BARR-COEXACTNESS FOR METRIC COMPACT HAUSDORFF SPACES

MARCO ABBADINI AND DIRK HOFMANN

Abstract. Compact metric spaces form an important class of metric spaces, but the category that they define lacks many important properties such as completeness and cocompleteness. In recent studies of "metric domain theory" and Stone-type dualities, the more general notion of a (separated) metric compact Hausdorff space emerged as a metric counterpart of Nachbin's compact ordered spaces. Roughly speaking, a metric compact Hausdorff space is a metric space equipped with a *compatible* compact Hausdorff topology (which does not need to be the induced topology). These spaces maintain many important features of compact metric spaces, and, notably, the resulting category is much better behaved. Moreover, one can use inspiration from the theory of Nachbin's compact ordered spaces to solve problems for metric structures.

In this paper we continue this line of research: in the category of separated metric compact Hausdorff spaces we characterise the regular monomorphisms as the embeddings and the epimorphisms as the surjective morphisms. Moreover, we show that epimorphisms out of an object X can be encoded internally on X by their kernel metrics, which are characterised as the continuous metrics below the metric on X; this gives a convenient way to represent quotient objects. Finally, as the main result, we prove that its dual category has an algebraic flavour: it is Barr-exact. While we show that it cannot be a variety of finitary algebras, it remains open whether it is an infinitary variety.

Contents

2020 *Mathematics Subject Classification.* 06F30, 54E45, 54F05, 18E08.

Key words and phrases. Metric compact Hausdorff space, Stone-type duality, Nachbin space, compact ordered space, regular category, exact category, quantale enriched category.

The first author's research was supported by the UK Research and Innovation (UKRI) under the UK government's Horizon Europe funding guarantee (Project "DCPOS", grant number EP/Y015029/1) and by the Italian Ministry of University and Research through the PRIN project n. 20173WKCM5 *Theory and applications of resource sensitive logics*.

The second author acknowledges support by Fundação para a Ciência e a Tecnologia (FCT) under CIDMA Grants UIDB/04106/2020 (https://doi.org/10.54499/UIDB/04106/2020) and UIDP/04106/2020 (https://doi.org/10.54499/UIDP/04106/2020).

1. Introduction

Since the early work of Pontrjagin [Pon34] and Stone [Sto36], it is known that the dual of many categories of topology have an algebraic flavour: the category of compact Hausdorff Abelian groups is dually equivalent to the category of Abelian groups, and the category of Boolean spaces (particular compact Hausdorff spaces) is dually equivalent to the category of Boolean algebras. Extending the latter fact, Duskin [Dus69] observed that the dual of the category CH of compact Hausdorff spaces and continuous maps is algebraic over Set. From a more concrete perspective, it is essentially shown in [Gel41] that $\mathsf{CH}^{\mathsf{op}}$ is equivalent to the category of commutative C^*-algebras and homomorphisms, and this category is indeed algebraic over Set with respect to the unit ball functor, as shown in [Neg71]. We also recall that, by the work of Stone [Sto38] and Priestley [Pri70, Pri72], the category of Priestley spaces (Boolean spaces with a compatible partial order) and continuous monotone maps is dually equivalent to the category of distributive lattices and homomorphisms and therefore its dual category is also a variety. Somewhat surprisingly, a similar investigation for related structures such as Nacbin's compact ordered spaces [Nac65] was carried out only recently. In [HNN18] it is shown that the dual of the category PosCH of compact ordered spaces and continuous monotone maps is a quasivariety over Set, and in [Abb19, AR20] it is finally shown that $\mathsf{PosCH}^{\mathsf{op}}$ is also exact, and hence a variety. In this paper we extend this line of research to include also metric structures.

With no doubt, the class of compact metric spaces (those metric spaces whose induced topology is compact) is an important class of metric spaces; however, together with non-expansive maps, this class forms a poorly behaved category. Firstly, we cannot even form the coproduct of two singleton spaces, a shortcoming we can easily overcome by allowing the distance ∞. This modification allows us also to consider the sup-metric on an infinite product, which is indeed the product metric. However, in general, the product metric does not induce the product topology and is therefore not necessarily compact. For instance, for the two-element space $2 = \{0, 1\}$ with distance 1 between 0 and 1, in $2^{\mathbb{N}}$ the distance between two different points is also 1 and therefore, while the topological power $2^{\mathbb{N}}$ is compact, the topology induced by the metric is discrete and hence non-compact. We take this discrepancy as a motivation to consider not only metric spaces with an *induced* compact (Hausdorff) topology but rather equipped with a *compatible* compact Hausdorff topology.

This notion is also inspired by Nachbin's definition of a compact ordered space (see [Nac65] and also [Tho09]), where a compact Hausdorff space is equipped with a *compatible* order relation. In fact, ordered sets can be seen as a special case of metric structures by just dropping the symmetry axiom from the definition of a metric space. Accordingly, to include also the ordered case in our investigation, we consider here *metric spaces* in a more general sense: a metric d on a set X is a map $d\colon X \times X \to [0, \infty]$ which is only required to satisfy

$$d(x, x) = 0 \quad \text{and} \quad d(x, z) \leq d(x, y) + d(y, z)$$

for all $x, y, z \in X$. These two axioms are analogous to the reflexivity and the transitivity conditions of a preorder. Under this analogy, the metric counterpart of anti-symmetry requires that, for all $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies $x = y$; a metric space (X, d) where d has this property is called *separated*. Let us note that the analogy between metric and order structures can be made more precise in the setting of enriched categories. For instance, the transitivity condition of a preorder and the triangular inequality of a metric space can be both seen as an instance of the composition law of a category; similarly, reflexivity and the condition $d(x, x) = 0$ correspond to the identity law of a category; see [Law73] for details.

In this paper we are interested in the category of compact Hausdorff spaces equipped with a *compatible* separated metric (called separated metric compact Hausdorff spaces here, see Section 2 for details) and continuous non-expansive maps. This category constitutes a natural common roof for the category of compact ordered spaces and continuous monotone maps as well as the category of compact metric spaces and non-expansive maps. Moreover, it has more pleasant properties than the latter one as it is, for instance, complete and cocomplete (see [Tho09]). Various properties and constructions of compact metric spaces and of compact ordered spaces can be naturally extended to this category. For instance, every metric compatible with some compact Hausdorff topology is Cauchy complete (see [HR18]), and [HN20] introduces a Hausdorff functor on this category combing naturally the Hausdorff metric and the Vietoris topology. We also point out that this type of spaces proved to be useful in an extension of Stone-type dualities and of the notion of continuous lattice to metric structures (see [GH13, HN18, HN23]).

Motivated by the corresponding results for compact Hausdorff spaces and compact ordered spaces, in this note we investigate the algebraic character of the dual of the category of separated metric compact Hausdorff spaces and continuous non-expansive maps. Our main result shows that this category is *Barr-exact*. To achieve this, in Section 3 we characterise surjective morphisms with domain X in terms of a certain metric on X, which should be seen as a metric counterpart to the fact that surjective maps with domain X are essentially in bijection with equivalence relations on X. In order to prove (co)regularity, in Section 4 we investigate pushouts of embeddings and also show that the embeddings are precisely the regular monomorphism and that the epimorphisms are precisely the surjective morphisms. Finally, in Section 5 we prove that equivalence (co)relations are effective.

2. Separated metric compact Hausdorff spaces

We start by recalling that in this paper the designation "metric" has a quite inclusive meaning.

Definition 2.1. A *metric* is a map $d\colon X \times X \to [0, \infty]$ which is only required to satisfy

$$d(x, x) = 0 \quad \text{and} \quad d(x, z) \leq d(x, y) + d(y, z),$$

for all $x, y, z \in X$. Furthermore, we say that d is *separated* whenever $d(x, y) = 0 = d(y, x)$ implies $x = y$, for all $x, y \in X$.

Without further ado, let us introduce the main objects of interest: separated metric compact Hausdorff spaces.

Definition 2.2 ([HR18])**.** A *(separated) metric compact Hausdorff space* X is a compact Hausdorff space X together with a (separated) metric $d\colon X \times X \to [0, \infty]$ that is continuous with respect to the upper topology of $[0, \infty]$.

We recall that the open subsets of the upper topology on $[0, \infty]$ are generated by the sets $]u, \infty]$; hence, the non-empty closed subsets of $[0, \infty]$ are of the form $[0, u]$, with $u \in [0, \infty]$. Throughout this paper we will often make use of the fact that, with respect to the upper topology, every non-empty compact subset of $[0,\infty]$ has a smallest element. The space $[0,\infty]$ with the upper topology should be seen as the metric counterpart of the Sierpiński space $\{0,1\}$ with $\{1\}$ closed. Accordingly, the notion of a separated metric compact Hausdorff space is the metric counterpart of Nachbin's compact ordered spaces. A more general approach to ordered and metric (and other) topological structures is given in [Tho09], see Remark 2.8.

The continuity with respect to the upper topology of $[0,\infty]$ of a function $f\colon X \to [0,\infty]$, with X a topological space, is also known as *lower semicontinuity* of f (for instance, see [Bou98, IV.6.2]), and is equivalently described by the pointwise formula (for $x_0 \in X$)

$$f(x_0) \leq \liminf_{x \to x_0} f(x)$$

(or, equivalently, $f(x_0) = \liminf_{x \to x_0} f(x)$). Therefore, the compatibility between the metric and the topology in Definition 2.2 amounts to the requirement that for all $x_0, y_0 \in X$ we have

$$d(x_0,y_0) \leq \liminf_{\substack{x \to x_0 \\ y \to y_0}} d(x,y).$$

Example 2.3. Every classical metric space whose induced topology is compact can be viewed as a separated metric compact Hausdorff space. More generally, every separated metric space whose induced symmetric topology is compact can be viewed as a separated metric compact Hausdorff space (see [HR18]). Here we recall from [Fla97] that the *topology symmetrically induced by a metric* $d\colon X \times X \to [0,\infty]$ is generated by the right and left open balls (with $x_0 \in X$ and $u \in [0,\infty]$)

$$B_r(x_0,u) \coloneqq \{x \in X \mid d(x_0,x) < u\} \quad \text{and} \quad B_l(x_0,u) \coloneqq \{x \in X \mid d(x,x_0) < u\}.$$

This topology is also generated by the symmetric open balls

$$B(x_0,u) \coloneqq B_r(x_0,u) \cap B_l(x_0,u) = \{x \in X \mid d(x_0,x) < u \text{ and } d(x,x_0) < u\},$$

with $x_0 \in X$ and $u \in [0,\infty]$ (see [Fla97, Theorem 4.8]).

Next, we provide an example of a metric space whose induced topology is not compact, but which admits a natural compatible compact Hausdorff topology.

Example 2.4. The interval $[0,\infty]$ becomes a separated metric space with the metric d defined by

$$d(u,v) = \begin{cases} |v-u| & \text{if } u,v < \infty, \\ 0 & \text{if } u = v = \infty, \\ \infty & \text{otherwise.} \end{cases}$$

This metric induces the topology on $[0,\infty]$ generated by the symmetric open balls (with $u \in [0,\infty]$ and $\varepsilon > 0$)

$$\{v \in [0,\infty] \mid d(u,v) < \varepsilon\}.$$

We emphasise that this topology on $[0,\infty]$ is not compact: for instance, the sequence $(n)_{n\in\mathbb{N}}$ does not have a convergent subsequence. However, we may consider the *Lawson topology* [GHK$^+$03] on $[0,\infty]$ which is generated by the basic open subsets $[0,u[$ and $]u,\infty]$, with $u \in [0,\infty]$. With respect to this topology, the interval $[0,\infty]$ is a compact Hausdorff space, and together with the metric d it becomes a separated metric compact Hausdorff space.

In a metric compact Hausdorff space, the compatibility between the metric and the topology is weaker than the one when considering the topology induced by the metric. Roughly speaking, in the latter case open balls are open and, consequently, closed balls are closed, whereby in a metric compact Hausdorff space closed balls are closed, but open balls need not be open. For example, take any compact Hausdorff space with the discrete metric, which assigns distance 1 to each pair of distinct points. Every singleton is an "open ball" of radius $\frac{1}{2}$, but it might fail to be open. The following result makes this relationship more precise.

Proposition 2.5. *Let X be a compact Hausdorff space and d be a separated metric on X. Then d symmetrically induces the topology of X if and only if the map*

$$d\colon X \times X \longrightarrow [0, \infty]$$

is continuous with respect to the lower topology on $[0, \infty]$.

Proof. If the topology on X is symmetrically induced by the metric d, then, for every $u \in [0, \infty]$, the set

$$\{(x, y) \in X \times X \mid d(x, y) < u\} = \bigcup_{z,\, u_1+u_2 \leq u} B_l(z, u_1) \times B_r(z, u_2)$$

is open in $X \times X$. Hence, $d\colon X \times X \to [0, \infty]$ is continuous with respect to the lower topology on $[0, \infty]$. Conversely, assume now that the metric is continuous with respect to the lower topology on $[0, \infty]$. Since d is separated, the topology induced by d is Hausdorff. Moreover, by continuity of d, the sets

$$\{y \in X \mid d(x, y) < u\} \quad \text{and} \quad \{y \in X \mid d(y, x) < u\}$$

are open also in the given compact Hausdorff topology, and hence both topologies coincide. □

Corollary 2.6. *Let X be a compact Hausdorff space and d be a separated metric on X that does not attain the value ∞. Then d symmetrically induces the topology of X if and only if the map*

$$d\colon X \times X \longrightarrow [0, \infty[$$

is continuous with respect to the Euclidean topology on $[0, \infty[$.

