Carathéodory-Equivalence, Noether Theorems, and Tonelli Full-Regularity in the Calculus of Variations and Optimal Control[1]

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Abstract

We study, in a unified way, the following questions related to the properties of Pontryagin extremals for optimal control problems with unrestricted controls: i) How the transformations, which define the equivalence of two problems, transform the extremals? ii) How to obtain quantities which are conserved along any extremal? iii) How to assure that the set of extremals include the minimizers predicted by the existence theory? These questions are connected to: i) the Carathéodory method which establishes a correspondence between the minimizing curves of equivalent problems; ii) the interplay between the concept of invariance and the theory of optimality conditions in optimal control, which are the concern of the theorems of Noether; iii) regularity conditions for the minimizers and the work pioneered by Tonelli.

0.1 Introduction

For more than three centuries, the calculus of variations played a central role stimulating the development of mathematics and the development of physics. Today, the calculus of variations, and its natural generalization known as the theory of optimal control, remain relevant and useful, generating new exciting and deep questions. There is a substantial progress in fundamental issues of both theory and applications [9, 23]. In this paper we will address some of these questions.

We study properties of the extremals and minimizers for various problems of the calculus of variations and optimal control. We are particularly interested in problems which may appear quite different but still can be reduced to the same problem if one uses appropriate transformations. Such problems are said to be equivalent. It turns out that for the equivalent problems there is a direct relation between admissible state-control pairs and the value for the cost functionals. In particular, after *solving* one problem, it is then straightforward to obtain the solutions for all the equivalent problems from the transformations which define the equivalence.

The standard scheme to solve a problem in the calculus of variations or optimal control proceeds along the following three steps. First we prove that a solution to the problem exists. Second we assure the applicability of necessary optimality conditions. Finally we apply the necessary conditions which identify the extremals (the candidates). Further elimination, if necessary, identifies the minimizer(s) of the problem. As pointed out by L. C. Young [34], although both the calculus of variations and optimal control have born from the study of necessary optimality conditions, any such theory is "naive" until the existence of minimizers is assured. The process leading to existence theorems was introduced by Leonida Tonelli, in the years 1911-1915, through the so called *direct method*. It turns out that, even for the simplest problem of the calculus of variations, the hypotheses of the existence theory do not imply those of the necessary optimality conditions. This is to say that all the three steps in the above procedure are indeed crucial: it does not make sense to apply necessary optimality conditions if no solution to the problem exists; and it may be the case that a solution exist but fails to satisfy the standard necessary optimality conditions such as the Euler-Lagrange equations or the Pontryagin maximum principle. Therefore, regularity conditions are also an essential step in the process of solving a problem in the calculus of variations or optimal control. They close the gap between existence and optimality theories, assuring that all the minimizers are indeed extremals [20].

The study of equivalent problems in the calculus of variations and optimal control is not enough. It is also important to know how the extremals of the problems are related. In the terminology of Constantin Carathéodory [6, §227], two problems of the calculus of variations are said to be equivalent if the respective Lagrangians differ by a total derivative. The importance of this equivalence concept is due to the fact that it implies that the Euler-Lagrange equations are identical for both problems. One can say that for Carathéodory is the correspondence between the extremals, and not that of the problems, the key concept to define equivalence. To the best of our knowledge this concept of Carathéodory-equivalence has not been previously explored, or even considered, in the optimal control context. Here we will be mainly interested in the following trivial but important remark: two Carathéodory-equivalent problems have "the same" *conservation laws*. This is not necessarily the case for equivalent problems: equivalence does not imply Carathéodory-equivalence and the other way around. Surprising enough, when one restrict attention to the abnormal extremals, the two concepts seem to be quite the same.

Conjecture 1. *Two problems of optimal control are equivalent if, and only if, they are abnormal-Carathéodory-equivalent.*

We will show in Section 0.2 the validity of Conjecture 1 for equivalent problems under transformations of the type of Gamkrelidze [10, §8.5] and under a time-reparameterization introduced by the author in [27].

Conservation laws, that is, conserved quantities along the extremals of the problem, are obtained in the calculus of variations with the help of the famous symmetry theorems of Emmy Noether. These classical results are known as the (first) Noether theorem and the second Noether theorem, and explain the correspondence between invariance of the problem with respect to a family of transformations and the existence of conservation laws. The first Noether theorem establishes the existence of ρ conservation laws of the Euler-Lagrange differential equations when the Lagrangian L is invariant under a family of transformations containing ρ parameters. The second Noether theorem establishes the existence of $k(m+1)$ conservation laws when the Lagrangian is invariant under a family of transformations which, rather than dependence on parameters, as in the first theorem, depend upon k arbitrary functions and their derivatives up to order m. Extensions of the first theorem for

the Pontryagin extremals of optimal control problems are available in [27, 24, 25]. In Section 0.3 we provide a rather general formulation of the first Noether theorem which envolves all the peculiarities of previous results. For optimal control versions of the second theorem we refer the reader to [26]. We will argue in Section 0.4 that the conservation laws obtained from the use of Noether's first theorem play an important role in the acquisition of regularity conditions.

Further extensions and related results are possible. Due to the restrictions on the volume of the paper we can not provide them here. The reader can find more details, complete and detailed proofs, illustrative examples, additional material and a complete list of references, in the author's thesis [28], available in Portuguese.

0.2   Carathéodory-Equivalence

In this section we address, under two types of transformations involving change of the time-variable t, the following question: *How the transformations, which define the equivalence of two problems, affect the Pontryagin extremals?* First we will need to introduce the problem considered in optimal control theory, and to give a characterization of the Pontryagin extremals.

The objective of the paper is to study some properties of the minimizing trajectories for general problems of optimal control in the case where controls are unconstrained, like in the calculus of variations. We will be considering, without any loss of generality, problems in the Lagrange form. This is indeed a general problem, and Bolza type problems or Mayer type problems can be put easily in this form. We look for a pair $(x(\cdot), u(\cdot))$, satisfying a control dynamical equation described by a system of ordinary differential equations

$$\dot{x}(t) = \varphi\left(t, x(t), u(t)\right) , \tag{1}$$

in such a way the pair $(x(\cdot), u(\cdot))$ minimizes a given integral functional:

$$I\left[x(\cdot), u(\cdot)\right] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min .$$

This problem is denoted by (P). The state trajectories $x(\cdot)$ are assumed to be absolute continuous functions and the admissible controls $u(\cdot)$ to be Lebesgue integrable:

$$x(\cdot) \in W_{1,1}\left(\left[a, b\right]; \mathbb{R}^n\right) , \quad u(\cdot) \in L_1\left(\left[a, b\right]; \mathbb{R}^r\right) .$$

For simplicity of exposition, we will assume that the Lagrangian L and function φ are C^1-smooth with respect to all variables. The results of the paper are valid for all kinds of boundary conditions one may want to consider. For this reason, boundary conditions are not considered in our formulation of the optimal control problem. We remark that, *a priori*, optimal controls may be unbounded and that the problems of the calculus of variations, like the basic problem of the calculus of variations or the problems with high-order derivatives, can be reduced in the obvious way to the Lagrange problem (P).

