The Noether Principle of Optimal Control

Delfim F. M. Torres

delfim@mat.ua.pt

Department of Mathematics
University of Aveiro
3810–193 Aveiro, Portugal
http://www.mat.ua.pt/delfim

Optimal Control

$$I[x(\cdot), u(\cdot)] = \int_a^b L\,(t, x(t), u(t))\; \mathrm{d}t \longrightarrow \min$$

$$\dot{x}\,(t) = \varphi\,(t, x\,(t)\,, u\,(t)) \qquad \text{(P)}$$

$$(x(a), x(b)) \in \mathcal{F}$$

$$x\,(\cdot) \in W_{1,1}\,([a,b]\,;\,\mathbb{R}^n)\;, \qquad u\,(\cdot) \in L_\infty\,([a,b]\,;\,\Omega \subseteq \mathbb{R}^r),$$

$$C^1 \ni L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}\,, \quad C^1 \ni \varphi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$$

Fundamental Problem of the Calculus of Variations (CV)
(A Particular Case: $\varphi = u$, $r = n$, $\Omega = \mathbb{R}^n$)

$$I[x(\cdot)] = \int_a^b L\,(t, x(t), \dot{x}(t))\; \mathrm{d}t \longrightarrow \min$$

$$x\,(\cdot) \in W_{1,\infty}\,([a,b]\,;\,\mathbb{R}^n)$$

Pontryagin Maximum Principle (PMP)

If $(x(\cdot),\, u(\cdot))$ is a minimizer of (P), then $\exists\ (\psi_0,\, \psi(\cdot)) \neq 0$, $\psi_0 \leq 0$, $\psi(\cdot) \in W^n_{1,1}$, such that the quadruple $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ is a (Pontryagin) extremal : it satisfies

★ the **Hamiltonian system** $\dot{x} = \dfrac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\dfrac{\partial H}{\partial x}$

★ The **maximality condition**

$$H\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \max_{v \in \mathbb{R}^r} H\left(t, x(t), v, \psi_0, \psi(t)\right);$$

with the Hamiltonian $H(t, x, u, \psi_0, \psi) = \psi_0\, L\,(t, x, u) + \psi \cdot \varphi\,(t, x, u)$.

Def. A function $C(t, x, u, \psi_0, \psi)$ constant along every extremal,

$$C\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \text{constant}\,, \tag{1}$$

is called a Constant of Motion (CM); (1) is the corresponding Conservation Law (CL).

Extremals of the CV ($\dot{x} = u$)

★ Hamiltonian: $H = \psi_0 L + \psi \cdot u$;

★ Hamiltonian system: $\dot{x} = u$, $\dot{\psi} = -\psi_0 \dfrac{\partial L}{\partial x}$;

★ maximality condition: $\psi = -\psi_0 \dfrac{\partial L}{\partial u}$

➜ No abnormal extremals exist for the fundamental problem CV

➜ If $x(\cdot)$ is a minimizer, it satisfies the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial u} (t,\, x(t),\, \dot{x}(t)) = \frac{\partial L}{\partial x} (t,\, x(t),\, \dot{x}(t)) \,. \qquad (2)$$

➜ $u(\cdot) \in L_\infty \Rightarrow x(\cdot) \in W_{1,\infty}$: the Euler-Lagrange equations (2) are valid for minimizers in the class of Lipschitzian functions

Def. A solution $x(\cdot)$ of (2) is an (Euler-Lagrange) extremal.

Def. A quantity $C\,(t, x(t), \dot{x}(t))$ preserved along every extremal $x(\cdot)$ is a CM; $C\,(t, x(t), \dot{x}(t)) = \text{constant}$ the corresponding CL.