Proof. If d symmetrically induces the topology on X, then by Proposition 2.5 and Example 2.3, $d\colon X \times X \to [0, \infty[$ is continuous with respect to the lower and the upper topology on $[0, \infty[$, and hence also with respect to the Euclidean topology. On the other hand, if $d\colon X \times X \to [0, \infty[$ is continuous with respect to the Euclidean topology, then d is also continuous with respect to the lower topology on $[0, \infty[$ and therefore, by Proposition 2.5, it symmetrically induces the topology on X. □

Remark 2.7. Proposition 2.5 is a generalisation of the following well-known fact for partially ordered topological spaces. Let X be a compact Hausdorff space and $E \subseteq X \times X$ a partial order on X. If E is open in $X \times X$, then E is also closed and, moreover, the topology of X is generated by the sets $\downarrow x$ and $\uparrow x$, with $x \in X$ (see also [Nac65, Theorem 5]).

Remark 2.8. The definition of a metric compact Hausdorff space was pulled out of the hat in Definition 2.2. However, it is not an arbitrary condition, but rather the specialisation to the metric setting of a more general notion. To explain the rationale behind it, the starting point is the observation made in [Tho09] that the ultrafilter monad on Set can be naturally extended to a monad on the category of preordered sets and monotone maps and on the category of (Lawvere) metric spaces and non-expansive maps, respectively. In the preordered case, the algebras for this monad are precisely Nachbin's preordered compact Hausdorff spaces [Nac65], and therefore the algebras in the metric case constitute a natural metric counterpart to preordered compact Hausdorff spaces. It is shown in [HR18] that this algebraic description is equivalent to the condition in Definition 2.2.

A map $f\colon X \to X'$ between metric spaces (X, d) and (X', d') is *non-expansive* whenever $d'(f(x), f(y)) \leq d(x, y)$, for all $x, y \in X$. A morphism of metric compact Hausdorff spaces is a map that is both non-expansive and continuous.

Definition 2.9. We denote the category of separated metric compact Hausdorff spaces and morphisms by $\mathsf{MetCH}_{\mathsf{sep}}$.

Below we collect some important properties of $\mathsf{MetCH}_{\mathsf{sep}}$; for more details we refer to [Tho09, GH13, HR18, HN20].

Theorem 2.10. *The category* $\mathsf{MetCH}_{\mathsf{sep}}$ *is complete and cocomplete. In particular:*

1. *The limit of a diagram* $D\colon \mathsf{I} \to \mathsf{MetCH}_{\mathsf{sep}}$ *is given by the limit* $(p_i\colon X \to D(i))_{i\in\mathsf{I}}$ *in* CH *equipped with the* sup-metric
$$d(x, y) = \sup_{i\in\mathsf{I}} d_i(p_i(x), p_i(y)), \quad (x, y \in X),$$
where d_i *denotes the metric of the space* $D(i)$*.*
2. *The coproduct* $X = X_1 + X_2$ *of metric compact Hausdorff spaces* X_1 *and* X_2 *with metrics* d_1 *and* d_2*, respectively, is given by their disjoint union equipped with the coproduct topology and the coproduct metric, that is, the metric* d *defined by*
$$d(x, y) = \begin{cases} d_1(x, y) & \text{if } x, y \in X_1, \\ d_2(x, y) & \text{if } x, y \in X_2, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.11. *The category* $\mathsf{MetCH}_{\mathsf{sep}}$ *has (surjection,embedding)-factorisations, that is, every morphism* $f\colon X \to Y$ *in* $\mathsf{MetCH}_{\mathsf{sep}}$ *can be factorised as* $f = e \cdot g$ *where* g *is a surjective morphism and* e *is an embedding. Here, an embedding in* $\mathsf{MetCH}_{\mathsf{sep}}$ *is an injective morphism such that the metric on the domain is the restriction of the metric on the codomain.*

In Section 4 we show that in $\mathsf{MetCH}_{\mathsf{sep}}$ the surjective morphisms are precisely the epimorphisms and the embeddings are precisely the regular monomorphisms (see Propositions 4.7 and 4.8).

Remark 2.12 (Compact ordered spaces as separated metric compact Hausdorff spaces)**.** We recall from [Nac65] that a *compact ordered space* (also called *compact pospace* or *Nachbin space*) consists of a compact space X together with a partial order $\leq$ on X so that the set $\{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$; such a space X is automatically Hausdorff. The category of compact ordered spaces and monotone and continuous maps is denoted by PosCH.

Every compact ordered space X can be thought of as a separated metric compact Hausdorff space with the same compact Hausdorff space and the metric d defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x \leq y, \\ \infty & \text{otherwise.} \end{cases}$$

In fact, compact ordered spaces can be identified with the separated metric compact Hausdorff spaces whose metric takes values in $\{0,\infty\}$. Since every map between compact ordered spaces is monotone if and only if it is non-expansive with respect to the corresponding metrics, we obtain a fully faithful functor

$$\mathsf{PosCH} \longrightarrow \mathsf{MetCH}_{\mathsf{sep}}.$$

This functor has a right adjoint which leaves maps unchanged and sends a separated metric compact Hausdorff space X (with metric d) to the compact ordered space X with the same topology and the partial order given by

$$x \leq y \quad \text{whenever} \quad d(x,y) = 0.$$

In particular, we conclude that PosCH is closed in $\mathsf{MetCH}_{\mathsf{sep}}$ under colimits, and it is easy to see that PosCH is also closed in $\mathsf{MetCH}_{\mathsf{sep}}$ under limits. The unit interval $[0,1]$ is a cogenerator in PosCH and PosCH is well-powered; therefore, the Special Adjoint Functor Theorem guarantees that $\mathsf{PosCH} \to \mathsf{MetCH}_{\mathsf{sep}}$ has also a left adjoint.

$$\mathsf{PosCH} \underset{\bot}{\overset{\bot}{\longrightarrow}} \mathsf{MetCH}_{\mathsf{sep}}$$

Remark 2.13 (Symmetrisation)**.** Every metric space X with metric d can be symmetrised by putting $d_s(x,y) = \max\{d(x,y), d(y,x)\}$ (see [Law73]). This metric is compatible with the topology: if $d\colon X \times X \to [0,\infty]$ is continuous, then so is $d_s\colon X \times X \to [0,\infty]$, since the map

$$\max\colon [0,\infty] \times [0,\infty] \longrightarrow [0,\infty]$$

is continuous with respect to the upper topology. In fact, this construction defines a right adjoint to the full embedding

$$\mathsf{MetCH}_{\mathsf{sep,sym}} \longrightarrow \mathsf{MetCH}_{\mathsf{sep}}$$

of the category $\mathsf{MetCH}_{\mathsf{sep,sym}}$ of symmetric separated metric compact Hausdorff spaces and continuous non-expansive maps into $\mathsf{MetCH}_{\mathsf{sep}}$. Similarly to Remark 2.12, we conclude that the category $\mathsf{MetCH}_{\mathsf{sep,sym}}$ is closed under limits and colimits in $\mathsf{MetCH}_{\mathsf{sep}}$. Therefore the inclusion functor $\mathsf{MetCH}_{\mathsf{sep,sym}} \to \mathsf{MetCH}_{\mathsf{sep}}$ has also a left adjoint by the Adjoint Functor Theorem (the solution set condition is trivial).

Below we briefly indicate an example where metric compact Hausdorff spaces played a crucial role.

Example 2.14. It is well-known that every classical metric space with compact *induced* topology is Cauchy complete. However, to infer Cauchy completeness of a metric space it suffices to show that there is a *compatible* compact Hausdorff topology, that is, the metric space is part of a metric compact Hausdorff space. To give a trivial example, consider an arbitrary product of classical metric spaces with compact induced topology. The product metric does not in general induce a compact topology; however, this metric is Cauchy complete because the product topology is compact Hausdorff and compatible. Less trivially, for every metric space (X, d), the space UX of all ultrafilters on X equipped with the metric

$$h(\mathcal{U}, \mathcal{V}) = \sup_{(A \in \mathcal{U},\, B \in \mathcal{V})} \inf_{(x \in A,\, y \in B)} d(x, y)$$

is Cauchy complete because the Zariski topology on UX (which is independent of d) is compatible with the metric h (see [HR18]).

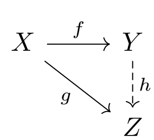
The rest of the paper is devoted to assessing a further feature of $\mathsf{MetCH}_{\mathsf{sep}}$, which exposes an algebraic flavour of the dual category $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$. We recall (see [Bor94], for instance) that a category C is *regular* whenever C is finitely complete, has pushouts of kernel pairs and regular epimorphisms are pullback-stable. Moreover, C is *Barr-exact* whenever it is regular and every internal equivalence relation is effective, *i.e.*, a kernel pair. Being Barr-exact expresses an algebraic trait: for example, every variety of (possibly infinitary) algebras is Barr-exact; on the other hand, the category of topological spaces and continuous maps is not. Barr-exactness distinguishes varieties from quasivarieties and can be seen as a categorical way to express the property of being "closed under quotients of congruences". The main result of this paper states that the category $\mathsf{MetCH}_{\mathsf{sep}}$ is Barr-coexact, that is, $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ is Barr-exact. Since $\mathsf{MetCH}_{\mathsf{sep}}$ is complete and cocomplete, to obtain this result we show that

- the regular monomorphisms and the epimorphisms in $\mathsf{MetCH}_{\mathsf{sep}}$ are precisely the embeddings and the surjective morphisms, respectively,
- embeddings are stable under pushouts, and
- equivalence corelations are effective.

3. Quotient objects and continuous submetrics

In this paper we will study equivalence relations in $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ by looking at their dual in $\mathsf{MetCH}_{\mathsf{sep}}$, which are particular epimorphisms. In this section we prepare the ground by giving a description of surjective morphisms in $\mathsf{MetCH}_{\mathsf{sep}}$ (= epimorphisms by Proposition 4.8 below) which is similar to the presentation of surjections out of a set by equivalence relations.

Definition 3.1 (Quotient object)**.** We let $\mathbf{Q}(X)$ denote the class of surjective morphisms of separated metric compact Hausdorff spaces with domain X. We consider $\mathbf{Q}(X)$ as the full subcategory of the *coslice category* $X \downarrow \mathsf{MetCH}_{\mathsf{sep}}$ of $\mathsf{MetCH}_{\mathsf{sep}}$ over X, which is the category whose objects are the morphisms in $\mathsf{MetCH}_{\mathsf{sep}}$ with domain X and whose morphisms from an object $f\colon X \to Y$ to an object $g\colon X \to Z$ are the morphisms $h\colon Y \to Z$ in $\mathsf{MetCH}_{\mathsf{sep}}$ such that the following triangle commutes.



Since surjective morphisms are epimorphisms in $\mathsf{MetCH}_{\mathsf{sep}}$, the category $\mathbf{Q}(X)$ is actually a preordered class:

$$f \leq g \quad \iff \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g} & \downarrow \exists h \\ & & Z. \end{array}$$

Roughly speaking, thinking of a quotient object as a partition, the preorder on $\mathbf{Q}(X)$ is the "finer than" preorder.

There is a standard way in which a partially ordered set is obtained from a preordered set, i.e. by identifying elements of an equivalence class. In the same fashion, from $\mathbf{Q}(X)$ we obtain a partially ordered class (in fact, a set) $\tilde{\mathbf{Q}}(X)$, whose elements we call *quotient objects* of X. Explicitly, a quotient object is an equivalence class of surjective morphisms of separated metric compact Hausdorff spaces with domain X, where two surjective morphisms $f\colon X \twoheadrightarrow Y$ and $g\colon X \twoheadrightarrow Z$ are equivalent if and only if there is an isomorphism $h\colon Y \to Z$ such that $hf = g$; moreover, the equivalence class of $f\colon X \twoheadrightarrow Y$ is below the equivalence class of $g\colon X \twoheadrightarrow Z$ if and only if there is a morphism $h\colon Y \to Z$ such that $hf = g$. With a little abuse of notation, we take the liberty to refer to an element of $\tilde{\mathbf{Q}}(X)$ just with one of its representatives.

Our next goal is to encode quotient objects of X internally on X. To make a parallelism: by [Eng89, The Alexandroff Theorem 3.2.11][1], in the category CH of compact Hausdorff spaces and continuous functions, a surjective morphism $f\colon X \twoheadrightarrow Y$ is encoded by the equivalence relation $\sim_f \coloneqq \{(x, y) \in X \times X \mid f(x) = f(y)\}$; the equivalence relation $\sim_f$ is closed, and, in fact, there is a bijection between equivalence classes of surjective morphisms of compact Hausdorff spaces with domain X and closed equivalence relations on X. There are also analogous versions for Boolean spaces, namely *Boolean relations*[2] [GH09, Lemma 1, Chapter 37], for Priestley spaces, namely *lattice preorders* [CLP91, Definition 2.3][3], and for Nachbin's compact ordered spaces [AR20, Lemma 11].

In the case of separated metric compact Hausdorff spaces, we encode a quotient object $f\colon X \twoheadrightarrow Y$ of X via a certain (possibly non-separated) metric κ_f on X, as follows.

Definition 3.2 (Kernel metric)**.** Given a morphism $f\colon X \to Y$ in $\mathsf{MetCH}_{\mathsf{sep}}$, we set, for $x_1, x_2 \in X$,

$$\kappa_f(x_1, x_2) \coloneqq d_Y(f(x_1), f(x_2)).$$

We call the function $\kappa_f\colon X \times X \to [0, \infty]$ the *kernel metric* of f.

