EXPONENTIATION FOR UNITARY STRUCTURES

DIRK HOFMANN

Abstract. Motivated by the observation that both pretopologies and preapproach limits can be characterized as those convergence relations which have a unit for a suitable composition, we introduce the category $\mathrm{Alg}_u(\mathsf{T};\mathbf{V})$ of reflexive and unitary lax algebras, for a symmetric monoidal closed lattice $\mathbf{V}$ and a **Set**-monad $\mathsf{T}=(T,e,m)$. For $\mathsf{T}=\mathsf{U}$ the ultrafilter monad, we characterize exponentiable morphisms in $\mathrm{Alg}_u(\mathsf{U};\mathbf{V})$. Further, we give a sufficient condition for an object to be exponentable in the category $\mathrm{Alg}(\mathsf{U};\mathbf{V})$ of reflexive and transitive lax algebras. This specializes to known and new results for pretopological, preapproach and approach spaces.

Introduction

The category **Top** of topological spaces and continuous maps is undoubtedly the most important category in topology. However, many categorically defined constructions either cannot be carried out in **Top** or destroy properties of spaces or maps. In order to perform these constructions, topologists move (temporarily) outside **Top** into larger better behaved environments such as the category **PrTop** of pretopological spaces or the category **PsTop** of pseudotopological spaces. A pseudotopology on a set X is most easily described by a convergence relation $\mathfrak{x}\to x$ between ultrafilters $\mathfrak{x}$ on X and points $x\in X$, where only the principal ultrafilter $\dot{x}$ is required to converge to x. In [2] Barr showed that a pseudotopology is a topology exactly if it satisfies in addition the transitivity axiom

$$\mathfrak{X}\to\mathfrak{x}\to x\Rightarrow m_X(\mathfrak{X})\to x,$$

for all $x\in X$, $\mathfrak{x}\in UX$ and $\mathfrak{X}\in U^2X$. Here m_X is the X-component of the multiplication m of the ultrafilter monad $\mathsf{U}=(U,e,m)$, and the relation $\to$ is naturally extended to a relation between U^2X and UX. In fact, the latter defines an extension of the **Set**-monad U to a lax monad on the 2-category **Rel** of relations; and a topology is exactly a lax Eilenberg-Moore algebra for this extension. In [14] we observed that pretopologies can be described in a very

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similar way, namely as those convergence relations which satisfy, for all $x \in X$ and $\mathfrak{X} \in U^2X$,

$$\overset{\bullet}{x} \to x \qquad \text{and} \qquad \mathfrak{X} \to \overset{\bullet}{x} \;\Rightarrow\; m_X(\mathfrak{X}) \to x.$$

After introducing the Kleisli composition $a * b = a \cdot Ub \cdot m_X^\circ$, for a relation a from UY to Z and a relation b from UX to Y, we see that the latter axiom is equivalent to $e_X^\circ * a = a$. Here f° denotes the inverse image relation of a funtion $f : X \to Y$, that is, $yf^\circ x$ if $y = f(x)$. Since one always has $a * e_X^\circ = a$, we obtain that pretopologies are exactly those pseudotopologies for which e_X° is a unit for the Kleisli composition.

Combining the approach of [2] with [16], in several papers (see [6, 7, 9, 11], for instance) we develop the theory of lax Eilenberg-Moore algebras for extensions of **Set**-monads T to lax monads on bicategories of the form $\mathrm{Mat}(\mathbf{V})$, for a monoidal closed lattice (category) $\mathbf{V}$. This theory encompasses many important categories of topology: besides **Top** also the category **Met** of (generalized) metric spaces and the category **Ap** of approach spaces [17]. The description of pretopologies above opens the door to incorporate also this concept into our theory. In Section 2 we define the category $\mathrm{Alg}_u(\mathsf{T};\mathbf{V})$ of reflexive and *unitary* lax algebras. We will show that, under mild conditions on $\mathbf{V}$, $\mathrm{Alg}_u(\mathsf{U};\mathbf{V})$ is the extensional topological hull of the category $\mathrm{Alg}(\mathsf{U};\mathbf{V})$ of reflexive and transitive lax algebras, generalizing known properties of the embeddings $\mathbf{Top} \hookrightarrow \mathbf{PrTop}$ (see [13]) and $\mathbf{Ap} \hookrightarrow \mathbf{PrAp}$ (see [19]).

The main purpose of this work is the study of exponentiability. Pisani [22] characterized exponentiable objects in **Top** by an interpolation property: a topological space X is exponentiable if and only if, for all $\mathfrak{X} \in U^2X$ and all $x \in X$, $m_X(\mathfrak{X}) \to x$ implies the existence of $\mathfrak{x} \in UX$ such that $\mathfrak{X} \to \mathfrak{x} \to x$. Hence for pretopological spaces we would expect that a space X is exponentiable if and only if, for all $\mathfrak{X} \in U^2X$ and all $x \in X$, $m_X(\mathfrak{X}) \to x$ implies $\mathfrak{X} \to \overset{\bullet}{x}$. It is not hard to see that this condition is equivalent to each point having a smallest neighborhood, which in turn is equivalent to the convergence structure being induced by a reflexive (binary) relation on X. In [21] exponentiable pretopological spaces are characterized as those which satisfy these (equivalent) conditions. Motivated by this example, in Section 3 we formulate a condition and indeed prove that it characterizes exponentiable morphisms in $\mathrm{Alg}_u(\mathsf{U};\mathbf{V})$ (Theorem 3.5), under mild conditions on $\mathbf{V}$. Our result covers the characterization of exponentiable morphisms in **PrTop** obtained in [24], of exponentiable objects in **PrAp** obtained in [20], and it gives a new characterization of exponentiable morphisms in **PrAp**.

The techniques used so far can be partially applied to transitive structures. Section 4 is devoted to this subject. Our main result here (Theorem 4.3) gives

a sufficient condition for an object in $\mathrm{Alg}(\mathsf{U};\mathbf{V})$ to be exponentiable. This theorem applies in particular to approach spaces.

1. Categories of lax algebras

1.1. **Hypothesis.** Throughout we assume that a complete lattice $\mathbf{V}$ equipped with a symmetric tensor product $\otimes$ is given. Moreover, this tensor has a unit element k and a right adjoint hom; that is, for each $\alpha, \beta, \gamma \in \mathbf{V}$,

$$\alpha \otimes \beta \leq \gamma \iff \beta \leq \hom(\alpha, \gamma).$$

Our two principal examples are the Boolean algebra $\mathbf{2} = \{\mathsf{false} \vdash \mathsf{true}\}$ with tensor provided by "and" & and neutral element true; and the extended real half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ ordered by the "greater or equal"-relation $\geq$, with tensor defined by addition (where $\infty + x = x + \infty = \infty$) and 0 the neutral element. Its right adjoint is given by truncated minus: $\hom(x, y) = \max\{y - x, 0\}$.

1.2. **V-matrices.** The category $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-*matrices* (see [4, 11]) has sets as objects, and a morphism $r : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{V})$ is a $\mathbf{V}$-matrix $r : X \times Y \to \mathbf{V}$. Composition of $\mathbf{V}$-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is defined as matrix multiplication

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

and the identity arrow $\mathrm{id}_X : X \nrightarrow X$ is the $\mathbf{V}$-matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\perp$ of $\mathbf{V}$. The complete order of $\mathbf{V}$ induces a complete order on $\mathrm{Mat}(\mathbf{V})(X, Y)$: for $\mathbf{V}$-matrices $r, r' : X \nrightarrow Y$ we define

$$r \leq r' \ :\iff \ \forall x \in X \ \forall y \in Y \ \ r(x, y) \leq r'(x, y).$$

Hence $\mathrm{Mat}(\mathbf{V})$ is actually a 2-*category*. Moreover, it has an order-preserving *involution* $\_^\circ$ sending each $r : X \nrightarrow Y$ to its transpose $r^\circ : Y \nrightarrow X$, defined by $r^\circ(y, x) = r(x, y)$. Returning to our main examples, we have $\mathrm{Mat}(\mathbf{2}) \cong \mathbf{Rel}$ while $\mathrm{Mat}(\overline{\mathbb{R}}_+)$ is the 2-category whose morphisms $a : X \nrightarrow Y$ are generalized distances $a : X \times Y \to \overline{\mathbb{R}}_+$ with composition given by

$$b \cdot a(x, z) = \inf\{a(x, y) + b(y, z) \mid y \in Y\};$$

$\mathrm{id}_X : X \nrightarrow X$ is the discrete distance sending the diagonal to 0 and all other pairs (x, x') to ∞.

The category $\mathbf{Set}$ can be naturally embedded into $\mathrm{Mat}(\mathbf{V})$ by leaving objects unchanged and sending each map $f : X \to Y$ to the $\mathbf{V}$-matrix

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

In the sequel we will write $f : X \to Y$ rather then $f : X \nrightarrow Y$ for a $\mathbf{V}$-matrix induced by a **Set**-map in the sense above. We remark that each $f : X \to Y$ satisfies the inequalities $\mathrm{id}_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq \mathrm{id}_Y$, i.e. f is left adjoint to f°.

1.3. **The ultrafilter monad.** Lax algebras are defined relative to a lax extension of a **Set**-monad $\mathsf{T} = (T, e, m)$ to $\mathrm{Mat}(\mathbf{V})$. In this article we concentrate on the identity monad $\mathsf{1} = (\mathrm{Id}, \mathrm{id}, \mathrm{id})$ and the ultrafilter monad $\mathsf{U} = (U, e, m)$. Recall that the latter monad is induced by the dual adjunction

$$\mathbf{Bool} \overset{\eta}{\Rightarrow} \underset{\mathrm{hom}(\_,2)}{\overset{\mathrm{hom}(\_,2)}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{Set}.$$

Explicitly, the ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ assigns to each set X the set UX of ultrafilters on X and to each function $f : X \to Y$ the function $Uf : UX \to UY$ which takes an ultrafilter $\mathfrak{x} \in UX$ to the (ultra)filter generated by its f-image $\{f[A] \mid A \in \mathfrak{x}\}$. The natural transformations $e : \mathrm{Id} \to U$ and $m : U^2 \to U$ are given by

$$e_X(x) = \dot{x} = \{A \subset X \mid x \in A\} \qquad \text{and} \qquad m_X(\mathfrak{X}) = \{A \subset X \mid A^\# \in \mathfrak{X}\},$$

for all sets X, $\mathfrak{X} \in U^2X$ and $x \in X$. Here $A^\#$ denotes the set $\{\mathfrak{a} \in UX \mid A \in \mathfrak{a}\}$. In the sequel we will extend this notation to filters and write $\mathfrak{f}^\#$ for $\{A^\# \mid A \in \mathfrak{f}\}$. The set of all proper filters on a set X we will denote by FX.

