Erratum to "Cauchy convergence in $\mathcal{V}$-normed categories" [Adv. Math. 470 (2025) 110247. https://doi.org/10.1016/j.aim.2025.110247]

Maria Manuel Clementino*[1], Dirk Hofmann†[2], and Walter Tholen[3]

[1]University of Coimbra, CMUC, Department of Mathematics, Portugal, mmc@mat.uc.pt
[2]University of Aveiro, CIDMA, Department of Mathematics, Portugal, dirk@ua.pt
[3]York University, Toronto, Canada, tholen@yorku.ca

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We correct a faulty argument given in the justification of Corollary 7.5 of "Cauchy convergence in $\mathcal{V}$-normed categories" and clarify that it does not have any impact on the validity of other results stated in the paper either.

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Corollary 7.5 of [1] states that the category Met_∞ of all (Lawvere) metric spaces with arbitrary maps $f : X \to Y$ as morphisms, $\mathcal{R}_+$-normed by $|f| = \sup_{x,x' \in X} \log^\circ(\frac{Y(fx,fx')}{X(x,x')})$, is Cauchy cocomplete. While this claim is correct, the arguments given involve the incorrect identity $\mathcal{B}_{\log^\circ}(\mathcal{R}_\times\text{-}\mathsf{Lip}) = \mathsf{Met}_\infty$. Instead of $\mathcal{R}_\times$-Lip, one should consider the category $\mathbb{M}$ whose objects are sets X equipped with a mere function $X \times X \to [0,\infty]$, and whose morphisms are arbitrary maps $f : X \to Y$, $\mathcal{R}_\times$-normed by their Lipschitz value $\mathrm{L}(f) = \sup_{x,x'} \frac{Y(fx,fx')}{X(x,x')}$. The second part of the proof of Theorem 7.1 of [1] shows that $\mathbb{M}$ is Cauchy cocomplete. Therefore, by Proposition 7.4 of [1], the $\mathcal{R}_+$-normed category $\mathcal{B}_{\log^\circ}(\mathbb{M})$ is also Cauchy cocomplete, and it contains Met_∞ as a full subcategory, with the same norm.

It now suffices to show that Met_∞ is closed in $\mathcal{B}_{\log^\circ}(\mathbb{M})$ under taking normed colimits of Cauchy sequences, and this follows with an adaptation of the first part of the proof of Theorem 7.1 of [1]. Indeed, for a Cauchy sequence $s = (\, X_m \xrightarrow{s_{m,n}} X_n \,)_{m \le n}$ in Met_∞, one shows that its normed colimit $X \in \mathbb{M}$, structured by

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(7.ii) of [1], satisfies the triangle inequality, as follows. Given any $\varepsilon, \eta > 0$, one chooses $K \in \mathbb{N}$ such that $\varepsilon, \eta \geq \log^\circ \mathrm{L}(s_{m,n})$ or, equivalently, $e^\varepsilon, e^\eta \geq \mathrm{L}(s_{m,n})$, for all $n \geq m \geq K$. Then, following the same steps as in the proof of Theorem 7.1, one shows that $e^\varepsilon X(x,y) + e^\eta X(y,z) \geq X(x,z)$ holds for all $x, y, z \in X$, which implies the triangle inequality.

The incorrect identity $\mathcal{B}_{\log^\circ}(\mathcal{R}_\times\text{-}\mathsf{Lip}) = \mathsf{Met}_\infty$ reappears in Facts 8.2 of [1] in preparation for Theorem 8.4, but its use can be avoided in the same way as in the justification of Corollary 7.5. Indeed, Facts 8.2 become correct when one replaces $\mathcal{R}_\times\text{-}\mathsf{Lip}$ by the above $\mathcal{R}_\times$-normed category $\mathbb{M}$ and observes that in part (3) the thus ensuing functor $U : \mathsf{SNVec}_\infty \to \mathcal{B}_{\log^\circ}(\mathbb{M})$ actually takes values in Met_∞, so that U may be regarded as a functor $\mathsf{SNVec}_\infty \to \mathsf{Met}_\infty$ as stated. Therefore also the proof of Theorem 8.4 remains intact without change. However, we must retract the comments made in the first four lines of Remarks 8.5 of [1], which have no impact on other parts of [1].

References

[1] M.M. Clementino, D. Hofmann, W. Tholen. Cauchy convergence in $\mathcal{V}$-normed categories. Advances in Mathematics 470 (2025) 110247. https://doi.org/10.1016/j.aim.2025.110247 .