Conservation Laws, Reduction, and Symmetries

Solving the Hamiltonian system by the elimination of the control, with the aid of the maximality condition, is typically difficult

✔ It is worthwhile to look for circumstances which make the resolution process easier

★ The existence of a CL may be used for reducing the $2n$ dimensional Hamiltonian system to a $(2n-2)$ dimensional set

★ With a sufficiently large number of (independent) CLs one can solve the problem completely

➜ CLs follow from the invariance of the problems (symmetry theorems of Emmy Noether, 1918)

General Method for Constructing CLs in the CV

Def. If $C^1 \ni h^s(t,x) = (h_t(t,s), h_x(x,s)) : [a,b] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $s \in (-\varepsilon, \varepsilon)$; $h^0(t,x) = (t,x)$ for all $(t,x) \in [a,b] \times \mathbb{R}^n$;

$$\int_{h_t(a,0)}^{h_t(\beta,0)} L\left(t^s, h_x^s\left(x(t^s)\right), \frac{\mathrm{d}}{\mathrm{d}t^s} h_x^s\left(x(t^s)\right)\right) \mathrm{d}t^s = \int_a^\beta L\left(t, x(t), \dot{x}(t)\right) \mathrm{d}t,$$

for $t^s = h_t(t,s)$, all $s \in (-\varepsilon, \varepsilon)$, all $\beta \in [a,b]$, and all $x(\cdot)$; then the problem of the CV is said to be invariant under the symmetry h^s.

Emmy Noether's Theorem (1918)

If the problem is invariant under the symmetry h^s, then

$$\psi(t) \cdot \frac{\partial}{\partial s} h_x\left(x(t), s\right)\Big|_{s=0} - H\left(t, x(t), \dot{x}(t), \psi(t)\right) \frac{\partial}{\partial s} h_t\left(t, s\right)\Big|_{s=0}$$

is a Constant of Motion.

Example. Time invariance $(h_t^s = t + s,\ h_x^s = x) \Rightarrow H = const$
State invariance $(h_t^s = t,\ h_x^s = x + s) \Rightarrow \psi = const$

The Universal Principle of Emmy Noether

Invariance $\Rightarrow$ Existence of a CL

Classical Formulations

Well Known:

★ Invariant problems of the CV defined on a manifold M;

★ Problems of the CV with multiple integrals;

★ Invariance with respect to families of transformations depending on several parameters;

★ Invariance of the Lagrangian up to addition of an exact differential $\mathrm{d}\Phi(t, x, s)$, with Φ linear on the parameter s.

Not Well Known:

➜ Noether's theorem is still valid for maps h^s depending on $\dot{x}$.

The Universal Principle of Emmy Noether

Invariance ⇒ Existence of a CL

Recent Formulations

★ Autonomous Hamiltonian control systems (van der Schaft, '81)

★ (Higher-order) Supermechanics (Cariñena & Figueroa, '94)

★ Discrete systems (e.g. cellular automata) (Baez & Gilliam, '94)

✔ Nonsmooth Calculus of Variations (DT, 2003)

Optimal Control:

★ $x \mapsto h^s(x)$ (van der Schaft, '87; Sussmann, '95; Jurdjevic, '97; Blankenstein & van der Schaft, 2001)

✔ $(t, x) \mapsto (h_t^s(t, x, u), h_x^s(t, x, u))$ + quasi-invariance + nonlinear gauge term $d\Phi(t, x, u, s)$ (DT, 2002, 2003)

✔ Discrete Optimal Control (DT, 2003)

Noether Theorem $\mathbf{x} \mapsto \mathbf{h^s(t, x, u)}$

Definition. Problem (P) is quasi-invariant under $h^s \in C^1$, $|s| < \varepsilon$, $h^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, up to $\Phi^s(t, x, u) \in C^1$ if $h^0(t, x, u) = x$ and there exists $u^s(\cdot) \in L_\infty([a, b]; \mathbb{R}^r)$ s.t.