Both the calculus of variations and optimal control theory have born from the study of first-order necessary optimality conditions: the Euler-Lagrange equations, in the case of the calculus of variations, which appear in the Euler's celebrated monograph of 1744, and the Pontryagin maximum principle, in the case of optimal control, which appear ~ 1956. The Pontryagin maximum principle gives conditions under which all the minimizers are Pontryagin extremals.

Definition 1. *The quadruple* $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$, $\psi_0 \leq 0$, $\psi(\cdot) \in W_{1,1}$, $(\psi_0, \psi(\cdot)) \neq 0$, *is a Pontryagin extremal if it satisfies:*

the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \psi} , \quad \dot{\psi} = -\frac{\partial H}{\partial x} ;$$

the maximality condition

$$H\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \max_{u \in \mathbb{R}^r} H\left(t, x(t), u, \psi_0, \psi(t)\right) ;$$

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with the Hamiltonian $H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u)$. *An extremal is said to be abnormal when ψ_0 vanishes and normal otherwise.*

The first equation in the Hamiltonian system is just the control equation (1). The second is known as the *adjoint system.*

For the basic problem of the calculus of variations one has $\dot{x} = u$ and the Hamiltonian is given by $H = \psi_0 L + \psi \cdot u$. From the adjoint system we obtain

$$\dot{\psi} = -\psi_0 \frac{\partial L}{\partial x} , \tag{2}$$

while from the maximality condition one gets

$$\psi = -\psi_0 \frac{\partial L}{\partial u} . \tag{3}$$

The maximum principle asserts that ψ_0 and $\psi(\cdot)$ do not vanish simultaneously and it comes immediately, from (3), that no abnormal minimizers exist for the basic problem of the calculus of variations. From equalities (2) and (3) one concludes that if $x(\cdot)$ is a minimizer it satisfies the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial u}(t, x(t), \dot{x}(t)) = \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) . \tag{4}$$

A solution $x(\cdot)$ of (4) is called an (Euler-Lagrange) extremal. The classical conditions [18], assuring that all minimizers are extremals, assume that the Lagrangian L and function φ are continuous with respect to all variables and continuously differentiable with respect to the state variables x; while the optimal controls are assumed to be essentially bounded: $L(\cdot, x, \cdot), \varphi(\cdot, x, \cdot) \in C$, $L(t, \cdot, u), \varphi(t, \cdot, u) \in C^1$, $u(\cdot) \in L_\infty$. For the basic problem of the calculus of variations, this means that the Euler-Lagrange equations are valid for minimizers in the class of Lipschitzian functions. Proving general versions of the maximum principle under weaker hypotheses is still very much in progress [22]. Conditions with $u(\cdot) \in L_1$ do exist, but they postulate growth conditions on the Lagrangian L and functions φ. For example, the following conditions follow easily from Berkovitz's [3] or Clarke's version of the maximum principle [8]:

$$\left\|\frac{\partial L}{\partial x}\right\| \le c\,|L| + k , \quad \left\|\frac{\partial \varphi_i}{\partial x}\right\| \le c\,|\varphi_i| + k , \tag{5}$$

$i = 1, \ldots, n$, for some constants $c > 0$ and k.

The following theorem provides an interesting property of the Pontryagin extremals.

Theorem 1 ([29]). *Let $F(t, x, u, \psi_0, \psi)$, $F : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}_0^- \times \mathbb{R}^n \to \mathbb{R}$, be a continuous differentiable function with respect to t, x, and ψ, for fixed u. If there exists a function $G(\cdot) \in L_1([a, b]; \mathbb{R})$ such that*

$$\left\|\nabla_{(t, x, \psi)} F(t, x(t), u(s), \psi_0, \psi(t))\right\| \le G(t) \quad (s, t \in [a, b]) ,$$

and for almost all t's in the interval $[a, b]$ the condition

$$F(t, x(t), u(t), \psi_0, \psi(t)) = \max_{v \in \mathbb{R}^r} F(t, x(t), v, \psi_0, \psi(t))$$

is true along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the optimal control problem (P), then $t \to F(t, x(t), u(t), \psi_0, \psi(t))$ is absolutely continuous and the equality

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} \doteq \frac{\partial F}{\partial t} + \{F, H\}$$

holds along the extremals, where $\{F, H\}$ denotes the Poisson bracket of the functions F and H, and on the left-hand side we have the total derivative with respect to t, and on the right-hand side partial derivatives.

If one chooses F in Theorem 1 to be the Hamiltonian H, one gets:

Theorem 2. *If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal, then the function $H(t, x(t), u(t), \psi_0, \psi(t))$ is an absolutely continuous function of t and satisfies the equality*

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t, x(t), u(t), \psi_0, \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi_0, \psi(t)) . \tag{6}$$

Equation (6) corresponds, for the basic problem of the calculus of variations, to the classical DuBois-Reymond necessary condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L(t, x(t), \dot{x}(t)) - \dot{x}(t) \cdot \frac{\partial L}{\partial u}(t, x(t), \dot{x}(t)) \right] = \frac{\partial L}{\partial t}(t, x(t), \dot{x}(t)) . \tag{7}$$

From our Theorem 1, a necessary and sufficient condition for a function F to be a conservation law is immediately obtained.

Corollary 1. *Under the conditions of Theorem 1, $F(t, x, u, \psi_0, \psi)$ is constant along every Pontryagin extremal of the problem if, and only if,*

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} = 0 .$$

Corollary 1 is very useful for the characterization of optimal control problems with given conserved quantities along the Pontryagin extremals. For example, if one wants to find a problem for which the function

$$F = H\psi x \tag{8}$$

is constant in t along the respective extremals, a necessary and sufficient condition is given by the relation

$$\psi x \frac{\partial H}{\partial t} + \psi H \frac{\partial H}{\partial \psi} - H x \frac{\partial H}{\partial x} = 0 .$$

One such problem is therefore

$$\int_a^b L(u(t)) \, \mathrm{d}t \longrightarrow \min , \tag{9}$$
$$\dot{x}(t) = \varphi(u(t)) x(t) .$$

Usually, the form of the problem is already known and conditions are sought in such a way that the problem as some "good" properties. Let us see one such situation. The following problem is related to the study of cubic polynomials on Riemannian manifolds:

$$\int_0^T \sum_{i=1}^n (u_i(t))^2 \, \mathrm{d}t \longrightarrow \min ,$$
$$\begin{cases} \dot{x}_1(t) = x_2(t) , \\ \dot{x}_2(t) = \sum_{i=1}^n X_i(x_1(t)) u_i(t) . \end{cases} \tag{10}$$

The problem is autonomous and from (6) we know that the respective Hamiltonian H is conserved along the extremals. Determination of the explicit solutions to problem (10) is a difficult task and is, in general, an open problem. However, the extremals can be explicitly computed if a new independent conserved quantity is found. The question is: what kind of conditions shall we impose on the vector fields X_i in order to obtain the new conserved quantity? To answer this question one needs to solve a characterization problem as before. Let $F = k_1 \psi_1 x_1 + k_2 \psi_2 x_2$ (k_1 and k_2 constants). This is called in the literature a *momentum map*. Using the relation given by Corollary 1 one gets

$$k_1 \psi_1 x_2 + k_2 \psi_2 (X_1(x_1)u_1 + \cdots + X_n(x_1)u_n) - k_1 x_1 \psi_2 (X_1'(x_1)u_1 + \cdots + X_n'(x_1)u_n) - k_2 x_2 \psi_1 = 0 .$$

This condition is trivially satisfied if $k_1 = k_2$ and $X_i'(x_1)x_1 = X_i(x_1)$, $i = 1, \ldots, n$. We have just proved the following proposition.

