 \cdot r_1 \leq r_3^{\circ}$ since $1_{X_1} \leq r_1 \cdot r_1^{\circ}$ and $1_{X_2} \leq r_3^{\circ} \cdot r_3$. Since $r_2 \leq \hat{r}_2 = \bigvee_{n \in \mathbb{N}} r_2 \cdot (r_2^{\circ} \cdot r_2)^n$, we show that $r_3 \cdot r_2 \cdot (r_2^{\circ} \cdot r_2)^n \cdot r_1 \leq 1_X$, for all $n \in \mathbb{N}$. We proceed by induction on n, with the base case $n = 0$ being the hypothesis of the lemma. Assuming that the assertion is true for $n \in \mathbb{N}$, we calculate

$$r_3 \cdot r_2 \cdot (r_2^\circ \cdot r_2)^n \cdot r_2^\circ \cdot r_2 \cdot r_1 \leq r_1^\circ \cdot (r_2^\circ \cdot r_2)^n \cdot r_2^\circ \cdot r_3^\circ = (r_3 \cdot r_2 \cdot (r_2^\circ \cdot r_2)^n \cdot r_1)^\circ \leq 1_X^\circ = 1_X.$$

◀