★ $$\int_a^\beta L\left(t, h^s(t, x(t), u(t)), u^s(t)\right) dt$$
$$= \int_a^\beta \left(L(t, x(t), u(t)) + \frac{d}{dt}\Phi^s(t,\, x(t),\, u(t)) + \delta\left(t, x(t), u(t), s\right)\right) dt\,,$$

★ $$\frac{d}{dt} h^s(t, x(t), u(t)) + \delta\left(t, x(t), u(t), s\right) = \varphi(t, h^s(t, x(t), u(t)), u^s(t))\,,$$

where $\delta\left(t, x, u, s\right)$ denote terms for which $\left.\frac{\partial \delta}{\partial s}\right|_{s=0} = 0 \ \forall\ (t, x, u)$.

Theorem. If (P) is quasi-invariant under h^s up to Φ^s, then

$$\psi(t) \cdot \frac{\partial}{\partial s}\, h^s(t,\, x(t),\, u(t))\big|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s(t,\, x(t),\, u(t))\big|_{s=0} = \text{constant}\,.$$

Proof of the Noether Theorem $\mathbf{x} \mapsto \mathbf{h^s(t, x, u)}$

★ From the the invariance definition, differentiating with respect to s and then setting $s = 0$ we obtain:

$$\frac{d}{dt}\frac{\partial}{\partial s}\,\Phi^s\big|_{s=0} = \frac{\partial L}{\partial x}\cdot\frac{\partial}{\partial s}\,h^s\big|_{s=0} + \frac{\partial L}{\partial u}\cdot\frac{\partial}{\partial s}\,u^s(t)\big|_{s=0}\,, \tag{3}$$

$$\frac{d}{dt}\frac{\partial}{\partial s}\,h^s\big|_{s=0} = \frac{\partial \varphi}{\partial x}\cdot\frac{\partial}{\partial s}\,h^s\big|_{s=0} + \frac{\partial \varphi}{\partial u}\cdot\frac{\partial}{\partial s}\,u^s(t)\big|_{s=0}\,. \tag{4}$$

★ $s \mapsto \psi_0\, L(t,\, x(t),\, u^s(t)) + \psi(t)\cdot\varphi(t,\, x(t),\, u^s(t))$ attains its maximum for $s = 0$. Therefore

$$\psi_0\frac{\partial L}{\partial u}\cdot\frac{\partial}{\partial s}\,u^s(t)\big|_{s=0} + \psi(t)\cdot\frac{\partial \varphi}{\partial u}\cdot\frac{\partial}{\partial s}\,u^s(t)\big|_{s=0} = 0. \tag{5}$$

★ Using (5) to simplify the expression $\psi_0(3) + \psi(t)\cdot(4)$ one gets

$$\dot\psi(t)\cdot\frac{\partial}{\partial s}\,h^s\big|_{s=0} + \psi(t)\cdot\frac{d}{dt}\frac{\partial}{\partial s}\,h^s\big|_{s=0} + \psi_0\frac{d}{dt}\frac{\partial}{\partial s}\,\Phi^s\big|_{s=0}$$
$$= \frac{d}{dt}\left(\psi(t)\cdot\frac{\partial}{\partial s}\,h^s\big|_{s=0} + \psi_0\frac{\partial}{\partial s}\,\Phi^s\big|_{s=0}\right) = 0\,.$$

An Example from the Calculus of Variations

$$\int_a^b (u(t))^2 \, dt \longrightarrow \min, \quad \dot{x}(t) = u(t) \quad (n = r = 1).$$

★ Invariance under $h^s(t,x) = x + st$ (problem autonomous but the state transformation h^s is depending also on t) up to $\Phi^s(t,x) = s^2 t + 2sx$ (nonlinear gauge term) with $u^s(t) = u(t) + s$:

$$\frac{d}{dt} h^s(t, x(t)) = \dot{x}(t) + s = \varphi(u^s(t)),$$

$$\int_a^\beta L(u^s(t)) \, dt = \int_a^\beta (u(t) + s)^2 \, dt = \int_a^\beta \Big((u(t))^2 + s^2 + 2su(t) \Big) \, dt$$

$$= \int_a^\beta \left(L(u(t)) + \frac{d}{dt} \Phi^s(t, x(t)) \right) dt.$$

$\Longrightarrow$ $\psi(t)t + 2\psi_0 x(t) = \text{const}$ along the Pontryagin extremals, that is, $\dot{x}(t)t - x(t)$ is constant along the Euler-Lagrange extremals.