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Proposition 1. *If the homogeneity condition*

$$X_i\left(\lambda x_1\right) = \lambda X_i(x_1)\,, \quad i = 1,\,\ldots,\,n\,,\,\forall\,\lambda > 0\,, \tag{11}$$

holds, then

$$\psi_1(t)x_1(t) + \psi_2(t)x_2(t)$$

is constant in $t \in [0, T]$ along any extremal of the problem (10).

We shall elaborate more on this issue later, in Section 0.3, in relation with Noether's theorems (cf. Example 6).

After this short introduction to Pontryagin extremals and their characterization, we are now in conditions to study how the extremals are affected when one transforms problem (P). Let us consider the following optimal control problem:

$$J\left[t(\cdot), z(\cdot), v(\cdot)\right] = \int_{\tau_a}^{T} \Upsilon\left(t(\tau), z(\tau), v(\tau)\right) L\left(t(\tau), z(\tau), v(\tau)\right) \mathrm{d}\tau \longrightarrow \min,$$

$$\begin{cases} t'(\tau) &= \Upsilon\left(t(\tau), z(\tau), v(\tau)\right) \\ z'(\tau) &= \Upsilon\left(t(\tau), z(\tau), v(\tau)\right)\varphi\left(t(\tau), z(\tau), v(\tau)\right) \end{cases} \tag{12}$$

$$v : \mathbb{R} \to \mathbb{R}^r\,,$$

$$t(\tau_a) = a\,, \quad t(T) = b\,,$$

where $\Upsilon(\cdot, \cdot, \cdot)$ is a strictly positive continuously differentiable function,

$$C^1 \ni \Upsilon(t, z, v) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^+\,,$$

and T is free. Problem (12) is obtained from (P) by entering a new time variable τ, related with with t by the relation

$$\tau(t) = \tau_a + \int_a^t \frac{1}{\Upsilon\left(\theta, x(\theta), u(\theta)\right)}\mathrm{d}\theta\,, \quad t \in [a, b]\,. \tag{13}$$

Compared with (P), problem (12) has one more state variable. Namely, its state variables are $t(\cdot)$ and $z(\cdot)$. We note that the problem is autonomous: both the Lagrangian and the right-hand side of the control system do not depend directly on τ. Thereafter, the admissible set of problem (12) is invariant with respect to translations on the time variable τ. For the concrete situation wherein $\tau_a = 0$ and $\Upsilon(t, z, v) = \frac{1}{L(t,z,v)}$, one obtains the transformation introduced by R. V. Gamkrelidze [10, Chap. 8], of the Lagrange problem (P) into the autonomous time optimal problem.

We denote by H the Hamiltonian associated with problem (P), and by $\mathcal{H}$ the Hamiltonian associated with (12):

$$H\left(t, x, u, \psi_0, \psi\right) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u)\,,$$

$$\mathcal{H}\left(t, z, v, p_0, p_t, p_z\right) = \left[p_0 L(t, z, v) + p_t + p_z \cdot \varphi(t, z, v)\right]\Upsilon(t, z, v)\,.$$

As far as $\mathcal{H}$ does not depend on τ, it follows from Theorem 1 that $\mathcal{H} \equiv const$ along an extremal. The following theorems assert that the extremals of problem (P) are related to those extremals of problem (12) for which this constant is zero.

Theorem 3 ([28]). *Let $(t(\tau), z(\tau), v(\tau), p_0, p_t(\tau), p_z(\tau))$, $\tau \in [\tau_a, T]$, be a Pontryagin extremal of problem* (12) *with*

$$\mathcal{H}\left(t(\tau), z(\tau), v(\tau), p_0, p_t(\tau), p_z(\tau)\right) = 0\,. \tag{14}$$

Then $(x(t), u(t), \psi_0, \psi(t)) = (z(\tau(t)), v(\tau(t)), p_0, p_z(\tau(t)))$, $t \in [a, b]$, where $\tau(\cdot)$ is the inverse function of $t(\cdot)$, is a Pontryagin extremal of (P). Moreover, the value for the functionals coincide:

$$I\left[x(\cdot), u(\cdot)\right] = J\left[t(\cdot), z(\cdot), v(\cdot)\right]\,. \tag{15}$$

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Theorem 4 ([28]). *Let $(x(t), u(t), \psi_0, \psi(t))$, $t \in [a, b]$, be a Pontryagin extremal of problem (P). Then, with $t(\cdot)$ the inverse function of (13),*

$$(t(\tau), z(\tau), v(\tau), p_0, p_t(\tau), p_z(\tau)) = (t(\tau), x(t(\tau)), u(t(\tau)), \psi_0, -H\left(t(\tau), x(t(\tau)), u(t(\tau)), \psi_0, \psi(t(\tau))\right), \psi(t(\tau))) \,,$$

$\tau \in [\tau_a, T]$, is a Pontryagin extremal of (12) which satisfies equalities (14) and (15).

We remark that the correspondence given by Theorems 3 and 4 keeps the normality or abnormality of the extremals ($\psi_0 = p_0$).

Similar correspondences between the extremals, to the one established by Theorems 3 and 4, can be obtained under different transformations of the problem (P). One such transformation is based on an idea of time reparameterization used by the author in [27]. The idea generalizes a well known time reparameterization that has proved to be useful in many different contexts of the calculus of variations and optimal control (see references in [27]). Considering t as a dependent variable, we introduce a one to one Lipschitzian transformation $[a, b] \ni t \longmapsto \tau \in [a, b]$, $\frac{d}{d\tau} t(\tau) > 0$, such that

$$L\left(t, x(t), u(t)\right) dt = L\left(t(\tau), x(t(\tau)), u(t(\tau))\right) \frac{dt(\tau)}{d\tau} d\tau \,,$$

$$\frac{d}{d\tau} x\left(t(\tau)\right) = \frac{dx\left(t(\tau)\right)}{dt} \frac{dt(\tau)}{d\tau} = \varphi\left(t(\tau), x\left(t(\tau)\right), u\left(t(\tau)\right)\right) \frac{dt(\tau)}{d\tau} \,.$$

In this way, if one consider the notations $z(\tau) = x(t(\tau))$ and $w(\tau) = u(t(\tau))$, problem (P) takes the form

$$K\left[t(\cdot), z(\cdot), v(\cdot), w(\cdot)\right] = \int_a^b L\left(t(\tau), z(\tau), w(\tau)\right) v(\tau) d\tau \longrightarrow \min$$

$$\begin{cases} t'(\tau) = v(\tau) \\ z'(\tau) = \varphi\left(t(\tau), z(\tau), w(\tau)\right) v(\tau) \end{cases} \tag{P_τ}$$

$$t(a) = a \,, \quad t(b) = b \,,$$

$$t(\cdot) \in W_{1,\infty}\left([a, b]; [a, b]\right) \,, \quad z(\cdot) \in W_{1,1}\left([a, b]; \mathbb{R}^n\right) \,,$$

$$v(\cdot) \in L_\infty\left([a, b]; \mathbb{R}^+\right) \,, \quad w(\cdot) \in L_1\left([a, b]; \mathbb{R}^r\right) \,.$$

For the new transformed problem (P_τ), the state variables are $t(\tau)$ and $z(\tau)$ while the controls are $v(\tau)$ and $w(\tau)$. The fact that the control variable $v(\cdot)$ takes only strictly positive values, assure that $t(\tau)$ has an inverse function $\tau(t)$.

