Lipschitzian Regularity of Minimizers for Optimal Control Problems

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Lagrange Problem of Optimal Control with unrestricted controls

$$\int_a^b L\left(t,\, x\left(t\right),\, u\left(t\right)\right)\, dt \to \min$$

$$\dot{x}\left(t\right) = \varphi\left(t,\, x\left(t\right),\, u\left(t\right)\right) \qquad (P)$$

$$x\left(a\right) = x_a, \quad x\left(b\right) = x_b$$

$$u\left(\cdot\right) \in L_1\left(\left[a,\, b\right];\, \mathbb{R}^m\right), \qquad x\left(\cdot\right) \in W_{1,1}\left(\left[a,\, b\right];\, \mathbb{R}^n\right)$$

$$C^1 \quad \ni \quad L : \left[a,\, b\right] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R},$$

$$C^1 \quad \ni \quad \varphi : \left[a,\, b\right] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

"Tonelli" Existence Theorem for (P)

Problem (P) has an absolute minimum in the space $u(\cdot) \in L_1$, provided that there exist at least one admissible pair $(x(\cdot),\, u(\cdot))$, and the following conditions are satisfied for all $(t,\, x,\, u)$.

★ **Coercivity:** there exists a positive function $\theta : \mathbb{R}_0^+ \to \mathbb{R}^+$, such that

$$\lim_{r \to +\infty} \frac{\theta\,(r)}{r} = +\infty,$$

$$L\,(t,\, x,\, u) \geq \theta\,(\|\varphi(t,\, x,\, u)\|)\,,$$

$$\lim_{\|u\| \to +\infty} \|\varphi(t,\, x,\, u)\| = +\infty.$$

★ **Convexity:** $L\,(t,\, x,\, u)$ and $\varphi\,(t,\, x,\, u)$ are convex w.r.t. u.

Necessary Optimality Condition for (P)
Pontryagin Maximum Principle (PMP)

If $(x(\cdot), u(\cdot))$ is a minimizer of (P) and $u(\cdot) \in L_\infty,$ then

$\exists\ (\psi_0, \psi(\cdot)) \neq 0,\ \psi(\cdot) \in W_{1,1}$, s.t. $(x(\cdot), \psi_0, \psi(\cdot), u(\cdot))$ satisfies:

★ the **Hamiltonian system** $\dot{x} = \dfrac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\dfrac{\partial H}{\partial x}$, with the

Hamiltonian $H = \psi_0 L(t, x, u) + \langle \psi, \varphi(t, x, u) \rangle$;

★ The **maximality condition**

$$H(t, x(t), \psi_0, \psi(t), u(t)) = \sup_{u \in \mathbb{R}^m} H(t, x(t), \psi_0, \psi(t), u)$$

for almost all $t \in [a, b]$.

Discrepancy

Necessary Conditions $u(\cdot) \in L_\infty$	$\longleftrightarrow$	Existence Theorems $u(\cdot) \in L_1$

★ For minimizers predicted by existence theory, necessary optimality conditions may fail to be valid.

★ In the class of controls where optimality conditions are valid (measurable bounded controls) existence is not guaranteed.

★ **What can be done:** postulate conditions beyond those of the Existence Theorem, assuring that all minimizing controls are bounded.

Lipschitzian Regularity Conditions

Values of Optimal Controls can be unbounded and fail to satisfy the PMP

J. Ball & V. Mizel Problem (1985):

$$\int_0^1 \left(\left| x^3 - t^2 \right|^2 |u|^{14} + \varepsilon\, |u|^2 \right) dt \to \min$$

$$\dot{x}(t) = u(t)$$

$$x(0) = 0, \quad x(1) = k.$$

★ Hypotheses of Tonelli's Existence Theorem are satisfied

★ F. H. Clarke & R. B. Vinter proved that for certain choices of constants k and ε, we have the unique optimal control

$$u(t) = k\, t^{-1/3}$$

★ PMP (Euler-Lagrange eq. in integral form) is not satisfied since $\dot{\psi}(t) = L_x(t, x(t), \dot{x}(t)) = c\, t^{-4/3}$ is not integrable