We will prove that κ_f is indeed a (not necessarily separated) metric on X, which justifies the nomenclature *kernel metric* (Lemma 3.5).

Example 3.3. If Y is a compact Hausdorff space equipped with the metric assigning distance ∞ to any pair of distinct points, and $f\colon X \to Y$ is a morphism of

[1] The reader is warned that, in [Eng89], by 'compact space' is meant what we here call a compact *Hausdorff* space.

[2] Sometimes called *Boolean equivalences.*

[3] Lattice preorders are also called *Priestley quasiorders* ([Sch02, Definition 3.5]), or *compatible quasiorders.*

separated metric compact Hausdorff spaces, then

$$\kappa_f(x,y) = \begin{cases} \infty & \text{if } f(x) \neq f(y), \\ 0 & \text{if } f(x) = f(y). \end{cases}$$

Therefore we may identify κ_f with the closed equivalence relation

$$\sim_f = \{(x,y) \in X \times X \mid f(x) = f(y)\} = \{(x,y) \in X \times X \mid \kappa_f(x,y) = 0\},$$

that is, the specialisation to compact Hausdorff spaces of this approach is precisely the one discussed before Definition 3.2.

Definition 3.2 will be relevant especially for f a surjective morphism. The idea is that a surjective morphism f can be completely recovered from κ_f (up to an isomorphism). In order to establish an inverse for the assignment that maps f to its kernel metric κ_f, we investigate the properties satisfied by kernel metrics: these properties are precisely being a continuous metric below the given metric (Theorem 3.14).

Definition 3.4 (Continuous submetric)**.** Let X be a separated metric compact Hausdorff space with metric d. A *continuous submetric* γ on X is a (not necessarily separated) continuous metric $\gamma\colon X \times X \to [0,\infty]$ with respect to the upper topology of $[0,\infty]$ which is below the given metric d, i.e., for all $x, y \in X$, $\gamma(x,y) \leq d(x,y)$.

Lemma 3.5. *The kernel metric $\kappa_f\colon X \times X \to [0,\infty]$ of a morphism $f\colon X \to Y$ in $\mathsf{MetCH}_{\mathsf{sep}}$ is a continuous submetric on X.*

Proof. For all $x \in X$ we have $\kappa_f(x,x) = d_Y(f(x), f(x)) = 0$. Moreover, for all $x, y, z \in X$ we have

$$\kappa_f(x,y) = d_Y(f(x), f(y)) \leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) = \kappa_f(x,y) + \kappa_f(y,z).$$

Therefore, κ_f is a (possibly non-separated) metric. The function $\kappa_f\colon X \times X \to [0,\infty]$ is continuous (with respect to the upper topology) because it is the composite of the two continuous functions

$$X \times X \xrightarrow{f \times f} Y \times Y \xrightarrow{d_Y} [0,\infty].$$

Since f is non-expansive, for all $x, y \in X$ we have $\kappa_f(x,y) = d_Y(f(x), f(y)) \leq d_X(x,y)$, and therefore κ_f is below d_X. □

Remark 3.6. We shall now see how one recovers a surjective morphism f from its kernel metric κ_f. Let (X,d) be a separated metric compact Hausdorff space and let γ be a continuous submetric on X. Then take the separation-reflection $X/{\sim_\gamma}$ of X with respect to γ (for instance, see [HN20]). We recall that $X/{\sim_\gamma}$ is the quotient of X with respect to the equivalence relation $\sim_\gamma$ defined by

$$x \sim_\gamma y \iff \gamma(x,y) = \gamma(y,x) = 0;$$

the topology is the quotient topology, and the metric is $d_{X/{\sim_\gamma}}([x],[y]) = \gamma(x,y)$. Since γ is below d, the function $X \twoheadrightarrow X/{\sim_\gamma}$ is non-expansive: $d_{X/{\sim_\gamma}}([x],[y]) = \gamma(x,y) \leq d(x,y)$. Moreover, it is continuous because $X/{\sim_\gamma}$ is equipped with the quotient topology. Thus, the function $X \twoheadrightarrow X/{\sim_\gamma}$ is a surjective morphism in $\mathsf{MetCH}_{\mathsf{sep}}$.

Definition 3.7. For X a separated metric compact Hausdorff space, we let $\mathbf{S}(X)$ denote the set of continuous submetrics on X. We equip $\mathbf{S}(X)$ with the pointwise partial order, i.e. $\gamma_1 \leq \gamma_2$ whenever, for all $x, y \in X$, $\gamma_1(x,y) \leq \gamma_2(x,y)$.

Our goal, met in Theorem 3.14, is to prove that the assignments

$$\begin{aligned}\tilde{\mathbf{Q}}(X) &\longrightarrow \mathbf{S}(X) \\ \big(f\colon X \twoheadrightarrow Y\big) &\longmapsto \kappa_f\end{aligned} \qquad\qquad \begin{aligned}\mathbf{S}(X) &\longrightarrow \tilde{\mathbf{Q}}(X) \\ \gamma &\longmapsto \big(X \twoheadrightarrow X/{\sim_\gamma}\big)\end{aligned}$$

establish a dual isomorphism between the partially ordered sets of quotient objects of X and of continuous submetrics on X. This will allow us to work with $\mathbf{S}(X)$ instead of $\tilde{\mathbf{Q}}(X)$.

3.0.1. *The dual adjunction between the coslice category over X and $\mathbf{S}(X)$.* For the rest of this section, we fix a separated metric compact Hausdorff space X. In Theorem 3.14 below, we will prove the correspondence between quotient objects on X and continuous submetrics on X. To do so, we start by establishing in Lemma 3.10 below a dual adjunction between the coslice category $X \downarrow \mathsf{MetCH_{sep}}$ and the poset $\mathbf{S}(X)$ regarded as a category. From this dual adjunction, we will obtain a dual equivalence between $\mathbf{Q}(X)$ and $\mathbf{S}(X)$ by restricting to the fixed objects, and then an isomorphism between $\tilde{\mathbf{Q}}(X)$ and $\mathbf{S}(X)$ via a certain quotient.

Notation 3.8. We let

$$\begin{aligned}G\colon (X \downarrow \mathsf{MetCH_{sep}}) &\longrightarrow \mathbf{S}(X) \\ \big(f\colon X \to Y\big) &\longmapsto \kappa_f\end{aligned}$$

denote the assignment sending f to its kernel metric κ_f as described in Definition 3.2. Note that κ_f belongs to $\mathbf{S}(X)$ by Lemma 3.5. This assignment can be extended to morphisms so that G becomes a contravariant functor: given $f\colon X \to Y$ and $g\colon X \to Z$ in $X \downarrow \mathsf{MetCH_{sep}}$, and given $h\colon Y \to Z$ such that $g = hf$, we set $G(h)$ as the unique morphism in $\mathbf{S}(X)$ from κ_g to κ_f.

We let

$$\begin{aligned}F\colon \mathbf{S}(X) &\longrightarrow (X \downarrow \mathsf{MetCH_{sep}}) \\ \gamma &\longmapsto \big(X \twoheadrightarrow X/{\sim_\gamma}\big)\end{aligned}$$

denote the assignment described in Remark 3.6. This assignment can be extended to morphisms so that F becomes a contravariant functor: given $\gamma_1, \gamma_2 \in \mathbf{S}(X)$ such that $\gamma_2 \leq \gamma_1$, F maps the unique morphism from γ_2 to γ_1 to the morphism of separated metric compact Hausdorff spaces

$$\begin{aligned}X/{\sim_{\gamma_1}} &\longrightarrow X/{\sim_{\gamma_2}} \\ [x]_{\sim_{\gamma_1}} &\longmapsto [x]_{\sim_{\gamma_2}}.\end{aligned}$$

It is easily seen that the functor $GF\colon \mathbf{S}(X) \to \mathbf{S}(X)$ is the identity functor. We let η denote the identity natural transformation from the identity functor $1_{\mathbf{S}}$ on $\mathbf{S}$ to itself.

For every morphism $f\colon X \to Y$ in $\mathsf{MetCH_{sep}}$, we have a morphism in $\mathbf{Q}(X)$

$$\begin{aligned}\varepsilon_f\colon X/{\sim_{\kappa_f}} &\longrightarrow Y \\ [x] &\longmapsto f(x)\end{aligned}$$

from $FG(f)$ to f.

Claim 3.9. *ε is a natural transformation.*

Proof of claim. Let $f\colon X \to Y$ and $g\colon X \to Z$ be elements of $X \downarrow \mathsf{MetCH_{sep}}$, and let $h\colon Y \to Z$ be such that $g = hf$. We shall prove that the following diagram commutes.

$$\begin{array}{ccc} (X \xrightarrow{\pi} X/{\sim_{\kappa_f}}) & \xrightarrow{\varepsilon_f} & (X \xrightarrow{f} Y) \\ {\scriptstyle FG(h)}\downarrow & & \downarrow{\scriptstyle h} \\ (X \xrightarrow{\pi} X/{\sim_{\kappa_g}}) & \xrightarrow{\varepsilon_g} & (X \xrightarrow{g} Z) \end{array}$$

The commutativity of the diagram above amounts to the commutativity of the following one.

$$\begin{array}{ccc} X/{\sim_{\kappa_f}} & \xrightarrow{\varepsilon_f} & Y \\ {\scriptstyle FG(h)}\downarrow & & \downarrow{\scriptstyle h} \\ X/{\sim_{\kappa_g}} & \xrightarrow{\varepsilon_g} & Z \end{array}$$

For every $x \in X$ we have

$$h(\varepsilon_f([x]_{\sim_{\kappa_f}})) = h(f(x)) = g(x) = \varepsilon_g([x]_{\sim_{\kappa_g}}) = \varepsilon_g(FG(h)(x)).$$

This proves our claim. ☐

Lemma 3.10. *The functor*

$$F\colon \mathbf{S}(X)^{\mathsf{op}} \longrightarrow (X \downarrow \mathsf{MetCH_{sep}})$$

is left adjoint to the functor

$$G\colon (X \downarrow \mathsf{MetCH_{sep}}) \longrightarrow \mathbf{S}(X)^{\mathsf{op}},$$

with unit η and counit ε.

Proof. It remains to prove that the triangle identities hold. One triangle identity is trivial because every diagram in a category arising from a partially ordered set commutes. We now set the remaining triangle identity. Let $\gamma \in \mathbf{S}(X)$. We shall prove that the following diagram commutes.

$$\begin{array}{ccc} F(\gamma) & \xrightarrow{F(\eta_\gamma)} & FGF(\gamma) \\ & {\scriptstyle 1_{F(\gamma)}}\searrow & \downarrow{\scriptstyle \varepsilon_{F(\gamma)}} \\ & & F(\gamma) \end{array}$$

Since GF is the identity functor, and η is the identity natural transformation, the commutativity of the diagram amounts to the fact that $\varepsilon_{F(\gamma)}$ is the identity on $F(\gamma)$, which is not hard to see. ☐

We recall that a morphism in a coslice category $X \downarrow \mathsf{C}$ is an isomorphism in $X \downarrow \mathsf{C}$ if and only if it is an isomorphism in C. Then, we have the following.

Lemma 3.11. *Given an object $f\colon X \to Y$ of $X \downarrow \mathsf{MetCH_{sep}}$, the component of the counit ε at f is an isomorphism if and only if f is surjective.*

Proof. For every $f\colon X \to Y$, the function $\varepsilon_f\colon X/{\sim_\gamma} \to Y$ is an embedding because, for all $x, y \in X$, we have

$$d([x],[y]) = \kappa_f(x,y) = d_Y(f(x), f(y)) = d_Y(\varepsilon_f([x]), \varepsilon_f([y])).$$

Therefore, ε_f is an isomorphism if and only if it is surjective. ☐

We state the following for future reference.

Lemma 3.12. *Let $f\colon X \to Y$ and $g\colon X \to Z$ be morphisms of separated metric compact Hausdorff spaces, and suppose that f is surjective. Then, the condition $\kappa_g \leq \kappa_f$ holds if and only if there is a continuous non-expansive map $h\colon Y \to Z$ such that the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle h} \\ & & Z \end{array}$$

Proof. By Lemmas 3.10 and 3.11. □

We recall that the preordered class $\mathbf{Q}(X)$ of surjective morphisms with domain X is a full subcategory of $X \downarrow \mathsf{MetCH}_{\mathsf{sep}}$.

Lemma 3.13. *The restrictions of the functors F and G to $\mathbf{S}(X)$ and $\mathbf{Q}(X)$ are dual quasi-inverses.*

Proof. By Lemma 3.10, the functor $F\colon \mathbf{S}(X) \to (X \downarrow \mathsf{MetCH}_{\mathsf{sep}})$ is left adjoint to $G\colon (X \downarrow \mathsf{MetCH}_{\mathsf{sep}}) \to \mathbf{S}(X)$. For every $\gamma \in \mathbf{S}(X)$, the component of the unit η at γ is the identity morphism; in particular, it is an isomorphism. As observed in Lemma 3.11, the component of the counit ε at an element $f\colon X \to Y$ of $X \downarrow \mathsf{MetCH}_{\mathsf{sep}}$ is an isomorphism if and only if $f\colon X \to Y$ is surjective (Lemma 3.11). □

We obtain now the main result of this section: for any separated metric compact Hausdorff space X, quotient objects of X bijectively correspond to continuous submetrics on X.

