A Computer Algebra Package for Determining Symmetries and Conservation Laws in the Calculus of Variations

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Organization of the Presentation

Computer Algebra Systems (CAS)

Modern CAS are extremely versatile and powerful, allowing one to approach complex problems in a symbolic way

- They put at our disposal a big collection of mathematical knowledge, from many different areas
- They permit, through scientific programming languages of very high level, to enrich available mathematical knowledge

CAS potentialities are still for exploring...

- Many initiatives and projects are appearing. An example is the use of CAS in the resolution of concrete control problems: `http://anadrasis.math.auth.gr/cacsd/Home.htm`

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Resolution of problems of the Calculation of Variations

The dynamic optimization problems studied in the CV are usually solved with the help of Euler-Lagrange equations (EL)

- EL equations are differential equations of 2nd order (or higher)
- EL equations are, in general, nonlinear and difficult to solve

- A way of simplifying EL equations consists of obtaining first integrals or Conservation Laws (CL)
  - CL allow one to lower the order of the EL equations and, in extreme cases, to integrate them completely
- Given a problem of the CV, how to obtain CL?

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Noether's Theorem and our purpose

How to obtain Conservation Laws?

In 1918, Emmy Noether investigated this subject. She proved that if the integral functional one wants to minimize or maximize is invariant under a certain group of transformations (Symmetries), then there exist explicit formulas for the CL

- Noether's theorem associate the existence of CL to the existence of Variational Symmetries (VS).
- Difficulty: to obtain the VS

Our purpose

To show how the symbolic computational facilities of a CAS can be very useful in the Calculus of Variations (CV)

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The Higher-Order Calculus of Variations

Problem of Dynamic Optimization:

$$J[\mathbf{x}(\cdot)] = \int_a^b L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(m)}(t))\,\mathrm{d}t \rightarrow min \qquad (1)$$

We call Lagrangian to function $L : [a, b] \times \mathbb{R}^{n\times(m+1)} \rightarrow \mathbb{R}$

- $t \in \mathbb{R}$;
- $\mathbf{x}(t) = [x_1(t)\ x_2(t) \cdots x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$;
- $\mathbf{x}^{(i)}(t) = [\frac{\mathrm{d}^i x_1(t)}{\mathrm{d}t^i}\ \frac{\mathrm{d}^i x_2(t)}{\mathrm{d}t^i} \cdots \frac{\mathrm{d}^i x_n(t)}{\mathrm{d}t^i}]^{\mathrm{T}} \in \mathbb{R}^n$, $i = 1, \ldots, m$.

Problem (1) is usually solved with the help of Euler-Lagrange differential equations ($\frac{\partial L}{\partial \mathbf{x}} = [\frac{\partial L}{\partial x_1}\ \frac{\partial L}{\partial x_2}\ \cdots\ \frac{\partial L}{\partial x_n}]$):

$$\frac{\partial L}{\partial \mathbf{x}} + \sum_{i=1}^{m} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\frac{\partial L}{\partial \mathbf{x}^{(i)}}\right) = \mathbf{0}.$$

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Conservation Laws

A function $t \to \phi(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(k)}(t))$, $k < 2m$, preserved along all the solutions of the EL eq, is called a first integral. Equation $\phi(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(k)}(t)) = \text{const}$ is called a Conservation Law (CL).

- CL allow one to lower the order of the EL equations;
- With a sufficiently high number of independent CL, it is even possible to explicitly determine the extremals.

Example ($n = m = 1$)

$$J[x(\cdot)] = \int_a^b \left(\dot{x}^2(t) - x^2(t)\right) \mathrm{d}t \longrightarrow \min$$

$$\begin{cases} \dot{x}(t)\cos(t) + x(t)\sin(t) & = c_1 \\ -\dot{x}(t)\sin(t) + x(t)\cos(t) & = c_2 \end{cases} \Rightarrow x(t) = c_1 \sin(t) + c_2 \cos(t) .$$

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Variational Symmetries

Invariance Definition

The problem of the CV is said to be invariant under a one-parameter group of transformations $h^s(t, \mathbf{x}) = (t^s, \mathbf{x}^s) : [a, b] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, if:

$$\int_\alpha^\beta L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(m)}(t))\, \mathrm{d}t = \int_{\alpha^s}^{\beta^s} L(t^s, \mathbf{x}^s(t^s), \dot{\mathbf{x}}^s(t^s), \ldots, \mathbf{x}^{s(m)}(t^s))\, \mathrm{d}t^s,$$

in any subinterval $[\alpha, \beta] \subseteq [a, b]$.

- In the conditions of the previous definition, the one-parameter transformations $h^s(t, \mathbf{x})$ constitute a variational symmetry of the integral functional to minimize.