1.4. **Extending to** $\mathrm{Mat}(\mathbf{V})$**.** By a *lax extension* of a monad $\mathsf{T} = (T, e, m)$ on **Set** to $\mathrm{Mat}(\mathbf{V})$ we mean an extension of the endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ to $\mathrm{Mat}(\mathbf{V})$ which satisfies

(1) $Tb \cdot Ta \leq T(b \cdot a)$,
(2) $a \leq a' \;\Rightarrow\; Ta \leq Ta'$,
(3) $e_Y \cdot a \leq Ta \cdot e_X$,
(4) $m_Y \cdot T^2a \leq Ta \cdot m_X$,
(5) $(Ta)^\circ = T(a^\circ)$ (and we write Ta°),

for all $a, a' : X \nrightarrow Y$ and $b : Y \nrightarrow Z$. Note that we have automatically equality in (1) if $a = f$ is a **Set**-map. The identity monad $\mathsf{1} = (\mathrm{Id}, \mathrm{id}, \mathrm{id})$ on **Set** can be obviously "extended" to the identity monad on $\mathrm{Mat}(\mathbf{V})$. In [7, 11] are shown equivalent ways how to extend U to $\mathrm{Mat}(\mathbf{V})$ provided that $\mathbf{V}$ is $\sqsubset$-*atomic*. By that we mean that there is a binary relation $\sqsubset$ on $\mathbf{V}$ which satisfies

- $\alpha \sqsubset \beta \leq \gamma \;\Rightarrow\; \alpha \sqsubset \gamma$ and
- $\alpha = \bigvee\{\alpha_0 \in \mathbf{V} \mid \alpha_0 \sqsubset \alpha, \alpha_0 \text{ is a } \sqsubset\text{-atom}\}$,

for all $\alpha, \beta, \gamma \in \mathbf{V}$. Here an element $\alpha_0 \in \mathbf{V}$ is called an $\sqsubset$-*atom* if, for each $S \subset \mathbf{V}$, $\alpha_0 \sqsubset \bigvee S$ implies $\alpha_0 \leq s$ for some $s \in S$. We will write $\mathrm{At}(\alpha)$ for the

set of all atoms $\alpha_0 \sqsubset \alpha$. It is easy to see that $\mathbf{2}$ is $\leq$-atomic and $\overline{\mathbb{R}}_+$ is $>$-atomic. For $a : X \nrightarrow Y$, $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$ one defines

$$Ua(\mathfrak{x}, \mathfrak{y}) := \bigwedge_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{y}}} \bigvee_{\substack{x \in A \\ y \in B}} a(x, y) = \bigvee \{\gamma \in \mathbf{V} \mid \mathfrak{x}(Ua_\gamma)\mathfrak{y}\}.$$

In the latter formula a_γ denotes the relation defined by

$$xa_\gamma y :\Longleftrightarrow a(x, y) \geq \gamma;$$

and for a relation $r : X \nrightarrow Y$ and ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$ we define

$$\mathfrak{x}(Ur)\mathfrak{y} :\Longleftrightarrow r[\mathfrak{x}] \subset \mathfrak{y} \iff r^\circ[\mathfrak{y}] \subset \mathfrak{x}.$$

In [7] it is shown that it holds indeed $\bigwedge_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{y}}} \bigvee_{\substack{x \in A \\ y \in B}} a(x, y) = \bigvee \{\gamma \in \mathbf{V} \mid \mathfrak{x}(Ua_\gamma)\mathfrak{y}\}$, for all ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$. The argument used there still works when we substitute $\mathfrak{y}$ by a filter base $\mathfrak{f}$, hence we also have $\bigwedge_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{f}}} \bigvee_{\substack{x \in A \\ y \in B}} a(x, y) = \bigvee \{\gamma \in \mathbf{V} \mid a_\gamma^\circ[\mathfrak{f}] \subset \mathfrak{x}\}$. Besides the properties listed above, this extension satisfies

(6) $Ub \cdot Ua = U(b \cdot a)$ provided that $\otimes = \wedge$,
(7) $Ug \cdot Ua = U(g \cdot a)$ and $Ub \cdot Uf = U(b \cdot f)$,
(8) $m_Y \cdot U^2 a = Ua \cdot m_X$,

for all $a : X \nrightarrow Y$, $b : Y \nrightarrow Z$, $f : X \to Y$ and $g : Y \to Z$. We remark that the last property is implied by the following useful fact: for each map $f : X \to Y$, $\mathfrak{x} \in UX$ and $\mathfrak{Y} \in U^2Y$ with $Uf(\mathfrak{x}) = m_Y(\mathfrak{Y})$, there exists $\mathfrak{X} \in U^2X$ such that $m_X(\mathfrak{X}) = \mathfrak{x}$ and $U^2 f(\mathfrak{X}) = \mathfrak{Y}$.

Thought a monad might have more than one extension to $\mathrm{Mat}(\mathbf{V})$ (see [9]), in this article we only consider the extension of U (resp. $\mathsf{1}$) mentioned in this subsection.

1.5. **Extension Lemma.** This Lemma plays an essential role in many proofs using ultrafilters (see [22], for instance). It basically states that if a filter is in relation with an ultrafilter, then this filter can be extended to an ultrafilter with the same property. We shall now extend this lemma to $\mathbf{V}$-matrices, provided that $\mathbf{V}$ fulfils an additional condition. Concretely, we require that the atoms $\alpha \sqsubset \gamma$ below an element γ behave well, either collectively – $\mathrm{At}(\gamma)$ is up-directed – or individually – each $\alpha \in \mathrm{At}(\gamma)$ is connected. Here a subset $A \subset \mathbf{V}$ is called up-directed if any two elements of A have an upper bound. An element $\alpha \in \mathbf{V}$ is connected if $\alpha \leq \beta_1 \vee \beta_2$ implies $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.

Lemma. *Let $a : X \times Y \to \mathbf{V}$, $\mathfrak{x} \in UX$ and $\mathfrak{f} \in FY$. Moreover, assume that $\mathbf{V}$ is $\sqsubset$-atomic and that $\alpha \in \mathbf{V}$ is such that*

$$\alpha \leq \bigvee \{\gamma \in \mathbf{V} \mid a_\gamma^\circ[\mathfrak{f}] \subset \mathfrak{x}\} = \bigwedge_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{f}}} \bigvee_{\substack{x \in A \\ y \in B}} a(x, y),$$

with α connected or $\mathrm{At}(\alpha)$ *up-directed. Then there exists an ultrafilter $\mathfrak{y} \in UY$ such that $\mathfrak{f} \subset \mathfrak{y}$ and $Ua(\mathfrak{x}, \mathfrak{y}) \geq \alpha$.*

Proof. If $\alpha = \bot$, we choose any ultrafilter $\mathfrak{y}$ containing $\mathfrak{f}$. Assume $\alpha \neq \bot$. We define an ideal $\mathfrak{j}$ on Y by

$$\mathfrak{j} := \{B \subset Y \mid \exists \alpha_0 \in \mathrm{At}(\alpha) \;\; a^\circ_{\alpha_0}[B] \notin \mathfrak{x}\} = \{B \subset Y \mid \exists A \in \mathfrak{x} \bigvee_{\substack{x \in A \\ y \in B}} a(x,y) \not\geq \alpha\}.$$

Obviously, $\mathfrak{j} \cap \mathfrak{f} = \emptyset$, $\emptyset \in \mathfrak{j}$ and $\mathfrak{j}$ is down-directed. Let $B_1, B_2 \in \mathfrak{j}$. We consider first $\mathrm{At}(\alpha)$ up-directed. There exist $\alpha_1, \alpha_2 \in \mathrm{At}(\alpha)$ such that

$$a^\circ_{\alpha_1}[B_1] \notin \mathfrak{x} \qquad \text{and} \qquad a^\circ_{\alpha_2}[B_2] \notin \mathfrak{x}.$$

Let α_0 be any upper bound of α_1 and α_2. Then

$$a^\circ_{\alpha_0}[B_1 \cup B_2] = a^\circ_{\alpha_0}[B_1] \cup a^\circ_{\alpha_0}[B_2] \subset a^\circ_{\alpha_1}[B_1] \cup a^\circ_{\alpha_2}[B_2] \notin \mathfrak{x}$$

and therefore $B_1 \cup B_2 \in \mathfrak{j}$.
Assume now that α is connected. We choose A_1 and A_2 with $\bigvee_{\substack{x \in A_i \\ y \in B_i}} a(x,y) \not\geq \alpha$, for $i \in \{1, 2\}$. Connectedness of α implies

$$\bigvee_{\substack{x \in A_1 \\ y \in B_1}} a(x,y) \vee \bigvee_{\substack{x \in A_2 \\ y \in B_2}} a(x,y) \not\geq \alpha.$$

From

$$\bigvee_{\substack{x \in A_1 \cap A_2 \\ y \in B_1 \cup B_2}} a(x,y) \leq \bigvee_{\substack{x \in A_1 \\ y \in B_1}} a(x,y) \vee \bigvee_{\substack{x \in A_2 \\ y \in B_2}} a(x,y)$$

it follows $\bigvee_{\substack{x \in A_1 \cap A_2 \\ y \in B_1 \cup B_2}} a(x,y) \not\geq \alpha$, hence $B_1 \cup B_2 \in \mathfrak{j}$.
The Prime Ideal Theorem (see Section 2 of [15]) guarantees the existence of an ultrafilter $\mathfrak{y} \in UY$ such that $\mathfrak{f} \subset \mathfrak{y}$ and $\mathfrak{y} \cap \mathfrak{j} = \emptyset$, that is, $Ua(\mathfrak{x}, \mathfrak{y}) \geq \alpha$. □

Corollary. *Assume that $\mathbf{V}$ is $\sqsubset$-atomic and that, for each $\gamma \in \mathbf{V}$,* $\mathrm{At}(\gamma)$ *is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected. Let $a : X \times Y \to \mathbf{V}$, $\mathfrak{x} \in UX$ and $\mathfrak{f} \in FY$. Then, for each $\alpha \in \mathrm{At}(\bigvee\{\gamma \in \mathbf{V} \mid a^\circ_\gamma[\mathfrak{f}] \subset \mathfrak{x}\})$, there exists an ultrafilter $\mathfrak{y}_\alpha$ such that $\mathfrak{f} \subset \mathfrak{y}_\alpha$ and $Ua(\mathfrak{x}, \mathfrak{y}_\alpha) \geq \alpha$.*

For $\mathbf{V} = \overline{\mathbb{R}}_+$ or $\mathbf{V} = \mathbf{2}$, both possible conditions above are fulfilled. Note that $\mathbf{V} = \mathbf{2}$ is a special case of $\mathbf{V} = \mathbf{2}^X$, the power set of a set X, ordered by inclusion $\subset$ with $\otimes = \cap$. Then $\mathbf{2}^X$ is $\subset$-atomic, each atom is connected but in general $\mathrm{At}(A)$ is not up-directed.