Theorem 3.14. *For every separated metric compact Hausdorff space X, the assignments*

$$\begin{array}{rcl} \tilde{\mathbf{Q}}(X) & \longrightarrow & \mathbf{S}(X) \\ \big(f\colon X \twoheadrightarrow Y\big) & \longmapsto & \kappa_f \end{array} \qquad\qquad \begin{array}{rcl} \mathbf{S}(X) & \longrightarrow & \tilde{\mathbf{Q}}(X) \\ \gamma & \longmapsto & \big(X \twoheadrightarrow X/{\sim_\gamma}\big) \end{array}$$

establish a dual isomorphism between the posets of quotient objects of X and of continuous submetrics on X.

Proof. By Lemma 3.13. □

4. Coregularity for separated metric compact Hausdorff spaces

The purpose of this section is to prove that the category $\mathsf{MetCH}_{\mathsf{sep}}$ is coregular, i.e., that $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ is regular. Recall from Section 2 that one of the conditions of coregularity is that regular monomorphisms are preserved under pushouts. For this reason, we start by investigating pushouts of embeddings, which will be proved to characterise regular monomorphisms. We start with a technical lemma, which will be useful in our description of pushouts of embeddings (Proposition 4.2 and Corollary 4.3).

Lemma 4.1. *Let $i\colon A \hookrightarrow X$ be an embedding and $f\colon A \to B$ a morphism in $\mathsf{MetCH}_{\mathsf{sep}}$. Let $\iota_B\colon B \hookrightarrow B + X$ and $\iota_X\colon X \hookrightarrow B + X$ denote the coproduct maps. The following describes a continuous submetric γ on $B + X$:*

(1) For $b, b' \in B$,

$$\gamma(\iota_B(b), \iota_B(b')) = d_B(b, b').$$

(2) For $x, x' \in X$,

$$\gamma(\iota_X(x), \iota_X(x')) \\ = \min\Big\{ d_X(x, x'),\ \inf_{a,a' \in A} \big(d_X(x, i(a)) + d_B(f(a), f(a')) + d_X(i(a'), x') \big) \Big\}.$$

(3) For $b \in B$ and $x \in X$,

$$\gamma(\iota_B(b), \iota_X(x)) = \inf_{a \in A} \big(d_B(b, f(a)) + d_X(i(a), x) \big),$$
$$\gamma(\iota_X(x), \iota_X(x)) = \inf_{a \in A} \big(d_X(x, i(a)) + d_B(f(a), b) \big).$$

Moreover, for all $a \in A$, we have

$$\gamma(\iota_B f(a), \iota_X i(a)) = \gamma(\iota_X i(a), \iota_B f(a)) = 0.$$

Before starting the proof, we mention that the interest for the metric γ in Lemma 4.1 comes from the fact that it describes the pushout of the diagram $B \xleftarrow{f} A \xhookrightarrow{i} X$ (see Corollary 4.3 below).

Proof of Lemma 4.1. For $x \in X$, $b \in B$ and $a \in A$, we write x for $\iota_X(x)$, b for $\iota_B(b)$, and a for $i(a)$ and $\iota_X(i(a))$.

From (3) in Lemma 4.1, it is clear that, for all $a \in A$,

$$\gamma(f(a), a) = \gamma(a, f(a)) = 0.$$

Case analysis shows immediately that γ is below d_{B+X}. To prove continuity of $\gamma\colon (B + X) \times (B + X) \to [0, \infty]$ with respect to the upper topology of $[0, \infty]$, it is enough to prove that it is continuous over each of its four pieces $B \times B$, $B \times X$, $X \times B$ and $X \times X$. It is clear that γ is continuous over $B \times B$. We show continuity of $B \times X$; the other cases are similar. The function

$$\begin{aligned} B \times X &\longrightarrow [0, \infty] \\ (b, x) &\longmapsto \inf_{a \in A} \big(d_B(b, f(a)) + d_X(a, x) \big) \end{aligned}$$

can be written as the composite of the following two functions

$$\begin{aligned} B \times X &\longrightarrow [0, \infty]^A \\ (b, x) &\longmapsto \big(a \mapsto d_B(b, f(a)) + d_X(a, x) \big) \end{aligned} \qquad \begin{aligned} \inf\colon [0, \infty]^A &\longrightarrow [0, \infty] \\ f &\longmapsto \inf_{a \in A} f(a), \end{aligned}$$

where $[0, \infty]^A$ is the exponential in the category of topological spaces, which is the set of continuous functions from A to $[0, \infty]$ equipped with the compact-open topology, and which exists because A is a compact Hausdorff space (see [EH01], for instance). The first function is continuous because it is the transpose of the function

$$\begin{aligned} A \times B \times X &\longrightarrow [0, \infty] \\ (a, b, x) &\longmapsto d_B(b, f(a)) + d_X(a, x), \end{aligned}$$

which is continuous because d_B, d_X, f, and $+\colon [0, \infty]^2 \to [0, \infty]$ are continuous. The function $\inf\colon [0, \infty]^A \to [0, \infty]$ is continuous because, for every $u \in [0, \infty]$, we have

$$\begin{aligned}
\inf{}^{-1}(]u,\infty]) &= \{\varphi \in [0,\infty]^A \mid \inf \varphi > u\} && \text{by definition of inf} \\
&= \{\varphi \in [0,\infty]^A \mid \exists v > u \text{ s.t. } \inf \varphi \geq v\} && \text{since } \varphi[A] \text{ is compact} \\
&= \bigcup_{v>u} \{\varphi \in [0,\infty]^A \mid \varphi[A] \subseteq]v,\infty]\},
\end{aligned}$$

and this set is open in $[0,\infty]^A$. Hence, their composite $B \times X \to [0,\infty]$ is continuous, as desired.

Let us now prove that γ is a metric.

It is immediate that for all $z \in B + X$ we have $\gamma(z,z) = 0$.

We now prove by cases that γ satisfies the triangle inequality.

Clearly, for all $b, b', b'' \in B$, we have

$$\gamma(b,b'') = d_B(b,b'') \leq d_B(b,b') + d_B(b',b'') = \gamma(b,b') + \gamma(b',b''),$$

proving a case of the triangle inequality.

Let $x, x', x'' \in X$, and let us prove $\gamma(x,x'') \leq \gamma(x,x') + \gamma(x',x'')$. We have

$$\begin{aligned}
\gamma(x,x'') &= \min\left\{ d_X(x,x'),\ \inf_{a,a' \in A} \big(d_X(x,a) + d_B(f(a),f(a')) + d_X(a',x'')\big)\right\} \\
&\leq d_X(x,x'') \\
&\leq d_X(x,x') + d_X(x',x'').
\end{aligned}$$

Moreover, for all $a_0, a_0' \in A$, we have

$$\begin{aligned}
\gamma(x,x'') &= \min\left\{ d_X(x,x'),\ \inf_{a,a' \in A} \big(d_X(x,a) + d_B(f(a),f(a')) + d_X(a',x'')\big)\right\} \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_0')) + d_X(a_0',x'') \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_0')) + d_X(a_0',x') + d_X(x',x'').
\end{aligned}$$

Moreover, for all $a_1, a_1' \in A$, we have

$$\begin{aligned}
\gamma(x,x'') &= \min\left\{ d_X(x,x'),\ \inf_{a,a' \in A} \big(d_X(x,a) + d_B(f(a),f(a')) + d_X(a',x'')\big)\right\} \\
&\leq d_X(x,a_1) + d_B(f(a_1),f(a_1')) + d_X(a_1',x'') \\
&\leq d_X(x,x') + d_X(x',a_1) + d_B(f(a_1),f(a_1')) + d_X(a_1',x'').
\end{aligned}$$

Moreover, for all $a_0, a_0', a_1, a_1' \in A$ we have

$$\begin{aligned}
&\gamma(x,x'') \\
&= \min\left\{ d_X(x,x'),\ \inf_{a,a' \in A} \big(d_X(x,a) + d_B(f(a),f(a')) + d_X(a',x'')\big)\right\} \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_1')) + d_X(a_1',x'') \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_0')) + d_B(f(a_0'),f(a_1)) + d_B(f(a_1),f(a_1')) + d_X(a_1',x'') \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_0')) + d_X(a_0',a_1) + d_B(f(a_1),f(a_1')) + d_X(a_1',x'') \\
&\leq d_X(x,a_0) + d_B(f(a_0),f(a_0')) + d_X(a_0',x') + d_X(x',a_1) + d_B(f(a_1),f(a_1')) + d_X(a_1',x'').
\end{aligned}$$

Therefore, putting the four previous displays together, we obtain

$$\begin{aligned}
&\gamma(x,x'')\\
&\leq \min\Big\{d_X(x,x')+d_X(x',x''),\\
&\inf_{a_0,a_0'\in A}\big(d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')+d_X(x',x'')\big),\\
&\inf_{a_1,a_1'\in A}\big(d_X(x,x')+d_X(x',a_1)+d_B(f(a_1),f(a_1'))+d_X(a_1',x'')\big),\\
&\inf_{a_0,a_0',a_1,a_1'\in A}\big(d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')+d_X(x',a_1)+\\
&\qquad d_B(f(a_1),f(a_1'))+d_X(a_1',x'')\big)\Big\}\\
&=\min\Big\{d_X(x,x'),\ \inf_{a_0,a_0'\in A}\big(d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')\big)\Big\}\\
&\quad+\min\Big\{d_X(x',x''),\ \inf_{a_1,a_1'\in A}\big(d_X(x',a_1)+d_B(f(a_1),f(a_1'))+d_X(a_1',x'')\big)\Big\}\\
&=\gamma(x,x')+\gamma(x',x'').
\end{aligned}$$

This proves another case of the triangle inequality.

Let $x\in X$ and $b,b'\in B$, and let us prove $\gamma(x,b')\leq\gamma(x,b)+\gamma(b,b')$. We have

$$\begin{aligned}
\gamma(x,b')&=\inf_{a\in A}\big(d_X(x,a)+d_B(f(a),b')\big)\\
&\leq\inf_{a\in A}\big(d_X(x,a)+d_B(f(a),b)+d_B(b,b')\big)\\
&=\left(\inf_{a\in A}\big(d_X(x,a)+d_B(f(a),b)\big)\right)+d_B(b,b')\\
&=\gamma(x,b)+\gamma(b,b').
\end{aligned}$$

This proves another case of the triangle inequality. Similarly, one proves $\gamma(b',x)\leq\gamma(b',b)+\gamma(b,x)$.

Now, let $x,x'\in X$ and $b\in B$, and let us prove $\gamma(x,b)\leq\gamma(x,x')+\gamma(x',b)$. For all $a_1\in A$ we have

$$\begin{aligned}
\gamma(x,b)&=\inf_{a\in A}\big(d_X(x,a)+d_B(f(a),b)\big)\\
&\leq d_X(x,a_1)+d_B(f(a_1),b)\\
&\leq d_X(x,x')+d_X(x',a_1)+d_B(f(a),b).
\end{aligned}$$

Moreover, for all $a_0,a_0',a_1\in A$, we have

$$\begin{aligned}
\gamma(x,b)&=\inf_{a\in A}\big(d_X(x,a)+d_B(f(a),b)\big)\\
&\leq d_X(x,a_0)+d_B(f(a_0),b)\\
&\leq d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_B(f(a_0'),f(a_1))+d_B(f(a_1),b)\\
&\leq d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_A(a_0',a_1)+d_B(f(a_1),b)\\
&= d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',a_1)+d_B(f(a_1),b)\\
&\leq d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')+d_X(x',a_1)+d_B(f(a_1),b).
\end{aligned}$$

By combining the two displayed inequalities above, we get

$$\begin{aligned}
&\gamma(x,b)\\
&\leq \min\Big\{\inf_{a_1\in A}\big(d_X(x,x')+d_X(x',a_1)+d_B(f(a_1),b)\big),\\
&\inf_{a_0,a_0',a_1\in A}\big(d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')+d_X(x',a_1)+d_B(f(a_1),b)\big)\Big\}\\
&=\min\Big\{d_X(x,x'),\inf_{a_0,a_0'\in A}\big(d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')\big)\Big\}\\
&\quad+\inf_{a_1\in A}\big(d_X(x',a_1)+d_B(f(a_1),b)\big)\\
&=\gamma(x,x')+\gamma(x',b).
\end{aligned}$$

This proves another case of the triangle inequality. Similarly, one proves $\gamma(b,x)\leq\gamma(b,x')+\gamma(x',x)$.

Now, let $x,x'\in X$ and $b\in B$, and let us prove $\gamma(x,x')\leq\gamma(x,b)+\gamma(b,x')$. For all $a_0,a_0'\in A$ we have

$$\begin{aligned}
\gamma(x,x')&=\min\Big\{d_X(x,x'),\inf_{a,a'\in A}\big(d_X(x,a)+d_B(f(a),f(a'))+d_X(a',x')\big)\Big\}\\
&\leq d_X(x,a_0)+d_B(f(a_0),f(a_0'))+d_X(a_0',x')\\
&\leq d_X(x,a_0)+d_B(f(a_0),b)+d_B(b,f(a_0'))+d_X(a_0',x').
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(x,x')&\leq\inf_{a_0,a_0'\in A}\big(d_X(x,a_0)+d_B(f(a_0),b)+d_B(b,f(a_0'))+d_X(a_0',x')\big)\\
&=\inf_{a_0\in A}\big(d_X(x,a_0)+d_B(f(a_0),b)\big)+\inf_{a_0'\in A}\big(d_B(b,f(a_0'))+d_X(a_0',x')\big)\\
&=\gamma(x,b)+\gamma(b,x').
\end{aligned}$$

This proves another case of the triangle inequality.