Necessary and Sufficient Condition of Invariance

Theorem (Torres'2004)

The higher-order problem of the CV is invariant under a one-parameter group of transformations $h^s(t, \mathbf{x})$ if, and only if,

$$\frac{\partial L}{\partial t} T \quad + \quad \sum_{i=0}^{m} \frac{\partial L}{\partial \mathbf{x}^{(i)}} \cdot \mathbf{p}^i + L \frac{\mathrm{d}T}{\mathrm{d}t} = 0,$$

$$\begin{aligned} \textit{with} \quad \mathbf{p}^0 &= \mathbf{X}, \\ \mathbf{p}^{i+1} &= \frac{\mathrm{d}\mathbf{p}^i}{\mathrm{d}t} - \mathbf{x}^{(i+1)} \frac{\mathrm{d}T}{\mathrm{d}t}, \; i = 0, \ldots, m-1, \end{aligned}$$

where $T \equiv T(t, \mathbf{x})$ and $\mathbf{X} \equiv [X_1(t, \mathbf{x}) \; X_2(t, \mathbf{x}) \cdots X_n(t, \mathbf{x})]^T$ are the so-called infinitesimal generators: $T(t, \mathbf{x}) = \left.\frac{\partial h_t^s}{\partial s}\right|_{s=0}$, $X_i(t, \mathbf{x}) = \left.\frac{\partial h_{x_i}^s}{\partial s}\right|_{s=0}$, $i = 1, \ldots, n$.

Expanding the total derivatives, one gets:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}},$$

$$\frac{\mathrm{d}\mathbf{p}^i}{\mathrm{d}t} = \frac{\partial \mathbf{p}^i}{\partial t} + \sum_{k=0}^{i} \frac{\partial \mathbf{p}^i}{\partial \mathbf{x}^{(k)}} \cdot \mathbf{x}^{(k+1)},$$

$$\textit{with} \quad \frac{\partial \mathbf{p}^i}{\partial \mathbf{x}^{(k)}} = \left[\frac{\partial \mathbf{p}^i}{\partial x_1^{(k)}} \; \frac{\partial \mathbf{p}^i}{\partial x_2^{(k)}} \; \cdots \; \frac{\partial \mathbf{p}^i}{\partial x_n^{(k)}} \right] = \begin{bmatrix} \frac{\partial p_1^i}{\partial x_1^{(k)}} & \frac{\partial p_1^i}{\partial x_2^{(k)}} & \cdots & \frac{\partial p_1^i}{\partial x_n^{(k)}} \\ \frac{\partial p_2^i}{\partial x_1^{(k)}} & \frac{\partial p_2^i}{\partial x_2^{(k)}} & \cdots & \frac{\partial p_2^i}{\partial x_n^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_n^i}{\partial x_1^{(k)}} & \frac{\partial p_n^i}{\partial x_2^{(k)}} & \cdots & \frac{\partial p_n^i}{\partial x_n^{(k)}} \end{bmatrix}.$$

Corollary

Corollary for the fundamental problem of the CV ($m = 1$)

The necessary and sufficient condition takes the well-known form:

$$\frac{\partial L}{\partial t} T + \frac{\partial L}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \left(\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} - \dot{\mathbf{x}} \frac{\mathrm{d}T}{\mathrm{d}t} \right) + L \frac{\mathrm{d}T}{\mathrm{d}t} = 0,$$

$$\textit{with} \quad \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{\partial \mathbf{X}}{\partial t} + \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}}$$

- As we will see, Noether's theorem asserts that for the determination of the Conservation Laws one just need to know the infinitesimal generators T and $\mathbf{X}$.

The presented theorem, besides serving as a test to the existence of symmetries, establish an algorithm for the determination of the correspondent infinitesimal generators T and $\mathbf{X}$.

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Algorithm for obtaining the Variational Symmetries

The necessary and sufficient invariance condition is a differential equation in the $n+1$ unknown functions T, X_1, X_2, $\ldots$, X_n (the generators), that we intend to determine:

1. Substitute Lagrangian L and its partial derivatives with the respective values, obtaining a polynomial in the $n \times m$ variables $\dot{x}_1, \ldots, \dot{x}_n$, $x_1^{(2)}, \ldots, x_n^{(2)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}$.
2. Solve the system of differential equations that is obtained by placing all the coefficients of the polynomial equal to zero.

- The number of terms of the obtained polynomial can be greater than the number of unknowns in the problem $(= n+1) \Rightarrow$ not all functionals of the CV admit symmetries;
- Application of the algorithm, namely when one works with n and m greater than 1, involves a very high number of analytic calculations... $\rightsquigarrow$ better to use a CAS!