Next theorem shows how to construct an extremal of (P) given an extremal of problem (P_τ).

Theorem 5 ([30]). *Let $(t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot))$ be an extremal of (P_τ). Then*

$$(x(\cdot), u(\cdot), \psi_0, \psi(\cdot)) = (z\left(\tau(\cdot)\right), w\left(\tau(\cdot)\right), p_0, p_z\left(\tau(\cdot)\right)) \,,$$

where $\tau(\cdot)$ is the inverse function of $t(\cdot)$, is an extremal of (P) with the same value for the functional: $I\left[x(\cdot), u(\cdot)\right] = K\left[t(\cdot), z(\cdot), v(\cdot), w(\cdot)\right]$.

The transformed problem (P_τ) is autonomous, and we already know that the corresponding Hamiltonian is constant along the extremals. As before, for the Gamkrelidze-type transformations, to a Pontryagin extremal of the original problem (P) one can correspond an extremal of the transformed problem for which the Hamiltonian vanishes.

Theorem 6 ([30]). *Let $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ be a Pontryagin extremal of (P). Then, for all $v(\cdot) \in L_\infty\left([a, b]; \mathbb{R}^+\right)$ such that $\int_a^b v(\theta) d\theta = b - a$, the 7-uple $(t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot))$, defined by*

$$t(\tau) = a + \int_a^\tau v(\theta) d\theta \,,$$

$$z(\tau) = x\left(t(\tau)\right) \,, \quad w(\tau) = u\left(t(\tau)\right) \,,$$

$$p_0 = \psi_0 \,, \quad p_z(\tau) = \psi(t(\tau)) \,,$$

$$p_t(\tau) = -H\left(t(\tau), x(t(\tau)), u(t(\tau)), \psi_0, \psi(t(\tau))\right) \,,$$

is a Pontryagin extremal of (P_τ) giving a zero value for the respective Hamiltonian. Moreover,

$$K\left[t(\cdot), z(\cdot), v(\cdot), w(\cdot)\right] = I\left[x(\cdot), u(\cdot)\right] .$$

From Theorems 5 and 6 the following corollary is trivially obtained.

Corollary 2. *If $(\tilde{x}(t), \tilde{u}(t))$ is a minimizer of (P), then the quadruple*

$$\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau), \tilde{w}(\tau)\right) = (\tau, \tilde{x}(\tau), 1, \tilde{u}(\tau))$$

furnishes a minimizer to (P_τ).

Once again, Theorems 5 and 6 establish a correspondence between the abnormal extremals of the original and transformed problem.

Corollary 3. *If no abnormal extremals exist for the problem (P), then no abnormal extremals exist for the problem (P_τ) too. If no abnormal extremals exist for the problem (P_τ), then no abnormal extremals exist for the problem (P) too.*

We shall see in Section 0.4 that one can obtain regularity conditions, assuring that all minimizing controls of (P), predicted by Tonelli's existence theorem, are Pontryagin extremals, from the applicability conditions of the Pontryagin maximum principle to the transformed problems. The proof relies on certain conserved quantities along the Pontryagin extremals. These conserved quantities are addressed in the following section.

0.3   Noether Symmetry Theorems

We now turn our attention to the following question: *How to obtain quantities which are conserved along the Pontryagin extremals?* This is an important, profound, and far-reaching litigation. Such conserved quantities can be used to lower the order of the Hamiltonian system of differential equations and simplify the resolution of the optimal control problem. They are also important for many other reasons. In the calculus of variations they have been used to synthesise the Lavrentiev phenomenon [13] while in control, to analyze the stability and controllability of nonlinear control systems, they are used for the system decomposition in terms of simpler lower dimensional subsystems [12]. These are just few examples, but many other applications are possible: proving existence of minimizers, solving the Hamilton-Jacobi-Bellman equation, etc. We show that conserved quantities along the extremals are also a useful tool to prove Lipschitzian regularity of the minimizing trajectories.

We will obtain some generalizations of the well known theorems of E. Noether, providing a connection between such conserved quantities and the invariance of the problems in optimal control. The theory of this connection, as it appears in many branches of classical theoretical physics, constitutes one of the most beautiful chapters of the calculus of variations.

The universal principle described by Noether's theorems of 1918, asserts that the invariance of a problem with respect to a family of transformations implies the existence of conserved quantities along the Euler-Lagrange extremals.

Definition 2. *If $C^1 \ni h^s(t, x) = (h_t^s(t), h_x^s(x)) : [a, b] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $s \in (-\varepsilon, \varepsilon)$; $h^0(t, x) = (t, x)$ for all $(t, x) \in [a, b] \times \mathbb{R}^n$;*

$$\int_{h_t^0(a)}^{h_t^0(b)} L\left(t^s, h_x^s\left(x(t^s)\right), \frac{\mathrm{d}}{\mathrm{d}t^s} h_x^s\left(x(t^s)\right)\right) \mathrm{d}t^s = \int_a^b L\left(t, x(t), \dot{x}(t)\right) \mathrm{d}t ,$$

for $t^s = h_t^s(t)$, all $s \in (-\varepsilon, \varepsilon)$, and all $x(\cdot)$; then the basic problem of the calculus of variations is said to be invariant under h^s.

Theorem 7 (First Noether's Theorem). *If the basic problem of the calculus of variations is invariant under h^s, then*

$$\psi(t) \cdot \frac{\partial}{\partial s} h_x^s\left(x(t)\right)|_{s=0} - H\left(t, x(t), \dot{x}(t), \psi(t)\right) \frac{\partial}{\partial s} h_t^s\left(t\right)|_{s=0} \tag{16}$$

is constant in t along every extremal.

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We recall that for the basic problem of the calculus of variations one has $\psi_0 = -1$ and $\psi = \frac{\partial L}{\partial u}$. Quantity (16) is then equivalent to

$$\frac{\partial L}{\partial u}\left(t, x(t), \dot{x}(t)\right) \cdot \frac{\partial}{\partial s}\, h_x^s\left(x(t)\right)|_{s=0} + \left[L\left(t, x(t), \dot{x}(t)\right) - \frac{\partial L}{\partial u}\left(t, x(t), \dot{x}(t)\right) \cdot \dot{x}(t)\right] \frac{\partial}{\partial s}\, h_t^s\left(t\right)|_{s=0}. \qquad (17)$$

If the Lagrangian L does not involve the time variable t explicitly, one has invariance relative to translation with respect to time: one can choose $h_t^s(t) = t + s$ and $h_x^s(x) = x$ in Definition 2. It follows from Theorem 7 that the corresponding Hamiltonian is a first integral of the Euler-Lagrange equations. At the light of (17), this is nothing more than the classical 2nd Erdmann condition,

$$L\left(x(t), \dot{x}(t)\right) - \frac{\partial L}{\partial u}\left(x(t), \dot{x}(t)\right) \cdot \dot{x}(t) \equiv \text{constant}, \qquad (18)$$

which is a first-order necessary optimality condition for the autonomous basic problem of the calculus of variations. Condition (18) can also be obtained as a straight corollary from the DuBois-Reymond necessary condition (7) discussed in Section 0.2, and one can already guess the interplay between the concept of Carathéodory equivalence and Noether theorems. Such relation is the central key to obtain our generalizations. This is in contrast with the classical proof of Theorem 7, which is based on the so called "general variational formula" [11, pp. 172–198]. We claim that the proof based on Carathéodory approach is more far-reaching than the traditional procedure. We will obtain a version of Theorem 7 to the optimal control setting with several extensions and improvements (cf. Theorem 9 below).