Finally, let $x\in X$ and $b,b'\in B$, and let us prove $\gamma(b,b')\leq\gamma(b,x)+\gamma(x,b')$. For all $a_0,a_0'\in A$ we have

$$\begin{aligned}
\gamma(b,b')&=d_B(b,b')\\
&\leq d_B(b,f(a_0))+d_B(f(a_0),f(a_0'))+d_B(f(a_0'),b')\\
&\leq d_B(b,f(a_0))+d_A(a_0,a_0')+d_B(f(a_0'),b')\\
&=d_B(b,f(a_0))+d_X(a_0,a_0')+d_B(f(a_0'),b')\\
&\leq d_B(b,f(a_0))+d_X(a_0,x)+d_X(x,a_0')+d_B(f(a_0'),b').
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(b,b')&\leq\inf_{a_0,a_0'\in A}\big(d_B(b,f(a_0))+d_X(a_0,x)+d_X(x,a_0')+d_B(f(a_0'),b')\big)\\
&=\inf_{a_0\in A}(d_B(b,f(a_0))+d_X(a_0,x))+\inf_{a_0'\in A}\big(d_X(x,a_0')+d_B(f(a_0),b')\big)\\
&=\gamma(b,x)+\gamma(x,b').
\end{aligned}$$

This proves the last case of the triangle inequality. □

The following result, together with the corollary that follows it, describes pushouts along embeddings in $\mathsf{MetCH}_{\mathsf{sep}}$.

Proposition 4.2 (Description of a pushout along an embedding)**.** *Consider an embedding $i\colon A \hookrightarrow X$ and a morphism $f\colon X \to B$ in $\mathsf{MetCH}_{\mathsf{sep}}$, and their pushout as displayed below.*

$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ B & \underset{j}{\longrightarrow} & P \end{array}$$

(1) For $b, b' \in B$,

$$d_P(j(b), j(b')) = d_B(b, b').$$

(2) For $x, x' \in X$,

$$\begin{aligned} &d_P(g(x), g(x')) \\ &= \min\left\{ d_X(x, x'),\ \inf_{a,a' \in A} \big(d_X(x, i(a)) + d_B(f(a), f(a')) + d_X(i(a'), x')\big) \right\}. \end{aligned}$$

(3) For $b \in B$ and $x \in X$,

$$\begin{aligned} d_P(j(b), g(x)) &= \inf_{a \in A} \big(d_B(b, f(a)) + d_X(i(a), x)\big), \\ d_P(g(x), j(b)) &= \inf_{a \in A} \big(d_X(x, i(a)) + d_B(f(a), b)\big). \end{aligned}$$

Proof. For $a \in A$ we write a for $i(a)$. We start by proving the inequality $\leq$ for all the equations in the statement.

(1, $\leq$). Since j is non-expansive, for all $b, b' \in B$ we have

$$d_P(j(b), j(b')) \leq d_B(b, b').$$

(2, $\leq$). Let $x, x' \in X$. Since g is non-expansive, we have

$$d_P(g(x), g(x')) \leq d_X(x, x').$$

Moreover, for all $a, a' \in A$, we have

$$\begin{aligned} &d_P(g(x), g(x')) \\ &\leq d_P(g(x), jf(a)) + d_P(jf(a), jf(a')) + d_P(jf(a'), g(x')) && \text{(by the triangle ineq.)} \\ &= d_P(g(x), g(a)) + d_P(jf(a), jf(a')) + d_P(g(a'), g(x')) && \text{(since } j \circ f = g \circ i) \\ &\leq d_X(x, a) + d_B(f(a), f(b)) + d_X(a', x'), \end{aligned}$$

where the last inequality follows from the fact that g and j are non-expansive. We conclude

$$\begin{aligned} &d_P(g(x), g(x')) \\ &\leq \min\left\{ d_X(x, x'),\ \inf_{a,a' \in A} \big(d_X(x, a) + d_B(f(a), f(a')) + d_X(a', x')\big) \right\}. \end{aligned}$$

(3, $\leq$). Let $b \in B$ and $x \in X$. For all $a \in A$ we have

$$\begin{aligned} &d_P(j(b), g(x)) \\ &\leq d_P(j(b), jf(a)) + d_P(g(a), g(x)) && \text{(by triangle ineq. and since } j \circ f = g \circ i) \\ &\leq d_B(b, f(a)) + d_X(a, x), \end{aligned}$$

and, similarly,

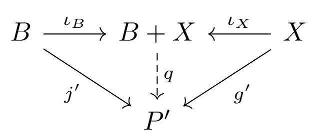
$$\begin{aligned} & d_P(g(x), j(b)) \\ & \leq d_P(g(x), g(a)) + d_P(jf(a), j(b)) \quad \text{(by triangle ineq. and since } g \circ i = j \circ f) \\ & \leq d_X(x, (a)) + d_B(f(a), b), \end{aligned}$$

Therefore,

$$\begin{aligned} d_P(j(b), g(x)) &= \inf_{a \in A} \big(d_B(b, f(a)) + d_X(a, x)\big), \\ d_P(g(x), j(b)) &= \inf_{a \in A} \big(d_X(x, a) + d_B(f(a), b)\big). \end{aligned}$$

This proves the inequality $\leq$ for all the equations in the statement.

We next prove the converse inequalities ($\geq$). Let $\iota_B \colon B \hookrightarrow B + X$ and $\iota_X \colon X \hookrightarrow B + X$ denote the coproduct injections. Let $q \colon B + X \twoheadrightarrow P'$ be the quotient object of $B + X$ associated (under the correspondence in Theorem 3.14) to the continuous submetric γ on $B + X$ described in Lemma 4.1. Set $j' \coloneqq q \circ \iota_B$ and $g' \coloneqq q \circ \iota_X$.



By the last part of Lemma 4.1, for all $a \in A$ we have

$$\gamma(\iota_B f(a), \iota_X(a)) = \gamma(\iota_X(a), \iota_B f(a)) = 0,$$

and thus

$$d_{P'}(j' f(a), g'(a)) = d_{P'}(q\iota_B f(a), q\iota_X(a)) = 0$$

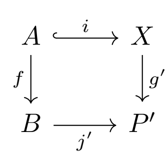
and

$$d_{P'}(g'(a), j' f(a)) = d_{P'}(q\iota_X(a), q\iota_B f(a)) = 0.$$

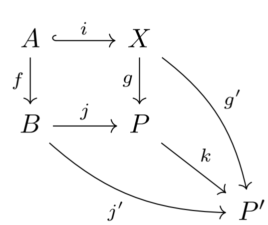
Since P is separated, from these two inequalities it follows that

$$j' f(a) = g'(a).$$

Therefore, the following diagram commutes.



Therefore, by the universal property of the pushout P, there is a unique morphism $k \colon P \to P'$ making the following diagram commute.



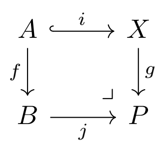
Since k is non-expansive, we have, for example, for all b, b',

$$
\begin{aligned}
d_P(j(b), j(b')) &\geq d_{P'}(kj(b), kj(b')) && \text{(since } k \text{ is non-expansive)} \\
&= d_{P'}(j'(b), j'(b')) && \text{(since } kj = j'\text{)} \\
&= d_{P'}(q\iota_B(b), q\iota_B(b')) && \text{(since } j' = q\iota_B\text{)} \\
&= \gamma(\iota_B(b), \iota_B(b')) && \text{(by the definition of } q \text{ in terms of } \gamma\text{)} \\
&= d_B(b, b') && \text{(by the definition of } \gamma\text{).}
\end{aligned}
$$

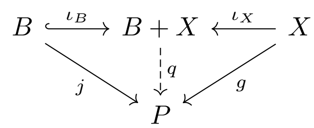
This proves one of the four inequalities to be proven. The other three are similar. □

We rewrite Proposition 4.2 in terms of a kernel metric, which should make clearer why these results describe pushouts of embeddings along arbitrary morphisms.

Corollary 4.3. *Consider an embedding $i\colon A \hookrightarrow X$ and a morphism $f\colon X \to B$ in $\mathsf{MetCH}_{\mathsf{sep}}$, and their pushout as displayed below.*



Let $q\colon B + X \to P$ be the unique morphism making the following diagram commute.



Then q is surjective, and the kernel metric κ_q on $B + X$ associated with q is as follows.

(1) *For $b, b' \in B$,*
$$\kappa_q(\iota_B(b), \iota_B(b')) = d_B(b, b').$$
(2) *For $x, x' \in X$,*
$$\kappa_q(\iota_X(x), \iota_X(x')) = \min\left\{ d_X(x, x'),\ \inf_{a, a' \in A} \big(d_X(x, a) + d_B(f(a), f(a')) + d_X(a', x')\big) \right\}.$$
(3) *For $b \in B$ and $x \in X$,*
$$\kappa_q(\iota_B(b), \iota_X(x)) = \inf_{a \in A} \big(d_B(b, f(a)) + d_X(a, x)\big).$$
$$\kappa_q(\iota_X(x), \iota_X(x)) = \inf_{a \in A} \big(d_X(x, a) + d_B(f(a), b)\big).$$

Proof. By the (surjective, embedding)-factorisation in $\mathsf{MetCH}_{\mathsf{sep}}$ (see Section 2), every coequaliser is surjective, and hence q is surjective. Everything else follows from Proposition 4.2. □

Corollary 4.4 (of Proposition 4.2)**.** *In $\mathsf{MetCH}_{\mathsf{sep}}$, the pushout of an embedding along any morphism is an embedding.*

We specialise Proposition 4.2 to the case of a pushout of two embeddings.

Corollary 4.5 (of Proposition 4.2)**.** *Consider embeddings $f_0 \colon X \hookrightarrow Y_0$ and $f_1 \colon X \hookrightarrow Y_1$ in $\mathsf{MetCH}_{\mathsf{sep}}$ and their pushout as displayed below.*

$$\begin{array}{ccc} X & \overset{f_1}{\hookrightarrow} & Y_1 \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_1} \\ Y_0 & \underset{\lambda_0}{\longrightarrow} & P \end{array}$$

The maps λ_0 and λ_1 are embeddings. Moreover, for all $i, j \in \{0,1\}$, $u \in Y_i$ and $v \in Y_j$,

$$d_P(\lambda_i(u), \lambda_j(v)) = \begin{cases} d_{Y_i}(u,v) & \text{if } i = j, \\ \inf_{x \in X} \big(d_{Y_i}(u, f_i(x)) + d_{Y_j}(f_j(x), v)\big) & \text{if } i \neq j. \end{cases}$$

It is known that, in a regular category, any pullback square consisting entirely of regular epimorphisms is also a pushout square. This follows from (the dual of) the main result of [Rin72] (cf. also [FS90, §1.565] or [CKP93, Remark 5.3]). Since we are aiming to prove that the category $\mathsf{MetCH}_{\mathsf{sep}}$ is coregular (and that the embeddings are precisely the regular monomorphisms), the following result should not come as a surprise.

Lemma 4.6. *A pushout square in $\mathsf{MetCH}_{\mathsf{sep}}$ consisting entirely of embeddings is also a pullback.*

Proof. Let us give names to the morphisms in the pushout square.

$$\begin{array}{ccc} X & \overset{f_1}{\hookrightarrow} & Y_1 \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_1} \\ Y_0 & \underset{\lambda_0}{\hookrightarrow} & P \end{array}$$

Assume that we have $y_0 \in Y_0$ and $y_1 \in Y_1$ with $\lambda_0(y_0) = \lambda_1(y_1)$, and let us prove that there is (a necessarily unique) $x \in X$ such that $y_0 = f_0(x)$ and $y_1 = f_1(x)$. By Corollary 4.5, we have

$$0 = d_P(\lambda_0(y_0), \lambda_1(y_1)) = \inf_{x \in X} \big(d_{Y_0}(y_0, f_0(x)) + d_{Y_1}(f_1(x), y_1)\big).$$

The set

$$C \coloneqq \{d_{Y_0}(y_0, f_0(x)) + d_{Y_1}(f_1(x), y_1) \mid x \in X\}$$

is compact in $[0,\infty]$ with respect to the upper topology because it is the image of the compact space X under the continuous function

$$\begin{aligned} X &\longrightarrow [0,\infty] \\ x &\longmapsto d_{Y_0}(y_0, f_0(x)) + d_{Y_1}(f_1(x), y_1). \end{aligned}$$

Since C is compact and $\inf C = 0$, we have $0 \in C$, and therefore there is $x \in X$ such that $d_{Y_0}(y_0, f_0(x)) + d_{Y_1}(f_1(x), y_1) = 0$, and hence $d_{Y_0}(y_0, f_0(x)) = d_{Y_1}(f_1(x), y_1) = 0$. Similarly, there is $x' \in X$ such that $d_{Y_1}(y_1, f_1(x')) = d_{Y_0}(f_0(x'), y_0) = 0$. Therefore,

$$d_{Y_0}(f_0(x'), f_0(x)) \leq d_{Y_0}(f_0(x'), y_0) + d_{Y_0}(y_0, f_0(x)) = 0,$$

and hence $d_X(x', x) = 0$ since f_0 is an embedding. Similarly, one shows $d_X(x, x') = 0$. Therefore, since X is separated, $x = x'$. Consequently, we have also $d_{Y_1}(y_1, f_1(x)) = 0$ and $d_{Y_0}(f_0(x), y_0) = 0$, and by separation we get $y_0 = f_0(x)$ and $y_1 = f_1(x)$.