Lipschitzian Regularity Conditions for the Basic problem of the Calculus of Variations

$$\int_a^b L\left(t,\, x\left(t\right),\, u\left(t\right)\right)\, dt \to \min, \quad \dot{x}\left(t\right) = u\left(t\right)$$

★ **L. Tonelli - C. B. Morrey:** $\|L_x\| + \|L_u\| \leq c\, |L| + r \quad (c > 0)$

★ **S. Bernstein**, $n = 1$; F. H. Clarke & R. B. Vinter, $n > 1$:

$$\left\| L_{uu}^{-1}\, \left(L_x - L_{ut} - L_{ux}\, u\right) \right\| \leq c \left(\|u\|^3 + 1 \right), \quad (L_{uu} > 0)$$

★ **F. H. Clarke & R. B. Vinter:**

- (P) autonomous: $L = L(x,\, u)$
- $|L_t| \leq c\, |L| + k\left(t\right), \quad (k\left(\cdot\right) \in L_1)$
- $\|L_x\| \leq c\, |L| + k\left(t\right)\, \|L_u\| + m\left(t\right), \quad (k\left(\cdot\right),\, m\left(\cdot\right) \in L_1)$

★ **R. B. Vinter:** $L\left(t, \cdot, \cdot\right)$ convex for each t

Lipschitzian Regularity for problems of the Calculus of Variations with higher-order derivatives

$$\int_a^b L\left(t, x\left(t\right), u\left(t\right)\right) dt \to \min, \quad \dot{x}\left(t\right) = A\, x\left(t\right) + B\, u\left(t\right)$$

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

★ **F.H. Clarke & R.B. Vinter (1990):** 'Tonelli-Morrey condition': $\quad \|L_x\| \leq c\, \left(L + \|u\|\right) + \gamma\left(t\right)\, r\left(x\right)$

$$\gamma(\cdot) \text{ integrable}, \quad r(\cdot) \text{ locally bounded}$$

Lipschitzian Regularity of Optimal Controls

F.H. Clarke & R.B. Vinter (1990): Tonelli-Morrey type condition for the Lagrange Problem of Optimal Control with linear time invariant dynamics:

$$\int_a^b L\left(t, x\left(t\right), u\left(t\right)\right) dt \to \min$$

$$\dot{x}\left(t\right) = A\, x\left(t\right) + B\, u\left(t\right) + d\left(t\right)$$

- The result is obtained reformulating the problem as a problem in the Calculus of Variations with higher-order derivatives.

Lagrange Problem of Optimal Control with control affine dynamics

$$\int_a^b L\left(t,\, x\left(t\right),\, u\left(t\right)\right)\, dt \to \min$$

$$\dot{x}\left(t\right) = \varphi\left(t,\, x\left(t\right),\, u\left(t\right)\right) := f\left(t,\, x\left(t\right)\right) + g\left(t,\, x\left(t\right)\right) u\left(t\right)$$

$$x\left(a\right) = x_a, \quad x\left(b\right) = x_b \qquad (P)$$

$$u\left(\cdot\right) \in L_1\left(\left[a,\, b\right];\, \mathbb{R}^m\right), \quad x\left(\cdot\right) \in W_{1,1}\left(\left[a,\, b\right];\, \mathbb{R}^n\right)$$

$$\begin{aligned} C^1 &\ni \quad L : \left[a,\, b\right] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \\ C^1 &\ni \quad f : \left[a,\, b\right] \times \mathbb{R}^n \to \mathbb{R}^n, \\ C^1 &\ni \quad g : \left[a,\, b\right] \times \mathbb{R}^n \to \mathbb{R}^{n\times m} \end{aligned}$$