An Example of Quasi-Invariance $(n = 3, r = 2)$

$$\int_a^b (u_1(t))^2 + (u_2(t))^2 \, \mathrm{d}t \longrightarrow \min\,, \qquad \begin{cases} \dot{x}_1(t) = u_1(t) \\ \dot{x}_2(t) = u_2(t) \\ \dot{x}_3(t) = u_2(t)\,(x_2(t))^2 \end{cases}$$

★ Problem is quasi-invariant $(u_1^s = u_1 + s,\ u_2^s = u_2 + s)$ under

$$\left(h_{x_1}^s(t, x_1), h_{x_2}^s(t, x_2), h_{x_3}^s(t, x_2, x_3)\right) = \left(x_1 + st, x_2 + st, x_3 + x_2^2 st\right)$$

up to $\Phi^s(t, x_1, x_2) = 2s\,(x_1 + x_2) + 2s^2 t$

$$\begin{aligned} \varphi_3\left(h_{x_2}^s(t, x_2), u_2^s\right) &= (u_2 + s)\,(x_2 + st)^2 \\ &= u_2 x_2^2 + s\left(x_2^2 + 2x_2 u_2 t\right) + \left(u_2 t^2 + 2x_2 t\right) s^2 + t^2 s^3 \\ &= \frac{\mathrm{d}}{\mathrm{d}t} h_{x_3}^s(t, x_2, x_3) + \delta(t, x_2, u_2, s) \end{aligned}$$

$\Longrightarrow$ $2\psi_0\,(x_1(t) + x_2(t)) + \psi_1(t)t + \psi_2(t)t + \psi_3(t)\,(x_2(t))^2\, t$ is a CM

Noether Theorem $(\mathbf{t},\mathbf{x}) \mapsto (\mathbf{h}^{\mathbf{s}}_{\mathbf{t}}(\mathbf{t},\mathbf{x},\mathbf{u}), \mathbf{h}^{\mathbf{s}}_{\mathbf{x}}(\mathbf{t},\mathbf{x},\mathbf{u}))$

Def. Problem (P) is quasi-invariant under $h^s = (h^s_t, h^s_x) \in C^1$ up to $\Phi^s(t,x,u) \in C^1\left([a,b],\mathbb{R}^n,\mathbb{R}^r;\mathbb{R}\right)$, if $h^0(t,x,u) = (t,x)$ and there exists $u^s(\cdot) \in L_\infty\left([a,b];\mathbb{R}^r\right)$ such that for $t^s = h^s_t(t,x(t),u(t))$

★ $$\int_{h^s_t(a,x(a),u(a))}^{h^s_t(\beta,x(\beta),u(\beta))} L\left(t^s, h^s_x(t^s,x(t^s),u(t^s)), u^s(t^s)\right) dt^s$$

$$= \int_a^\beta \left(L(t,x(t),u(t)) + \frac{d}{dt}\Phi^s(t,x(t),u(t)) + \delta\left(t,x(t),u(t),s\right)\right) dt\,,$$

★ $$\frac{d}{dt^s} h^s_x(t^s,x(t^s),u(t^s)) + \delta = \varphi(t^s,\, h^s_x(t^s,x(t^s),u(t^s)),\, u^s(t^s))\,.$$

Theorem. If (P) is quasi-invariant under h^s up to Φ^s, then

$$\psi(t)\cdot\frac{\partial}{\partial s}\, h^s_x(t,\, x(t),\, u(t))\big|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s(t,\, x(t),\, u(t))\big|_{s=0}$$
$$- H\left(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t)\right) \frac{\partial}{\partial s}\, h^s_t(t,\, x(t),\, u(t))\big|_{s=0} = \text{const}\,.$$

Proof of the general Noether Theorem (idea)

Time Reparameterization (Weierstrass, 1872)

$$J\left[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)\right] = \int_{\tau_a}^{\tau_b} L\left(t(\tau),\, z(\tau),\, w(\tau)\right) v(\tau)\, \mathrm{d}\tau \longrightarrow \min$$

$$\begin{cases} t'(\tau) = v(\tau), \quad v(\tau) > 0 \\ z'(\tau) = \varphi\left(t(\tau),\, z(\tau),\, w(\tau)\right) v(\tau) \end{cases} \qquad (P_\tau)$$

Partial Carathéodory-Equivalence

Proposition. Let $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ be a Pontryagin extremal of (P). Then, for all $v(\cdot) \in L_\infty\left([\tau_a, \tau_b]; \mathbb{R}^+\right)$, s.t. $\int_{\tau_a}^{\tau_b} v(\theta)\, d\theta = b - a$, $(t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot))$ defined by $t(\tau) = a + \int_{\tau_a}^{\tau} v(\theta)\, d\theta$, $z(\tau) = x(t(\tau))$, $w(\tau) = u(t(\tau))$, $p_0 = \psi_0$, $p_z(\tau) = \psi(t(\tau))$, $p_t(\tau) = -H\left(t(\tau), z(\tau), w(\tau), p_0, p_z(\tau)\right)$, is an extremal of (P_τ) .

➜ Proof follows by application of our previous theorem to (P_τ).

Conservation Laws are a Useful Tool in Control

Typical application is to lower the order of the Hamiltonian system of differential equations, and simplify the resolution of the optimal control problem (van der Schaft, '87 '99), but CLs are also important for many other reasons, e.g.:

- ✔ to analyze the stability and controllability of nonlinear control systems (Respondek, '82; Grizzle & Marcus, '85)
- ✔ to characterize problems with the Lavrentiev phenomenon (Heinricher & Mizel, '88)
- ✔ to prove existence of minimizers (Clarke '93)
- ✔ solving the Hamilton-Jacobi-Bellman equation (Jurdjevic, '97)
- ✔ to establish Lipschitzian regularity of the minimizing trajectories (DT, 2002)

A Quasi-Invariance Necessary Condition

We have considered the existence of the parameter family of transformations h^s. How to obtain these transformations?

Theorem. If problem (P) is quasi-invariant under the one-parameter family of transformations $h^s(t,x,u)$, then

$$\frac{d}{dt}\left.\frac{\partial \Phi^s}{\partial s}\right|_{s=0} = \frac{\partial L}{\partial t}\left.\frac{\partial h_t^s}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial x}\cdot\left.\frac{\partial h_x^s}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial u}\cdot\left.\frac{\partial u^s}{\partial s}\right|_{s=0} + L\frac{d}{dt}\left.\frac{\partial h_t^s}{\partial s}\right|_{s=0}$$

$$\frac{d}{dt}\left.\frac{\partial h_x^s}{\partial s}\right|_{s=0} = \frac{\partial \varphi}{\partial t}\left.\frac{\partial h_t^s}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial x}\cdot\left.\frac{\partial h_x^s}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial u}\cdot\left.\frac{\partial u^s}{\partial s}\right|_{s=0} + \varphi\frac{d}{dt}\left.\frac{\partial h_t^s}{\partial s}\right|_{s=0}$$

➜ The proof follows easily from the definition of quasi-invariance

✔ The conditions are useful to determine the transformations under which a given optimal control problem is quasi-invariant

✔ The conditions are also useful to characterize or classify optimal control problems which have a given quasi-invariance property

A Simple Example ($n = 2$, $r = 1$)
Problem (P) with $L = u^2$, $\varphi_1 = 1 + x_2^2$ and $\varphi_2 = u$

$$\int_a^b (u(t))^2 \, dt \longrightarrow \min\,, \qquad \begin{cases} \dot{x}_1(t) = 1 + (x_2(t))^2 \\ \dot{x}_2(t) = u(t) \end{cases}$$

★ From previous necessary condition we obtain the one-parameter transformation $h^s = \left(h_t^s, h_{x_1}^s, h_{x_2}^s\right)$: $h_t^s = t(1-2s)$, $h_{x_1}^s = x_1 + 2s(t - 2x_1)$, $h_{x_2}^s = x_2(1-s)$, with $u^s = u(1+s)$.

★ The problem is quasi-invariant but not invariant:

$$L(u^s)\,\frac{d}{dt}h_t^s = u^2\,(1+s)^2\,(1-2s) = u^2 - (3+2s)\,u^2 s^2 = L + \delta(u,s)\,,$$

$$\varphi_1\left(h_{x_2}^s\right)\,\frac{d}{dt}h_t^s = \frac{d}{dt}\left[x_1 + 2s(t-2x_1)\right] + \left(5x_2^2 - 2x_2^2 s\right)s^2 = \frac{d}{dt}h_{x_1}^s + \delta$$

$$\varphi_2(u^s)\,\frac{d}{dt}h_t^s = u(1+s)(1-2s) = u(1-s) - 2us^2 = \frac{d}{dt}h_{x_2}^s + \delta(u,s)\,.$$

$\Longrightarrow$ $2\psi_{x_1}\,(t - 2x_1(t)) - \psi_{x_2}(t)x_2(t) + 2Ht$ is a CM

Characterization of Optimal Control Problems

$$\int_0^T \sum_{i=1}^n (u_i(t))^2 \, \mathrm{d}t \longrightarrow \min\,, \qquad \begin{cases} \dot{x}_1(t) = x_2(t)\,, \\ \dot{x}_2(t) = \sum_{i=1}^n X_i\,(x_1(t))\, u_i(t)\,. \end{cases}$$

Question. What kind of conditions shall we impose on the vector fields X_i in order to obtain a new CM? (H is a trivial CM)

Proposition 1. The homogeneity condition $X_i\,(\lambda x_1) = \lambda X_i(x_1)$, $\forall\,\lambda > 0$, implies quasi-invariance under $h_t^s = t$, $h_{x_1}^s = \mathrm{e}^s x_1$, $h_{x_2}^s = \mathrm{e}^s x_2$, $u_i^s = u_i$. Then the following CL holds:

$$\psi_1(t) x_1(t) + \psi_2(t) x_2(t) \equiv \text{constant}\,.$$

Proposition 2. If $X_i\,(\lambda x_1) = \lambda^\alpha X_i\,(x_1)$, $\alpha \in \mathbb{R} \setminus \{1\}$, then one has quasi-invariance under $t^s = \mathrm{e}^{-2s} t$, $h_{x_1}^s\,(x_1(t^s)) = \mathrm{e}^{\frac{3}{\alpha-1}s} x_1(t)$, $h_{x_2}^s\,(x_2(t^s)) = \mathrm{e}^{\left(\frac{3\alpha}{\alpha-1}-1\right)s} x_2(t)$, $u_i^s(t^s) = \mathrm{e}^s u_i(t)$, and the CL holds:

$$\psi_1(t) \frac{3}{\alpha-1} x_1(t) + \psi_2(t) \left(\frac{3\alpha}{\alpha-1} - 1\right) x_2(t) + 2Ht \equiv \text{constant}\,.$$

Conclusions

A more general version of Noether's theorem, enlarging the scope of application of previous optimal control results, is obtained:

- ✔ We cover both normal and abnormal cases
- ✔ We deal with transformations of the variables up to a (nonlinear) gauge term
- ✔ The parameter transformations may depend on time, state, and control variables
- ✔ We deal with quasi-invariant and not necessarily invariant optimal control problems
- ✔ We derive conditions which help to obtain parameter transformations under which the problem is quasi-invariant

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