This proves that the pushout is a pullback in Set. Since all involved maps are embeddings, it is also a pullback in $\mathsf{MetCH}_{\mathsf{sep}}$. □

Proposition 4.7. *In* $\mathsf{MetCH}_{\mathsf{sep}}$*, regular monomorphisms = embeddings.*

Proof. A regular mono in $\mathsf{MetCH}_{\mathsf{sep}}$ is in particular a regular mono of compact Hausdorff spaces and of metric spaces (since the forgetful functors are right adjoint and hence preserve limits), which implies that it is an embedding.

Conversely, let $i\colon X \hookrightarrow Y$ be an embedding. By Corollary 4.4, the pushout of i along itself consists entirely of embeddings.

$$\begin{array}{ccc} X & \xhookrightarrow{\;i\;} & Y_1 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_1} \\ Y_0 & \xhookrightarrow[\;\lambda_0\;]{} & P \end{array}$$

Hence, by Lemma 4.6 it is also a pullback. Therefore, i is the equaliser of λ_0 and λ_1. □

Proposition 4.8. *In* $\mathsf{MetCH}_{\mathsf{sep}}$*, epimorphisms = surjective morphisms.*

Proof. Clearly, every surjection is an epimorphism. For the converse direction, let f be an epimorphism, and consider its (surjection, embedding)-factorisation (Theorem 2.11):

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow\!\!\!\!\searrow \quad \nearrow{\scriptstyle i} & \\ & Z & \end{array}$$

By Proposition 4.7, i is a regular mono. Since f is an epimorphism, i is also an epimorphism, and therefore an isomorphism. Consequently, f is surjective. □

The following is an immediate consequence of Corollary 4.4 and Proposition 4.7.

Corollary 4.9. *In* $\mathsf{MetCH}_{\mathsf{sep}}$*, the pushout of a regular monomorphism along any morphism is a regular monomorphism.*

We are finally able to prove the main result of the section.

Theorem 4.10. $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ *is a regular category.*

Proof. By Theorems 2.10 and 2.11, $\mathsf{MetCH}_{\mathsf{sep}}$ is complete and cocomplete and has (surjections, embeddings)-factorisation. By Propositions 4.7 and 4.8, surjections = epimorphisms, and embeddings = regular monos. Therefore, $\mathsf{MetCH}_{\mathsf{sep}}$ has (epimorphisms, regular monomorphisms)-factorisation. Finally, by Corollary 4.9, in $\mathsf{MetCH}_{\mathsf{sep}}$ the pushout of a regular monomorphism along any morphism is a regular monomorphism. □

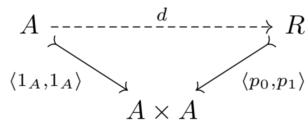
5. Barr-coexactness for separated metric compact Hausdorff spaces

In this section we show that the dual of the category $\mathsf{MetCH}_{\mathsf{sep}}$ is Barr-exact.

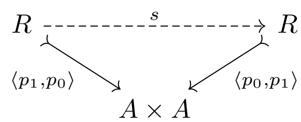
Notation 5.1. Given morphisms $f_0\colon X \to Y_0$ and $f_1\colon X \to Y_1$, the unique morphism induced by the universal property of the product is denoted by $\langle f_0, f_1\rangle\colon X \to Y_0 \times Y_1$. Similarly, given morphisms $f_0\colon X_0 \to Y$ and $f_1\colon X_1 \to Y$, the coproduct map is denoted by $\binom{g_0}{g_1}\colon X_0 + X_1 \to Y$.

Let C be a category with finite limits, and A an object of C. An *(internal) binary relation* on A is a subobject $\langle p_0, p_1\rangle\colon R \rightarrowtail A \times A$, (or, equivalently, a pair of jointly monic maps $p_0, p_1\colon R \rightrightarrows A$). A binary relation $\langle p_0, p_1\rangle\colon R \rightarrowtail A \times A$ on A is called

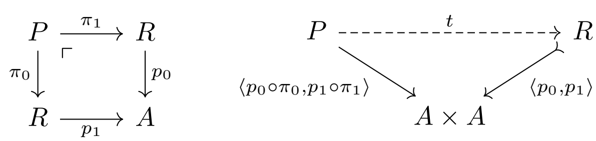
***reflexive*:** provided that there is a morphism $d\colon A \to R$ such that the following diagram commutes;



***symmetric*:** provided that there is a morphism $s\colon R \to R$ such that the following diagram commutes;



***transitive*:** provided that, if the left-hand diagram below is a pullback square, then there is a morphism $t\colon P \to R$ such that the right-hand diagram commutes.



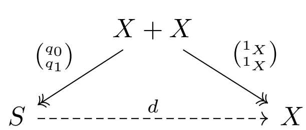
An *(internal) equivalence relation* on A is a reflexive symmetric transitive binary relation on A.

Definition 5.2. An equivalence relation $R \rightrightarrows A$ is *effective* if it is a kernel pair.

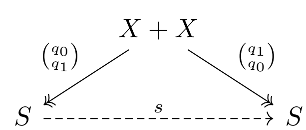
Remark 5.3. In a regular category with coequalisers, an equivalence relation is effective if and only if it is the kernel pair of its coequaliser.

We recall that a *Barr-exact category* is a regular category where every equivalence relation is effective. In this section we provide a description of equivalence relations in the category $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$, and we exploit it to prove that equivalence relations in $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ are effective.

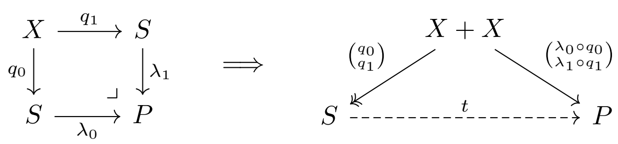
Recall that a binary relation on an object A of a category C is a subobject of $A \times A$. Dualising this definition, given a separated metric compact Hausdorff space X, we call a *binary corelation* on X a quotient object $\binom{q_0}{q_1}\colon X + X \twoheadrightarrow S$ of the separated metric compact Hausdorff space $X + X$. We recall from Theorem 2.10 that $X + X$ is the disjoint union of two copies of X equipped with the coproduct topology and the coproduct metric. A binary corelation on X is called respectively *reflexive*, *symmetric*, *transitive* provided it satisfies the properties:



reflexivity



symmetry



transitivity

An *equivalence corelation* on X is a reflexive symmetric transitive binary corelation on X. The key observation is that, since quotient objects of $X+X$ are in bijection with certain metrics on $X+X$, equivalence corelations are more manageable than their duals.

Definition 5.4. We call *binary continuous submetric* on a separated metric compact Hausdorff space X an element of $\mathbf{S}(X+X)$, i.e. a continuous metric on $X+X$ which is below the coproduct metric on $X+X$.

Theorem 3.14 establishes a bijective correspondence between binary continuous submetrics on X (i.e., elements of $\mathbf{S}(X+X)$) and binary corelations on X (i.e., elements of $\tilde{\mathbf{Q}}(X+X)$).

Definition 5.5. A binary continuous submetric on a separated metric compact Hausdorff space X is called a *reflexive* (resp. *symmetric*, *transitive*, *equivalence*) continuous submetric if the corresponding binary corelation on X is reflexive (resp. symmetric, transitive, equivalence).

Notation 5.6. We denote the elements of $X+X$ by (x,i), where x varies in X and i varies in $\{0,1\}$. Furthermore, i^* stands for $1-i$; for example, $(x,1^*)=(x,0)$.

As we will prove, every equivalence continuous submetric γ on a separated metric compact Hausdorff space X is obtained as follows: consider a closed subset Y of X and let γ be the greatest metric on $X+X$ that extends the coproduct metric of $X+X$ and that satisfies $d((y,0),(y,1))=\gamma((y,1),(y,0))=0$ for every $y\in Y$.

Lemma 5.7. *A binary continuous submetric γ on a separated metric compact Hausdorff space X is reflexive if and only if, for all $x,y\in X$ and $i,j\in\{0,1\}$,*

$$d_X(x,y)\leq\gamma((x,i),(y,j)).$$

Proof. Let $\binom{q_0}{q_1}\colon X+X\twoheadrightarrow S$ be the binary corelation associated with γ. By the definition of a reflexive continuous submetric, γ is reflexive if and only if $\binom{q_0}{q_1}\colon X+X\twoheadrightarrow S$ is below $\binom{1_X}{1_X}\colon X+X\twoheadrightarrow X$ in the poset $\tilde{\mathbf{Q}}(X+X)$. By Theorem 3.14, this is equivalent to $\kappa_{\binom{1_X}{1_X}}\leq\gamma$. Given $(x,i),(y,j)\in X+X$, we have

$$\kappa_{\binom{1_X}{1_X}}((x,i),(y,j))=d(x,y).$$

It follows that the binary continuous submetric γ is reflexive if and only if, for all $x,y\in X$ and $i,j\in\{0,1\}$, $d_X(x,y)\leq\gamma((x,i),(y,j))$. □

Remark 5.8. Note that any reflexive continuous submetric γ on X satisfies, for all $x,y\in X$ and $i\in\{0,1\}$,

$$\gamma((x,i),(y,i))=d(x,y).$$

Indeed, the inequality $\geq$ follows from Lemma 5.7, while the inequality $\leq$ holds because γ is below the coproduct metric of $X+X$.

Lemma 5.9. *A binary continuous submetric γ on a separated metric compact Hausdorff space X is symmetric if and only if, for all $x, y \in X$ and $i, j \in \{0,1\}$, we have*

$$\gamma((x,i),(y,j)) = \gamma((x,i^*),(y,j^*)).$$

Proof. Let $\binom{q_0}{q_1}\colon X + X \twoheadrightarrow S$ be the binary corelation associated with γ. By the definition of a symmetric continuous submetric, γ is symmetric if and only if $\binom{q_0}{q_1}\colon X + X \twoheadrightarrow S$ is below $\binom{q_1}{q_0}\colon X + X \twoheadrightarrow S$ in $\tilde{\mathbf{Q}}(X + X)$. By Theorem 3.14, this happens exactly when $\kappa_{\binom{q_1}{q_0}} \leq \gamma$. For all $(x,i),(y,j) \in X + X$, we have

$$\kappa_{\binom{q_1}{q_0}}((x,i),(y,j)) = \gamma((x,i^*),(y,j^*)).$$

Therefore, the binary continuous submetric γ is symmetric if and only if, for all $x, y \in X$ and $i, j \in \{0,1\}$, $\gamma((x,i),(y,j)) \leq \gamma((x,i^*),(y,j^*))$. Now use the fact that the statement with $\leq$ is equivalent to the statement with $=$ (since $i^{**} = i$). □

Remark 5.10. $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ is not a Mal'cev category (i.e., a finitely complete category where every reflexive relation is an equivalence relation), because not every reflexive relation is an equivalence relation. Indeed, the following provides an example of a reflexive non-symmetric continuous submetric on a one-point space $\{*\}$ (with $d(*,*) = 0$):

$$\gamma((*,0),(*,0)) = \gamma((*,0),(*,1)) = \gamma((*,1),(*,1)) = 0$$

and

$$\gamma((*,1),(*,0)) = \infty.$$

This corresponds to the surjection $\{(*,0),(*,1)\} \to \{a_1, a_2\}$, $(*,i) \mapsto a_i$, where $d(a_i, a_j)$ is ∞ if $i = 1$ and $j = 0$, and 0 otherwise.

Lemma 5.11. *A reflexive continuous submetric γ on a separated metric compact Hausdorff space X is transitive if and only if, for all $x, y \in X$ and all $i \in \{0,1\}$, we have*

$$\gamma((x,i),(y,i^*)) = \inf_{z \in X} \big(\gamma((x,i),(z,i^*)) + \gamma((z,i),(y,i^*))\big).$$

Proof. Let $\binom{q_0}{q_1}\colon X+X \twoheadrightarrow S$ be the binary corelation associated with γ. To improve readability, we write $[x,i]$ instead of $\binom{q_0}{q_1}(x,i)$. By definition of transitivity, the binary continuous submetric γ is transitive if and only if, given a pushout square in $\mathsf{MetCH}_{\mathsf{sep}}$ as in the left-hand diagram below, there is a morphism $t\colon S \to P$ such that the right-hand diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{q_0} & S \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\ S & \xrightarrow[\lambda_0]{} & P \end{array} \qquad\qquad \begin{array}{ccccc} & & X + X & & \\ & {\scriptstyle \binom{q_0}{q_1}}\swarrow & & \searrow{\scriptstyle \binom{\lambda_0 \circ q_0}{\lambda_1 \circ q_1}} & \\ S & & \overset{t}{\dashrightarrow} & & P \end{array}$$

By Lemma 3.12, such a t exists precisely when

$$\kappa_{\binom{\lambda_0 \circ q_0}{\lambda_1 \circ q_1}} \leq \kappa_{\binom{q_0}{q_1}},$$

i.e., when, for every $(x,i),(y,j) \in X + X$,

$$d_P\left(\begin{pmatrix}\lambda_0 \circ q_0 \\ \lambda_1 \circ q_1\end{pmatrix}(x,i), \begin{pmatrix}\lambda_0 \circ q_0 \\ \lambda_1 \circ q_1\end{pmatrix}(y,j)\right) \leq \gamma((x,i),(y,j)).$$

The former equals $d_P(\lambda_i([x,i]),\lambda_j([y,j]))$. Recall that γ is reflexive provided q_0 and q_1 are both sections of a morphism $d\colon S \to X$. In particular, q_0 and q_1 are regular monomorphisms in $\mathsf{MetCH}_{\mathsf{sep}}$. Thus, by Corollary 4.5,

$$d_P(\lambda_i([x,i]),\lambda_j([y,j])) = \begin{cases} \gamma(x,y) & \text{if } i=j, \\ \inf_{z\in X}\big(\gamma((x,i),(z,j)) + \gamma((z,i),(y,j))\big) & \text{if } i\neq j. \end{cases}$$

This finishes the proof. □

All told, we obtain a characterisation of equivalence continuous submetrics.