Our approach to Lipschitzian Regularity

Step 1: Reduction to a time-optimal problem

We introduce a new time variable

$$\tau(t) = \int_a^t L(\theta, x(\theta), u(\theta))\, d\theta, \quad t \in [a, b]$$

and consider $t(\tau)$ (inverse function of $\tau(t)$) and $z(\tau) = x(t(\tau))$ as state trajectories; $v(\tau) = u(t(\tau))$ as the control. We obtain:

$$T \to \min,$$

$$\begin{cases} \dot{t}(\tau) = \dfrac{1}{L(t(\tau), z(\tau), v(\tau))} \\[2ex] \dot{z}(\tau) = \dfrac{\varphi(t(\tau), z(\tau), v(\tau))}{L(t(\tau), z(\tau), v(\tau))} \end{cases}, \qquad \begin{array}{c} v : \mathbb{R} \to \mathbb{R}^m \\ t(0) = a,\ t(T) = b \\ z(0) = x_a,\ z(T) = x_b \end{array}$$

Step 2: Compactification of the Control Set (following R. V. Gamkrelidze)

★ We compactify the space $\mathbb{R}^m$ of admissible controls, adding the point ∞: $\overline{\mathbb{R}^m} \approx S^m$

$$\pi : S^m \quad \to \quad \overline{\mathbb{R}^m} \qquad \text{(stereographic projection)}$$

★ Now $\left\{ \left(\dfrac{1}{L(t,\, z,\, v)}, \dfrac{\varphi(t,\, z,\, v)}{L(t,\, z,\, v)} \right) : v \in \overline{\mathbb{R}^m} \right\}$ is compact.

★ We define 'equivalent' (extended) optimal control problem:

$$T \to \min, \qquad \begin{cases} \dot{t}(\tau) = \dfrac{1}{L(t,\, z,\, \pi(w))} \\[2ex] \dot{z}(\tau) = \dfrac{\varphi(t,\, z,\, \pi(w))}{L(t,\, z,\, \pi(w))} \end{cases}, \quad w \in S^m$$

$$\boxed{u \text{ bounded} \Leftrightarrow v \text{ bounded} \Leftrightarrow w \neq \text{north pole}}$$

Step 3
Applicability of PMP to the extended problem

To apply the Pontryagin Maximum Principle to the compactified problem, we need to assure that the respective extension of the right-hand side is C^1 with respect to t and z. The only problem is the continuously differentiability at the north pole.

For that we require the following **growth** condition to be fulfilled:

$$\exists\, \gamma > 0,\ \ \beta < 2,\ \ \mu \geq \max\{\beta - 2, -2\} \ \text{ and } \eta \in \mathbb{R} \text{ s.t.}$$

$$\boxed{\left(|L_t| + \|L_x\| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_x - L_x\,\varphi\|\,\right)\ \|u\|^{\mu} \leq \gamma\, L^{\beta} + \eta}$$

Step 4: PMP and Lipschitzian Regularity

★ Maximality Condition:

$$0 \leq c = \sup_{v \in \overline{R^m}} \frac{\psi_t + \langle \psi_z, f(t, z) + g(t, z)\, v \rangle}{L(t, z, v)}$$

★ It turns out that $c = 0$ corresponds to an abnormal extremal control of the original problem. Thus, for minimizers which are not abnormal extremal controls, there must be $c > 0$.

★ From the coercivity condition

$$\lim_{\|v\| \to +\infty} \frac{\psi_t + \langle \psi_z, f(t, z) + g(t, z)\, v \rangle}{L(t, z, v)} = 0$$

$$\boxed{c > 0 \Longrightarrow \|v\| - \text{bounded}}$$

Main Theorem*

Under the hypotheses:

★ $g\,(t,\,x)$ has rank m for all t and x;

★ (coercivity) $\exists\ \theta : \mathbb{R}_0^+ \to \mathbb{R}$ and $\zeta \in \mathbb{R}$ s.t. for all $(t,\,x,\,u)$

$$L\,(t,\,x,\,u) \geq \theta\,(\|u\|) > \zeta\ ; \quad \textit{and} \quad \lim_{r \to +\infty} \frac{r}{\theta\,(r)} = 0;$$

★ (growth condition) $\exists$ constants $\gamma,\,\beta,\,\eta$ and μ, with $\gamma > 0,\ \beta < 2$ and $\mu \geq \max\{\beta - 2,\, -2\}$, s.t. for all $(t,\,x,\,u)$

$$\left(|L_t| + \|L_x\| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_x - L_x\,\varphi\|\,\right)\,\|u\|^\mu \leq \gamma\,L^\beta + \eta;$$

all minimizers $u\,(\cdot)$ of the Lagrange problem (P), which are not abnormal extremal controls, are essentially bounded on $[a,\,b]$.

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* *Lipschitzian Regularity of Minimizers for Optimal Control Problems with Control-Affine Dynamics*, Appl. Math. Optim., 41 (2000), pp. 237–254.

Corollaries to the Calculus of Variations
Corollary 1: Basic Problem

$$\boxed{\begin{array}{c}\displaystyle\int_a^b L(t,x,\dot{x})\,dt \to \min \\ x(\cdot)\in W_{1,1}\end{array}} \quad \Leftrightarrow \quad \boxed{\begin{array}{c}\displaystyle\int_a^b L(t,x,u)\,dt \to \min \\ \dot{x}=u, \quad u(\cdot)\in L_1\end{array}}$$

Growth Condition $(\gamma > 0, \quad \beta < 2, \quad \mu \geq \max\{\beta - 1, -1\})$

$$\boxed{(|L_t| + \|L_x\|)\ \|u\|^{\mu} \leq \gamma\, L^{\beta} + \eta}$$

implies that all minimizers $x(\cdot)$ pRedicted by Tonelli's Existence theorem are Lipschitzian:

$$\boxed{x(\cdot)\in W_{1,\infty}} \quad \Leftrightarrow \quad \boxed{u(\cdot)\in L_{\infty}}$$

Corollaries to the Calculus of Variations
Corollary 2: Higher-Order Derivatives

$$\boxed{\begin{gathered}\int_a^b L\left(t, x, \dot{x}, \ldots, x^{(m)}\right) dt \to \min \\ x(\cdot) \in W_{m,1}\end{gathered}} \quad \Leftrightarrow \quad \boxed{\begin{gathered}\int_a^b L\left(t, \xi, u\right) dt \to \min \\ \dot{\xi} = A\,\xi + B\,u \\ \xi(\cdot) \in W_{1,1},\ u(\cdot) \in L_1\end{gathered}}$$

Growth Condition $(\gamma > 0, \quad \beta < 2, \quad \mu \ge \max\{\beta - 1,\ -1\})$

$$\boxed{\begin{gathered}\left(|L_t| + \|L_{x^{(i)}}\|\right) \left\|x^{(m)}\right\|^{\mu} \le \gamma\, L^{\beta} + \eta \\ \forall\, i \in \{0,\ \ldots,\ m-1\}\end{gathered}} \quad \Leftrightarrow \quad \boxed{\begin{gathered}\left(|L_t| + \|L_{\xi}\|\right) \|u\|^{\mu} \le \\ \le \gamma\, L^{\beta} + \eta\end{gathered}}$$

implies that all minimizers $x(\cdot)$ predicted by Tonelli's Existence theorem belong to $W_{m,\infty}$: $\boxed{x(\cdot) \in W_{m,\infty}} \quad \Leftrightarrow \quad \boxed{u(\cdot) \in L_\infty}$

Final Remarks

Our approach:

- ✔ Is applicable to a broad class of problems of Optimal Control with control affine dynamics.
- ✔ Even for simpler dynamics, we are able to deal with examples which are not covered by previously known regularity conditions.

Much work remains to be done. E.g.:

- ✍ Problems with dynamics which is nonlinear in control.
- ➜ Problems with state constraints.
- ➜ Problems with multiple-integrals.