Theorem 7 comprises all theorems on first integrals known to classical and quantum mechanics, field theories, and has deep implications in the general theory of relativity. For example, in mechanics (18) correspond to the energy integral of conservative systems, a conservation law first discovered by Leonhard Euler in 1744; while applying Noether's principle to the Lagrangian describing a system of point masses, one obtains conservation of linear momentum or angular momentum, corresponding, respectively, to invariance under spatial translation or spatial rotation.[1]

As already mentioned, Noether's theorem can be considered as an universal principle. Well known classical formulations include invariant problems of the calculus of variations defined on a manifold M; problems of the calculus of variations with multiple integrals; invariance notions with respect to more than one parameter; families of maps depending upon arbitrary functions (second Noether theorem); invariance of the Lagrangian up to addition of an exact differential $d\Phi(t, x, s)$, with Φ linear on the parameter s. In the original paper [17], Noether explains that the derivatives of the state trajectories $x(\cdot)$ may also occur in the family of transformations h^s. However, this possibility has been forgotten in the literature of the calculus of variations. From our point of view, this possibility is very interesting: it means that the parameter transformations h^s may also depend on the control variables.

Recent formulations, in other contexts than the calculus of variations, include the ones obtained by van der Schaft [32] for autonomous Hamiltonian control systems with inputs and outputs; the results of Cariñena and Figueroa [7] for (higher-order) supermechanics; and the discrete versions, in which time proceeds in integer steps, obtained by Baez and Gilliam [1]. In the optimal control setting, the important relation between invariance of the problem under a parameter family of transformations, and the existence of preserved quantities along the Pontryagin extremals, was established by Djukic [31], Sussmann [21], Jurdjevic [14, Ch. 13], Blankenstein & van der Schaft [4], and Torres [27]. Our purpose here is to provide a Noether type theorem to generic problems of optimal control in a broader sense, enlarging the scope of its application. Our results will be formulated under a weak notion of invariance which admits several parameters, equalities up to first order terms in the parameters, addition of an exact differential, not necessarily linear with respect to the parameters, and a family of transformations which may also depend on the control variables. We will make use of a technique different from the classical one. This technique, introduced by the author in [27], does not need to use transversality conditions as happens in the classical proof. For this reason, the results will be valid even in the situation when we do not know the boundary conditions.

We begin with a Noether theorem with no transformation of the time variable.

---

[1]In this context the Hamiltonian multiplier $\psi(\cdot)$ represent the generalized momentum of the system.

Definition 3. *Let* $h^s : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, $s = (s_1, \ldots, s_\rho)$, $\|s\| = \sqrt{\sum_{k=1}^{\rho} (s_k)^2} < \varepsilon$, *be a ρ-parametric family of C^1 transformations which for $s = 0$ reduce to identity:*

$$h^0(t, x, u) = x, \quad \forall (t, x, u) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^r.$$

If there exists a function $\Phi^s(t, x, u) \in C^1([a,b], \mathbb{R}^n, \mathbb{R}^r; \mathbb{R})$ and for all $s = (s_1, \ldots, s_\rho)$, $\|s\| < \varepsilon$, and admissible $(x(\cdot), u(\cdot))$ there exists a control $u^s(\cdot) \in L_\infty([a,b]; \mathbb{R}^r)$ such that:

$$\int_a^\beta L\left(t, h^s(t, x(t), u(t)), u^s(t)\right) \mathrm{d}t = \int_a^\beta \left(L(t, x(t), u(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi^s(t, x(t), u(t)) + \delta\left(t, x(t), u(t), s\right)\right) \mathrm{d}t \quad (19)$$

for all $\beta \in [a,b]$;

$$\frac{\mathrm{d}}{\mathrm{d}t} h^s(t, x(t), u(t)) + \delta\left(t, x(t), u(t), s\right) = \varphi(t, h^s(t, x(t), u(t)), u^s(t)); \quad (20)$$

where $\delta(t, x, u, s)$ denote terms which go to zero faster than $\|s\|$ for each t, x, u, i.e.,

$$\lim_{\|s\| \to 0} \frac{\delta(t, x, u, s)}{\|s\|} = 0,$$

then problem (P) is said to be quasi-invariant under $h^s(t, x, u)$ up to $\Phi^s(t, x, u)$.

The following examples illustrate some of the new possibilities.

As far as in Definition 3 Φ^s may depend on the parameters in a nonlinear way, one may cover new situations even for the basic problem of the calculus of variations. Example 1 illustrate this issue. We also note that in the example the state-transformation h^s depend not only on the state variable x but also on time t.

Example 1 $(n = r = 1)$. *Consider the following basic problem of the calculus of variations:*

$$\int_a^b (u(t))^2 \, \mathrm{d}t \longrightarrow \min,$$

$$\dot{x}(t) = u(t).$$

In this case we have $L = u^2$ and $\varphi = u$. The problem is invariant, in the sense of Definition 3, under the one-parameter transformation $h^s(t, x) = x + st$ $(h^0(t, x) = x)$. Indeed, we observe that for $\Phi^s(t, x) = s^2 t + 2sx$ and $u^s(t) = u(t) + s$ $(u^0(t) = u(t))$ one obtains:

$$\int_a^\beta L(u^s(t)) \, \mathrm{d}t = \int_a^\beta (u(t) + s)^2 \, \mathrm{d}t = \int_a^\beta \left((u(t))^2 + s^2 + 2su(t)\right) \mathrm{d}t$$

$$= \int_a^\beta \left(L(u(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi^s(t, x(t))\right) \mathrm{d}t;$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h^s(t, x(t)) = \dot{x}(t) + s = \varphi(u^s(t)).$$

The optimal control problem in Example 2 is quasi-invariant under a one-parameter family of transformations up to an exact differential.

Example 2 $(n = 3, \ r = 2)$. *Let us consider the following optimal control problem:*

$$\int_a^b (u_1(t))^2 + (u_2(t))^2 \, \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = u_1(t), \\ \dot{x}_2(t) = u_2(t), \\ \dot{x}_3(t) = \dfrac{u_2(t)(x_2(t))^2}{2}. \end{cases}$$

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One has $L(u_1, u_2) = u_1^2 + u_2^2$, $\varphi(x_2, u_1, u_2) = \left(u_1, u_2, \frac{u_2 x_2^2}{2}\right)^T$ and direct computations show that the problem is quasi-invariant under

$$h^s(t, x_1, x_2, x_3) = \left(h_{x_1}^s(t, x_1), h_{x_2}^s(t, x_2), h_{x_3}^s(t, x_2, x_3)\right)$$
$$= \left(x_1 + st, x_2 + st, x_3 + \frac{1}{2}x_2^2 st\right)$$

up to $\Phi^s(x_1, x_2) = 2s(x_1 + x_2)$: $h^0(t, x_1, x_2, x_3) = (x_1, x_2, x_3)$ and by choosing $u_1^s = u_1 + s$ and $u_2^s = u_2 + s$ $(u_1^0 = u_1, u_2^0 = u_2)$ we obtain

$$\int_a^\beta L(u_1^s(t), u_2^s(t))\, dt = \int_a^\beta (u_1(t) + s)^2 + (u_2(t) + s)^2\, dt$$
$$= \int_a^\beta \left[(u_1(t)^2 + u_2(t)^2) + 2s(u_1(t) + u_2(t)) + 2s^2\right] dt$$
$$= \int_a^\beta \left[L(u_1(t), u_2(t)) + \frac{d}{dt}(\Phi^s(x_1(t), x_2(t))) + \delta(s)\right] dt,$$

that is, equation (19) is satisfied with $\delta(s) = 2s^2$;

$$\varphi_1(u_1^s(t)) = u_1(t) + s = \frac{d}{dt}(x_1(t) + st) = \frac{d}{dt}h_{x_1}^s(t, x_1(t)),$$
$$\varphi_2(u_2^s(t)) = u_2(t) + s = \frac{d}{dt}(x_2(t) + st) = \frac{d}{dt}h_{x_2}^s(t, x_2(t)),$$

$$\varphi_3\left(h_{x_2}^s(t, x_2(t)), u_2^s(t)\right) = \frac{(u_2(t) + s)(x_2(t) + st)^2}{2}$$
$$= \frac{u_2(t) x_2(t)^2}{2} + \frac{1}{2}s\left(x_2(t)^2 + 2x_2(t)u_2(t)t\right) + \frac{(u_2(t)t^2 + 2x_2(t)t)s^2 + t^2 s^3}{2}$$
$$= \frac{d}{dt}\left(x_3(t) + \frac{1}{2}x_2(t)^2 st\right) + \delta(t, x_2(t), u_2(t), s)$$
$$= \frac{d}{dt}h_{x_3}^s(t, x_2(t), x_3(t)) + \delta(t, x_2(t), u_2(t), s),$$

and therefore (20) is also satisfied.

Theorem 8. *If (P) is quasi-invariant under the transformations $h^s(t, x, u)$ up to $\Phi^s(t, x, u)$, in the sense of Definition 3, then the ρ quantities*

$$\psi(t) \cdot \frac{\partial}{\partial s_k} h^s(t, x(t), u(t))|_{s=0} + \psi_0 \frac{\partial}{\partial s_k} \Phi^s(t, x(t), u(t))|_{s=0}, \quad (k = 1, \ldots, \rho),$$

are constant in t along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem.

Example 3. *From Theorem 8 it follows that $\psi(t)t + 2\psi_0 x(t)$ is constant in t along the extremals of the basic problem of the calculus of variations considered in Example 1. We know that $\psi(t) = \frac{\partial L}{\partial u}(t, x(t), u(t))$ and $\psi_0 = -1$, so the conclusion is that $\dot{x}(t)t - x(t)$ is a first integral of the Euler-Lagrange differential equations.*

Example 4. *One concludes from Theorem 8 that $2\psi_0(x_1(t) + x_2(t)) + \psi_1(t)t + \psi_2(t)t + \frac{1}{2}\psi_3(t)(x_2(t))^2 t$ is constant along all the Pontryagin extremals of the problem considered in Example 2.*

Example 5. *Problem (9) is invariant under the one-parameter family of transformations $h^s(x) = e^s x$ (cf. Definition 3 with $u^s = u$ and $\Phi^s \equiv 0$). We get from Theorem 8 that*

$$\psi(t)x(t) \equiv constant, \tag{21}$$

$t \in [a, b]$, along any Pontryagin extremal of problem (9).

Example 6. *Under the condition* (11) *problem* (10) *is invariant under* $h_{x_1}^s = \mathrm{e}^s x_1$, $h_{x_2}^s = \mathrm{e}^s x_2$, *with* $u_i^s = u_i$ *and* $\Phi^s \equiv 0$. *Proposition 1 follows from Theorem 8. Other conclusions are also possible if one imposes different conditions on the vector fields* X_i *(cf. Example 7).*

We now generalize the invariance notion given by Definition 3, in order to admit the possibility of a ρ-parametric transformation of the independent variable t.

Definition 4. *Let* $h^s(t, x, u) = (h_t^s(t, x, u), h_x^s(t, x, u))$, $h_t^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ *and* $h_x^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ *($\|s\| < \varepsilon$), be a ρ-parametric family of C^1 transformations which for $s = 0$ satisfies $h^0(t, x, u) = (t, x)$ for all triple $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^r$. If there exists a function $\Phi^s(t, x, u) \in C^1([a, b], \mathbb{R}^n, \mathbb{R}^r; \mathbb{R})$ and for all $s = (s_1, \ldots, s_\rho)$ and for all admissible pair $(x(\cdot), u(\cdot))$ there exists a control $u^s(\cdot) \in L_\infty([a, b]; \mathbb{R}^r)$ such that:*

$$
\int_{h_t^s(a, x(a), u(a))}^{h_t^s(\beta, x(\beta), u(\beta))} L\left(t^s, h_x^s(t^s, x(t^s), u(t^s)), u^s(t^s)\right) \mathrm{d}t^s
$$
$$
= \int_a^\beta \left(L(t, x(t), u(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi^s(t, x(t), u(t)) + \delta\left(t, x(t), u(t), s\right) \right) \mathrm{d}t,
$$

for all $\beta \in [a, b]$;

$$
\frac{\mathrm{d}}{\mathrm{d}t^s} h_x^s(t^s, x(t^s), u(t^s)) + \delta\left(t, x(t), u(t), s\right) = \varphi(t^s, h_x^s(t^s, x(t^s), u(t^s)), u^s(t^s)),
$$

for $t^s = h_t^s(t, x(t), u(t))$; then the problem (P) is said to be quasi-invariant under transformations $h^s(t, x, u)$ up to $\Phi^s(t, x, u)$.

What follows is a more general version of the first Noether theorem which admits transformations of the time-variable. Theorem 9 gives ρ conservation laws when the optimal control problem (P) is quasi-invariant up to Φ^s under a family of transformations with ρ parameters.

Theorem 9 (First Noether Theorem for Optimal Control). *If (P) is quasi-invariant under the transformations $(h_t^s(t, x, u), h_x^s(t, x, u))$ up to $\Phi^s(t, x, u)$, in the sense of Definition 4, then*

$$
\psi(t) \cdot \frac{\partial}{\partial s_k} h_x^s(t, x(t), u(t))|_{s=0} + \psi_0 \frac{\partial}{\partial s_k} \Phi^s(t, x(t), u(t))|_{s=0}
$$
$$
- H\left(t, x(t), u(t), \psi_0, \psi(t)\right) \frac{\partial}{\partial s_k} h_t^s(t, x(t), u(t))|_{s=0}
$$

is constant in t along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem $\forall k = 1, \ldots, \rho$.

The proof of Theorem 9 is done by reduction to the situation of Theorem 8. If (P) is quasi-invariant under $(h_t^s(t, x, u), h_x^s(t, x, u))$ up to $\Phi^s(t, x, u)$, in the sense of Definition 4, then the problem (P_τ) introduced in Section 0.2 is quasi-invariant under $h^s(t, z, w) = (h_t^s(t, z, w), h_x^s(t, z, w))$ up to $\Phi^s(t, z, w)$ in the sense of Definition 3. The pretended conclusion is then obtained from Theorem 6.

<u>Remark</u> **1.** *Every autonomous problem is invariant under $h_t^s = t + s$ and $h_x^s = x$ (autonomous problems are time-invariant). It follows from Theorem 9 that the corresponding Hamiltonian H is constant along the Pontryagin extremals (cf. equality (6)). For the time-invariant problem (9), this fact, together with the conservation law (21), gives an alternative explanation for (8) to be constant along the extremals of the problem.*

Example 7. *Under the hypotheses $X_i(\lambda x_1) = \lambda^\alpha X_i(x_1)$, $\alpha \in \mathbb{R} \setminus \{1\}$, problem (10) is invariant under*

$$
t^s = h_t^s(t) = \mathrm{e}^{-2s} t, \ h_{x_1}^s(x_1(t^s)) = \mathrm{e}^{\frac{3}{\alpha-1}s} x_1(t), \ h_{x_2}^s(x_2(t^s)) = \mathrm{e}^{\left(\frac{3\alpha}{\alpha-1}-1\right)s} x_2(t),
$$

with $u_i^s(t^s) = \mathrm{e}^s u_i(t)$ and $\Phi^s \equiv 0$. We conclude from Theorem 9 that

$$
\psi_1(t)\frac{3}{\alpha-1}x_1(t) + \psi_2(t)\left(\frac{3\alpha}{\alpha-1} - 1\right)x_2(t) + 2Ht \equiv constant
$$

holds along any extremal of the problem (10).

It is interesting to note that Theorem 9 cover both normal and abnormal situations. This is an important issue because abnormal minimizers while nonexistent for the basic problem of the calculus of variations, in general Lagrange problem they may occur frequently. This is the case, for example, for the problems in Sub-Riemannian Geometry.

Even for the basic problem of the calculus of variations the results are new and provide new information. As far as the notions of invariance, conserved quantity along the extremals and reduction belong to the most important tools in the study of classical mechanics, it is not surprising that the conserved quantities obtained by Theorem 9 may be very useful in practice. We remark that solving the Hamiltonian system by the elimination of the control, with the aid of the maximality condition, is typically a difficult task. The existence of such conserved quantities are a circumstance which may make the resolution process easier and are often useful for purposes of analyzing a nonlinear control system. We shall see in the next section that the conserved quantities obtained by Theorem 9 are also useful to establish Lipschitzian regularity of the minimizing trajectories.

0.4 Tonelli Full-Regularity

In this section we address the question: *How to assure that the set of extremals include the minimizers predicted by the existence theory?*

It is easy to find examples of the optimal control problem (P), very simple in aspect, for which the application of the Pontryagin maximum principle gives a unique function $u(\cdot)$ which is not an optimal control. This happens because the optimal solution does not exist. One cannot conclude that we have found the solution unless we know *a priori* that a solution really exists.

A general existence theory for the calculus of variations has been introduced by Leonida Tonelli, in a series of Italian papers, as from 1911, when he was 26. The first general existence theorem for optimal control was given by Filippov. The original paper, in Russian, appeared in 1959. There exist now an extensive literature on the existence of solutions to problems of optimal control. The following set of conditions, of the type of Tonelli, guarantee the existence of minimizer for our problem (P). It is called a Tonelli type existence theorem because for the basic problem of the calculus of variations one has $\varphi = u$ and the theorem coincides with the classical Tonelli existence theorem.

Theorem 10 ("Tonelli" Existence Theorem for (P)). *Problem (P) has an absolute minimum in the space $u(\cdot) \in L_1$, provided that there exist at least one admissible pair, and the following conditions are satisfied for all (t, x, u):*

- *Coercivity: there exists a function $\theta : \mathbb{R}_0^+ \to \mathbb{R}$, bounded below, such that*

$$\lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty \, ,$$
$$L(t, x, u) \geq \theta(\|\varphi(t, x, u)\|) \, ,$$
$$\lim_{\|u\| \to +\infty} \|\varphi(t, x, u)\| = +\infty \, ;$$

- *Convexity: $L(t, x, u)$ and $\varphi(t, x, u)$ are convex with respect to u.*

Roughly speaking, the theorem asserts that under convexity and coercivity, a solution exists in the class of integrable controls. For the basic problem of the calculus of variations one has $\dot{x} = u$ and this mean that existence is given in the class of absolutely continuous functions, possible with unbounded derivative. We note that the assumptions on the solution for the derivation of the necessary optimality conditions have more regularity than the one considered here. For example, for the basic problem of the calculus of variations, the biggest class for which the Euler-Lagrange equation is valid is the class of Lipschitzian functions, that is, the class of absolutely continuous functions having essentially bounded derivative. The steps for the derivation of the Euler-Lagrange equations can no longer be justified in the class of absolutely continuous functions. So the central question, which immediately comes to mind, is the following: How different is the problem with controls in L_1 from the problem with the controls in L_∞? It seems that in order to apply the standard approach to solving optimization problems, one needs an intermediate step between existence, which is proved for controls

in L_1, and standard classical necessary conditions, which are valid for controls in L_∞. Is this intermediate step really necessary? Is this a technical phenomenon or does it reflect a fundamental difficulty? Can the solution predicted by Tonelli's existence theorem be irregular and fail to satisfy the Pontryagin maximum principle? Tonelli proved, for the basic problem of the calculus of variations in the scalar case ($n = 1$), that bad behaviour is only possible in a closed set of measure zero. It turns out that, as has been shown by F. H. Clarke and R. B. Vinter [33, Ch. 11], that the result is general.

Theorem 11 ("Tonelli" Regularity). *Assume that the Tonelli Existence Hypotheses are satisfied. Take any minimizer $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ of (P). Then there exists a closed subset $\Omega \subset [a, b]$ of zero measure with the following property: for any $\tau \in [a, b] \setminus \Omega$, $\tilde{u}(\tau)$ is essentially bounded on a relative neighborhood of τ.*

The theorem asserts that the optimal solutions predicted by Tonelli's existence theorem satisfy the Pontryagin maximum principle or the Euler-Lagrange equations everywhere, except possibly at the points of a closed exceptional set Ω of measure zero. Tonelli, against the general opinion that, at least for "reasonable" problems, all minimizers predicted by his existence theorem are extremals, conjectured the possibility of the Ω set to be nonempty and the possibility of such a minimizer $\tilde{u}(\cdot)$ to be unbounded and fail to be an extremal. It has been proved in 1984 by J. M. Ball and V. J. Mizel [2] that in general the Ω set can not be taken to be empty, even for very "reasonable" problems. Even for polynomial Lagrangians and linear dynamics, minimizers predicted by Tonelli's existence theorem may fail to be Pontryagin (or Euler-Lagrange) extremals. Given this possibility, the natural question to ask now is the following: How to exclude the possibility of bad behaviour? How to obtain full-regularity ($\Omega = \emptyset$)? This is achieved by postulating conditions beyond those of Tonelli's existence theorem, assuring that all optimal controls are essentially bounded. These conditions close the gap between the hypotheses arising in the existence theory and those of necessary optimality conditions, assuring that the solutions can be identified via the Pontryagin maximum principle. As far as φ is bounded on bounded sets, it also follows that the optimal trajectory is Lipschitzian and, similarly, the Hamiltonian adjoint multipliers ψ of the Pontryagin maximum principle turn out to be Lipschitzian either. Thus, full-regularity justifies searching for minimizers among extremals and establishes a weaker form of the maximum principle in which the Hamiltonian adjoint multipliers are not required to be absolutely continuous but merely Lipschitzian. With the Lipschitzian regularity in hand, other regularity properties follow easily, like C^1 or C^2, or even C^∞, imposing some more additional conditions. Full-regularity conditions also precludes occurrence of undesirable phenomena, like the Lavrentiev one, making possible the implementation of efficient discretization schemes and algorithms for numerical computation of the optimal controls. Again, such undesirable phenomena are possible even when the Lagrangian is a polynomial and the control system is linear [15].

The regularity condition one most often finds, implying that the minimizing controls are bounded, was suggested by Tonelli himself. Tonelli and Morrey proved that under the growth conditions

$$\|L_x\| + \|L_u\| \leq c\,|L| + k\,, \tag{22}$$

$c > 0$ and k constants, points of bad behaviour cannot occur for the basic problem of the calculus of variations: under conditions (22) all optimal controls predicted by Tonelli's existence theorem are bounded and the corresponding minimizing trajectories Lipschitzian. The conditions impose global growth hypothesis on the derivatives of the Lagrangian L with respect to the state and control variables. F. H. Clarke and R. B. Vinter have shown that the bound on the derivatives of the Lagrangian with respect to the control variables can be discarded, and that regularity conditions

$$\|L_x\| \leq c\,|L| + k\,,$$

of the type of Tonelli-Morrey, hold not only for the basic problem but universally in the calculus of variations [33, Ch. 11]. We will see that Tonelli-Morrey-type regularity conditions apply in fact more generally: they hold in the generic context of optimal control. They hold even when the dynamics are nonlinear both in the state and control variables.

The literature on regularity conditions for the problems of the calculus of variations is now vast, but for the problems of optimal control, if one excludes the special cases that can be easily recast as problems in the calculus of variations, regularity conditions are a rarity. The first results appeared in 2000 and treat the case of control-affine dynamics:

Theorem 12 ([19]). *For the Lagrange Problem of Optimal Control (P) with control affine dynamics, $\varphi = f(t, x) + g(t, x) u$, if $g(t, x)$ has complete rank r for all t and x; the coercivity condition holds; and $\exists\, \gamma > 0$, $\beta < 2$, η, and $\mu \geq \max\{\beta - 2, -2\}$, such that*

$$\left(|L_t| + |L_{x^i}| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_{x^i} - L_{x^i}\,\varphi\| \right) \|u\|^{\mu} \leq \gamma\, L^{\beta} + \eta\,, \tag{23}$$

then all the minimizers $\tilde{u}(\cdot)$ of the problem, which are not abnormal extremal controls, are essentially bounded on $[a, b]$.

The proof of Theorem 12 is based on the reduction of problem (P) to problem (12) with $\Upsilon(t, z, v) = \frac{1}{L(t,z,v)}$; on the subsequent Gamkrelidze's compactification of the space of admissible controls [10, §8.5]; on the abnormal-Carathéodory-equivalence given by Theorems 3 and 4; and utilization of the classic Pontryagin maximum principle and the time-invariance property of problem (12). We remark that conditions (23) are not of the type of Tonelli-Morrey, and even for the basic problem of the calculus of variations one can cover new situations [19, 20]. The only drawback is that in order to cover new situations the regularity conditions become harder to verify. For the case of control-affine dynamics this is not a problem, and it is possible to deal pretty well with the conditions. In [19] other conditions, not so general as (23), more strong, but more easy to check in practice, were obtained. Theorem 12 admit a generalization for Lagrange problems with dynamics which are nonlinear in control, introducing generalized controls and making a reduction of the nonlinear dynamics to the control affine case by relaxation, a technique introduced by R. V. Gamkrelidze. The only problem, with this nice approach, is that the conditions become cumbersome. We must not forget that checking regularity conditions is a preliminary step in the process of solving a problem, and that, by definition, regularity conditions must be simple to verify. To go to the general nonlinear case with verifiable regularity conditions, a new technique is needed. Such technique was introduced by the author in [30], showing that Tonelli-Morrey type conditions work universally in optimal control:

Theorem 13 ([30]). *Coercivity plus the growth conditions: there exist constants $c > 0$ and k such that*

$$\begin{aligned}
\left|\frac{\partial L}{\partial t}\right| &\leq c\,|L| + k\,, \quad & \left\|\frac{\partial L}{\partial x}\right\| &\leq c\,|L| + k\,, \\
\left\|\frac{\partial \varphi}{\partial t}\right\| &\leq c\,\|\varphi\| + k\,, \quad & \left\|\frac{\partial \varphi_i}{\partial x}\right\| &\leq c\,|\varphi_i| + k \quad (i = 1, \ldots, n)\,;
\end{aligned} \tag{24}$$

imply that all minimizers $\tilde{u}(\cdot)$ of (P), which are not abnormal extremal controls, are essentially bounded on $[a, b]$.

The Lipschitzian regularity conditions (24) are obtained using the applicability conditions (5) of the Pontryagin maximum principle to the auxiliary problem (P_τ) introduced in Section 0.2. Theorem 13 is then proved using the theorem of Emmy Noether and the established abnormal-Carathéodory-equivalence between problems (P) and (P_τ). The theorem covers the general optimal control problem (P), providing conditions of the type of Tonelli-Morrey under which non-abnormal optimal controls are bounded. This guarantees the Lipschitzian regularity of the non-abnormal minimizing trajectories and that all minimizers are Pontryagin extremals.

Corollary 4. *Under the hypotheses of Theorem 13, all minimizers of (P) are Pontryagin extremals.*

We remark that convexity is not required in Theorems 12 and 13 in order to establish the Lipschitzian regularity of the (non-abnormal) minimizing trajectories $\tilde{x}(\cdot)$. This fact is important because existence theorems without the convexity assumptions are a question of great interest.

It is also possible to obtain new regularity conditions, which are not of the type of Tonelli-Morrey, for the generic nonlinear problem (P) [30].

I found pertinent to quote Constantin Carathéodory addressing the question of *The Beginning of Research in the Calculus of Variations* [5]: "I will be glad if I have succeeded in impressing the idea that it is not only pleasant and entertaining to read at times the works of the old mathematical authors, but that this may occasionally be of use for the actual advancement of science."

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