Theorem 5.12. *A binary continuous submetric γ on a separated metric compact Hausdorff space X is an equivalence continuous submetric if and only if for all $x,y\in X$ and all $i,j\in\{0,1\}$ we have*

$$d_X(x,y) \leq \gamma((x,i),(y,j)) = \gamma((x,i^*),(y,j^*))$$

and

$$\gamma((x,i),(y,i^*)) = \inf_{z\in X}\big(\gamma((x,i),(z,i^*)) + \gamma((z,i),(y,i^*))\big).$$

Proof. By Lemmas 5.7, 5.9 and 5.11. □

Remark 5.13. The category CH of compact Hausdorff spaces, which can be identified with the full subcategory of PosCH defined by the symmetric objects, is co-Mal'cev. One might then suspect that $\mathsf{MetCH}_{\mathsf{sep,sym}}$ is coMal'cev too. However, this is not the case: $\mathsf{MetCH}_{\mathsf{sep,sym}}^{\mathsf{op}}$ is not a Mal'cev category, as there are reflexive internal relations in $\mathsf{MetCH}_{\mathsf{sep,sym}}^{\mathsf{op}}$ that are neither symmetric nor transitive. An example is the following: let $\{a,b\}$ be a two-element discrete space, with metric $d(a,b)=d(b,a)=0$ (and self-distances equal to 0). Consider the following binary continuous submetric γ on X (i.e. a continuous metric below d_{X+X}): self-distances are 0, and all other distances are ∞, except for the distances from $(a,0)$ to $(b,1)$ and from $(b,1)$ to $(a,0)$ which are 1. Comparing this to the "ordered" case, it seems that what breaks being Mal'cev is the possibility of having more than just two possible values for the distances.

Dualising Definition 5.2, we say that an equivalence corelation $\binom{q_0}{q_1}\colon X+X \twoheadrightarrow S$ on a separated metric compact Hausdorff space X (and so the corresponding equivalence continuous submetric) is *effective* provided it coincides with the cokernel pair of its equaliser. That is, provided that the following is a pushout square in $\mathsf{MetCH}_{\mathsf{sep}}$,

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle q_1} \\ X & \xrightarrow[q_0]{} & S \end{array}$$

where $i\colon A\hookrightarrow X$ is the equaliser of $q_0,q_1\colon X \rightrightarrows S$ in $\mathsf{MetCH}_{\mathsf{sep}}$.

Notation 5.14. Given a separated metric compact Hausdorff space X and a closed subspace Y of X, we define the function $\gamma^A\colon (X+X)\times(X\times X)\to[0,\infty]$ as follows: for all $x,y\in X$ and $i\in\{0,1\}$ we set

$$\gamma^A((x,i),(y,i)) \coloneqq d(x,y),$$

and

$$\gamma^A((x,i),(y,i^*)) \coloneqq \inf_{a\in A}\big(d(x,a)+d(a,y)\big).$$

Lemma 5.15. *Let X be a separated metric compact Hausdorff space, let A be a closed subspace of X, equipped with the induced topology and metric. The binary continuous submetric on X associated with the pushout in $\mathsf{MetCH}_{\mathsf{sep}}$ of the inclusion $A \hookrightarrow X$ along itself is γ^A.*

Proof. This is an immediate consequence of Corollary 4.5. □

Lemma 5.16. *Let γ be an equivalence continuous submetric on a separated metric compact Hausdorff space X, and set*

$$A \coloneqq \{a \in X \mid \gamma((a,0),(a,1)) = 0\} = \{a \in X \mid \gamma((a,1),(a,0)) = 0\}.$$

Then γ is effective if and only if for all $x, y \in X$ and $i \in \{0,1\}$, we have

$$\gamma((x,i),(y,i^*)) = \inf_{a \in A} \big(d_X(x,a) + d_X(a,y)\big).$$

Proof. Let us endow A with the induced topology and induced metric. Denoting by $\binom{q_0}{q_1}\colon X + X \twoheadrightarrow S$ the binary corelation on X associated with γ, we have

$$A = \{a \in X \mid d(q_0(a), q_1(a)) = 0 = d(q_1(a), q_0(a))\} = \{a \in X \mid q_0(a) = q_1(a)\}.$$

Therefore, the embedding $A \hookrightarrow X$ is the equaliser of $q_0, q_1\colon X \rightrightarrows S$ in $\mathsf{MetCH}_{\mathsf{sep}}$. Therefore, the binary continuous submetric γ is effective if and only if the following diagram is a pushout in $\mathsf{MetCH}_{\mathsf{sep}}$.

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle q_1} \\ X & \xrightarrow[q_0]{} & S \end{array}$$

In turn, by Lemma 5.15, this is equivalent to saying that $\gamma = \gamma^A$. By definition of γ^A, for all $x, y \in X$ and $i \in \{0,1\}$ we have

$$\gamma^A((x,i),(y,i)) = d(x,y),$$

and

$$\gamma^A((x,i),(y,i^*)) = \inf_{a \in A} \big(d(x,a) + d(a,y)\big).$$

By Remark 5.8, $\gamma((x,i),(y,i)) = d(x,y)$. Therefore, γ is effective if and only if for all $x, y \in X$ and $i \in \{0,1\}$, we have

$$\gamma((x,i),(y,i^*)) = \inf_{a \in A} \big(d(x,a) + d(a,y)\big),$$

as desired. □

Lemma 5.17. *Let X be a compact Hausdorff space. Let $\rho\colon X \times X \to [0,\infty]$ be a continuous function with respect to the upper topology of $[0,\infty]$, and suppose that for all $x, y \in X$ we have*

$$\rho(x,y) = \inf_{z \in X} \big(\rho(x,z) + \rho(z,y)\big).$$

Then, setting $A \coloneqq \{x \in X \mid \rho(x,x) = 0\}$, we have

$$\rho(x,y) = \inf_{a \in A} \big(\rho(x,a) + \rho(a,y)\big).$$

Proof. By the triangle inequality, it is enough to prove the inequality $\geq$.

Fix $x, y \in X$, and let us prove $\rho(x,y) \geq \inf_{a \in A} \big(\rho(x,a) + \rho(a,y)\big)$.

If $\rho(x,y) = \infty$, the inequality is trivial. Let us then assume that $\rho(x,y) < \infty$. It is enough to show that the set

$$\{a \in A \mid \rho(x,y) = \rho(x,a) + \rho(a,y)\}$$

is nonempty; for this, it is enough to prove that this set is a codirected intersection of closed nonempty sets.

Let $\mathcal{V}$ be the set of closed subsets of X, and, for each $\lambda \in (0,\infty]$, set

$$\mathcal{W}_\lambda \coloneqq \{K \in \mathcal{V} \mid \exists u, v \in K.\ \rho(u,v) \leq \lambda,\ \rho(x,u) + \rho(u,v) + \rho(v,y) = \rho(x,y)\}.$$

Let $\mathcal{F}$ be the set of finite closed covers of X, i.e. the set of finite subsets $\mathcal{A}$ of $\mathcal{V}$ such that $\bigcup \mathcal{A} = X$. For every $\mathcal{A} \in \mathcal{F}$ and $\lambda \in (0,\infty]$, we set

$$D_\lambda^{\mathcal{A}} \coloneqq \bigcup_{K \in \mathcal{A} \cap \mathcal{W}_\lambda} K.$$

We will prove that the set $\{D_\lambda^{\mathcal{A}} \mid \mathcal{A} \in \mathcal{F},\ \lambda \in (0,\infty]\}$ is a codirected set of closed nonempty sets with intersection

$$\{a \in A \mid \rho(x,y) = \rho(x,a) + \rho(a,y)\};$$

an application of compactness will then give the desired result.

For all $\mathcal{A} \in \mathcal{F}$ and $\lambda \in (0,\infty]$, the set $D_\lambda^{\mathcal{A}}$ is closed because it is a finite union of closed sets.

The set $\{D_\lambda^{\mathcal{A}} \mid \mathcal{A} \in \mathcal{F},\ \lambda \in (0,\infty]\}$ is codirected because $D_\infty^{\{X\}}$ belongs to it and, for all $\mathcal{A}, \mathcal{A}' \in \mathcal{F}$ and $\lambda, \lambda' \in (0,\infty]$,

$$D_{\min\{\lambda,\,\lambda'\}}^{\{K \cap K' \mid K \in \mathcal{A},\, K' \in \mathcal{A}'\}} \subseteq D_\lambda^{\mathcal{A}} \cap D_{\lambda'}^{\mathcal{A}'}.$$

We show that, for all $\mathcal{A} \in \mathcal{F}$ and $\lambda \in (0,\infty]$, the set $D_\lambda^{\mathcal{A}}$ is nonempty. We denote with $\sharp S$ the cardinality of a set S. Pick any natural number l such that $\frac{l}{\sharp\mathcal{A}} > 1$ and $\frac{\rho(x,y)}{\frac{l}{\sharp\mathcal{A}} - 1} \leq \lambda$. Note that, having fixed $x, y \in X$, we can use nonemptiness of X, and so for all $u, v \in X$ there is $z \in X$ such that $\rho(u,v) = \rho(u,z) + \rho(z,v)$, i.e. the infimum in the hypothesis of the lemma is a minimum. Thus, there are $z_1, \dots, z_l \in X$ such that

$$\rho(x,y) = \rho(x,z_1) + \rho(z_1,z_2) + \dots + \rho(z_{l-1},z_l) + \rho(z_l,y).$$

Since every z_i belongs to some $K \in \mathcal{A}$, we have $\sum_{K \in \mathcal{A}} \sharp(K \cap \{z_1,\dots,z_l\}) \geq l$. Therefore, the average of $\sharp(K \cap \{z_1,\dots,z_l\})$ for K ranging in $\mathcal{A}$ is greater than or equal to $\frac{l}{\sharp\mathcal{A}}$. (The average makes sense because, by nonemptiness of X, $\sharp\mathcal{A} \neq 0$.) Therefore, there is $K \in \mathcal{A}$ with $\sharp(K \cap \{z_1,\dots,z_l\}) \geq \frac{l}{\sharp\mathcal{A}}$. Let $z_{i_1}, \dots, z_{i_n}$ (with $i_1 < \dots < i_n$) be an enumeration of the elements of $K \cap \{z_1,\dots,z_l\}$. Note that $n \geq \frac{l}{\sharp\mathcal{A}} > 1$ and so $n \geq 2$. We have

$$\begin{aligned}\rho(x,y) &= \rho(x,z_1) + \rho(z_1,z_2) + \dots + \rho(z_{l-1},z_l) + \rho(z_l,y)\\ &\geq \rho(z_{i_1},z_{i_2}) + \dots + \rho(z_{i_{n-1}},z_{i_n}).\end{aligned}$$

Therefore, the average of $\rho(z_{i_j},z_{i_j})$ for j ranging in $\{1,\dots,n-1\}$ is less than or equal to $\frac{\rho(x,y)}{n-1}$. (The average makes sense since $n \geq 2$ and so $n-1 \geq 1$.) Therefore, there is $j \in \{1, \ldots, n-1\}$ such that

$$\rho(z_{i_j}, z_{i_{j+1}}) \leq \frac{\rho(x,y)}{n-1},$$

and so

$$\rho(z_{i_j}, z_{i_{j+1}}) \leq \frac{\rho(x,y)}{n-1} \leq \frac{\rho(x,y)}{\frac{l}{\sharp\mathcal{A}} - 1} \leq \lambda.$$

We have

$$\begin{aligned}\rho(x,y) &\leq \rho(x, z_{i_j}) + \rho(z_{i_j}, z_{i_{j+1}}) + \rho(z_{i_{j+1}}, y) && \text{by triangle inequality}\\ &\leq \rho(x, z_1) + \rho(z_1, z_2) + \cdots + \rho(z_{l-1}, z_l) + \rho(z_l, y) && \text{by triangle inequality}\\ &= \rho(x,y),\end{aligned}$$

and hence

$$\rho(x,y) = \rho(x, z_{i_j}) + \rho(z_{i_j}, z_{i_{j+1}}) + \rho(z_{i_{j+1}}, y).$$

Therefore, $K \in \mathcal{W}_\lambda$. Thus, $\varnothing \neq K \subseteq D_\lambda^{\mathcal{A}}$, and hence $D_\lambda^{\mathcal{A}} \neq \varnothing$.

We now prove

$$\bigcap_{\mathcal{A}\in\mathcal{F},\, \lambda\in(0,\infty]} D_\lambda^{\mathcal{A}} = \{a \in A \mid \rho(x,y) = \rho(x,a) + \rho(a,y)\}. \tag{1}$$

The inclusion $\supseteq$ is immediate.

Let us prove the converse inclusion, i.e. $\subseteq$. Let $z \in \bigcap_{\mathcal{A}\in\mathcal{F},\, \lambda\in(0,\infty]} D_\lambda^{\mathcal{A}}$.