1.6. **Lax algebras.** Let T be a **Set**-monad laxly extended to $\mathrm{Mat}(\mathbf{V})$. Our main interest lies in the study of categories with objects *lax algebras* (X, a) – that is: sets X equipped with a structure $a : TX \nrightarrow X$ in $\mathrm{Mat}(\mathbf{V})$ – and *lax homomorphisms* $f : (X, a) \to (Y, b)$ between them, i.e. maps $f : X \to Y$ satisfying $f \cdot a \leq b \cdot Tf$. A $\mathbf{V}$-matrix $a : TX \nrightarrow X$ is called

(1) *reflexive* if $1_X \leq a \cdot e_X$,
(2) *transitive* if $a \cdot Ta \leq a \cdot m_X$.

Note that these axioms are just lax versions of the Eilenberg-Moore axioms of a T-algebra. Expressed componentwise, they read as

(1') $k \leq a(e_X(x), x)$,
(2') $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$,

for all $\mathfrak{X} \in U^2X$, $\mathfrak{x} \in UX$ and $x \in X$. In the sequel we shall say that a lax algebra (X, a) is reflexive, transitive, ..., if a is so. We denote by $\mathrm{Alg}(T; \mathbf{V})$ the category of lax algebras and lax homomorphisms, by $\mathrm{Alg}(T, e; \mathbf{V})$ its full subcategory of reflexive lax algebras and by $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$ its full subcategory of reflexive and transitive lax algebras. If $\mathsf{T} = \mathsf{1}$, a reflexive lax algebra is a reflexive $\mathbf{V}$-graph and a reflexive and transitive lax algebra is a $\mathbf{V}$-category; moreover, a lax homomorphism is a $\mathbf{V}$-functor (see [16]). We write $\mathbf{V}$-**RGph** instead of $\mathrm{Alg}(\mathrm{Id}, \mathrm{id}; \mathbf{V})$ and $\mathbf{V}$-**Cat** instead of $\mathrm{Alg}(\mathsf{1}; \mathbf{V})$. Note that $\mathbf{V}$ equipped with the $\mathbf{V}$-matrix $\hom : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ is a $\mathbf{V}$-category.

We obtain full embeddings $\mathrm{Alg}(\mathsf{T}; \mathbf{V}) \hookrightarrow \mathrm{Alg}(T, e; \mathbf{V}) \hookrightarrow \mathrm{Alg}(T; \mathbf{V})$. All three categories are topological (in the sense of [1]) over **Set** with respect to their canonical forgetful functor, and in each case the initial structure a on a set X with respect to a source $(f_i : X \to (X_i, a_i))_{i \in I}$ is given by

$$a = \bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot Tf_i.$$

Therefore we see that all inclusion functors above have a left adjoint. We also remark that T extends to a functor $T : \mathrm{Alg}(T; \mathbf{V}) \to \mathrm{Alg}(T; \mathbf{V})$ which preserves initial morphisms $f : (X, a) \to (Y, b)$[1] since $T(f^\circ \cdot b \cdot Tf) = Tf^\circ \cdot Tb \cdot T^2 f$.

1.7. **Examples.** A $\mathbf{2}$-category is just a preordered set and a $\mathbf{2}$-functor is a order-preserving map, whereby an $\overline{\mathbb{R}}_+$-category is a (generalized) metric space and an $\overline{\mathbb{R}}_+$-functor is a non-expansive map.

In [2] Barr describes the topological convergence relations $\mathfrak{x} \to x$ between ultrafilters $\mathfrak{x}$ on a set X and points $x \in X$ as exactly those relations which satisfy the lax Eilenberg-Moore axioms for the extension of U to $\mathbf{Rel} \cong \mathrm{Mat}(\mathbf{2})$. In other words: $\mathrm{Alg}(\mathsf{U}; \mathbf{2}) \cong \mathbf{Top}$.

[1] that is: a is initial for $f : X \to (Y, b)$

Approach spaces were introduced by Lowen (see [17]) as a common framework for both topological and metric structures. Being precise, an approach space is a pair (X, δ) consisting of a set X and a function $\delta : \mathbf{2}^X \times X \to \overline{\mathbb{R}}_+$ subject to

(1) $\delta(\{x\}, x) = 0$,
(2) $\delta(\emptyset, x) = \infty$,
(3) $\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\}$,
(4) $\delta(A, x) \leq \delta(A^{(\varepsilon)}, x) + \varepsilon$;

for each $A, B \subset X$, $x \in X$ and $\varepsilon \in \overline{\mathbb{R}}_+$. Here $A^{(\varepsilon)} = \{x \in X \mid \delta(A, x) \leq \varepsilon\}$. A function $\delta : \mathbf{2}^X \times X \to \overline{\mathbb{R}}_+$ satisfying the axioms above is called *approach distance* on X. If δ takes only the values 0 and ∞, these are the axioms of a topological closure operator considering $\overline{A} = A^{(0)}$. For $\delta : \mathbf{2}^X \times X \to \overline{\mathbb{R}}_+$ and $\delta' : \mathbf{2}^Y \times Y \to \overline{\mathbb{R}}_+$, a map $f : X \to Y$ is called *non-expansive* if $\delta(A, x) \geq \delta'(f[A], f(x))$ for each $A \subset X$ and $x \in X$. Approach spaces and non-expansive maps are the objects and morphisms of the category $\mathbf{Ap}$. Each $\delta : \mathbf{2}^X \times X \to \overline{\mathbb{R}}_+$ defines a map $a : UX \times X \to \overline{\mathbb{R}}_+$ by

$$a(\mathfrak{x}, x) = \sup_{A \in \mathfrak{x}} \delta(A, x),$$

and vice versa, each $a : UX \times X \to \overline{\mathbb{R}}_+$ defines a function $\delta : \mathbf{2}^X \times X \to \overline{\mathbb{R}}_+$ by

$$\delta(A, x) = \inf_{A \in \mathfrak{x}} a(\mathfrak{x}, x).$$

In [6] it is shown that the functions $a : UX \times X \to \overline{\mathbb{R}}_+$ coming from a approach distance are precisely those satisfying the lax Eilenberg-Moore axioms for the lax extension of U to $\mathrm{Mat}(\overline{\mathbb{R}}_+)$. Hence we have $\mathrm{Alg}(\mathsf{U}; \overline{\mathbb{R}}_+) \cong \mathbf{Ap}$.

1.8. **Kleisli composition.** Given a lax extension of a **Set**-monad T to $\mathrm{Mat}(\mathbf{V})$, the *Kleisli "category"* has as objects sets and a morphism from X to Y is a $\mathbf{V}$-matrix $a : TX \nrightarrow Y$. Composition is given by *Kleisli composition*

$$b * a := b \cdot Ta \cdot m_X^\circ,$$

$$\begin{tikzcd}
TX \arrow[d, "a"] & TY \arrow[d, "b"] & \mapsto & TX \arrow[r, "m_X^\circ"] \arrow[ddr, "b*a"'] & T^2X \arrow[d, "Ta"] \\
Y & Z & & & TY \arrow[d, "b"] \\
& & & & Z
\end{tikzcd}$$

for all $b : TY \nrightarrow Z$ and $a : TX \nrightarrow Y$. For each $a : TX \nrightarrow Y$ it holds $a * e_X^\circ = a$ and $e_Y^\circ * a \geq a$, that is, the $\mathbf{V}$-matrix e_X° is a lax identity for this composition. Moreover, we have

$$c * (b * a) \leq (c * b) * a \qquad \text{or} \qquad c * (b * a) \geq (c * b) * a$$

provided that $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ is a functor or m extends to a (strict) natural transformation respectively. Of course, for $\mathsf{T} = \mathsf{1}$ the Kleisli category coincides with $\mathrm{Mat}(\mathbf{V})$. Reflexivity and transitivity of $a : TX \nrightarrow X$ can be now equivalently expressed by the inequalities

$$e_X^\circ \leq a, \qquad\qquad a * a \leq a.$$

1.9. **Elementary structures.** To derive properties of a lax algebra (X, a), it will be often useful to "isolate" values $\gamma = a(\mathfrak{x}, x)$. For the ultrafilter monad U and $\sqsubset$-atomic $\mathbf{V}$, this can be done using the following structure. Let X be a set and $\mathfrak{x} \in X$, $x \in X$ and $\gamma \in \mathbf{V}$. We put $Y = X_1 + X_2$, where $X_1 = X_2 = X$, and define a structure $a^\gamma_{\mathfrak{x},x} : UY \times Y \to \mathbf{V}$ by putting

$$a^\gamma_{\mathfrak{x},x}(\mathfrak{y}, y) = \begin{cases} k & \text{if } \mathfrak{y} = \overset{\bullet}{y}, \\ \gamma & \text{if } \mathfrak{y} = \mathfrak{x} \in UX_1 \text{ and } y = x \in X_2, \\ \perp & \text{else,} \end{cases}$$

for each $\mathfrak{y} \in UY$ and $y \in Y$. In this subsection we will always consider $\mathfrak{x}$ as an ultrafilter on the component X_1 of Y and $x \in X_2$. It is clear from the definition that $a^\gamma_{\mathfrak{x},x}$ is reflexive. We are now going to show that it is also transitive. In order to see that, let $\mathfrak{Y} \in U^2Y$, $\mathfrak{y} \in UY$ and $y \in Y$. We consider first the case $\mathfrak{y} = \overset{\bullet}{y}$. If

$$Ub(\mathfrak{Y}, \overset{\bullet}{y}) = \bigwedge_{\mathcal{B} \in \mathfrak{Y}} \bigvee_{\mathfrak{b} \in \mathcal{B}} b(\mathfrak{b}, y) > \perp,$$

it follows $\{\overset{\bullet}{y}, \mathfrak{x}\} \in \mathfrak{Y}$ which implies $\mathfrak{Y} = \overset{\bullet}{\overset{\bullet}{y}}$ or $\mathfrak{Y} = \overset{\bullet}{\mathfrak{x}}$. Note that the latter case is only possible if $y = x$. Hence in both cases we have

$$Ub(\mathfrak{Y}, \overset{\bullet}{y}) = b(m_Y(\mathfrak{Y}), y).$$

We now consider $\mathfrak{y} = \mathfrak{x}$ and $y = x$. Here

$$Ub(\mathfrak{Y}, \mathfrak{x}) = \bigwedge_{\substack{\mathcal{B} \in \mathfrak{Y} \\ A \in \mathfrak{x}}} \bigvee_{\substack{\mathfrak{b} \in \mathcal{B} \\ z \in A}} b(\mathfrak{b}, z) > \perp$$

implies $\mathfrak{Y} = Ue_Y(\mathfrak{x})$ and therefore

$$Ub(\mathfrak{Y}, \mathfrak{x}) \otimes b(\mathfrak{x}, x) = k \otimes \gamma = \gamma = b(m_Y(\mathfrak{Y}), y).$$

2. Unitary structures

2.1. **Pretopological spaces.** Pretopological spaces were essentially introduced by Choquet in [5]; for more informations see also [3, 13]. A *pretopology* a on a set X is a relation $\mathfrak{x} \to x$ between ultrafilters and points which fulfils, for each $x \in X$ and $\mathfrak{x} \in UX$,

$$\overset{\bullet}{x} \to x \qquad\qquad \text{and} \qquad\qquad \mathfrak{x} \to x \text{ whenever } \bigcap_{\mathfrak{y} \to x} \mathfrak{y} \subset \mathfrak{x}.$$

This can be equivalently expressed by saying that to each point $x \in X$ is associated a neighborhood filter $\mathcal{V}(x) \subset \overset{\bullet}{x}$ and defining $\mathfrak{x} \to x$ precisely if $\mathcal{V}(x) \subset \mathfrak{x}$. A *pretopological space* is a set X equipped with a pretopology; together with continuous maps they form the category **PrTop**. Recall that the Zariski closure on UX is defined by $\mathfrak{x} \in \overline{\mathcal{A}} :\Longleftrightarrow \bigcap \mathcal{A} \subset \mathfrak{x}$, for all $\mathfrak{x} \in UX$ and $\mathcal{A} \subset UX$. Hence the second condition above just states that the set $\{\mathfrak{y} \in UX \mid \mathfrak{y} \to x\}$ is closed in UX with respect to the Zariski closure. In [14] we observed that this is equivalent to $e_X^\circ * a = a$.

2.2. **Preapproach spaces** [18, 19]**.** A *preapproach limit* a on a set X is a function $a : UX \times X \to \overline{\mathbb{R}}_+$ which fulfils, for each $x \in X$ and $\mathfrak{x} \in UX$,

$$a(\overset{\bullet}{x}, x) = 0 \qquad \text{and} \qquad a(\mathfrak{x}, x) \leq \sup\{a(\mathfrak{y}_i, x) \mid i \in I\} \text{ whenever } \bigcap_{i \in I} \mathfrak{y}_i \subset \mathfrak{x}.$$

A *preapproach space* is a pair (X, a) consisting of a set X and a preapproach limit a on X. **PrAp** denotes the category of preapproach spaces and non-expansive maps. As in the case of pretopological spaces, the second axiom is equivalent to $e_X^\circ * a = a$.

2.3. **Definition and basic properties.** Guided by the two previous examples, we introduce the following

Definition. A $\mathbf{V}$-matrix $a : TX \nrightarrow X$ is called *unitary* provided that $e_X^\circ * a \leq a$.

Therefore a unitary $\mathbf{V}$-matrix $a : TX \nrightarrow X$ satisfies $e_X^\circ * a = a$. We let $\mathrm{Alg}_u(\mathsf{T}; \mathbf{V})$ denote the category of reflexive and unitary lax algebras and lax homomorphisms. The two examples above tell us that $\mathrm{Alg}_u(\mathsf{U}, \mathbf{2}) \cong \mathbf{PrTop}$ and $\mathrm{Alg}_u(\mathsf{U}, \overline{\mathbb{R}}_+) \cong \mathbf{PrAp}$. The canonical forgetful functor $|\_| : \mathrm{Alg}_u(\mathsf{T}; \mathbf{V}) \to \mathbf{Set}$ is topological: the initial structure a on X with respect to a source $(f_i : X \to (X_i, a_i))_{i \in I}$ is also given by

$$a = \bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot Tf_i.$$

We infer that the full embeddings $\mathrm{Alg}(\mathsf{T}; \mathbf{V}) \hookrightarrow \mathrm{Alg}_u(\mathsf{T}; \mathbf{V})$ and $\mathrm{Alg}_u(\mathsf{T}; \mathbf{V}) \hookrightarrow \mathrm{Alg}(T, e; \mathbf{V})$ are right adjoints. In the latter case the reflector R has a simple description provided that T preserves composition of $\mathbf{V}$-matrices with **Set**-maps: $R(X, a) = (X, e_X^\circ * a)$.

2.4. **Embedding V-Cat.** Assume that $\mathbf{V}$ is $\sqsubset$-atomic and let X be a set. We define order-preserving maps

$$\{r : X \nrightarrow X\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \{a : UX \nrightarrow X\}$$

by putting

$$\phi(r) = e_X^\circ \cdot Ur \qquad \text{and} \qquad \psi(a) = a \cdot e_X.$$

Componentwise, this translates to

$$\phi(r)(\mathfrak{x}, x) = \bigwedge_{A \in \mathfrak{x}} \bigvee_{y \in A} r(y, x) \qquad \text{and} \qquad \psi(a)(y, x) = a(\overset{\bullet}{y}, x).$$

It follows that $r = \psi\phi(r)$ for each $r : X \nrightarrow X$. For each unitary a we have $a \geq \phi\psi(a)$: $\phi\psi(a) = e_X^\circ \cdot Ua \cdot Ue_X \leq e_X^\circ \cdot Ua \cdot m_X^\circ = e_X^\circ * a = a$. Further, for all $r : X \nrightarrow X$, $\mathfrak{x} \in UX$, $x \in X$ and $\gamma \in \mathbf{V}$,

$$\phi(r)(\mathfrak{x}, x) \geq \gamma \iff \forall \gamma_0 \in \mathrm{At}(\gamma)\ \ r_{\gamma_0}^\circ(x) \in \mathfrak{x} \iff \forall \gamma_0 \in \mathrm{At}(\gamma)\ \ \mathfrak{x}\phi(r_{\gamma_0})x.$$

Hence we have $\phi(r)_\gamma = \bigcap_{\gamma_0 \in \mathrm{At}(\gamma)} \phi(r_{\gamma_0})$, and therefore $\phi(r_\gamma) \leq \phi(r)_\gamma$. We also obtain $\phi(s \wedge r) = \phi(s) \wedge \phi(r)$ for all $r, s : X \nrightarrow X$: Since ϕ preserves the order, we have $\phi(s \wedge r) \leq \phi(s) \wedge \phi(r)$. Let $\mathfrak{x} \in UX$, $x \in X$ and $\gamma \in \mathrm{At}(\phi(s) \wedge \phi(r)(\mathfrak{x}, x))$. We have $s_\gamma^\circ(x) \in \mathfrak{x}$ and $r_\gamma^\circ(x) \in \mathfrak{x}$ and then

$$(s \wedge r)_\gamma^\circ(x) = s_\gamma^\circ(x) \cap r_\gamma^\circ(x) \in \mathfrak{x}.$$

Lemma. *For each $r, s : X \nrightarrow X$ and $a : UX \nrightarrow X$,*

(1) *$a \cdot m_X \leq e_X^\circ \cdot Ua$ implies $a \leq \phi\psi(a)$.*
(2) *$\phi(s) \cdot U\phi(r) \leq \phi(s \cdot r) \cdot m_X$.*
(3) *$e_X^\circ \cdot U\phi(r) = \phi(r) \cdot m_X$.*
(4) *$\phi(s) \cdot U\phi(r) = \phi(s \cdot r) \cdot m_X$ provided that $\otimes = \wedge$ and, for each $\gamma \in \mathbf{V}$, $\mathrm{At}(\gamma)$ is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected.*

Proof. (1) From $a \cdot m_X \leq e_X^\circ \cdot Ua$ we obtain

$$a = a \cdot m_X \cdot Ue_X \leq e_X^\circ \cdot Ua \cdot Ue_X = \phi\psi(a).$$

$$\begin{aligned} \phi(s) \cdot U\phi(r) &= e_X^\circ \cdot Us \cdot Ue_X^\circ \cdot U^2 r \\ &\leq e_X^\circ \cdot Us \cdot m_X \cdot U^2 r \\ &= e_X^\circ \cdot Us \cdot Ur \cdot m_X \\ &\leq e_X^\circ \cdot U(s \cdot r) \cdot m_X = \phi(s \cdot r) \cdot m_X. \end{aligned} \tag{2}$$

(3) Let $\mathfrak{X} \in U^2X$, $x \in X$ and $\gamma_0 \in \mathrm{At}(\phi(r)(m_X(\mathfrak{X}), x))$. It holds $r_{\gamma_0}^\circ(x) \in m_X(\mathfrak{X})$, and therefore $r_{\gamma_0}^\circ(x)^\# \in \mathfrak{X}$. Consequently, each $\mathcal{A} \in \mathfrak{X}$ contains some $\mathfrak{a}$ with $\phi(r)(\mathfrak{a}, x) \geq \gamma_0$. We obtain $Ua(\mathfrak{X}, \overset{\bullet}{x}) \geq \gamma_0$.
(4) We assume now that $\otimes = \wedge$ and that $\mathrm{At}(\gamma)$ is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected, for each $\gamma \in \mathbf{V}$. Let $\mathfrak{X} \in U^2X$, $x \in X$ and $\gamma_0 \in \mathrm{At}(\phi(s \cdot r)(m_X(\mathfrak{X}), x))$. As above we have $(s \cdot r)_{\gamma_0}^\circ(x)^\# \in \mathfrak{X}$. Hence for each $\gamma_1 \in \mathrm{At}(\gamma_0)$ it holds $(r_{\gamma_1}^\circ \cdot s_{\gamma_1}^\circ)(x)^\# \in \mathfrak{X}$. Consequently, for each $\mathcal{A} \in \mathfrak{X}$, there exist some $\mathfrak{a} \in \mathcal{A}$ and $y \in s_{\gamma_1}^\circ(x)$ such that $\phi(r)(\mathfrak{a}, y) \geq \gamma_1$. Corollary 1.5 implies that, for each

$\alpha \in \mathrm{At}(\gamma_1)$, there exists $\mathfrak{x}_\alpha \in UX$ with $U\phi(r)(\mathfrak{X}, \mathfrak{x}_\alpha) \geq \alpha$ and $\phi(s)(\mathfrak{x}_\alpha, x) \geq \gamma_1$. Therefore $\phi(s) \cdot U\phi(r)(\mathfrak{X}, x) \geq \gamma_1$. $\square$

The lemma above implies that ϕ is a lax homomorphism with respect to the Kleisli composition:

$$\phi(\mathrm{id}_X) = e_X^\circ \qquad \text{and} \qquad \phi(s) * \phi(r) \leq \phi(s \cdot r).$$

It is not hard to see that the same is true for ψ:

$$\psi(e_X^\circ) = \mathrm{id}_X \qquad \text{and} \qquad \psi(b) \cdot \psi(a) \leq \psi(b * a).$$

We conclude that $\phi(r)$ is unitary and $\phi(r)$ resp. $\psi(a)$ is reflexive or transitive if r resp. a is so. Hence ϕ and ψ induce pairs of adjoint functors

$$\begin{tikzcd} \mathbf{V}\text{-}\mathbf{RGph} \arrow[r, hook] & \mathrm{Alg}_u(\mathsf{U}; \mathbf{V}) \arrow[l, bend right, "\top"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathbf{V}\text{-}\mathbf{Cat} \arrow[r, hook] & \mathrm{Alg}(\mathsf{U}; \mathbf{V}) \arrow[l, bend right, "\top"'] \end{tikzcd},$$

where the left adjoint inclusion functor preserves finite limites. In particular, $\mathbf{V}$ becomes an $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$-object $(\mathbf{V}, h)$ with $h := \phi(\mathrm{hom})$. Finally, we can characterize those structures $a : UX \nrightarrow X$ coming from a $\mathbf{V}$-matrix $r : X \nrightarrow X$:

$$a = \phi\psi(a) \iff a \cdot m_X = e_X^\circ \cdot Ua.$$

2.5. **Extensional topological hulls.** Among other results it is shown in [13] that **PrTop** is the extensional topological hull of **Top**, that is, the smallest extensional topological category containing **Top** nicely. We present here the results and definitions necessary for this paper; for more information we refer to [13].

Let $\mathbf{A}$ be a category and $\mathcal{S}$ be a pullback-stable class of morphisms of $\mathbf{A}$. A *$\mathcal{S}$-partial map* from X to Y is a pair $(X \xleftarrow{s} Z \to Y)$ where $s \in \mathcal{S}$. We say that $\mathbf{A}$ has *$\mathcal{S}$-partial map classifiers* if, for every $Y \in \mathbf{A}$, there is an $\mathcal{S}$-morphism $\mathsf{true}_Y : Y \to Y^*$ such that every $\mathcal{S}$-partial map $(X \xleftarrow{s} Z \to Y)$ from X to Y can be uniquely completed so that the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "s"'] & Y \arrow[d, "\mathsf{true}_Y"] \\ X \arrow[r, dashed] & Y^* \end{tikzcd}$$

is a pullback. If $\mathbf{A}$ is a construct (i.e., $\mathbf{A}$ is equipped with a forgetful functor $|\_| : \mathbf{A} \to \mathbf{Set}$) and $\mathcal{S} = \{\text{embeddings}\}$, we simply say partial map instead of $\mathcal{S}$-partial map. A topological construct which has partial map classifiers is called *extensional*. By an *extensional topological hull* of a construct $\mathbf{A}$ we understand a full concrete embedding $E : A \hookrightarrow B$ such that $\mathbf{B}$ is an extensional topological construct and E is finally dense, and for each full, concrete, finally dense embedding $F : \mathbf{A} \to \mathbf{B}'$ with $\mathbf{B}'$ extensional there exists a unique full concrete embedding $G : \mathbf{B} \to \mathbf{B}'$ with $G \cdot E = F$. There is, up to isomorphism,

at most one extensional topological hull of a category $\mathbf{A}$. It is characterized by the following theorem.

Theorem. *The extensional topological hull* $\mathbf{B}$ *of a construct* $\mathbf{A}$ *is characterized by the following properties:*

(1) $\mathbf{B}$ *is an extensional topological construct.*
(2) $\mathbf{A}$ *is a finally dense full concrete subcategory of* $\mathbf{B}$*.*
(3) $\{X^* \mid X \in \mathbf{A}\}$ *is initially dense in* $\mathbf{B}$*.*

We will now use the theorem above to show that $\mathrm{Alg}_u(\mathsf{U};\mathbf{V})$ is the extensional topological hull of $\mathrm{Alg}(\mathsf{U};\mathbf{V})$, provided that $\mathbf{V}$ satisfies further properties.

2.6. **One-point extensions.** Assume that $\mathbf{V}$ is $\sqsubset$-atomic and let $(X,a) \in \mathrm{Alg}_u(\mathsf{U};\mathbf{V})$. We put

$$X^* = X + \{*\}$$

and extend a to X^* by defining

$$a(\mathfrak{x},*) = \top, \qquad \text{and} \qquad a(\overset{\bullet}{*},x) = \top,$$

for all $\mathfrak{x} \in UX$ and $x \in X^*$. It is easy to see that (X^*,a) is again unitary and that $(X,a) \hookrightarrow (X^*,a)$ classifies partial morphisms to (X,a). We conclude that $\mathrm{Alg}_u(\mathsf{U};\mathbf{V})$ is extensional.

2.7. **Initially dense object.** Assume that $\mathbf{V}$ is $\sqsubset$-atomic and $k = \top$. Let $(X,a) \in \mathrm{Alg}_u(\mathsf{U};\mathbf{V})$. For each $A \subset X$ and $x \in X$ we define a map $g_{A,x} : X \to \mathbf{V}^*$ by putting

$$g_{A,x}(y) = \begin{cases} \bigvee_{A\in\mathfrak{a}} a(\mathfrak{a},y) & \text{if } y \in A \cup \{x\}, \\ * & \text{else;} \end{cases}$$

for each $y \in X$. We are going to show that $g_{A,x}$ is a lax homomorphism. If $\bigvee_{A\in\mathfrak{a}} a(\mathfrak{a},x) = k$, then $g_{A,x}[X] \subset \{k,*\}$ and hence $g_{A,x}$ is a lax homomorphism. Otherwise, put $\gamma := \bigvee_{A\in\mathfrak{a}} a(\mathfrak{a},x) < k$ and let $\mathfrak{y} \in UX$ and $y \in X$. We only have to consider the case $Ug_{A,x}(\mathfrak{y}) = \overset{\bullet}{k}$ and $g_{A,x}(y) = \gamma$, that is: $A \in \mathfrak{y}$ and $y = x$. Hence we have

$$k \otimes a(\mathfrak{y},x) = a(\mathfrak{y},x) \le g_{A,x}(x)$$

from which follows

$$a(\mathfrak{y},x) \le \hom(k, g_{A,x}(x)) = h(Ug_{A,x}(\mathfrak{y}), g_{A,x}(x)).$$

Lemma. *Assume that* $k = \top$ *and, for each* $\gamma \in \mathbf{V}$*,* $\mathrm{At}(\gamma)$ *is up-directed or each* $\gamma_0 \in \mathrm{At}(\gamma)$ *is connected. For each* $(X,a) \in \mathrm{Alg}_u(\mathsf{U};\mathbf{V})$*,* $\mathfrak{x} \in UX$ *and* $x \in X$*,*

$$a(\mathfrak{x},x) = \bigwedge_{A\in\mathfrak{x}} h(Ug_{A,x}(\mathfrak{x}), g_{A,x}(x)).$$

Proof. Note that $Ug_{A,x}(\mathfrak{x}) = \overset{\bullet}{k}$ for each $\mathfrak{x} \in A^{\#}$, hence we have to show

$$a(\mathfrak{x}, x) \geq \bigwedge_{A \in \mathfrak{x}} g_{A,x}(x) = \bigwedge_{A \in \mathfrak{x}} \bigvee_{\mathfrak{a} \in A^{\#}} a(\mathfrak{a}, x) =: \gamma.$$

Applying Corollary 1.5 to $\mathfrak{x}^{\#}$ and $\overset{\bullet}{x}$, for each $\alpha \in \mathrm{At}(\gamma)$ there exists an ultrafilter $\mathfrak{X}_\alpha \in U^2X$ such that $\mathfrak{x}^{\#} \subset \mathfrak{X}_\alpha$ and $\alpha \leq Ua(\mathfrak{X}_\alpha, \overset{\bullet}{x})$. Since a is unitary, we have $\gamma \leq a(\mathfrak{x}, x)$. □

Corollary. *If $k = \top$ and, for each $\gamma \in \mathbf{V}$, $\mathrm{At}(\gamma)$ is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected, then $(\mathbf{V}^*, h)$ (see end of 2.4 and 2.6) is initially dense in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$.*

2.8. **Theorem.** $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ *is the extensional topological hull of* $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$ *provided that $k = \top$, $\mathbf{V}$ is $\sqsubset$-atomic and, for each $\gamma \in \mathbf{V}$, $\mathrm{At}(\gamma)$ is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected.*

Proof. Combine 1.9, 2.6 and 2.7 and apply Theorem 2.5. □

3. Exponentiable morphisms in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$

3.1. **Partial products.** Recall that, by definition, a morphism $f : X \to Y$ in a finitely complete category $\mathbf{X}$ is *exponentiable* if the pullback functor

$$X \times_Y \_ : \mathbf{X}/Y \to \mathbf{X}/Y$$

has a right adjoint. In [12] it is shown that this is equivalent to the existence of *partial products* over f, that is, for each $Z \in \mathbf{X}$ there is a diagram

$$\begin{tikzcd} Z & P \times_Y X \arrow[l, "\mathrm{ev}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ & P \arrow[r, "p"'] & Y \end{tikzcd}$$

such that for every diagram

$$\begin{tikzcd} Z & P' \times_Y X \arrow[l, "\mathrm{ev}'"'] \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & X \arrow[d, "f"] \\ & P' \arrow[r, "p'"'] & Y \end{tikzcd}$$

there exists a unique $t : P' \to P$ with $p \cdot t = p'$ and $\mathrm{ev} \cdot (\mathrm{id}_X \times_Y t) = \mathrm{ev}'$. Considering $P' = (1, e_1^\circ)$, we see that, in any of the categories $\mathrm{Alg}(U, e; \mathbf{V})$, $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ or $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$, P should have as underlying set

$$P = \{(s, y) \mid y \in Y, s : (X_y, a_y) \to (Z, c)\}$$

where (X_y, a_y) is the domain of the pullback of $y : (1, e_1^\circ) \to (Y, b)$ along f, $p : P \to Y$ is the projection map and $\mathrm{ev} : P \times_Y X \to Z$ is the evaluation map $(s, x) \mapsto s(x)$.

3.2. **Exponentiation in** $\mathrm{Alg}(U, e; \mathbf{V})$**.** In [10] it is shown that the category $\mathrm{Alg}(U, e; \mathbf{V})$ is locally cartesian closed provided that $\mathbf{V}$ is so, that is, a Heyting algebra. For $f : (X, a) \to (Y, b)$ and (Z, c) in $\mathrm{Alg}(U, e; \mathbf{V})$, the structure d on the partial product P is defined as (with $Q := P \times_Y X$)

$$d(\mathfrak{p}, (s, y)) = \bigvee \left\{ \gamma \in \mathbf{V} \;\middle|\; \begin{array}{l} \gamma \le b(Up(\mathfrak{p}), y) \;\; \& \\ \forall \mathfrak{q} \in U\pi_1^{-1}(\mathfrak{p}), x \in f^{-1}(y) \;\; a(U\pi_2(\mathfrak{q}), x) \wedge \gamma \le c(U\mathrm{ev}(\mathfrak{q}), s(x)) \end{array} \right\},$$

for every $\mathfrak{p} \in UP$ and $(s, y) \in P$.

3.3. **Partial products coincide.** Assume that $\mathbf{V}$ is $\sqsubset$-atomic and a Heyting algebra. Let $f : (X, a) \to (Y, b)$ and (Z, c) in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ and assume that the partial product (P, d) of (Z, c) over f in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ exists. In 3.1 we observed that it has the same underlying set as the partial product (P, d') of (Z, c) over f in $\mathrm{Alg}(U, e; \mathbf{V})$. An argument similar to the one in [8], 3.3, shows that also the structures d and d' coincide: we just replace the two-element $\mathbf{V}$-category $(\{0, 1\}, e)$ used in [8] by the elementary structure described in 1.9.

Note that the same argument shows that partial products taken in $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$ and $\mathrm{Alg}(U, e; \mathbf{V})$ coincide as well.

3.4. **Lemma.** *Assume that* $\mathbf{V}$ *is* $\sqsubset$*-atomic. For each pullback*

$$\begin{array}{ccc} (P, d) & \xrightarrow{\pi_1} & (X, a) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ (Y, b) & \xrightarrow[g]{} & (Z, c) \end{array}$$

in $\mathrm{Alg}(U; \mathbf{V})$*,* $\mathfrak{P} \in U^2P$*,* $x \in X$ *and* $y \in Y$ *with* $f(x) = g(y)$*,*

$$Ud(\mathfrak{P}, (\overset{\bullet}{x}, y)) = Ua(U^2\pi_1(\mathfrak{P}), \overset{\bullet}{x}) \wedge Ub(U^2\pi_2(\mathfrak{P}), \overset{\bullet}{y}).$$

Proof. Since $U\pi_1$ and $U\pi_2$ are lax homomorphisms, we only have to show

$$Ud(\mathfrak{P}, (\overset{\bullet}{x}, y)) \ge Ua(\mathfrak{X}, \overset{\bullet}{x}) \wedge Ub(\mathfrak{Y}, \overset{\bullet}{y}),$$

where $\mathfrak{X} = U^2\pi_1(\mathfrak{P})$ and $\mathfrak{Y} = U^2\pi_2(\mathfrak{P})$. Let $\alpha \in \mathrm{At}(Ua(\mathfrak{X}, \overset{\bullet}{x}))$ and $\beta \in \mathrm{At}(Ub(\mathfrak{Y}, \overset{\bullet}{y}))$. We have $\{\mathfrak{a} \in UX \mid a(\mathfrak{a}, x) \ge \alpha\} \in \mathfrak{X}$ and $\{\mathfrak{b} \in UX \mid b(\mathfrak{b}, y) \ge \beta\} \in \mathfrak{Y}$ and therefore $\{\mathfrak{d} \in UP \mid Ud(\mathfrak{d}, (x, y)) \ge \alpha \wedge \beta\} \in \mathfrak{P}$. $\square$

3.5. **Theorem.** *If* $\mathbf{V}$ *is* $\sqsubset$*-atomic and a Heyting algebra, then* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ *is exponentiable if and only if,*

$$(\diamond) \qquad \forall \mathfrak{X} \in U^2X \ \forall x \in X \ \ Ua(\mathfrak{X}, \overset{\bullet}{x}) \geq Ub(Uf(\mathfrak{X}), \overset{\bullet}{f(x)}) \wedge a(m_X(\mathfrak{X}), x).$$

Proof. We are first going to show that the condition $(\diamond)$ is sufficient for exponentiability. Assume that $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ satisfies $(\diamond)$ and let $(Z, c) \in \mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$. We have to show that the partial product (see 3.2)

$$\begin{tikzcd} (Z,c) & (Q, a \wedge d) \arrow[l, "\mathrm{ev}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (X,a) \arrow[d, "f"] \\ & (P,d) \arrow[r, "p"'] & (Y,b) \end{tikzcd}$$

of (Z, c) over f taken in $\mathrm{Alg}(U, e; \mathbf{V})$ actually lies in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$. To see that, let $\mathfrak{P} \in U^2P$ and $(s, y) \in P$. According to the definition of the structure d on P,

$$Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \leq d(m_P(\mathfrak{P}), (s, y))$$

holds precisely if

$$Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \leq b(Up(m_P(\mathfrak{P})), y)$$

and, for each $\mathfrak{q} \in UQ$ with $U\pi_1(\mathfrak{q}) = m_P(\mathfrak{P})$ and each $x \in f^{-1}(y)$,

$$Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \wedge a(U\pi_2(\mathfrak{q}), x) \leq c(U\,\mathrm{ev}(\mathfrak{q}), s(x)).$$

The first inequality follows from

$$Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \leq Ub(U^2p(\mathfrak{P}), \overset{\bullet}{y}) \leq b(m_Y(U^2p(\mathfrak{P})), y) = b(Up(m_P(\mathfrak{P})), y).$$

From $U\pi_1(\mathfrak{q}) = m_P(\mathfrak{P})$ follows that there exists a $\mathfrak{Q} \in U^2Q$ with (see remark at the end of 1.4)

$$m_Q(\mathfrak{Q}) = \mathfrak{q} \qquad \text{and} \qquad U^2\pi_1(\mathfrak{Q}) = \mathfrak{P}.$$

We obtain

$$\begin{aligned} c(U\,\mathrm{ev}(\mathfrak{q}), s(x)) &= c(m_Z(U^2\,\mathrm{ev}(\mathfrak{Q}), s(x)) \\ &\geq Uc(U^2\,\mathrm{ev}(\mathfrak{Q}), \overset{\bullet}{s(x)}) \\ &\geq U(a \wedge d)(\mathfrak{Q}, \overset{\bullet}{(s, x)}) \\ &= Ua(U^2\pi_2(\mathfrak{Q}), \overset{\bullet}{x}) \wedge Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \\ &\geq Ub(U^2(p \cdot \pi_1)(\mathfrak{Q}), \overset{\bullet}{y}) \wedge a(m_X(U^2\pi_2(\mathfrak{Q})), x) \wedge Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}) \\ &= a(U\pi_2(\mathfrak{q}), x) \wedge Ud(\mathfrak{P}, \overset{\bullet}{(s, y)}). \end{aligned}$$

To prove the necessity of condition ($\diamond$), let $f : (X, a) \to (Y, b)$ be exponentiable in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ and consider the partial product of $(\mathbf{V}^*, h)$ over f in $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$

$$\begin{tikzcd} (\mathbf{V}^*, h) & (Q, a \wedge d) \arrow[l, "\mathrm{ev}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (X, a) \arrow[d, "f"] \\ & (P, d) \arrow[r, "p"'] & (Y, b). \end{tikzcd}$$

For each $x \in X$ we define a lax homomorfism (with $y = f(x)$)

$$\begin{aligned} \delta_x : X_y &\to \mathbf{V}^* \\ z &\mapsto \begin{cases} k & \text{if } z = x, \\ * & \text{if } z \neq x. \end{cases} \end{aligned}$$

It induces a map $X \to P$, $x \mapsto (\delta_x, f(x))$; for every $A \subset X$, $\mathfrak{x} \in UX, \ldots$, we will write $A^\natural$, $\mathfrak{x}^\natural, \ldots$, for its image under this map.
For $\mathfrak{X} \in U^2X$ and $x_0 \in X$ we put $\mathfrak{Y} = U^2 f(\mathfrak{X})$ and $y_0 = f(x_0)$. We have to show that, for each $\mathcal{A} \in \mathfrak{X}$,

$$\bigvee_{\mathfrak{a} \in \mathcal{A}} a(\mathfrak{a}, x_0) \geq Ub(\mathfrak{Y}, y_0) \wedge a(m_X(\mathfrak{X}), x_0).$$

Given $\mathcal{A} \in \mathfrak{X}$, we define a lax homomorphism

$$\begin{aligned} \lambda_{\mathcal{A}} : X_{y_0} &\to \mathbf{V}^* \\ x &\mapsto \begin{cases} \bigvee_{\mathfrak{a} \in \mathcal{A}} a(\mathfrak{a}, x_0) & \text{if } x = x_0, \\ * & \text{else.} \end{cases} \end{aligned}$$

Claim 1. *For each $\mathcal{B} \in \mathfrak{X}$ and each atom $\beta \sqsubset Ub(\mathfrak{Y}, \overset{\bullet}{y_0})$, there exists $\mathfrak{a} \in \mathcal{B}$ such that $d(\mathfrak{a}^\natural, (\lambda_{\mathcal{A}}, y_0)) \geq \beta$.*
Proof (Claim 1). Since $Uf(\mathcal{A} \cap \mathcal{B}) \in \mathfrak{Y}$, there exists some $\mathfrak{a} \in \mathcal{A} \cap \mathcal{B}$ with $b(Uf(\mathfrak{a}), y_0) \geq \beta$. We will show that $d(\mathfrak{a}^\natural, (\lambda_{\mathcal{A}}, y_0)) \geq \beta$. To see that, let $\mathfrak{q} \in UQ$ with $U\pi_1(\mathfrak{q}) = \mathfrak{a}^\natural$. We have to show that

$$h(U\,\mathrm{ev}(\mathfrak{q}), \lambda_{\mathcal{A}}(x_0)) \geq \beta \wedge a(U\pi_2(\mathfrak{q}), x_0).$$

From $U\pi_1(\mathfrak{q}) = \mathfrak{a}^\natural$ follows $U\,\mathrm{ev}(\mathfrak{q}) = \overset{\bullet}{k}$ or $U\,\mathrm{ev}(\mathfrak{q}) = \overset{\bullet}{*}$. In the latter case we have $h(U\,\mathrm{ev}(\mathfrak{q}), \lambda_{\mathcal{A}}(x_0)) = k$. If $U\,\mathrm{ev}(\mathfrak{q}) = \overset{\bullet}{k}$, then $U\pi_2(\mathfrak{q}) = \mathfrak{a}$ and hence $\lambda_{\mathcal{A}}(x_0) \geq a(U\pi_2(\mathfrak{q}), x_0)$. This finishes the proof of Claim 1.
We obtain $d(m_P(\mathfrak{X}^\natural), (\lambda_{\mathcal{A}}, y_0)) \geq Ud(\mathfrak{X}^\natural, (\overset{\bullet}{\lambda_{\mathcal{A}}}, y_0)) \geq Ub(\mathfrak{Y}, \overset{\bullet}{y_0})$.
Claim 2. *There exists $\mathfrak{Q} \in U^2Q$ such that*

- $U^2\pi_1(\mathfrak{Q}) = \mathfrak{X}^\natural$,
- $U^2\pi_2(\mathfrak{Q}) = \mathfrak{X}$ *and*
- $\Delta = \{\mathfrak{d} \in UQ \mid \mathrm{ev}(\mathfrak{d}) = \overset{\bullet}{k}\} \in \mathfrak{Q}$.

Proof (Claim 2). We have to show that, for all $\mathcal{B}, \mathcal{C} \in \mathfrak{X}$,

$$U\pi_1^{-1}(\mathcal{B}^\natural) \cap U\pi_1^{-1}(\mathcal{C}) \cap \Delta \neq \emptyset.$$

We take any $\mathfrak{a} \in \mathcal{B} \cap \mathcal{C}$. For all $A, B \in \mathfrak{a}$ and $z \in A \cap B$ we have

$$(\delta_z, z) \in \pi_1^{-1}(A^\natural) \cap \pi_2^{-1}(B) \cap \{(\psi, x) \in Q \mid \psi(x) = k\},$$

hence $\{\pi_1^{-1}(A^\natural) \cap \pi_2^{-1}(B) \cap \{(\psi, x) \in Q \mid \psi(x) = k\} \mid A, B \in \mathfrak{a}\}$ is a filter on Q and any $\mathfrak{q} \in UQ$ containing it belongs to $U\pi_1^{-1}(\mathcal{B}^\natural) \cap U\pi_1^{-1}(\mathcal{C}) \cap \Delta$. This finishes the proof of Claim 2.
We conclude that

$$(d \wedge a)(m_Q(\mathfrak{Q}), (\lambda_{\mathcal{A}}, x_0)) \geq Ub(\mathfrak{Y}, y_0) \wedge a(m_X(\mathfrak{X}), x_0)$$

and therefore

$$\lambda_{\mathcal{A}}(x_0) = \hom(k, \lambda_{\mathcal{A}}(x_0)) = h(m_{\mathbf{V}}(U^2 \operatorname{ev}(\mathfrak{Q})), \lambda_{\mathcal{A}}(x_0)) \geq \\ (d \wedge a)(m_Q(\mathfrak{Q}), (\lambda_{\mathcal{A}}, x_0)) \geq Ub(\mathfrak{Y}, y_0) \wedge a(m_X(\mathfrak{X}), x_0).$$

□

3.6. **Corollary.** *Assume that* $\mathbf{V}$ *is* $\sqsubset$*-atomic and a Heyting algebra.*

(1) *An object* $(X, a) \in \mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ *is exponentiable if and only if* $a = \phi\psi(a)$.
(2) *Each* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$ *with exponentiable domain is exponentiable.*
(3) *Each initial* $f : (X, a) \to (Y, b)$ *is exponentiable in* $\mathrm{Alg}_u(\mathsf{U}; \mathbf{V})$.

Proof. Theorem 3.5 implies that (X, a) is exponentiable if and only if $a \cdot m_X = e_X^\circ \cdot Ua$ which, as shown in 2.4, is equivalent to $a = \phi\psi(a)$. If (X, a) is exponentiable we have

$$Ua(\mathfrak{X}, \overset{\bullet}{x}) \geq a(m_X(\mathfrak{X}), x) \geq Ub(Uf(\mathfrak{X}), \overset{\bullet}{f(x)}) \wedge a(m_X(\mathfrak{X}), x);$$

and if $i : (X, a) \hookrightarrow (Y, b)$ is initial

$$Ua(\mathfrak{X}, \overset{\bullet}{x}) \geq Ub(Uf(\mathfrak{X}), \overset{\bullet}{f(x)}) \geq Ub(Uf(\mathfrak{X}), \overset{\bullet}{f(x)}) \wedge a(m_X(\mathfrak{X}), x).$$

□

3.7. **Examples.** Exponentiable objects in **PrTop** and **PrAp** are characterized in [21] (see also [23]) and [20] respectively as exactly the finitely generated ones, i.e. the structure is induced by a reflexive function $r : X \times X \to \mathbf{2}$ resp. $d : X \times X \to \overline{\mathbb{R}}_+$. Hence both results are special cases of Corollary 3.6. For a pretopological space X, this is equivalent to each point x of X having a smallest neighborhood V. Note that any (ultra)filter which contains V necessarily contains all neighborhoods of x and hence converges to x. The corresponding map version of this result is proved in [24]. It states that a map $f : X \to Y$ between pretopological spaces is exponentiable if and only if it is

fibrewise finitely generated, that is, each $x \in X$ has a neighborhood V such that each tied filter $(\mathfrak{g}, f(x))$ which contains V converges to x. Here a tied filter $(\mathfrak{g}, y)$ is a filter $\mathfrak{g}$ together with a limit point y of $f[\mathfrak{g}]$. Of course, it is enough to consider only ultrafilters. Our Theorem 3.5 specializes to

Theorem 1. *A continuous map $f : X \to Y$ between pretopological spaces is exponentiable if and only if, for each $\mathfrak{X} \in U^2X$ and $x \in X$, $\mathfrak{X} \to \overset{\bullet}{x}$ whenever $m_X(\mathfrak{X}) \to x$ and $U^2f(\mathfrak{X}) \to \overset{\bullet}{f(x)}$.*

The condition on $f : X \to Y$ in the theorem above must be equivalent to $f : X \to Y$ being fibrewise finitely generated. This can be also seen directly as follows. Assume first that $f : X \to Y$ is fibrewise finitely generated and let $\mathfrak{X} \in U^2X$ and $x \in X$ be such that, with $\mathfrak{Y} := U^2f(\mathfrak{X})$ and $y := f(x)$, $m_X(\mathfrak{X}) \to x$ and $\mathfrak{Y} \to \overset{\bullet}{y}$. By hypothesis, there is some neighborhood V of x such that every ultrafilter $\mathfrak{a}$ on X with $V \in \mathfrak{a}$ and $Uf(\mathfrak{a}) \to y$ converges to x. From $m_X(\mathfrak{X}) \to x$ follows $V \in m_X(\mathfrak{X})$, hence $V^\# \in \mathfrak{X}$. Let $\mathcal{A} \in \mathfrak{X}$. We have $V^\# \cap \mathcal{A} \in \mathfrak{X}$, hence for some $\mathfrak{a} \in V^\# \cap \mathcal{A}$ it holds $Uf(\mathfrak{a}) \to y$ and therefore $\mathfrak{a} \to x$. We have shown that $\mathfrak{X} \to \overset{\bullet}{x}$. Assume now that $f : X \to Y$ is not fibrewise finitely generated. Hence there is some $x \in X$ such that, for each neighborhood V of x, there exists an ultrafilter $\mathfrak{a} \in UX$ such that $V \in \mathfrak{a}$, $Uf(\mathfrak{a}) \to y := f(x)$ and $\mathfrak{a} \nrightarrow x$. We define

$$\mathcal{A} := \{\mathfrak{a} \in UX \mid V \in \mathfrak{a} \;\&\; Uf(\mathfrak{a}) \to y \;\&\; \mathfrak{a} \nrightarrow x\}.$$

Then $\mathfrak{G} := \{W^\# \mid W \text{ is a neighborhood of } x\} \cup \{\mathcal{A}\}$ is a filter base on UX; applying 1.5 twice we obtain first an ultrafilter $\mathfrak{Y} \in U^2Y$ such that $Uf[\mathfrak{G}] \subset \mathfrak{Y}$ and $\mathfrak{Y} \to \overset{\bullet}{y}$, and then $\mathfrak{X} \in U^2X$ with $\mathfrak{G} \subset \mathfrak{X}$ and $U^2f(\mathfrak{X}) = \mathfrak{Y}$. Hence $m_X(\mathfrak{X}) \to x$ but $\mathfrak{X} \nrightarrow \overset{\bullet}{x}$.

Theorem 3.5 gives also a characterization of exponentiable maps in **PrAp**:

Theorem 2. *A non-expansive map $f : (X, a) \to (Y, b)$ between preapproach spaces is exponentiable if and only if, for each $\mathfrak{X} \in U^2X$ and $x \in X$,*

$$Ua(\mathfrak{X}, \overset{\bullet}{x}) \leq \max(Ub(U^2f(\mathfrak{X}), \overset{\bullet}{f(x)}), a(m_X(\mathfrak{X}), x)).$$

4. Exponentiability in $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$

4.1. **Preamble.** In this section we give a sufficient condition for an object to be exponentiable in $\mathrm{Alg}(\mathsf{U}; \mathbf{V})$. The proof will be very similar to the one of Theorem 3.5, we "just" substitute principal by arbitrary ultrafilters. Unfortunately, due to this complication, we are not able to derive a result about maps because we are only able to prove Lemma 4.2 – the corresponding version of Lemma 3.4 – for products.

Throughout this section we assume that $\mathbf{V}$ *is $\sqsubset$-atomic and a Heyting algebra and that, for each $\gamma \in \mathbf{V}$, $\mathrm{At}(\gamma)$ is up-directed or each $\gamma_0 \in \mathrm{At}(\gamma)$ is connected.*

4.2. **Lemma.** *Let*

$$\begin{tikzcd} & (P,d) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (X,a) & & (Y,b) \end{tikzcd}$$

be a product in $\mathrm{Alg}(U;\mathbf{V})$*. For every* $\mathfrak{P} \in U^2P$*,* $\mathfrak{x} \in UX$ *and* $\mathfrak{y} \in UY$*,*

$$\bigvee_{\substack{\mathfrak{p}\in UP:\ U\pi_1(\mathfrak{p})=\mathfrak{x}\\ U\pi_2(\mathfrak{p})=\mathfrak{y}}} Ud(\mathfrak{P},\mathfrak{p}) = Ua(U^2\pi_1(\mathfrak{P}),\mathfrak{x}) \wedge Ub(U^2\pi_2(\mathfrak{P}),\mathfrak{y}).$$

Proof. Let $\alpha \in \mathrm{At}(Ua(U^2\pi_1(\mathfrak{P}),\mathfrak{x}))$ and $\beta \in \mathrm{At}(Ub(U^2\pi_2(\mathfrak{P}),\mathfrak{y}))$. For $A \in \mathfrak{x}$ and $B \in \mathfrak{y}$, we have $a^\circ_\alpha[A] \in U^2\pi_1(\mathfrak{P})$ and $b^\circ_\beta[B] \in U^2\pi_2(\mathfrak{P})$ and therefore

$$d^\circ_{\alpha\wedge\beta}[A\times B] \supset U\pi_1^{-1}[a^\circ_\alpha[A]] \cap U\pi_2^{-1}[b^\circ_\beta[B]] \in \mathfrak{P}.$$

The Extension Lemma guarantees now the existence of $\mathfrak{p} \in UP$ such that $U\pi_1(\mathfrak{p}) = \mathfrak{x}$, $U\pi_2(\mathfrak{p}) = \mathfrak{y}$ and $Ud(\mathfrak{P},\mathfrak{p}) \geq \alpha \wedge \beta$. □

4.3. **Theorem.** *Under the conditions 4.1, an object* (X,a) *in* $\mathrm{Alg}(\mathsf{U};\mathbf{V})$ *is exponentiable provided that*

$(\diamond\diamond)$ $\forall \mathfrak{X} \in U^2X, x \in X\ \forall \gamma_1,\gamma_0 \in \mathbf{V}$

$$\bigvee_{\mathfrak{x}\in UX} (Ua(\mathfrak{X},\mathfrak{x}) \wedge \gamma_1) \otimes (a(\mathfrak{x},x) \wedge \gamma_0) \geq a(m_X(\mathfrak{X}),x) \wedge (\gamma_1 \otimes \gamma_0).$$

Proof. Assume that (X,a) fulfils $(\diamond\diamond)$ and let $(Y,b) \in \mathrm{Alg}_u(\mathsf{U};\mathbf{V})$. We will show that the structure d on P defined in 3.2 is transitive. To see that, we put $Q = P \times X$ and let $\mathfrak{P} \in U^2P$, $\mathfrak{p} \in UP$ and $s \in P$. According to the definition of d, we have to show that, for each $x \in X$ and $\mathfrak{q} \in UQ$ with $U\pi_1(\mathfrak{q}) = m_P(\mathfrak{P})$,

$$(Ud(\mathfrak{P},\mathfrak{p}) \otimes d(\mathfrak{p},s)) \wedge a(U\pi_2(\mathfrak{q}),x) \leq b(\mathrm{ev}(\mathfrak{q}),s(x)).$$

Assume now that $x \in X$ and $\mathfrak{q} \in UQ$ are given such that $U\pi_1(\mathfrak{q}) = m_P(\mathfrak{P})$. As in the proof of Theorem 3.5, for such $\mathfrak{q}$ there exists $\mathfrak{Q} \in UQ$ such that $m_Q(\mathfrak{Q}) = \mathfrak{q}$ and $U^2\pi_1(\mathfrak{Q}) = \mathfrak{P}$. The previous lemma implies that, for each $\mathfrak{x} \in UX$,

$$\bigvee_{\substack{\mathfrak{q}'\in UQ:\ U\pi_2(\mathfrak{q}')=\mathfrak{x}\\ U\pi_1(\mathfrak{q}')=\mathfrak{p}}} U(d\wedge a)(\mathfrak{Q},\mathfrak{q}') = Ud(\mathfrak{P},\mathfrak{p}) \wedge Ua(U^2\pi_2(\mathfrak{Q}),\mathfrak{x}).$$

Moreover, for each $\mathfrak{q}' \in UQ$ with $U\pi_2(\mathfrak{q}') = \mathfrak{x}$ and $U\pi_1(\mathfrak{q}') = \mathfrak{p}$ it holds

$$\begin{aligned} b(U\,\mathrm{ev}(\mathfrak{q}'),s(x)) &= b(m_Y(U^2\,\mathrm{ev}(\mathfrak{Q})),s(x)) \\ &\geq Ub(U^2\,\mathrm{ev}(\mathfrak{Q}),U\,\mathrm{ev}(\mathfrak{q}')) \otimes b(U\,\mathrm{ev}(\mathfrak{q}'),s(x)) \\ &\geq U(d\wedge a)(\mathfrak{Q},\mathfrak{q}) \otimes (d(\mathfrak{p},s) \wedge a(\mathfrak{x},x)). \end{aligned}$$

Combining the inequalities above we obtain

$$\begin{aligned} b(U\,\mathrm{ev}(q), s(x)) &\geq \bigvee_{\mathfrak{x}\in UX} \left(\bigvee_{\substack{\mathfrak{q}'\in UQ,\ U\pi_2(\mathfrak{q}')=\mathfrak{x} \\ U\pi_1(\mathfrak{q}')=\mathfrak{p}}} (U(d\wedge a)(\mathfrak{Q},\mathfrak{q}') \right) \otimes (d(\mathfrak{p},s)\wedge a(\mathfrak{x},x)) \\ &\geq \bigvee_{\mathfrak{x}\in UX} (Ud(\mathfrak{P},\mathfrak{p})\wedge Ua(U^2\pi_2(\mathfrak{Q}),\mathfrak{x}))\otimes (d(\mathfrak{p},s)\wedge a(\mathfrak{x},x)) \\ &\geq a(m_X(U^2\pi_2(\mathfrak{Q})),x)\wedge (Ud(\mathfrak{P},\mathfrak{p})\otimes d(\mathfrak{p},s)) \\ &= a(U\pi_2(\mathfrak{q}),x)\wedge (Ud(\mathfrak{P},\mathfrak{p})\otimes d(\mathfrak{p},s)). \end{aligned}$$

□

4.4. **Corollary.** *In addition to 4.1, assume that* $\otimes = \wedge$.

(1) (X,a) *in* $\mathrm{Alg}(\mathsf{U};\mathbf{V})$ *is exponentiable provided that* $a\cdot Ua = a\cdot m_X$.
(2) $(X,\phi(r))$ *is exponentiable in* $\mathrm{Alg}(\mathsf{U};\mathbf{V})$, *for each* $\mathbf{V}$*-category* (X,r).

Proof. (1) follows immediately from the theorem above. To see (2), let $(X,r)\in$ **V-Cat**. We apply Lemma 2.4 to $r\cdot r = r$ and obtain $\phi(r)\cdot U\phi(r) = \phi(r)\cdot m_X$. □

4.5. **Application to approach spaces.** As for metric spaces (see [8]), the condition ($\diamond\diamond$) can be simplified in the case of approach spaces.

Theorem. *Let* (X,a) *be an approach space. Then the following assertions are equivalent.*

(1) (X,a) *satisfies* ($\diamond\diamond$).
(2) *For* $\mathfrak{X}\in U^2X$, $x\in X$ *and* $\gamma_1,\gamma_0\in\overline{\mathbb{R}}_+$ *with* $\gamma_1+\gamma_0 = a(m_X(\mathfrak{X}),x)$,
$$\inf_{\mathfrak{x}\in UX}(Ua(\mathfrak{X},\mathfrak{x})\vee\gamma_1)\otimes(a(\mathfrak{x},x)\vee\gamma_0)\leq a(m_X(\mathfrak{X}),x).$$
(3) *For* $\mathfrak{X}\in U^2X$, $x\in X$ *with* $a(m_X(\mathfrak{X}),x)<\infty$ *and* $\gamma_1,\gamma_0\in[0,\infty)$ *with* $\gamma_1+\gamma_0 = a(m_X(\mathfrak{X}),x)$, *for each* $\varepsilon>0$ *there exists an ultrafilter* $\mathfrak{x}\in UX$ *such that*
$$Ua(\mathfrak{X},\mathfrak{x})\leq\gamma_1+\varepsilon \qquad \text{and} \qquad a(\mathfrak{x},x)\leq\gamma_0+\varepsilon.$$
(4) *For* $\gamma_1,\gamma_0\in[0,\infty)$ *and* $\varepsilon>0$,
$$a_{\gamma_0+\varepsilon}\cdot(Ua)_{\gamma_1+\varepsilon}\geq a_{\gamma_0+\gamma_1}\cdot m_X.$$

Corollary. $(X,\phi(r))$ *is exponentiable in* **Ap** *if and only if* (X,r) *is exponentiable in* **Met**.

Proof. Let (X,r) be exponentiable in **Met**. According to [8], this is equivalent to
$$\forall\gamma_0,\gamma_1\in\overline{\mathbb{R}}_+\ \forall\varepsilon>0\ \ r_{\gamma_0+\varepsilon}\cdot r_{\gamma_1+\varepsilon}\geq r_{\gamma_0+\gamma_1+\varepsilon}.$$

Let $\gamma_1, \gamma_0 \in [0,\infty)$ and $\varepsilon > 0$. We apply 2.4 and obtain

$$\phi(r_{\gamma_0+\varepsilon}) \cdot U\phi(r_{\gamma_1+\varepsilon}) \geq \phi(r_{\gamma_0+\gamma_1+\varepsilon}) \cdot m_X.$$

For the right hand side we have

$$\phi(r_{\gamma_0+\varepsilon}) \cdot U\phi(r_{\gamma_1+\varepsilon}) \leq \phi(r)_{\gamma_0+\varepsilon} \cdot U(\phi(r)_{\gamma_1+\varepsilon}) \leq \phi(r)_{\gamma_0+\varepsilon} \cdot (U\phi(r))_{\gamma_1+\varepsilon},$$

and for the left hand side $\phi(r_{\gamma_0+\gamma_1+\varepsilon}) \cdot m_X \geq \phi(r)_{\gamma_0+\gamma_1} \cdot m_X$. The reverse implication follows immediately from the fact that **Met** is a full and coreflective subcategory of **Ap** closed under finite limits (see 2.4). □

We finish this paper by exhibiting an exponentiable approach space whose structure is not induced by a metric or a topology, namely the *Sierpinski approach space* $\mathbb{P}$. Recall from [17] that $\mathbb{P}$ has as underlying set $\overline{\mathbb{R}}_+ = [0,\infty]$ and structure p defined by $p(\mathfrak{x}, x) := \hom(l(\mathfrak{x}), x) = \max\{x - l(\mathfrak{x}), 0\}$, for each ultrafilter $\mathfrak{x} \in U\mathbb{P}$ and each $x \in \mathbb{P}$ and $l(\mathfrak{x}) = \inf_{A\in\mathfrak{x}} \sup_{y\in A} y$. Since $\hom(\_, x)$ is a contravariant adjoint functor, it holds

$$\hom(\inf_{y\in A} y, x) = \sup_{y\in A} \hom(y, x)$$

for all $x \in \mathbb{P}$ and $A \subset \mathbb{P}$. If $x \in [0,\infty)$ and $A \neq \emptyset$, then we also have

$$\hom(\sup_{y\in A} y, x) = \inf_{y\in A} \hom(y, x).$$

In particular we see that the subspace $[0,\infty)$ of $\mathbb{P}$ is finitely generated. Therefore, in order to show that $\mathbb{P}$ is exponentiable, we only need to consider $\mathfrak{X} \in U^2\mathbb{P}$ and $x \in \mathbb{P}$ with $p(m_X(\mathfrak{X}), x) < \infty$, where $\mathfrak{X} = \dot{\dot{\infty}}$ or $x = \infty$. Since $x = \infty$ implies $l(m_X(\mathfrak{X})) = \infty$, we cover both cases by assuming from now on that $\mathfrak{X} \in U^2\mathbb{P}$ and $x \in \mathbb{P}$ are given with $l(m_X(\mathfrak{X})) = \infty$. Note that

$$\begin{aligned} l(m_X(\mathfrak{X})) = \infty &\iff \forall y \in [0,\infty)\ \ [y,\infty] \in m_X(\mathfrak{X}) \\ &\iff \forall y \in [0,\infty)\ \ [y,\infty]^{\#} \in \mathfrak{X}. \end{aligned}$$

Let $\mathfrak{f} := \{[y,\infty] \mid y \in [0,\infty)\}$. Then $\sup_{\mathcal{A}\in\mathfrak{X}, F\in\mathfrak{f}} \inf_{\mathfrak{a}\in\mathcal{A}, z\in F} p(\mathfrak{a}, z) = 0$. From Lemma 1.5 we infer that there exists some ultrafilter $\mathfrak{x} \in U\mathbb{P}$ with $\mathfrak{f} \subset \mathfrak{x}$ and $Up(\mathfrak{X}, \mathfrak{x}) = 0$. By construction we have $p(\mathfrak{x}, x) = 0$, which completes the proof.

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Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal

E-mail address: `dirk@mat.ua.pt`