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Noether's Theorem – Obtaining the Conservation Laws

Noether's Theorem (Torres'2004)

If the higher-order problem of the CV is invariant under a one-parameter group of transformations with infinitesimal generators T and $\mathbf{X}$, then

$$\sum_{i=1}^{m} \mathbf{\Psi}^i \cdot \mathbf{p}^{i-1} + \left(L - \sum_{i=1}^{m} \mathbf{\Psi}^i \cdot \mathbf{x}^{(i)} \right) T = \text{const},$$

$$\text{with} \quad \mathbf{\Psi}^m = \frac{\partial L}{\partial \mathbf{x}^{(m)}},$$

$$\mathbf{\Psi}^{i-1} = \frac{\partial L}{\partial \mathbf{x}^{(i-1)}} - \frac{\mathrm{d}\mathbf{\Psi}^i}{\mathrm{d}t}, \quad i = m, m-1, \ldots, 2$$

$$\frac{\mathrm{d}\mathbf{\Psi}^i}{\mathrm{d}t} = \frac{\partial \mathbf{\Psi}^i}{\partial t} + \sum_{k=0}^{2m-i} \left(\mathbf{x}^{(k+1)} \right)^{\mathrm{T}} \cdot \frac{\partial \mathbf{\Psi}^i}{\partial \mathbf{x}^{(k)}},$$

Noether's Theorem – Obtaining the Conservation Laws

$$\text{where} \quad \frac{\partial \boldsymbol{\Psi}^i}{\partial \mathbf{x}^{(k)}} = \begin{bmatrix} \frac{\partial \boldsymbol{\Psi}^i}{\partial x_1^{(k)}} \\ \frac{\partial \boldsymbol{\Psi}^i}{\partial x_2^{(k)}} \\ \vdots \\ \frac{\partial \boldsymbol{\Psi}^i}{\partial x_n^{(k)}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi_1^i}{\partial x_1^{(k)}} & \frac{\partial \psi_2^i}{\partial x_1^{(k)}} & \cdots & \frac{\partial \psi_n^i}{\partial x_1^{(k)}} \\ \frac{\partial \psi_1^i}{\partial x_2^{(k)}} & \frac{\partial \psi_2^i}{\partial x_2^{(k)}} & \cdots & \frac{\partial \psi_n^i}{\partial x_2^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \psi_1^i}{\partial x_n^{(k)}} & \frac{\partial \psi_2^i}{\partial x_n^{(k)}} & \cdots & \frac{\partial \psi_n^i}{\partial x_n^{(k)}} \end{bmatrix}$$

The Theorem of E. Noether establish a relationship between existence of Symmetries and the existence of Conservation Laws

Corollary for the fundamental problem of the CV ($m = 1$)

Previous CL takes the well-known form found in the literature:

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \mathbf{X} + \left(L - \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\mathbf{x}} \right) T = \text{const.}$$

Noether's Theorem – Obtaining the Conservation Laws

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Automatic determination of Conservation Laws

The CL are given by Noether's Theorem, using the generators T and $\mathbf{X}$ found by the method already described.

Summarizing

Given a problem of the CV, we obtain CL through a two-stage process:

1. We use the necessary and sufficient invariance condition to obtain all possible symmetries of the problem;
2. Next we use Noether's theorem to obtain the correspondent conservation laws.

Our contribution in this work

We define, using Maple CAS, function Symmetry that returns all possible symmetries of the problem; function Noether that calculates correspondent conservation laws.

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Maple Procedure: Symmetry

Symmetry(L, t, x, x1, x2, ..., xm)

Determines the symmetries of a Lagrangian of *n* dependent variables with higher-order derivatives.

L - Lagrangian expression;

t - name of the independent variable;

x - list of names of the dependent variables;

xi - list of names of ith order derivatives of dependent variables;

- Return: group of infinitesimal generators of the VS

Maple Procedure: Noether

Noether(L, t, x, x1, x2, ..., xm, S)

Given the infinitesimal generators, for a Lagrangian of several dependent variables and with derivatives of higher order, it gives the Conservation Law.

- L - Lagrangian expression;
- t - name of the independent variable;
- x - list of names of the dependent variables;
- xi - list of names for the ith derivatives of the dependent variables;
- S - set of infinitesimal generators of the VS (output of the procedure *Symmetry*).

- Return: conservation laws.

Maple Procedure: EulerLagrange

(Auxiliary procedure)

EulerLagrange(L, t, x, x1, x2, ..., xm)

Determines the system of Euler-Lagrange equations of a higher-order problem of the CV with several dependent variables.

- L - Lagrangian expression;
- t - name of the independent variable;
- x - list of names of the dependent variables;
- xi - list of names of ith order derivatives of the dependent variables;

- Return: system of Euler-Lagrange equations.

Example 1 ($n = m = 1$)

$L(t, x, \dot{x}) = t\dot{x}^2$

```
> L:= t*v^2;
```

$$L := tv^2$$

```
> Symmetry(L,t,x,v);
```

$$\{T(t,x) = (2\_C1 \ln(t) + \_C3)\, t,\ X(t,x) = \_C1\, x + \_C2\}$$

```
> subs(_C1=1,_C2=0,_C3=0,%);
```

$$\{T(t,x) = 2 \ln(t)\, t,\ X(t,x) = x\}$$

```
> CL := Noether(L,t,x,v,%);
```

$$CL := x(t)\, t \frac{d}{dt} x(t) - t^2 \left(\frac{d}{dt} x(t)\right)^2 \ln(t) = const$$

It is easy to verify the validity of the obtained CL...

```
> EulerLagrange(L,t,x,v);
```

$$\left\{-2\,\frac{d}{dt}x\left(t\right)-2\,t\,\frac{d^2}{dt^2}x\left(t\right)=0\right\}$$

```
> dsolve(%);
```

$$\{x(t)=\_C1+\_C2\ln(t)\}$$

```
> expand(subs(%,CL));
```

$$\_C2\_C1=const$$

As expected, substituting the extremals in the CL results in a true proposition.

Example 2 ($n = m = 2$)

$L(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \dot{x}_1^2 + \ddot{x}_2^2$

```
> L:=v[1]^2+a[2]^2;
```

$$L := {v_1}^2 + {a_2}^2$$

```
> Symmetry(L, t, [x[1],x[2]], [v[1],v[2]], [a[1],a[2]]);
```

$$\begin{aligned} \{\ T(t, x_1, x_2) &= \_C1\ t + \_C2, \\ X_1(t, x_1, x_2) &= 1/2\ \_C1\ x_1 + \_C5, \\ X_2(t, x_1, x_2) &= 3/2\ \_C1\ x_2 + \_C3\ t + \_C4\ \} \end{aligned}$$

```
> CL:=Noether(L, t, [x[1],x[2]], [v[1],v[2]], [a[1],a[2]], %);
```

$$
\begin{aligned}
CL := \quad & 2\,(1/2\,\_C1\,x_1(t) + \_C5)\,\frac{d}{dt}x_1(t) \\
- \quad & 2\,(3/2\,\_C1\,x_2(t) + \_C3\,t + \_C4)\,\frac{d^3}{dt^3}x_2(t) \\
+ \quad & 2\left(\_C3 + 1/2\,\_C1\,\frac{d}{dt}x_2(t)\right)\frac{d^2}{dt^2}x_2(t) \\
+ \quad & \left(-\left(\frac{d}{dt}x_1(t)\right)^2 - \left(\frac{d^2}{dt^2}x_2(t)\right)^2\right. \\
+ \quad & \left. 2\,\frac{d}{dt}x_2(t)\,\frac{d^3}{dt^3}x_2(t)\right)(\_C1\,t + \_C2) \\
= \quad & const
\end{aligned}
$$

The obtained CL is an equation of 3rd order. The EL equation is, in this case, a system of two differential equations of order 4.

Let us verify the validity of the CL just obtained...

```
> EulerLagrange(L, t, [x[1],x[2]], [v[1],v[2]], [a[1],a[2]]);
```

$$\left\{-2\,\frac{d^2}{dt^2}x_1(t)=0,\ 2\,\frac{d^4}{dt^4}x_2(t)=0\right\}$$

```
> dsolve(%);
```

$$\begin{aligned}\{\, x_1(t) &= \_C1\, t + \_C2,\\ x_2(t) &= 1/6\, \_C3\, t^3 + 1/2\, \_C4\, t^2 + \_C5\, t + \_C6 \,\}\end{aligned}$$

```
> expand(subs(%,CL));
```

$$2\,\_C1\,\_C5 - 3\,\_C3\,\_C1\,\_C6 + \_C1\,\_C5\,\_C4 - \_C4^2\,\_C2 + 2\,\_C3\,\_C5\,\_C2 = const$$

Substituting the extremals in the CL it results, as expected, a true proposition.

Conclusion

1. Starting from the classical results of Emmy Noether of the CV, it is possible to develop a systematic method to identify Symmetries and Conservation Laws;
2. We have used the scientific computational system `Maple 9` to define new functions that automatically determine the Symmetries and Conservation Laws of the Calculation of the Variations;
3. The new functionalities are of great practical usefulness.

Future Work

We intend to generalize our "Maple package" to Optimal Control

- The resolution of OC problems is usually based on the application of the Pontryagin Maximum Principle. The differential equations one gets are, in general, non-linear and difficult to solve...
- Emmy Noether's classical results can be generalized to the Optimal Control setting (Torres'2002)

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References

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