We first prove that $\rho(x,y) = \rho(x,z) + \rho(z,y)$. By way of contradiction, suppose this is not the case. Then, $\rho(x,y) < \rho(x,z) + \rho(z,y)$ (since the inequality $\geq$ holds by the triangle inequality). The function

$$\begin{aligned} f\colon X \times X &\longrightarrow [0,\infty]\\ (u,v) &\longmapsto \rho(x,u) + \rho(v,y)\end{aligned}$$

is continuous with respect to the upper topology of $[0,\infty]$ because ρ is such. Since $f(z,z) = \rho(x,z) + \rho(z,y) > \rho(x,y)$, there is an open neighbourhood U of z such that, for all $u, v \in U$, $f(u,v) > \rho(x,y)$. Then, since X is a compact Hausdorff space, there are closed subsets K and L of X such that $z \in K \subseteq U$, $z \notin L$ and $K \cup L = X$. We have $K \notin \mathcal{W}_\infty$ because for all $u, v \in K$ we have $\rho(x,u) + \rho(u,v) + \rho(v,y) = f(u,v) + \rho(u,v) \geq f(u,v) > \rho(x,y)$. From $K \notin \mathcal{W}_\infty$ and $z \notin L$ we deduce $z \notin D_\infty^{\{K,L\}}$, a contradiction. Thus, $\rho(x,y) = \rho(x,z) + \rho(z,y)$.

We now prove $z \in A$, i.e. $\rho(z,z) = 0$. By way of contradiction, suppose this is not the case. Choose λ such that $0 < \lambda < \rho(z,z)$. Then, since ρ is continuous, there is an open neighbourhood U of z such that for all $u, v \in U$ we have $\rho(u,v) > \lambda$. Then, since X is a compact Hausdorff space, there are closed subsets K and L of X such that $z \in K \subseteq U$, $z \notin L$ and $K \cup L = X$. We have $K \notin \mathcal{W}_\lambda$ because for all $u, v \in X$ we have $\rho(u,v) > \lambda$. From $K \notin \mathcal{W}_\lambda$ and $z \notin L$ we deduce $z \notin D_\lambda^{\{K,L\}}$, a contradiction. This proves $\rho(z,z) = 0$.

By compactness, $\bigcap_{\mathcal{A}\in\mathcal{F},\, \lambda\in(0,\infty]} D_\lambda^{\mathcal{A}}$ is nonempty and thus, by (1), there is $a \in A$ such that $\rho(x,y) = \rho(x,a) + \rho(a,y)$. □

Remark 5.18. Lemma 5.17 has the following corollary: given a closed idempotent endorelation $\prec$ on a compact Hausdorff space X, for every $x, y \in X$ with $x \prec y$ there is $a \in X$ such that $x \prec a \prec a \prec y$.

Theorem 5.19. *Every equivalence corelation in* $\mathsf{MetCH}_{\mathsf{sep}}$ *is effective.*

Proof. Let γ be an equivalence continuous submetric on a separated metric compact Hausdorff space X. For $x, y \in X$, set $\rho(x, y) \coloneqq \gamma((x, 0), (y, 1))$ $(= \gamma((x, 1), (y, 0))$, by Lemma 5.9). Set

$$A \coloneqq \{a \in X \mid \rho(a, a) = 0\}.$$

In view of Lemma 5.16, we shall show that, for all $x, y \in X$,

$$\rho(x, y) = \inf_{a \in A} \big(d_X(x, a) + d_X(a, y)\big).$$

Here are some properties of ρ.

(1) For all $x, y \in X$, $d(x, y) \leq \rho(x, y)$.
(2) For all $x, y, z \in X$, $\rho(x, y) \leq d(x, z) + \rho(z, y)$.
(3) For all $x, y, z \in X$, $\rho(x, y) \leq \rho(x, z) + d(z, y)$.
(4) The function $\rho \colon X \times X \to [0, \infty]$ is continuous with respect to the upper topology of $[0, \infty]$.
(5) For all $x, y \in X$, $\rho(x, y) = \inf_{z \in X} \big(\rho(x, z) + \rho(z, y)\big)$.

Indeed, (1) follows from Lemma 5.7, (2) and (3) follow from the triangle inequality of γ and the fact that γ is below the coproduct metric, (4) follows from the continuity of γ, and (5) follows from Lemma 5.11 and the transitivity of γ.

Therefore, we can apply Lemma 5.17 to ρ and obtain

$$\rho(x, y) = \inf_{a \in A} \big(\rho(x, a) + \rho(a, y)\big).$$

Moreover, for all $x \in X$ and $a \in A$, we have

$$\rho(x, a) \leq d(x, a) + \rho(a, a) = d(x, a) \leq \rho(x, a),$$

and hence $\rho(x, a) = d(x, a)$; similarly, $\rho(a, x) = d(a, x)$.

Therefore, for all $x, y \in X$,

$$\rho(x, y) = \inf_{a \in A} \big(\rho(x, a) + \rho(a, y)\big) = \inf_{a \in A} \big(d(x, a) + d(a, y)\big),$$

as desired. □

We finally reached the main result:

Theorem 5.20. $\mathsf{MetCH}_{\mathsf{sep}}$ *is Barr-coexact.*

Proof. By Theorem 4.10, $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ is a regular category. By Theorem 5.19, every equivalence relation in $\mathsf{MetCH}_{\mathsf{sep}}^{\mathsf{op}}$ is effective. □

Let us quickly point out that there is no hope of such a result without separation, as it is already visible in the preordered case.

Example 5.21 (Preorders are not Barr-exact)**.** Preorders do not have effective equivalence relations. Indeed, the two maps from a singleton $\{*\}$ to a two-element set $\{a, b\}$ with $a \leq b$ and $b \leq a$ form an equivalence corelation on $\{*\}$ which is not effective.

Theorem 5.20 shows an algebraic trait of the dual of $\mathsf{MetCH}_{\mathsf{sep}}$. As a negative result, we note that $\mathsf{MetCH}_{\mathsf{sep}}$ cannot be dually equivalent to a variety of *finitary* algebras, since, by [HN23, Corollary 4.30], every finitely copresentable object in $\mathsf{MetCH}_{\mathsf{sep}}$ is finite. However, the following remains open to us:

Question 5.22. *Is the category* $\mathsf{MetCH}_{\mathsf{sep}}$ *dually equivalent to a (possibly many-sorted) variety of (possibly infinitary) algebras?*

Having shown that the complete category $\mathsf{MetCH}_{\mathsf{sep}}$ is coexact, this problem amounts now (see [Bor94, AR94]) to the question of whether $\mathsf{MetCH}_{\mathsf{sep}}$ has a regular cogenerating set formed by regular injective objects.

6. The symmetric and the ordered cases

Let us end this paper with a few remarks about symmetric metrics and compact ordered spaces. Recall from Remark 2.13 that $\mathsf{MetCH}_{\mathsf{sep,sym}}$ denotes the full subcategory of $\mathsf{MetCH}_{\mathsf{sep}}$ defined by the symmetric objects (i.e. those satisfying $d(x,y) = d(y,x)$), and that the inclusion functor $\mathsf{MetCH}_{\mathsf{sep,sym}} \to \mathsf{MetCH}_{\mathsf{sep}}$ has a right adjoint and a left adjoint. This, together with the fact that $\mathsf{MetCH}_{\mathsf{sep}}$ is Barr-coexact (Theorem 5.20), implies immediately the following result.

Theorem 6.1. *The category* $\mathsf{MetCH}_{\mathsf{sep,sym}}$ *is Barr-coexact.*

Remark 6.2. Building on the results of Section 3, it is easy to see that, in the symmetric case, the class of equivalence classes of surjections going out from (X, d_X) is in bijection with the continuous *symmetric* submetrics on X.

Recall also from Remark 2.12 that the inclusion of the category PosCH into $\mathsf{MetCH}_{\mathsf{sep}}$ has a right adjoint and a left adjoint. This, together with the fact that $\mathsf{MetCH}_{\mathsf{sep}}$ is Barr-coexact (Theorem 5.20), implies immediately the main result of [AR20]:

Theorem 6.3. *The category* PosCH *is Barr-coexact.*

The proof in [AR20] of the result above involves an application of Zorn's lemma. In this paper we have illustrated a choice-free proof of this result, thanks to a choice-free proof of the fact that a closed idempotent relation on a compact Hausdorff space has enough reflexive elements (Remark 5.18).

Acknowledgement

We thank Guram Bezhanishvili for a discussion that made us realize an error in the proof of Lemma 5.17 in an earlier version of this paper.

References

[Abb19] Abbadini, M. The dual of compact ordered spaces is a variety. *Theory and Applications of Categories*, **34** (44):1401–1439, 2019. `arXiv:1902.07162 [math.LO]`.

[AR94] Adámek, J. and Rosický, J. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[AR20] Abbadini, M. and Reggio, L. On the axiomatisability of the dual of compact ordered spaces. *Applied Categorical Structures*, **28** (6):921–934, 2020. `arXiv:1909.01631 [math.CT]`.

[Bor94] Borceux, F. *Handbook of categorical algebra 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press (CUP), Cambridge, 1994. Categories and structures.

[Bou98] Bourbaki, N. *General topology. Chapters 1–4*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[CKP93] Carboni, A., Kelly, G. M., and Pedicchio, M. C. Some remarks on Mal'tsev and Goursat categories. *Appl. Categ. Structures*, **1** (4):385–421, 1993.

[CLP91] Cignoli, R., Lafalce, S., and Petrovich, A. Remarks on Priestley duality for distributive lattices. *Order*, **8** (3):299–315, 1991.

[Dus69] DUSKIN, J. Variations on Beck's tripleability criterion. In MACLANE, S., editor, *Reports of the Midwest Category Seminar III*, pages 74–129. Springer Berlin Heidelberg, 1969.

[EH01] ESCARDÓ, M. H. AND HECKMANN, R. Topologies on spaces of continuous functions. *Topology Proceedings*, **26** (2):545–564, 2001. Proceedings of the 16^{th} Summer Conference on General Topology and its Applications (New York).

[Eng89] ENGELKING, R. *General topology*, volume 6 of *Sigma series in pure mathematics*. Heldermann, Berlin, rev. and completed edition, 1989.

[Fla97] FLAGG, R. C. Quantales and continuity spaces. *Algebra Universalis*, **37** (3):257–276, 1997.

[FS90] FREYD, P. J. AND SCEDROV, A. *Categories, allegories*, volume 39 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1990.

[Gel41] GELFAND, I. Normierte Ringe. *Recueil Mathématique. Nouvelle Série*, **9** (1):3–24, 1941.

[GH09] GIVANT, S. AND HALMOS, P. *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics. Springer, New York, 2009.

[GH13] GUTIERRES, G. AND HOFMANN, D. Approaching metric domains. *Applied Categorical Structures*, **21** (6):617–650, 2013. `arXiv:1103.4744 [math.GN]`.

[GHK+03] GIERZ, G., HOFMANN, K. H., KEIMEL, K., LAWSON, J. D., MISLOVE, M. W., AND SCOTT, D. S. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[HN18] HOFMANN, D. AND NORA, P. Enriched Stone-type dualities. *Advances in Mathematics*, **330**:307–360, 2018. `arXiv:1605.00081 [math.CT]`.

[HN20] HOFMANN, D. AND NORA, P. Hausdorff Coalgebras. *Applied Categorical Structures*, **28** (5):773–806, 2020. `arXiv:1908.04380 [math.CT]`.

[HN23] HOFMANN, D. AND NORA, P. Duality theory for enriched Priestley spaces. *Journal of Pure and Applied Algebra*, **227** (3):107231, 2023. `arXiv:2009.02303 [math.CT]`.

[HNN18] HOFMANN, D., NEVES, R., AND NORA, P. Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces. *Theory and Applications of Categories*, **33** (12):276–295, 2018. `arXiv:1706.05292 [math.CT]`.

[HR18] HOFMANN, D. AND REIS, C. D. Convergence and quantale-enriched categories. *Categories and General Algebraic Structures with Applications*, **9** (1):77–138, 2018. `arXiv:1705.08671 [math.CT]`.

[Law73] LAWVERE, F. W. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matemàtico e Fisico di Milano*, **43** (1):135–166, 1973. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

[Nac65] NACHBIN, L. *Topology and Order*, volume 4 of *Van Nostrand Mathematical Studies*. D. Van Nostrand, Princeton, N.J.-Toronto, Ont.-London, 1965. Translated from the Portuguese by Lulu Bechtolsheim.

[Neg71] NEGREPONTIS, J. W. Duality in analysis from the point of view of triples. *Journal of Algebra*, **19** (2):228–253, 1971.

[Pon34] PONTRJAGIN, L. S. The theory of topological commutative groups. *The Annals of Mathematics*, **35** (2):361, 1934.

[Pri70] PRIESTLEY, H. A. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society*, **2** (2):186–190, 1970.

[Pri72] PRIESTLEY, H. A. Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society. Third Series*, **24** (3):507–530, 1972.

[Rin72] RINGEL, C. M. The intersection property of amalgamations. *J. Pure Appl. Algebra*, **2** (4):341–342, 1972.

[Sch02] SCHMID, J. Quasiorders and sublattices of distributive lattices. *Order*, **19** (1):11–34, 2002.

[Sto36] STONE, M. H. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, **40** (1):37–111, 1936.

[Sto38] STONE, M. H. Topological representations of distributive lattices and Brouwerian logics. *Časopis pro pěstování matematiky a fysiky*, **67** (1):1–25, 1938.

[Tho09] THOLEN, W. Ordered topological structures. *Topology and its Applications*, **156** (12):2148–2157, 2009.

School of Computer Science, University of Birmingham, B15 2TT Birmingham, UK.

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal.